w.r.t. the ranking $x_2 \succ x_1 \succ a$. This $\mathbf{G}$ contains a polynomial p in $\mathbb{Q}\,[x_1, a]$ having three factors.

The first two factors describe the fixed points $x_1 = 0$ and $x_1 = a + 1$. The third factor of p, let us call it p_3, corresponds to a truely three-periodic solution. To analyze how the roots of p_3 vary with a we compute the discriminant of p_3 w.r.t. x_1 (see [3, page 218]) to check for real solutions. The only real solutions in x_1 correspond to the zeros of the discriminant; however, these zeros render complex solutions in x_2. We conclude is that there is no $a \in \mathbb{R}$ such that (10) has a three-periodic real solution. □

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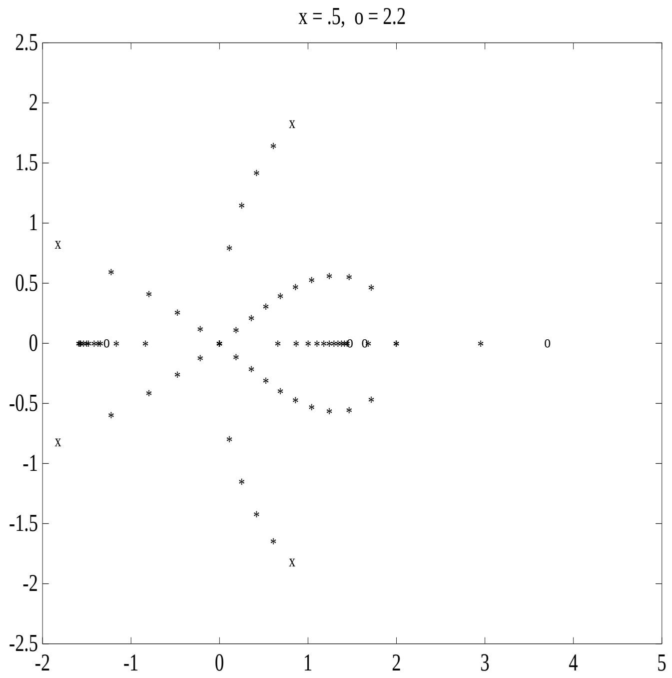

Figure 1: Root locus of p w.r.t. h in example 4.1.

method from real algebraic geometry. According to [3, page 116], if we compute the Gröbner base for $(p, \partial p/\partial x_1)$ w.r.t. the ranking $h \prec x_1$ we will obtain an $r \in \mathbb{Q}\,[h]$, and the roots of this r are the values of h for which p has double roots in x_1. Indirectly we get intervals for h within which p has a *real* solution in x_1. In our example

$$r = (4h^4 - 12h^3 + 9h^2 + 4h + 4)(h+1)(h-2)(h-1)^4 \tag{9}$$

The first factor has no real roots, so for $h = 1$ and $h = 2$ we have the only breakpoints (we do not consider $h < 0$). To interpret this fact we plot a root locus of p w.r.t. h for $h \in [0.5; 2.2]$ (see figure 1). The starting points are marked $(\times)$ and the end points $(\circ)$.

For $0 < h < 1$ we do not have any real roots in x_1, while for $h > 1$ there are at least two, one positive and one negative. For $h > 2$ we have two more real roots.

This conclude the example and answers our question. □

Example 4.2 Consider the system

$$x_{k+2} = x_{k+1}^2 - a x_k \tag{10}$$

which can be rewritten in state space form:

$$x_1(k+1) = x_1^2(k) - a x_2(k), \quad x_2(k+1) = x_1(k) \tag{11}$$

To examine all solutions to (10) of period three we compute the Gröbner base $\mathbf{G}$ for the ideal

$$I = (f_1^3(x_1, x_2) - x_1,\ f_2^3(x_1, x_2) - x_2) \tag{12}$$

the problem is simpler. An N-periodic solution to (4) satisfies the equation $x = f^N(x)$ where superscript denotes iteration. If $n = 1$ this equation can be analysed using numerical methods [8, section 14.3], [4, page 3], but for $n > 1$ the problem is quite complicated, and elimination theory is a great help. One advantage of using symbolic elimination instead of numerical methods is that we can handle parameters in the right hand side. What we do is that we compute the GB for

$$I = (x_1 - f_1^N(x_1, \ldots, x_n), \ldots, x_n - f_n^N(x_1, \ldots, x_n)) \tag{5}$$

If we are only interested in truely N-periodic solutions we may obtain these by first finding q-periodic solutions ($q < N$) and then divide the defining equation of the original problem with those of the q-periodic problem.

4 Examples

Let us give some concrete examples showing how to apply the algorithm. The computations involved were done in Maple [2].

