[20] B. Jakubczyk. Existence and uniqueness of realizations of nonlinear systems. *SIAM J. Contr. Optimiz.*, 18(4):445–471, July 1980.

[21] Yu. Manin. *Cubic Forms.* North Holland, second edition, 1986.

[22] A.M. Perdon, G. Conte, and C.H. Moog. Some canonical properties of nonlinear systems. In M.A. Kaashoek, J.H. van Schuppen, and A.C.M. Ran, editors, *Realization and Modelling in System Theory. Proc. Intl. Symp. MTNS-89*, volume I, pages 89–96, Amsterdam, 1990. Birkhäuser.

[23] M. Reid. *Undergraduate Algebraic Geometry*, volume 12 of *London Mathematical Society Student Texts.* Cambridge University Press, 1988.

[24] J.G. Semple and L. Roth. *Introduction to Algebraic Geometry.* Oxford Universit Press, 1985. First published 1949.

[25] J.R. Sendra and F. Winkler. Symbolic parametrization of curves. *J. Symb. Comp.*, 1991. (To appear).

[26] I.R. Shafarevich. *Basic Algebraic Geometry.* Springer, 1974.

[27] E.D. Sontag and Y. Wang. Input/output equations and realizability. In M.A. Kaashoek, J.H. van Schuppen, and A.C.M. Ran, editors, *Realization and Modelling in System Theory. Proc. Intl. Symp. MTNS-89*, volume I, pages 125–132, Amsterdam, 1990. Birkhäuser.

[28] A.J. van der Schaft. On realization of nonlinear systems described by higher-order differential equations. *Math. Systems Theory*, 19:239–275, 1987.

[29] B.L. van der Waerden. *Algebra I*, volume 12 of *Heidelberger Taschenbücher.* Springer, eighth edition, 1971. (In German).

[30] F. Verhulst. *Nonlinear Differential Equations and Dynamical Systems.* Universitext. Springer, 1990.

[31] J.C. Willems. Input-output state-space representations of finite-dimensional linear time-invariant systems. *Linear Algebra Appl.*, 50:581–608, April 1983.

[6] S. Diop. Differential algebraic decision methods and some applications to system theory. *Theoretical Comp. Sci.*, 1991. Special issue covering "Traitements Algébriques et Informatiques des Séries Formelles Non Commutative". Proc. Deuxièmes Journées - Seminaire, Paris, 1990.

[7] S. Diop. Elimination in control theory. *Math. Control Signals Systems*, 4(1):17–32, 1991.

[8] M. Fliess. Automatique et corps différentiels. *Forum Mathematicum*, 1:227–238, 1989.

[9] M. Fliess. What the Kalman state variable representation is good for. In *Proc. 29th CDC*, volume 3, pages 1282–1287, Honolulu, Hawaii, 1990. IEEE CSS.

[10] M. Fliess and M. Hasler. Questioning the classic state space description via circuit examples. In M.A. Kaashoek, J.H. van Schuppen, and A.C.M. Ran, editors, *Realization and Modelling in System Theory. Proc. Intl. Symp. MTNS-89*, volume I, pages 1–12, Amsterdam, 1990. Birkhäuser.

[11] K. Forsman. Conversion from state space to input-output description of nonlinear systems. Submitted for publication, 1991.

[12] M.I. Freedman and J.C. Willems. Smooth representation of systems with differential inputs. *IEEE Trans. Aut. Contr.*, AC-23(1):16–21, February 1978.

[13] R. Gebauer, M. Kalkbrenner, B. Wall, and F. Winkler. CASA - A computer algebra package for constructive algebraic geometry. In S.M. Watt, editor, *Proc. of ISSAC'91*, pages 403–410, Bonn, Germany, July 1991. ACM Press.

[14] T. Glad. Nonlinear state space and input output descriptions using differential polynomials. In J. Descusse, M. Fliess, A. Isidori, and D. Leborgne, editors, *New Trends in Nonlinear Control Theory*, pages 182–189. Springer, 1988.

[15] T. Glad. Differential algebraic modelling of nonlinear systems. In M.A. Kaashoek, J.H. van Schuppen, and A.C.M. Ran, editors, *Realization and Modelling in System Theory. Proc. Intl. Symp. MTNS-89*, volume I, pages 97–105, Amsterdam, 1990. Birkhäuser.

[16] W. Gröbner. *Moderne algebraische Geometrie. Die Idealtheoretischen Grundlagen.* Springer, 1949.

[17] R. Hartshorne. *Algebraic Geometry.* Springer, 1977.

[18] A. Isidori. *Nonlinear Control Systems.* CCES. Springer, second edition, 1989.

[19] C.G.J. Jacobi. De determinantibus functionalibus. *J. Reine Angew. Math.*, 22(4):319–359, 1841.

