"soften the infinitely steep walls" in (49). However as we have shown in the previous section (example 4.1) there is no guarantee that the deadzone criteria (2) performs any better than the conventional quadratic criteria in the situation (51). The value of a deadzone - although reasonable from an ad hoc point of view - can thus be said to be questionable.

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5 Deadzones and Set Membership Identification.

The traditional description of noise and disturbances influencing a system is to model them as stochastic processes. This leads to the conventional identification methods of maximum likelihood / least squares type. However, there may be reasons to reject this description of the disturbances; see also (Ljung, 1991). If they are measurement errors of quantization type all we can say is that they are bounded. This view has lead to the so called "unknown-but-bounded" approach to estimation (Schweppe, 1973). The idea is simply to accept all model parameter values that are consistent with a bounded noise assumption:

$$|e(t)| \leq c \tag{47}$$

without performing any averaging over the data. This could be described as

$$\hat{\vartheta} = \arg\min \frac{1}{N} \sum_{t=1}^{N} \ell_c(\varepsilon(t, \vartheta)) \tag{48}$$

where

$$\ell_c(\varepsilon) = \begin{cases} 0, & |\varepsilon| \leq c \\ \infty, & |\varepsilon| > c \end{cases} \tag{49}$$

or, equivalently,

$$\hat{\vartheta} \in D_N = \{ \vartheta \mid \sum_{t=1}^{N} \ell_c(\varepsilon(t, \vartheta)) = 0 \} = \{ \vartheta \mid \forall t : \ |\varepsilon(t, \vartheta)| \leq c \} \tag{50}$$

The estimate is thus a set, D_N, and the approach is often also called "set membership identification". The set is in practice not found by minimization of (48) but rather by linear programming techniques (Milanese and Tempo, 1985) direct calculation (Walter and Piet-Lahanier, 1987) or outer-bounding by ellipsoids (Fogel and Huang, 1982).

Now if (47) indeed holds for all disturbances this method works well as does the deadzone criterion (2) as found in section 2.

However even though there are several reasons to reject the traditional stochastic process description of disturbances, there are also several reasons to reject (47) as the sole description of the noise - i.e. that it possesses no averaging properties whatsoever. It can be argued that a better picture is to describe the noise as

$$e(t) = v(t) + w(t) \tag{51}$$

where $v(t)$ is subject to (47) and $w(t)$ has conventional averaging properties, i.e. in the linear regression case

$$\bar{\mathrm{E}}\varphi(t)w(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \mathrm{E}\varphi(t)w(t) = 0 \tag{52}$$

The conventional set-membership approach deals with (51) by extending the value c in (47) intil D_N in (50) becomes non-empty. This is quite a conservative approach. A seemingly more natural approach would be to use the deadzone criterion (2) i.e. to

If we differentiate both membra of (34) with respect to the parameter c we find that

$$0 = \frac{d}{dc}\bar{\mathrm{E}}\,\varphi\,[2g - \int_{c-g}^{c+g} F(x)\,dx] = \bar{\mathrm{E}}\,\varphi\left[2\varphi^{\mathrm{T}}\tilde{\vartheta}' - F(c+g)(1+\varphi^{\mathrm{T}}\tilde{\vartheta}') + F(c-g)(1-\varphi^{\mathrm{T}}\tilde{\vartheta}')\right] \tag{41}$$

We are only interested in the point $c = 0$:

$$\begin{aligned}\bar{\mathrm{E}}\,\varphi\Big[2\varphi^{\mathrm{T}}\tilde{\vartheta}'(0) - F(g)(1+\varphi^{\mathrm{T}}\tilde{\vartheta}'(0)) + (1-F(g))(1-\varphi^{\mathrm{T}}\tilde{\vartheta}'(0))\Big] &= \\ = \bar{\mathrm{E}}\,\varphi\Big[\varphi^{\mathrm{T}}\tilde{\vartheta}'(0) + 1 - 2F(g)\Big] = 0\end{aligned} \tag{42}$$

