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where we have left out the equations saying that $c_0 = c_2 = 0$. The next step is to compute a Gröbner base for (16). Since the indeterminates a, b enter into the coefficients we may to regard them as variables as well. We choose the ranking

$$\omega \prec a \prec b \prec c_1 \prec c_{3i} \prec c_{3r}$$

The important thing is that ω and a are the lowest ranked variables; then we will obtain an equation in ω and a only as an element of the Gröbner base. To make computations simpler we first note that f_1 has a factor c_1; we are not interested in solutions with $c_1 = 0$ so we divide f_1 by c_1. For the same reason we may divide f_2 by $ac_1\omega$. Having done this we use Maple to compute a Gröbner base $\mathbf{G}$ for the modified system. $\mathbf{G}$ contains 7 polynomials which are all quite large (42 lines of Maple output altogether) so we do not give it explicitly here. It contains one polynomial in ω only. If we substitute $z = \omega^2, K = a^2$ this polynomial is:

$$\begin{gathered} 3249\, z^5 + (549K - 6213)\, z^4 + (3754 - 567K)\, z^3 + \\ (159K - 842)\, z^2 + (53 - 13K)\, z - 1 \end{gathered} \tag{17}$$

A fifth order equation in z where K enters linearly in the coefficients. One can plot a root locus for this equation w.r.t. K. From it one can draw the conclusion that for $a = 0$ there are three real roots (in z) while any positive a gives us only one solution. This solution starts in 1 and moves towards the origin as a increases. We may expand the solution as a function of a^2 around 1 giving

$$\omega^2 \;=\; 1 + \alpha_1 a^2 + \alpha_2 a^4 + \alpha_3 a^6 + \ldots \tag{18}$$

The coefficients α_i can be determined by differentiating (17) w.r.t. a^2. This is quite easy to do in Maple, for instance, rendering

$$\alpha_1 \;=\; -\frac{1}{8}, \quad \alpha_2 \;=\; \frac{25}{256}, \quad \alpha_3 \;=\; -\frac{625}{4096}, \quad \ldots \tag{19}$$

$\square$

5 Other Applications.

Differential algebra is a field of mathematics attracting a lot of interest among nonlinear control theorists currently, see [8] and [11]. There are some close connections between the basic concepts of constructive differential algebra and Gröbner bases. See Forsman [9].

Gröbner bases have been used to see which state space models that are *input-output equivalent*, see e.g. Walter & al. [16].

Some *discrete event systems* can be modeled using multivariate polynomials over finite fields. Gröbner bases have already been used successfully in this theory, see Benvéniste et al. [2].

In *robotics* problems of geometric character can be solved using Gröbner bases; see [5] and [4].

It also possible to solve or prove the unsolvability of quite general control *design problems for linear SISO systems* with the help of Gröbner bases (Forsman [9]).

4 Finding Periodic Solutions with Gröbner Bases.

A commonly used method for finding periodic solutions of nonlinear differential equations is the method of harmonic balancing, described for instance in [1] and [12]. The basic idea is the following. Let there be given a differential equation on the form

$$f(\frac{d^n y}{dt^n}, \ldots, \frac{dy}{dt}, y, t) = 0 \tag{10}$$

in the unknown $y(t)$. Expand a candidate solution in Fourier series:

$$y(t) \sim \sum_{k=-\infty}^{\infty} c_k e^{-ik\omega t}, \qquad c_k \in \mathbb{C} \tag{11}$$

and evaluate (10) for this $y(t)$. This renders an infinite set of equations in the c_k and in ω. If we truncate the series (11)

$$y(t) = \sum_{k=-N}^{N} c_k e^{-ik\omega t} \tag{12}$$

and use the fact that $\bar{c}_k = c_{-k}$ we may represent (10) as a finite system of $N+1$ (complex) equations in $c_1, \ldots, c_N$ and ω

$$p_0(c_1, \ldots, c_N, \omega), \ \ldots, \ p_N(c_1, \ldots, c_N, \omega) \tag{13}$$

With this truncation the method is a special case of the Galërkin method for Banach spaces and is analyzed in [15]. It can be shown that if there is a periodic solution of the form (12) then, under certain conditions, there is a periodic solution to the original problem approximated by (12), see e.g. Glad and Ståhl [10].