A Maple session for example 4.1 looks as follows:

```
> with(linalg):
> read(par2sys);
> H:=[x^2-1,x*(x^2-1)]:
> f:=par2sys(H,[x]);
                                        2
                          f := [ 1/2 x  - 1/2 ]
>
```


## Summary

A special case of the realization problem has been investigated. It has been showed that a polynomial differential equation is possible to realize in rational explicit state space form iff the associated hypersurface is unirational. This gives a "lower bound" for the complexity of the problem: a very special case of the state space realization problem is equivalent to an unsolved problem in algebraic geometry. Some comments on systems with inputs as well as interpretations of the Lüroth problem have been made.



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## References


[1] S.S. Abhyankar. *Algebraic Geometry for Scientists an Engineers*, volume 35 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1990.

[2] M. Artin and D. Mumford. Some elementary examples of unirational varieties which are not rational. *Proc. London Math. Soc.*, 25:75–95, 1972.

[3] H. Blomberg and R. Ylinen. *Algebraic Theory for Multivariable Linear Systems*, volume 166 of *Mathematics in Science and Engineering*. Academic Press, 1983.

[4] G. Castelnuovo. Sulla razionalità delle involuzioni piane. *Math. Ann.*, 44:125–155, 1894.

[5] B. Char, K.O. Geddes, G.H. Gonnet, M.B. Monagan, and S.M. Watt. *Maple Reference Manual*. Symbolic Computation Group, Univ. of Waterloo, fifth edition, March 1988.

The resulting realization is

$$\dot{x} = \frac{1}{3}\sqrt[3]{u}, \quad y = x^3 \tag{20}$$

which is rational in $x$ over the field $\mathbb{Q}(\sqrt[3]{u})$. □

**Example 4.3** The DE $\dot{y}^m + y^m = 1$ is impossible to realize in rational state space form for all $m > 2$, for according to example 2.2 the corresponding curve is not rational, so by Lüroth's theorem (theorem 2.1) it is not unirational. □

As mentioned in section 2 a parametrization need not be surjective even if it is proper (example 2.3).

**Example 4.4** If we parametrize the surface

$$y_2 y_0 - y_1^3 = 0 \tag{21}$$

by

$$y_0 = x_1^2 x_2, \quad y_1 = x_1 x_2, \quad y_2 = x_1 x_2^2 \tag{22}$$

then the point $y_1 = y_2 = 0$, $y_0 = 1$ in (21) cannot be given by (22). If we on the other hand parametrize the surface as

$$y_0 = z_1^3, \quad y_1 = z_1 z_2, \quad y_2 = z_2^3 \tag{23}$$

the mentioned point is covered.

These phenomena should be compared to the results of Diop [7], stating that inequations may have to be added to the input-output relation in order to obtain system equivalence. □

The following Maple [5] procedure, `par2sys`, takes two input arguments, a list `H` containing the parametrization of the hypersurface and a list `vars` containing the names of the parameters. It returns the right hand side of a state space description. It presupposes the `linalg` package in Maple.

```
par2sys:=proc(H,vars)
    local A,b,i,n:
    n:=nops(H)-1:
    A:=matrix([convert(grad(H[1],vars),list)]):
    for i from 2 to n do A:=stack(A,grad(H[i],vars)) od;
    b:=vector([H[2..(n+1)]]);
    linsolve(A,b);
end:
```

can be realized for instance as

$$\dot{x} = -x^2, \quad y = x \tag{14}$$

or as

$$\dot{z} = -\frac{1}{2}z^3, \quad y = z^2 \tag{15}$$

The solution $y(t) = 1/(t+1)$ of (13) corresponds to a unique solution $x(t) = 1/(t+1)$ of (14) but to either of the two solutions $z(t) = \pm 1/\sqrt{4t+1}$ of (15). □

From a practical point of view there is a more important difference between the two realizations in example 3.2: (15) is stable but (14) is not. The reason is of course that we only consider $x, y, z \in \mathbb{R}$, so in (15) only *positive* initial values of $y$ are allowed.

One conclusion is that when we are comparing input-output relations with state space realizations we should not only add inequations to the I/O-relation as pointed out in [7] but also *inequalities* since we are dealing with real algebraic geometry.

# 4 Examples and a Computer Program

In this final section we give some further examples of rational realizations and briefly discuss how to handle inputs.