where we, as earlier, assumed that f is even. Recall that

$$g(\vartheta(c))\Big|_{c=0} = \tilde{\vartheta}_{LS}^{\mathrm{T}}\varphi + d$$

Hence

$$\bar{\mathrm{E}}[\varphi\varphi^{\mathrm{T}}\tilde{\vartheta}'(0) + \varphi - 2F(\tilde{\vartheta}_{LS}^{\mathrm{T}}\varphi + d)\varphi] = 0 \tag{43}$$

and

$$\tilde{\vartheta}'(0) = (\bar{\mathrm{E}}\varphi\varphi^{\mathrm{T}})^{-1}\,\bar{\mathrm{E}}[(2F(\tilde{\vartheta}_{LS}^{\mathrm{T}}\varphi + d) - 1)\varphi] \tag{44}$$

$$\tilde{\vartheta}_{LS} = -\,(\bar{\mathrm{E}}\varphi\varphi^{\mathrm{T}})^{-1}(\bar{\mathrm{E}}\varphi d) \tag{45}$$

Substituting this in (44) renders

$$\tilde{\vartheta}'(0) = (\bar{\mathrm{E}}\varphi\varphi^{\mathrm{T}})^{-1}\left[\bar{\mathrm{E}}\Big(2F(d - \bar{\mathrm{E}}(d\varphi^{\mathrm{T}})(\bar{\mathrm{E}}\varphi\varphi^{\mathrm{T}})^{-1}\varphi) - 1\Big)\varphi\right]$$

or, with $P = (\bar{\mathrm{E}}\varphi\varphi^{\mathrm{T}})^{-1}$,

$$\tilde{\vartheta}'(0) = P\,\bar{\mathrm{E}}\Big(2F(d - (\bar{\mathrm{E}}d\varphi)^{\mathrm{T}}\,P\varphi) - 1\Big)\varphi \tag{46}$$

Equation (46) is rather difficult to analyse in a general setting. What we may do is expanding F in a Taylor series to get some information. The following example was constructed by choosing d so that the first non-zero term of the Taylor expansion had the appropriate sign (to make $\tilde{\vartheta}'(0) > 0$).

Example 4.1 Consider the system

$$y(t+1) = -0.7y(t) + \vartheta u(t) + e(t) + d(t), \qquad \vartheta^0 = 1$$

where e is gaussian white noise with unit variance. Choose $u(t) = \sin(0.3t)$ as input and $d(t) = \sin(0.1t)$ as deterministic disturbance. Note that u and d are not correlated. Simulations in MATLAB now indicate that the LS-estimate of ϑ is *better* than the estimate obtained with a deadzone of width $c = 1$:

$$\hat{\vartheta}_{LS} = 0.96, \qquad \hat{\vartheta}_{DZ} = 0.92$$

These results were obtained with a data set consisting of 10000 samples. □

For any special case, the way to answer the opening question of this section is to determine the sign of the scalar product $\tilde{\vartheta}_{LS} \cdot \tilde{\vartheta}'(0)$. If and only if it is negative, the DZ has a positive effect on the LS estimate.

where $'$ denotes differentiation w.r.t. ϑ. So $A+B = 2g'\Gamma$ where

$$\Gamma \;=\; (g+c)F(-c-g) \;+\; (g-c)(1-F(c-g)) \;-\; \int_{-c-g}^{c-g} xf(x)\,dx \tag{37}$$

By partial integration,

$$\begin{aligned} \int_{-c-g}^{c-g} xf(x)\,dx \;&=\; \Big[xF(x)\Big]_{-c-g}^{c-g} \;-\; \int_{-c-g}^{c-g} F(x)\,dx \quad = \\ &= (c-g)F(c-g) + (c+g)F(-c-g) \;-\; \int_{-c-g}^{c-g} F(x)\,dx \end{aligned}$$

