Barycentric Coordinates

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October 23, 2012

Much of this lecture is drawn from the significantly more extensive article *Barycentric Coordinates in Olympiad Geometry*, which is [1] in the references. It is a featured article on AoPS and can be found at
http://www.artofproblemsolving.com/Resources/Papers/Bary_full.pdf.

1 Introduction

Cartesian coordinates are great: they allow you to translate an arbitrarily complicated geometry problem into a straightforward (albeit tedious) computation. However, the reference frame of Cartesian coordinates is a bit weird: you start with a pair of perpendicular lines and build everything from there. This works fine for certain problems (USAMO 2012/5, anyone?), but is otherwise awkward.

Barycentric coordinates follow the same theme, but are based off an arbitrary triangle, rather than a pair of perpendicular lines. The three vertices of the triangle play the role of the origin, and the sides of the triangle play the role of the axes.

In this lecture, we will present these tools "on the fly" as they appear in a single problem.

2 The Problem

(Almost TSTST 2012) Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. If N is the midpoint of segment PQ, show that $MN \parallel AD$.

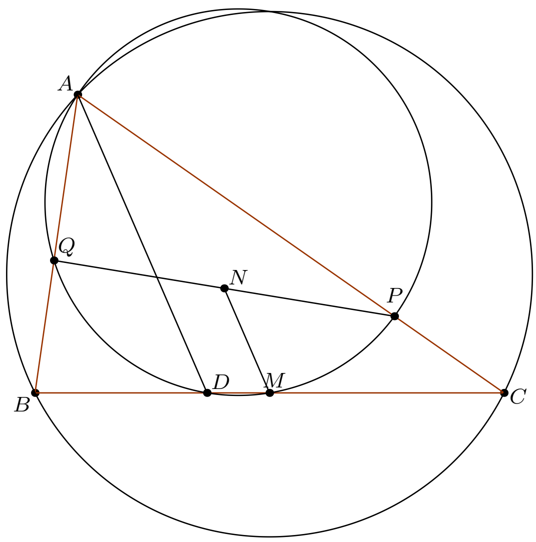

Figure 1: The Big Diagram

3 Theory & Technique

Fix $\triangle ABC$, labelled counterclockwise, and let $a = BC$, $b = CA$, $c = AB$. We'll let $[\mathcal{P}]$ denote the area of a polygon $\mathcal{P}$.

3.1 Setting the Stage

In barycentric coordinates, points are defined with three coordinates (instead of just two), but these coordinates must sum to 1. The point $P = (x, y, z)$ corresponds to the sum

$$\vec{P} = x\vec{A} + y\vec{B} + z\vec{C}.$$

Of course, we immediately have $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$. Because $x + y + z = 1$, the choice of the zero vector doesn't matter.

In Cartesian coordinates, the equation of a line is of the form $Ax + By + C = 0$. The analogous form in barycentric coordinates is as follows.

Theorem 1 (Line). *The equation of a line in barycentric coordinates is $ux+vy+wz = 0$, where u, v, w are real numbers, unique up to scaling.*

Question. What does the line $z = 0$ correspond to?

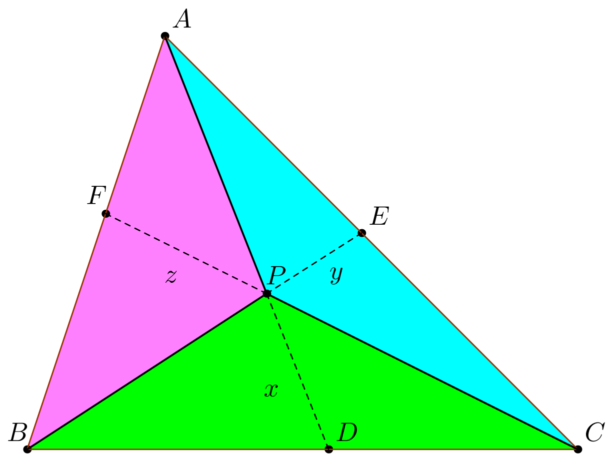

Figure 2: Areas and barycentrics

There's a second (equivalent) definition we can use: if $P = (x, y, z)$, then P is the point for which the areas[1] $[PBC]$, $[PCA]$, and $[PAB]$ are in the ratio $x : y : z$. For this reason, barycentric coordinates are sometimes called *areal coordinates*. See figure 2.