If f is polynomial, then so are the equations (13). Often, one is particularly interested in the oscillation frequency ω. To this end we compute a Gröbner base with ω as the lowest ranking variable.

We illustrate the idea by applying the method to the famous van der Pol equation. This equation has been treated extensively in the literature, see [15].

Example 4.1 Consider the van der Pol equation

$$\ddot{y} - a(1 - by^2)\dot{y} + y = 0 \tag{14}$$

and suppose that we are particularly interested in finding out what frequencies a periodic solution might have. Now choose $N = 3$, so that we are looking for solutions of the form

$$y(t) = \sum_{k=-3}^{3} c_k e^{-ik\omega t} \tag{15}$$

Using the notation $c_3 = c_{3r} + c_{3i} i$ and substituting (15) in (14) we get the following system of equations:

$$\left\{ \begin{array}{rcl} f_1 & = & -c_1\omega^2 - abc_1^2 c_{3i}\omega + c_1 \\ f_2 & = & 2abc_1c_{3r}^2\omega + abc_1^2c_{3r}\omega + 2abc_1c_{3i}^2\omega + abc_1^3\omega - ac_1\omega \\ f_3 & = & -9c_{3r}\omega^2 - 3abc_{3i}c_{3r}^2\omega - 3abc_{3i}^3\omega - 6abc_1^2c_{3i}\omega + \\ & & 3ac_{3i}\omega + c_{3r} \\ f_4 & = & 3abc_{3r}^3\omega - 9c_{3i}\omega^2 + 3abc_{3i}^2c_{3r}\omega + 6abc_1^2c_{3r}\omega - 3ac_{3r}\omega + \\ & & abc_1^3\omega + c_{3i} \end{array} \right. \tag{16}$$

The fact that the gradients are parallel provides us with two additional equations

$$\begin{cases} g_1 &= 6x_1 + 4x_2 - \lambda\,(36x_2x_1^2 - 12x_1 - 8x_2 + 16x_2^2x_1) \\ g_2 &= 4x_1 + 8x_2 - \lambda\,(12x_1^3 - 8x_1 + 16x_2x_1^2 - 16x_2) \end{cases} \tag{8}$$

Now we compute the Gröbner base for the four polynomials $\{V-d,\ V_xf,\ g_1,\ g_2\}$ w.r.t. the ranking

$$d \prec x_1 \prec \lambda \prec x_2$$

in Maple. This Gröbner base contains the polynomial

$$4d^4 - 147d^3 + 768d^2 + 2048d \tag{9}$$

having approximate solutions

$$0,\ -1.9223,\ 8.9657,\ 29.707$$

The smallest positive value of d for which there is a common solution to the system is thus 8.97... This is confirmed by the plot of figure 1 showing some level curves of V together with the curve $\dot{V} = 0$.

It should be added that in the two-dimensional case there are more effective methods than the one above, see Forsman [9]. □

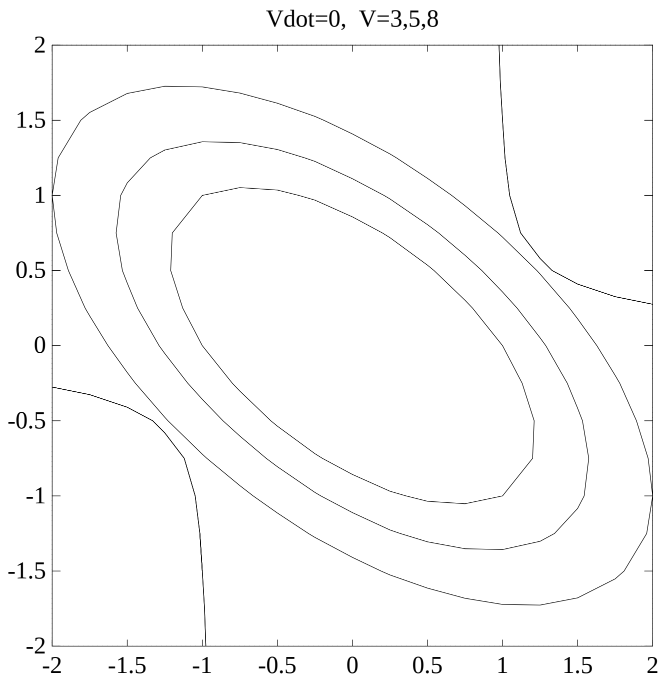

Figure 1: Level curves of V and $\dot{V}$.