From this we conclude that

$$\begin{aligned} \Gamma \;&=\; g-c+\int_{-c-g}^{c-g} F(x)\,dx \;=\; g-c+\int_{c+g}^{c-g} F(x)\,dx \;= \\ &=\; g-c-\int_{c-g}^{c+g} F(x)\,dx \;+\; \int_{0}^{c+g} (F(x)+1-F(x))\,dx \;=\; 2g-\int_{c-g}^{c+g} F(x)\,dx \end{aligned} \tag{38}$$

and finally we arrive at

$$\frac{d}{d\vartheta}\mathrm{E}\,\ell(\varepsilon) \;=\; g'\,(2g-\int_{c-g}^{c+g} F(x)\,dx) \tag{39}$$

To get (34) we have to consider the following a mathematical truth:

$$\frac{d}{d\vartheta}\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\mathrm{E}\,\ell(\varepsilon) \;=\; \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\frac{d}{d\vartheta}\mathrm{E}\,\ell(\varepsilon) \tag{40}$$

Whether there are (many) cases in which this is not so seems to be a difficult question, so we accept that as it is. $\square$

4.2 Going from LS to DZ.

We now know what equation to solve in order to obtain the minimizing ϑ for the criterion (2) when there are deterministic disturbances present, as in (32). An interesting question is: Suppose we have an LS-estimate of the parameters ϑ^0 in (32) and introduce a deadzone in the criterion, in what way will the estimate change? Mathematically this question can be put like this: Consider the estimate ϑ as a differentiable function of c, what is then $\tilde{\vartheta}'$ at the point $c=0$? In this section we will use equation (34) of the preceding theorem to answer this question. First we redefine $\tilde{\vartheta}$ slightly and note that

$$\frac{d}{dc}\tilde{\vartheta}(c) \;=\; \frac{d}{dc}(\vartheta(c)-\vartheta^0) \;=\; \frac{d}{dc}\vartheta(c)$$

4 Deterministic Disturbances.

In this section we aim to investigate what happens as a deterministic disturbance is present in the model. One might believe that in this case an LS-estimate will always improve if a deadzone is introduced. However, we will soon see that this is not true. Let us call the deterministic disturbance d. We then have:

$$y(t) \;=\; \varphi^{\mathrm{T}}(t)\vartheta^0 \;+\; e(t) \;+\; d(t) \tag{32}$$

and the residuals are

$$\varepsilon(t,\vartheta) \;= y(t) - \varphi^{\mathrm{T}}(t)\,\vartheta \;=\; \varphi^{\mathrm{T}}(t)\tilde{\vartheta} \;+\; d(t) \;+\; e(t) \tag{33}$$

These formulas are the analogues of (3) and (4). Let g denote the 'deterministic part' of (33):

$$g(t) \;:=\; \varphi^{\mathrm{T}}(t)\tilde{\vartheta} \;+\; d(t)$$

4.1 Minimizing the Criterion.

It is possible to derive an explicit expression for the criterion (2) in the case (32). It turns out that this expression is extensive and non-suggestive. Luckily, the derivative of the criterion is rather easy to compute, as long as f (the p.d.f. of the noise) is even. These computations result in the following equation.

Theorem 4.1 *We have that*

$$\frac{d}{d\vartheta}V(\vartheta) \;=\; 0 \qquad \Longleftrightarrow \qquad \bar{\mathrm{E}}\,\varphi\,[2\,g(\vartheta) \;-\; \int\limits_{c-g(\vartheta)}^{c+g(\vartheta)} F\,] \;=\; 0 \tag{34}$$

Proof. Recall the following formula from elementary calculus (see for example (Hildebrand, 1962) pp. 359-360):

$$\frac{d}{dx}\int\limits_{\varphi(x)}^{\psi(x)} f(x,y)\,dy \;=\; \int\limits_{\varphi(x)}^{\psi(x)} f_x(x,y)\,dy \;+\; f(x,\psi(x))\,\psi'(x) - f(x,\varphi(x))\,\varphi'(x) \tag{35}$$