3.2 The Centroid

The first thing we need to find in our original problem is the coordinates of the point M. While we're here, we will also look at the other two midpoints, and the centroid, denoted G.

Question. Determine the coordinates of the midpoints of BC, CA and AB.

Question. Determine the equation of the medians.

[1]These areas are signed, so when P lies on the opposite side as A of BC, the area $[PBC]$ is considered negative.

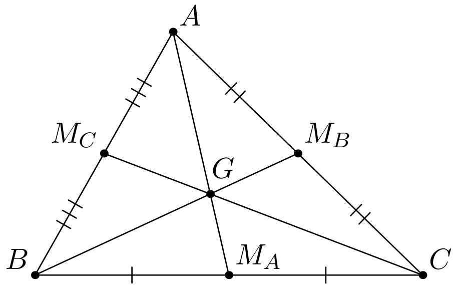

Figure 3: The centroid G of $\triangle ABC$. AM_A, BM_B and CM_C are medians.

Question. Determine the coordinates of the centroid G.

Notice that we managed to establish the existence of the point G almost by accident; these same ideas can be modified to yield a proof of Ceva's Theorem.

3.3 The Incenter

The next point we need to find the coordinates of is the point D. Along the way, we'll derive the angle bisector theorem.[2]

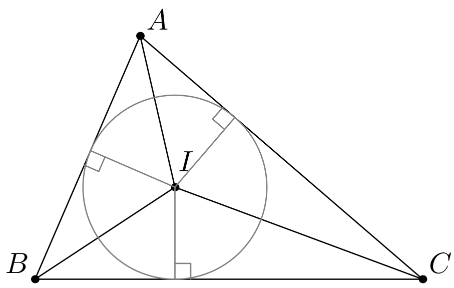

Figure 4: The incenter I of $\triangle ABC$. AI, BI and CI are angle bisectors.

Question. Show that $[IBC] : [ICA] : [IAB] = a : b : c$.

Using this, we can write $I = \left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right)$. At this point, writing the denominators is getting cubersome, so we will abbreviate this as just $I = (a : b : c)$. More generally, We will let $(x : y : z)$, where $x+y+z \neq 0$, denote the point $\left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z}\right)$.

Question. Can we plug $(a : b : c)$ instead of $\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right)$ into the line formula (Theorem 1) and still get valid results?

Question. What are the coordinates of point D?

Question. Deduce the angle bisector theorem from the coordinates of D.

3.4 The Circle

The next thing we need to do is find the equation of the circumcircle of $\triangle ADM$. Let's call it ω.

In Cartesian Coordinates, the general equation of a circle is $x^2+y^2+Ax+By+C = 0$. The analog in barycentric coordinates is as follows:

[2] Conversely, if you know the angle bisector theorem, you can find the coordinates of D pretty quickly. But shh!

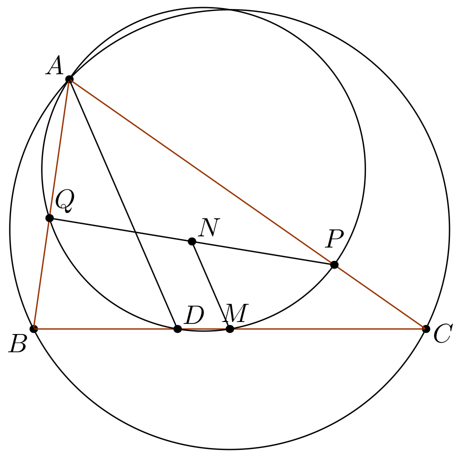

Figure 5: The Diagram Again

Theorem 2 (Circle)**.** *The equation of a circle in barycentric coordinates has the form*

$$-a^2yz - b^2zx - c^2xy + (x + y + z)(ux + vy + wz) = 0$$

Like its Cartesian analog, this isn't in general much fun to use. Thankfully, in barycentric coordinates, several terms die if any of $\{x, y, z\}$ are zero.

Question. Show that if a circle passes through $A = (1, 0, 0)$, then $u = 0$.

Question. Again, are we allowed to plug in $(a : b : c)$ instead of $\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right)$ into the circle formula?