The advantage of using Gröbner bases is even more striking in higher dimensions, where we have difficulties in seeing the geometry of the problem.

3 Gröbner Bases in Lyapunov Theory.

Suppose that we have the system

$$\dot{x}(t) = f(x(t)), \qquad x \in \mathbb{R}^n \tag{3}$$

where f is polynomial and we want to determine a stability region for the origin with a polynomial Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$. Since V is C^1 we have that $\dot{V} = V_x f$. Define two regions Ω and B_d in $\mathbb{R}^n$ as

$$\Omega = \{ x \in \mathbb{R}^n \,|\, V_x f < 0 \}, \qquad B_d = \{ x \in \mathbb{R}^n \,|\, V(x) \leq d \}$$

Thus the boundary ∂B_d is a level surface of V for each d. The following is an important problem:

How do we choose d in order to get B_d as large as possible while still inside of Ω ? This gives a region where we can guarantee stability, see [7], [15]. In e.g. [7] an efficient method for solving this problem is wanted. We claim that Gröbner bases provide such a method.

If V is positive definite the Lebesgue measure of B_d increases as d increases, and we may reformulate the problem as: which is the smallest d such that

$$\partial B_d \cap \partial\Omega \neq \emptyset \tag{4}$$

Now $\partial B_d = \mathcal{Z}(V - d)$ and $\partial\Omega = \mathcal{Z}(V_x f)$. We have two polynomial equations, so we can use Gröbner bases to eliminate one variable. This is not enough if we have a problem in $n > 2$ variables, since we will get a polynomial in $n - 1$ variables, the real solutions of which can be very hard to determine. However, since we are especially interested in *critical* values of d, we may use some extra information that we have. Namely, at the point were contact occurs the gradients of V and $\dot{V}$ w.r.t. x are *parallel*!

In this way we get a system of n equations in $n + 1$ variables

$$\begin{cases} V - d = & 0 \\ V_x f = & 0 \\ \nabla(V_x f) - \lambda \nabla V = & 0 \end{cases} \tag{5}$$

Let us demonstrate the method on an example.

Example 3.1 Consider a system of order two

$$\dot{x} = f(x) = \begin{pmatrix} -x_1 + 2x_1^2 x_2 \\ -x_2 \end{pmatrix} \tag{6}$$

and choose the Lyapunov function

$$V(x) = x^{\mathrm{T}} \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix} x = 3x_1^2 + 4x_1x_2 + 4x_2^2$$

This example is inspired by [7]. Now

$$\dot{V} = V_x f = -6x_1^2 + 12x_1^3 x_2 - 8x_1x_2 + 8x_1^2 x_2^2 - 8x_2^2 \tag{7}$$

Definition 2.3 The *manifold* of the ideal I is the set of solutions

$$\mathcal{Z}(I) = \{ x \in \mathbb{C}^n \mid \forall p \in I : p(x) = 0 \}$$

$\square$

Thus a system of equations $P = 0$ has a finite number of solutions iff $\mathcal{Z}(P)$ is zero dimensional.

When triangulating a system we eliminate variables one by one. A *ranking* of the variables tells us in what order we should eliminate them. For instance we can have the ranking

$$x_1 \prec x_2 \prec \ldots \prec x_{n-1} \prec x_n \tag{2}$$

meaning that x_j is to be eliminated before x_i if $j > i$. In this paper the ranking is (2), unless stated otherwise.

To do the elimination we have to be able to compare polynomials. This is possible if we have a ordering of the monomials of $\mathbb{R}[x_1, \ldots, x_n]$. This ordering is achieved if we say that a high ranking monomial should have a high degree in a high ranking variable. For details, consult Pauer and Pfeifhofer [13].

