

A Guessing Game: Mixtilinear Incircles

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Sometimes figuring out what to prove is harder than actually proving it!

An important skill for olympiad geometers is to be able to guess when three points are collinear, four points are concyclic, three lines are concurrent, and so on. Difficult geometry problems (on the level of IMO 3/6) often amount to finding two or three critical claims; each of these claims may be no harder to prove than an IMO 1/4, but making the right guesses of *what* to prove can turn out to be the core difficulty of the problem. For a fantastic example, see my solution to IMO 2011/6.

In this exercise, I'll put write down a configuration of several points, lines, and circles. Your job is to find as many "coincidences" as you can: nontrivial collinearities, equal angles, cyclic quadrilaterals, and so on and so forth. Then, see if you can prove them!

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1 The Configuration

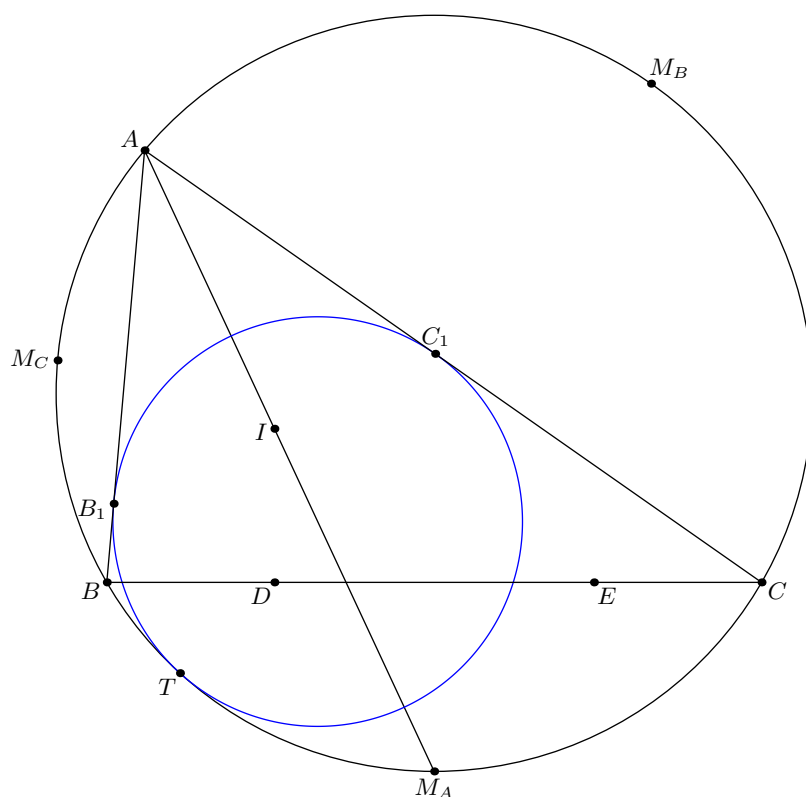
Let ABC be an acute triangle with incenter I and circumcircle Γ , and let D and E be the contact points of the incircle and A -excircle on BC . Let M_A, M_B, M_C be the midpoints of the arcs BC, CA, AB of Γ .

The **A -mixtilinear incircle** is the circle ω_A which is tangent to AB, AC, Γ at points B_1, C_1, T .

On the next page there is a bare-bones diagram with all these points, as well as some hints to get you started. However, before using it, **I encourage you to try and find as many things as you can using your own ruler and compass**. On an olympiad, you do not have the luxury of referring to a perfect, computer-drawn diagram! You can (and should) use more than one hand-drawn diagram.

2 Some Hints

Now that you've taken a look with the hand-drawn diagram, see if you can spot even more things in the following accurate computer-generated figure.