Now, for each sample

$$\mathrm{E}\,\ell(\varepsilon) \;=\; \int\limits_{-\infty}^{-c-g} (x+g+c)^2\,f(x)dx + \int\limits_{c-g}^{\infty} (x+g-c)^2\,f(x)dx \tag{36}$$

where g is a function of ϑ. We want to compute the derivative of (36) w.r.t. ϑ. In order to do this we use (35) on each of the terms in the right membrum:

$$A \;=\; \frac{d}{d\vartheta}\int\limits_{-\infty}^{-c-g(\vartheta)} (x+g(\vartheta)+c)^2 f(x)\,dx \;=\; 2g'\,[(g+c)F(-c-g) + \int\limits_{-\infty}^{-c-g} x f(x)\,dx]$$

$$B \;=\; \frac{d}{d\vartheta}\int\limits_{c-g(\vartheta)}^{\infty} (x+g(\vartheta)-c)^2 f(x)\,dx \;=\; 2g'\,[(g-c)(1-F(c-g)) + \int\limits_{c-g}^{\infty} x f(x)\,dx]$$

T_1 can be lower bounded:

$$T_1 \geq \frac{1}{3} f(c+1)/(a+b)^2 \tag{24}$$

But according to (22)

$$\frac{a^2(c)}{f(c+1)} < \frac{f^2(c)}{f(c+1)} \to 0, \quad c \to \infty \tag{25}$$

and

$$\lim_{c\to\infty} \frac{a(c)b(c)}{f(c+1)} = 0 \tag{26}$$

since

$$\lim_{c\to\infty} \frac{f(c+1)}{b(c)} = \lim_{c\to\infty} |\frac{f'(c+1)}{f(c+1)}| > 0 \tag{27}$$

according to l'Hospital's rule. The same argument can be used to show that

$$\lim_{c\to\infty} \frac{b(c)^2}{f(c+1)} = \lim_{c\to\infty} \frac{f(c+1)}{f'(c+1)} \cdot b(c) = 0 \tag{28}$$

We conclude that $T_1 \to \infty$ as $c \to \infty$.

Let us now study T_2 under the assumption (21). We have

$$\int_{c+1}^{\infty} (x-c)^2 f(x)dx \geq \int_{c+1}^{\infty} f(x)dx \tag{29}$$

so that

$$T_2 \geq \frac{b}{(a+b)^2} = \frac{1}{a^2/b + 2a + b} \tag{30}$$

If we can show that a^2/b tends to zero as $c \to \infty$ we are finished. This can be achieved by applying l'Hospital's rule and using condition (21) to get

$$\lim_{c\to\infty} \frac{a^2(c)}{b(c)} = \lim_{c\to\infty} \frac{f(c) - f(c+1)}{f(c+1)} a(c) = \lim_{c\to\infty} \frac{f(c)}{f(c+1)} a(c) = 0 \tag{31}$$

We conclude that $T_2 \to \infty$ as $c \to \infty$. □

The theorem above covers many of the interesting cases, e.g. that of the normal distribution:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$$

For this important special case straightforward computations show that

$$\kappa(\ell) = \frac{(\sigma^2 + c^2)(1 - \Phi(\frac{c}{\sigma})) - \frac{c\sigma}{\sqrt{2\pi}} e^{-c^2/2\sigma^2}}{2(1 - \Phi(\frac{c}{\sigma}))^2}$$

where

$$\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

supposing the integral in the denominator does not vanish. As expected, κ is strongly dependent on f.

We wish to prove that the variance tends to infinity as the width of the deadzone tends to infinity. Even if intuitively very clear, this result is not trivial to prove.

Here we give a theorem which covers many important cases. Different estimates of the quotient (20) are made in the different cases.

Definition 3.1 Let f be a continuously differentiable function. We say that f is *asymptotically decreasing* if

$$\exists N\,\forall x > N : \quad f'(x) < 0$$

$\square$

Theorem 3.1 *If f is asymptotically decreasing and either*

$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)} > 0 \tag{21}$$

or

$$\lim_{x\to\infty} \left|\frac{f'(x)}{f(x)}\right| > 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{f^2(x)}{f(x+1)} = 0 \tag{22}$$

then

$$\kappa(\ell) \to \infty \quad \text{as} \quad c \to \infty$$

in formula (20).