Example 4.1 *Euler's method.*
In Euler's method for solving differential equations numerically the time derivative $\dot{x}(t)$ is approximated by a difference quotient:

$$\frac{d}{dt}x(t) \simeq \frac{x(t+h) - x(t)}{h} \tag{6}$$

for some $h > 0$. Thus, if we approximate the system

$$\dot{x}_1 = -x_1 + x_1^2 x_2, \quad \dot{x}_2 = -2x_2 + x_1 x_2 \tag{7}$$

using (6) we get

$$\begin{cases} x_1(t+h) = (1-h)\, x_1(t) + h\, x_1^2(t) x_2(t) \\ x_2(t+h) = (1-2h)\, x_2(t) + h\, x_1(t) x_2(t) \end{cases} \tag{8}$$

Now we ask: *For what h are there truely two-periodic solutions?*

If we denote the right membrum of (8) by $f(x_1, x_2)$ and assume that (x_1, x_2) is a 2-periodic solution we only have to compute a Gröbner base for the ideal (5) w.r.t. the ranking $h \prec x_1 \prec x_2$. Computations can be simplified if we divide $x_1 - f_1^2$ by the factor hx_1 and analogously for $x_2 - f_2^2$. The GB thus obtained is quite large, consisting of four polynomials and taking more than 100 lines to print out. It contains one polynomial $P \in \mathbb{Q}\,[h][x_1]$. P contains factors representing equilibrium points as well, so we divide P by these factors. Stationary points have $x_1 = 0$ or $x_1 = 2$, and P is divisible by $x_1 - 2$. Let $p = P/(x_1 - 2)$ (we already excluded $x_1 = 0$).

So far we have proved that there is always a truely two periodic solution to the original system. If we are asking for *real* solutions only we can use some

2 Elimination Theory

Elimination theory consists in finding methods for eliminating variables in systems of polynomial equations in several variables. Modern elimination theory is entirely based on so called Gröbner bases.

The theory of Gröbner bases (GB) emerged about 25 years ago with the work of B. Buchberger. They have been used to solve problems in commutative algebra, and algorithms for computing GB have been implemented on computers [2].

For a detailed account of Gröbner bases we refer to [1] or [6]. Our main interest here is elimination, so we refrain from giving the details.

3 Finding Periodic Solutions

Consider a system of nonlinear difference equations:

$$\begin{cases} f_1(x_1(t+1), x_1(t), \ldots, x_n(t)) = 0 \\ \vdots \\ f_n(x_n(t+1), x_1(t), \ldots, x_n(t)) = 0 \end{cases} \tag{1}$$

where each f_i is a polynomial in $n+1$ variables. Note that we allow the components of $x(t+1)$ to occur *implicitly* in the equations; thus we do not require that (1) is a state space description.

A point ξ in state space is said to be *N-periodic* if for all t: $x(t) = \xi \Rightarrow x(t+N) = \xi$. Obviously an N-periodic point is also νN-periodic for every positive integer ν. An N-periodic point that is not q-periodic for any $q < N$ is said to be *truely N-periodic.*

To find N-periodic solutions we consider the ideal

$$I_N = (F_i(x^2, x^1),\ F_i(x^3, x^2),\ \ldots,\ F_i(x^1, x^N)), \quad i = 1, \ldots, n \tag{2}$$

where F_i denotes the i:th component of the vector

$$F(y, z) = (f_1(y_1, z_1, \ldots, z_n), \ldots\ f_n(y_n, z_1, \ldots, z_n))$$

and $x^i \in \mathbb{R}^n$. I_N is an ideal in $k[x_1^1, \ldots, x_n^1, \ldots, x_1^N, \ldots, x_n^N]$ generated by Nn polynomials. As the special case $N = 1$ we get the problem of finding all equilibrium points. As we compute the Gröbner base for I_N w.r.t. the ranking

$$x^N \succ x^{N-1} \succ \ldots \succ x^2 \succ x^1 \tag{3}$$

we obtain a polynomial in the variables x^1 only, if there is one, and we proceed by analysing this polynomial (see the examples below).

If the system (1) is in state space form

$$x(t+1) = f(x(t)), \quad x = (x_1, \ldots, x_n), \quad f : \mathbb{R}^n \to \mathbb{R}^n \tag{4}$$

An Algorithm for Finding Periodic Solutions to Systems of Nonlinear Difference Equations

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Abstract. We give an algorithm for finding all N-periodic solutions to systems of nonlinear difference equations. The algorithm is based on elimination theory, and in special Gröbner bases. It is applicable to polynomial systems and can handle indeterminate parameters in the system equations. The system is not required to be in state space form - implicit equations are allowed.

Keywords: discrete time systems, polynomial systems, periodic solutions, constructive commutative algebra

1 Introduction

Systems of nonlinear difference equations frequently arise in applications. In control theory some famous examples are furnished by the Kalman filtering updating equations, different adaptive control algorithms and recursive identification schemes. Of course they also arise when a continuous time control system is sampled. Nonlinear systems of difference equations without inputs are abundant in numerical analysis and optimization.

The general analysis of such systems is often very difficult; the approaches that can be taken include different kinds of approximations and ergodic theory. Some references are [7] and [5, Chapter 14].

The algorithm we propose in this paper stems from algebraic geometry, which implies that it can only be applied to systems with polynomial nonlinearities. It is based on what can be referred to as the correspondence of Gaussian elimination for polynomial equations, so called Gröbner bases.