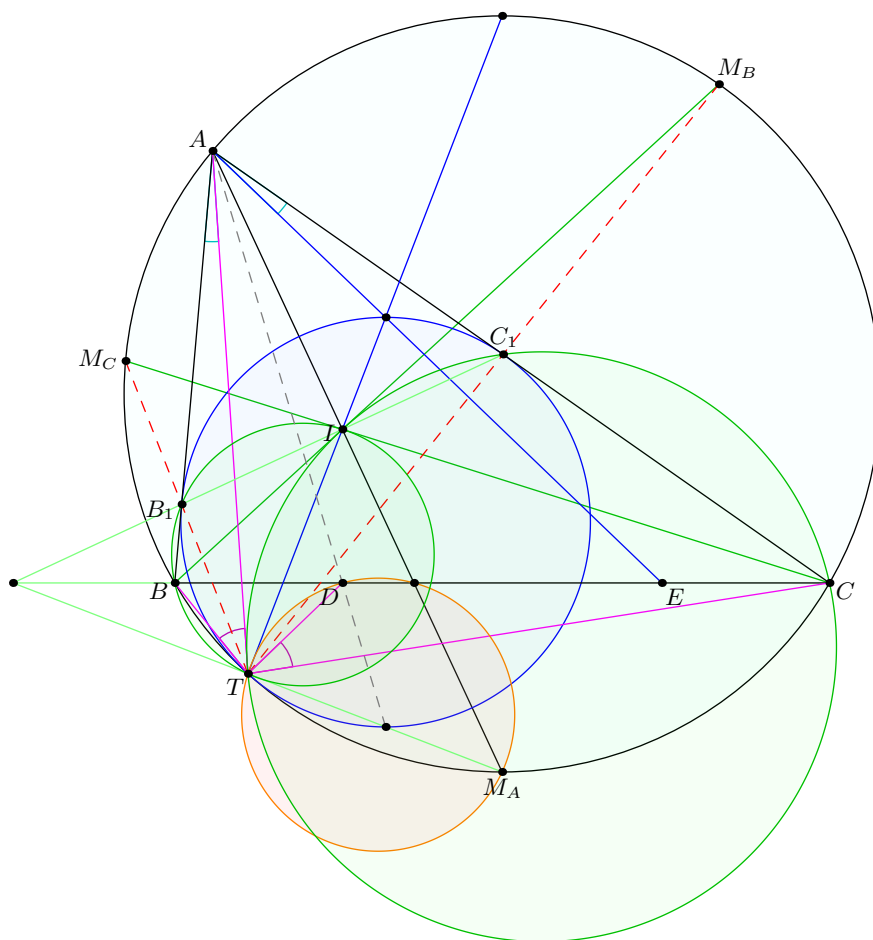


Here are some possible hints for things you could look for:

- There are at least three nontrivial collinearities among the labelled points.
- There are at least two nontrivial cyclic quadrilaterals among the labelled points.
- There are several nontrivial pairs of equal angles among the labelled points.
- There is at least one set of concurrent lines (which meet outside the triangle).
- Look at the “top” of the circumcircle.
- There are at least two lines tangent to some circumcircles.

My list of properties has ten items. When you want to see my answers, turn the page.

3 Answers



1. Points T , B_1 , M_C are collinear. Similarly, points T , C_1 , M_B are collinear.
2. Point I is the midpoint of B_1C_1 .
3. Ray TI passes through the “topmost” points of both ω_A and Γ (the point opposite M_A on Γ). In particular, AE and TI meet on ω_A .
4. $\angle BAT = \angle CAE$, and equivalent angles.
5. $\angle ATB = \angle CTD$, and equivalent angles.
6. $\angle ATM_C = \angle M_BTI$ (not shown in figure).
7. Quadrilaterals BB_1IT and CC_1IT are cyclic.
8. Lines BI and CI are tangents to these quadrilaterals the previously mentioned quadrilaterals.
9. The intersection of lines AI and BC lies on the circumcircle of $\triangle DTM_A$.
10. Lines BC , TM_A , B_1C_1 are concurrent.
11. Lines TM_A and AD meet on the mixtilinear incircle. (Thanks to my student H.W. for finding this one!)

This list is by no means exhaustive — there are more properties buried in here that I haven’t mentioned.

4 Sketches of Solutions

1. Consider the homothety at T sending ω_A to Γ .
2. You can use Pascal's Theorem on Γ . (The special case when $AB = AC$ was IMO 1978).
3. In $\triangle TB_1C_1$, TA is a symmedian and TI is a median. (This was given as a problem on Iran 2002.)
4. Inversion at A . (This was EGMO 2013, Problem 2.)
5. Reflect T across the perpendicular bisector of BC .
6. This just follows by symmedians.
7. Angle chasing (using above properties).
8. Angle chasing (using above properties).
9. Some more angle chasing, using $\angle ATB = \angle CTD$.
10. Use Pascal on Γ once again.
11. Two homotheties.

For more detailed discussion, you might consult <http://usamo.wordpress.com/2015/08/11/the-mixtilinear-incircle>.

Want to try this again? See what coincidences you can find if you draw in all three mixtilinear incircles. For example, what are their radical axes? Can you find some concurrent cevians?