Proof. We have that $\kappa(\ell) = T_1 + T_2$ where

$$T_1 = \frac{\int_c^{c+1} (x-c)^2 f(x)dx}{(\int_c^\infty f)^2}, \qquad T_2 = \frac{\int_{c+1}^\infty (x-c)^2 f(x)dx}{(\int_c^\infty f)^2}$$

Since both are positive it suffices to show that either T_1 or T_2 tends to ∞. Let us introduce the notations

$$a(c) = \int_c^{c+1} f, \quad b(c) = \int_{c+1}^\infty f$$

It is clear that a and b tend to zero as $c \to \infty$.

First we have a look at T_1. Assume that (22) holds. Since

$$\int_c^{c+1} (x-c)^2 f(x)\,dx \ge f(c+1) \int_c^{c+1} (x-c)^2\,dx = \frac{1}{3} f(c+1) \tag{23}$$

3 Variance.

We want to investigate what the asymptotic covariance matrix of the estimation error will look like when we use a deadzone, and how it depends on the width of the DZ. The following important fact can be found in e.g. (Ljung, 1987) p. 245:

$$\operatorname{Cov} \hat{\vartheta}_N \sim \frac{1}{N}\kappa(\ell)\,[\bar{\mathrm{E}}\psi_0\psi_0^{\mathrm{T}}]^{-1}, \quad \psi_0 = -\frac{d}{d\vartheta}\varepsilon(t,\vartheta)\Big|_{\vartheta=\vartheta_0} \tag{13}$$

where

$$\kappa(\ell) = \frac{\bar{\mathrm{E}}(\ell'(e))^2}{(\bar{\mathrm{E}}\ell''(e))^2} \tag{14}$$

This formula is valid if ℓ is twice continuously differentiable, which is not true for the squared deadzone. The trouble caused by the discontinuity in ℓ'' does not appear to be of that serious a kind, though. We assume that it is of no importance. If we disregard the time averaging property of $\bar{\mathrm{E}}$, which is the same as assuming that the p.d.f. of the noise does not vary in time, we obtain

$$\kappa(\ell) = \frac{\mathrm{E}\,(\ell'(e))^2}{(\mathrm{E}\,\ell''(e))^2} \tag{15}$$

Here ℓ is the squared deadzone (6), so

$$\ell'(\varepsilon) = \begin{cases} \varepsilon - c, & \varepsilon > c \\ 0, & |\varepsilon| \le c \\ \varepsilon + c, & \varepsilon < -c \end{cases} \tag{16}$$

and

$$\ell''(\varepsilon) = \begin{cases} 1, & |\varepsilon| > c \\ 0, & |\varepsilon| < c \\ \alpha, & |\varepsilon| = c \end{cases} \tag{17}$$

where α is a subdifferential, $\alpha = [0, 1]$. If we confine ourselves to even densities then

$$\mathrm{E}\,(\ell'(e))^2 = 2\int_c^\infty (x-c)^2 f(x)\,dx \tag{18}$$

and

$$\mathrm{E}\,\ell''(e) = 2\int_c^\infty f(x)\,dx \tag{19}$$

When we insert (19) and (18) in (14) we get

$$f \text{ even} \quad \Rightarrow \quad \kappa(\ell) = \frac{\int_c^\infty (x-c)^2 f(x)\,dx}{2\,(\int_c^\infty f(x)\,dx)^2} \tag{20}$$

Theorem 2.2 *Suppose that the noise and the input are bounded:*

$$\forall t: \quad |e(t)| \le B_1 < c, \quad |u(t)| \le B_2$$

and that the system is stable. Then

$$\hat{\vartheta} \supset \{\vartheta \mid c \ge \alpha(\vartheta)\, B_3 + \beta(\vartheta)\, B_2 + B_1\} \ne \{\vartheta_0\}$$

where $B_3 = \|y(t)\|_\infty$, $\alpha(\vartheta) = \|\tilde{\vartheta}_A\|_1$ *and* $\beta(\vartheta) = \|\tilde{\vartheta}_B\|_1$ *are defined via (10).*