We can get the equation of ω just as we might in Cartesian coordinates. We know from our earlier work that it passes through the points $A = (1 : 0 : 0)$, $M = (0 : 1 : 1)$ and $D = (0 : b : c)$. If we plug each of these in, we'll get three (linear) equations in three unknowns u, v, w

Question. Show that $u = 0$ and that

$$v = \frac{a^2c}{2(b+c)}, \quad w = \frac{a^2b}{2(b+c)}.$$

3.5 The Points P and Q

Question. Find two constraints on the coordinates of Q, and use these to solve for Q.

Question. Find the coordinates of P. (Hint: use symmetry!)

3.6 Finding N

To take the midpoint of two points, simply normalize them, and take the average of their coordinates. This is valid by the vector definition $x\vec{A} + y\vec{B} + z\vec{C}$.

Question. Why is it important that the coordinates be normalized?

Question. What is N?

3.7 The Area Formula

We're almost done now, because we've obtained the coordinates of all the points. All that's left to do is show that $MN \parallel AD$. But how do we do this? We need one last tool, the area formula.[3]

Theorem 3 (Area Formula)**.** *Let $P_i = (x_i, y_i, z_i)$ for $i = 1, 2, 3$. Then the area of $\triangle P_1P_2P_3$ is given by*

$$[P_1P_2P_3] = [ABC] \cdot \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

This area is positive when P_1, P_2 and P_3 are labelled counterclockwise, and negative otherwise.

Question. Is it important that the P_i are normalized?

Question. We're trying to prove that $MN \parallel AD$. What does this have to do with areas?

Question. Show that the problem is solved upon verifying

$$\begin{vmatrix} \frac{a^2}{4bc} & \frac{1}{2} - \frac{a^2b}{4bc(b+c)} & \frac{1}{2} - \frac{a^2c}{4bc(b+c)} \\ 0 & b & c \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & b & c \\ 1 & 0 & 0 \end{vmatrix}.$$

4 Summary

Here's a recap of the formulae that appeared:

	Cartesian	Barycentric
Definition	If $P = (x, y)$, then $\vec{P} = x\hat{i} + y\hat{j}$.	If $P = (x, y, z)$, where $x + y + z = 1$, then $\vec{P} = x\vec{A} + y\vec{B} + z\vec{C}$.
Origin	Arbitrary point $(0, 0)$.	Vertices of triangle $(1, 0, 0)$ etc.
Axes	Two perpendiculars $x, y = 0$.	Sides of triangle $x, y, z = 0$
Line	$Ax + By + C = 0$	$ux + vy + wz = 0$
Circle	$x^2 + y^2 + Ax + By + C = 0$	$-a^2yz - b^2zx - c^2xy$ $+ (x + y + z)(ux + vy + wz) = 0$
Area	$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$	$[ABC] \cdot \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

Table 1: Cartesian vs. Barycentric

[3] Can you show that this theorem implies the form for the equation of a line?

5 Additional Problems

1. (Mongolia TST 2000/6) In a triangle ABC, the angle bisector at A,B,C meet the opposite sides at A_1,B_1,C_1, respectively. Prove that if the quadrilateral $BA_1B_1C_1$ is cyclic, then
$$\frac{AC}{AB+BC} = \frac{AB}{AC+BC} + \frac{BC}{BA+AC}.$$

2. (USA TST 2012) In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that if P moves along side BC, the circumcircle of triangle ADE passes through a fixed point other than A.

3. (USA TST 2003/2) Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, PC intersect sides BC, CA, AB at D, E, F, respectively. Prove that
$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$$
if and only if P lies on at least one of the medians of triangle ABC. (Here $[XYZ]$ denotes the area of triangle XYZ.)

4. (Saudi Arabia TST 2012) Let $ABCD$ be a convex quadrilateral such that $AB = AC = BD$. The lines AC and BD meet at point O, the circles ABC and ADO meet again at point P, and the lines AP and BC meet at point Q. Show that $\angle COQ = \angle DOQ$.

References

[1] Max Schindler and C. "Barycentric Coordinates in Olympiad Geometry." `http://www.artofproblemsolving.com/Resources/Papers/Bary_full.pdf`.

[2] Zachary Abel. "Barycentric Coordinates." `http://zacharyabel.com/papers/Barycentric_A07.pdf`.