A Gröbner base is a generating set for an ideal, having certain properties which are appealing, algorithmically and otherwise. We refrain from giving the exact definition of Gröbner base, since it requires some machinery that will not enrichen this context. The definitions can be found in [4] or [13]. Instead we state some of the most important properties of Gröbner bases.

Theorem 2.1 *Let* $\mathbf{G}$ *be a Gröbner base for the ideal* I *w.r.t. the ranking (2). Then the following applies:*

1. $\mathcal{Z}(I) = \emptyset \;\Leftrightarrow\; \mathbf{G} = \{1\}$
2. $\dim(\mathcal{Z}(I)) = 0 \;\Leftrightarrow\;$ *Each variable appears alone in the highest ranking monomial of a polynomial in* $\mathbf{G}$*.*
3. $$I \cap \mathbb{R}[x_1] = \{0\} \quad \Leftrightarrow \quad \mathbf{G} \cap \mathbb{R}[x_1] = \emptyset$$

Proof. See for instance [3] and [13]. $\square$

Item no 3 above is the most important one in this paper.

We will sometimes be a little sloppy, saying "a Gröbner base for the set P" meaning, of course, the one for the ideal generated by P. Furthermore we make no distinction between equations and polynomials, every polynomial p corresponding to an equation $p = 0$.

What makes Gröbner bases so useful is that given a set P of polynomials, there exists an algorithm that computes the Gröbner base for (the ideal generated by) P in a finite number of steps (see Buchberger [3]). This algorithm has been implemented on computers in many versions and has been subject to different kinds of optimzation. In fact, most symbolic computation programs use Gröbner bases to solve systems of polynomial equations. Maple [6] and Macsyma both have functions for manipulating Gröbner bases explicitly. We will not give any comments on the algorithmical aspects of constructing Gröbner bases here.

Some references for Gröbner bases are [3], [4] and [13].

Abstract

This paper shows how Gröbner bases can be used to solve some common problems in nonlinear systems theory efficiently. These problems include finding critical levels of local Lyapunov functions and solving the equations that arise in the harmonic balancing method. The methods proposed are illustrated by some concrete examples in which the computer algebra system Maple is used for performing the necessary calculations.

1 Introduction.

Gröbner bases began to be studied seriously by mathematicians in the mid-70's. They have been used to solve difficult problems in commutative algebra constructively [14], and algorithms for constructing Gröbner bases have been implemented on computers [6]. However, they seem to have found little use in control applications sofar.

The problem of solving a system of nonlinear (algebraic) equations in several variables arises frequently in different applications. There is a general method for solving such systems in the case where the nonlinearities are of polynomial type. The knowledge about this method, based on Gröbner bases, seems to be concentrated to mathematics and computer science, though. In this paper we give some examples of how to use the method on problems in nonlinear control theory.

By "solving" such a system we will mean triangulating it, so that the minimal dimension of the solution becomes apparent, and so that we obtain an equation in one variable only, if the solution is zero-dimensional.

The problems we study are:

- Finding critical levels of local Lyapunov functions (addressed in [7])
- Solving equations obtained in harmonic balancing

2 What is a Gröbner Base?

We will need a few definitions from commutative algebra.

Definition 2.1 A *polynomial ring* over the real numbers is the system of all polynomials in some variables $x_1, \ldots, x_n$ with real coefficients. Notation: $\mathbb{R}[x_1, \ldots, x_n]$ □

Definition 2.2 Let $P = \{p_1, \ldots, p_m\}$ be a set of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$. The *ideal* generated by P is the set of polynomials f that can be written

$$f = \sum_{i=1}^{m} \alpha_i p_i$$

where the $\alpha_i \in \mathbb{R}[x_1, \ldots, x_n]$. □

P is said to be a *generating set* of the ideal, and we use the notation (P) for the ideal in question. Of course, an ideal can have several different generators. E.g.

$$\begin{aligned}(\{ x_1^2 - 1, x_2 + 1 \}) &= (\{ x_1^2 - 1, x_2 + 1, x_1^2 + x_2 \}) = \\ &= (\{ x_1^2 - 1, x_2 + x_1^2 \})\end{aligned} \tag{1}$$

Constructive Algebraic Geometry in Nonlinear Control.

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