Proof. The residuals are defined by equations (4) and (5). Now, if $|\varepsilon(t,\vartheta)| \le c$ for all t, we have that $\ell(\varepsilon(t,\vartheta)) \equiv 0$, which means that a sufficient condition for the estimate to be inconsistent is

$$|\varphi^{\mathrm{T}}(t)\, \tilde{\vartheta}| + B_1 \le c \tag{9}$$

Since we assumed that the input was bounded we can obtain an estimate of the first term in the following way: Partition the $\tilde{\vartheta}$ vector in elements corresponding to y and elements corresponding to u:

$$\tilde{\vartheta} = \begin{pmatrix} \tilde{\vartheta}_A \\ \hline \tilde{\vartheta}_B \end{pmatrix}, \tilde{\vartheta}_A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \tilde{\vartheta}_B = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} \tag{10}$$

and the regression vector φ analogously. If the true system is asymptotically stable the output will be bounded: $|y(t)| \le B_3$. (Of course, B_3 can be expressed in B_1 and B_2 if we know the true system parameters ϑ^0.) Hölder's inequality gives us an upper bound for the first term of (9):

$$|\, \varphi_A^{\mathrm{T}}(t)\, \tilde{\vartheta}_A| \le \| \begin{pmatrix} -y(t-1) \\ \vdots \\ -y(t-k) \end{pmatrix} \|_\infty \cdot \|\tilde{\vartheta}_A\|_1 \le B_3 \sum_{j=1}^{n} |\alpha_j| \tag{11}$$

If the B-part of $\varphi^{\mathrm{T}}\, \tilde{\vartheta}$ is estimated in the same way we get:

$$|\varphi^{\mathrm{T}}(t)\, \tilde{\vartheta}| \le \alpha B_3 + \beta B_2, \quad \alpha := \sum_{j=1}^{n} |\alpha_j|, \quad \beta := \sum_{j=1}^{k} |\beta_j|$$

Our conclusion is: since

$$|\varepsilon(t,\vartheta)| \le \alpha B_3 + \beta B_2 + B_1$$

a DZ width which is strictly greater than B_1 will give an estimate that is inconsistent. The set of parameter vectors in which the criterion is zero will be a superset of

$$\{\vartheta \mid c \ge \alpha(\vartheta)\, B_3 + \beta(\vartheta)\, B_2 + B_1\} \tag{12}$$

$\square$

Remark: It is easy to show that this is the best estimate that Hölder's inequality can produce with the information available.

- If the estimate is inconsistent, how much can it deviate from the true value of ϑ?
- How does the deadzone affect the variance of the estimate?
- What is the effect on the LS estimate when a deadzone is introduced, supposing there are unmodelled deterministic disturbances present?
- What are the relations between identification with a deadzone criterion and so called set membership identification?

Notational convention: From this point onwards we will use the notation $\bar{\mathrm{E}}$ defined by:

$$\bar{\mathrm{E}}\, x \;=\; \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \mathrm{E}\, x(t) \tag{7}$$

2 Consistency.

In some cases the DZ estimate of the parameters will be inconsistent. For instance, if the noise is bounded, the norm of the residuals may be zero for all time instants within a set (non-singleton) in the parameter space. Thus all members of this set are indistinguishable, and we only know that the true parameter is inside the set. The following two theorems say what is intuitively clear, namely that the estimate will be inconsistent if and only if the noise is bounded and the DZ is too wide.

Theorem 2.1 *Let f be the p.d.f. of the noise e in (3) and suppose that f is even. If f is not identically zero outside the interval $[-c, c]$ then V has a global minimum in ϑ_0. Shorter:*

$$\mathrm{supp}(f) \not\subset [-c,\, c] \quad \Rightarrow \quad \hat{\vartheta} \;=\; \vartheta_0$$

Proof. Define $v(b) \;:=\; \mathrm{E}\,\ell(e+b)$ i.e.

$$v(b) \;=\; \int_{-\infty}^{\infty} \ell(b+x) f(x)\, dx$$

It is sufficient to show that $v''(b) > 0$ and that $v'(0) = 0$. Straightforward computations show that $v'(0) = 0$. Furthermore

$$v''(b) \;=\; \int_{-\infty}^{-c-b} f(x)dx + \int_{c-b}^{\infty} f(x)dx \tag{8}$$

which is nonzero for all b iff $\mathrm{supp}(f) \;\not\subset [-c, c]$. $\square$

Remark: In the case of a density that is not even we get a messy implicit expression for the minimizing bias.

where ε are the prediction errors of a linear regression:

$$y(t) = \varphi^{\mathrm{T}}(t)\,\vartheta^0 + e(t) \tag{3}$$

$$\Rightarrow \quad \varepsilon(t,\vartheta) \;=\; y(t) - \hat{y}(t,\vartheta) \;=\; \varphi^{\mathrm{T}}(t)\,\tilde{\vartheta} \;+\; e(t) \tag{4}$$

where

$$\tilde{\vartheta} \;:=\; \vartheta^0 - \vartheta \tag{5}$$

Furthermore, ℓ is a quadratic norm which is zero within the interval $[-c, c]$:

$$\ell(x) \;=\; \begin{cases} \frac{1}{2}(x-c)^2, & x > c \\ 0, & |x| \leq c \\ \frac{1}{2}(x+c)^2, & x < -c \end{cases} \tag{6}$$

or, more compactly, $\ell(x) = \frac{1}{2}(\max(c, |x|) - c)^2$. A typical ℓ is displayed in figure 1.

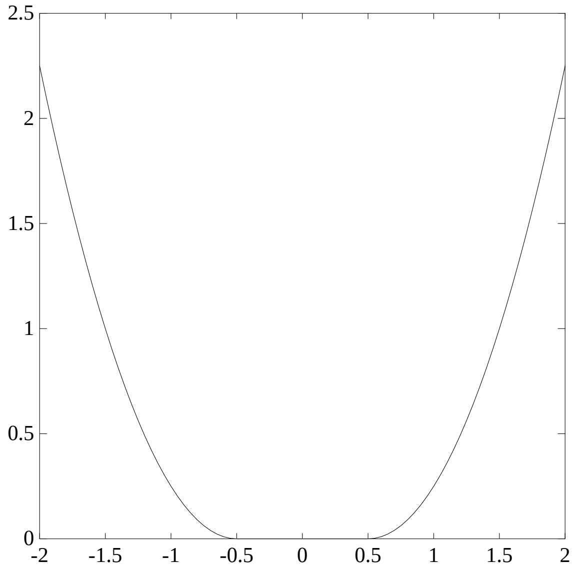

Figure 1: The norm of the criterion ($c = 0.5$) .

The use of such a deadzone is widespread in adaptive control and system identification. It appears in adaptive regulators used in the industry, but does also have theoretically interesting properties, for example in the stability theory of robust adaptive regulators. See e.g. (Egardt, 1976) or (Goodwin et al., 1985) who use the DZ for this purpose. Still, many properties of the DZ-estimate seem to have been neglected to some degree. The following questions are addressed in this paper:

- Under what circumstances will the DZ estimate be inconsistent?

On the Dead-zone in System Identification.

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Abstract. A prediction error method for parameter estimation in a dynamical system is studied.

$$\hat{\vartheta} = \arg\min_{\vartheta} \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathrm{E}\ell(\varepsilon(t,\vartheta))$$

where ε are the prediction errors of a linear regression. ℓ is a quadratic norm which is zero within an interval $[-c, c]$. This kind of a deadzone criterion is very common in robust adaptive control. The following problems are treated in this paper:

- When is the DZ estimate inconsistent, and what is the set of parameters which minimizes the criterion in the case of inconsistency?
- What happens to the variance of the estimate as the DZ is introduced?
- Does the deadzone give a better estimate than LS when there are unmodelled deterministic disturbances present?
- What are the relations between identification with a deadzone criterion and so called set membership identification?

Keywords: System identification, deadzone, robust adaptive control, set membership identification.

1 Introduction.

Consider a prediction error method for parameter estimation in a dynamical system:

$$\hat{\vartheta} = \arg\min_{\vartheta} V(\vartheta) \tag{1}$$

Here V is a deadzone (DZ) criterion:

$$V(\vartheta) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathrm{E}\,\ell(\varepsilon(t,\vartheta)) \tag{2}$$