**Example 4.1** Consider the differential equation $\dot{y}^2 - y^2 - y^3 = 0$. By example 2.1 we can choose

$$h_0 = x^2 - 1, \quad h_1 = x\,(x^2 - 1) \tag{16}$$

The procedure described in the proof of theorem 3.1 can now be applied to $h_0, h_1$ to render

$$\dot{x} = \frac{1}{2}(x^2 - 1), \quad y = x^2 - 1 \tag{17}$$

A Maple procedure called `par2sys` has been written to go from a parametrization of a hypersurface to a state space realization of a DE. This function is listed below. □

If we add an input in the previous example, so that for instance

$$\dot{y}^2 - y^2 - y^3 = u \tag{18}$$

it is not at all clear if there is a rational realization. The following examples indicates how inputs can be handled, in some cases. Note that the realization is now rational in $y$ only, $u$ will typically occur algebraically, but not rationally.

**Example 4.2** If we have a system $\dot{y}^3 - uy^2 = 0$ we may write

$$h_0 = x^3, \quad h_1 = \sqrt[3]{u}\,x^2 \tag{19}$$

An important question is of course what happens if we add control signals as well. If we try to apply the argument above to this case we will have to adjoin $u$ in an arbitrary way to the ground field; a simple example is provided by example 4.2. How this works generally is not clear.

Let us now via an example consider an interesting problem that occurs in realization theory.

**Example 3.1** Consider the curve $y_1^2 - y_0 = 0$ which can be parametrized as $y_0 = x^2$, $y_1 = x$. This corresponds to the realization

$$\dot{x} = 1/2, \quad y = x^2 \tag{8}$$

of the DE $\dot{y}^2 = y$. But this realization does not cover the solution $y_0 \equiv 0$! There are several different ways to interpret this fact; one is to note that original equation does not necessarily generate a prime differential ideal. In this case

$$[y_1^2 - y_0] \ni y_1 (2y_2 - 1) \tag{9}$$

The realization only represents one of the components of the (reducible) differential algebraic manifold.

In a sense we have an algebraic interpretation of the well known fact that a DE has a unique solution if the right hand side is Lipschitz continuous [30].

If we instead take the curve $y_1 - y_0^2 = 0$, parametrized as $y_0 = x$, $y_1 = x^2$ we obtain the realization

$$\dot{x} = \frac{1}{2}x^2, \quad y = x \tag{10}$$

Here all solutions to the original equation are also solutions to the realization (10). The original equation generates a prime differential ideal:

$$[y_1 - y_0^2] \in \operatorname{Spec} k\{x_1, \ldots, x_n\} \tag{11}$$

□

This problem can occur in contexts where the analysis is more complicated: for instance for the two equations

$$y_1^3 - y_0^2 = 0 \quad \text{and} \quad y_1^2 - y_0^3 = 0 \tag{12}$$

The Lüroth problem also has an immediate interpretation in realization theory. If the I/O-relation has been properly parametrized then there is a one-to-one correspondence between solutions to the I/O-equation and trajectories in state space, which is not necessarily the case for improper parametrizations.

**Example 3.2** The system

$$y_1 + y_0^2 = 0 \tag{13}$$

**Proof.** See Gröbner [16, page 35] or the original article by Jacobi [19]. □

**Theorem 3.1** *Consider a given irreducible polynomial $p \in k[y_0, \ldots, y_n]$, where $y_n$ occurs effectively. The differential equation $p = 0$ is realizable in the form (3) iff the hypersurface $Z(p)$ is unirational.*

**Proof.** Suppose that we have a rational state space realization of the system $p = 0$. Then

$$y_0 = h_0, \ldots, y_n = h_n, \quad h_i \in k(x_1, \ldots, x_n) \tag{4}$$

where $h_i$ is the $i$:th Lie-derivative of $h$ w.r.t. the vector field $f$. Thus the $h_i$ provide a rational parametrization of the hypersurface $Z(p)$.

Suppose that $Z(p)$ is unirational. A parametrization is of the form (4) according to the definition of unirational variety. In this case the $x$ may serve as state variables and the $f$ can be obtained by solving a linear system of equations. Namely, the relations that have to hold between the $h_i$ are

$$h_1 = f \cdot \nabla h_0, \quad \ldots, \quad h_n = f \cdot \nabla h_{n-1} \tag{5}$$

where $\nabla$ denotes the gradient w.r.t. $x$. With matrix notation

$$\begin{pmatrix} \nabla h_0 \\ \vdots \\ \nabla h_{n-1} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \tag{6}$$

The jacobian matrix of $h_0, \ldots, h_{n-1}$ has to be non-singular according to lemma 3.1, so we can solve for $f$. □

The main point above is that the realization is *explicit*. If we allow implicit realizations then, of course, all input-output equations are possible to realize in state space form.

It is worth noting that the realization obtained above is automatically minimal: this is why we assume that $y_n$ occurs effectively in $p$.

We have thus transformed a special case of the state space realization problem to one in algebraic geometry. The status of the algebro-geometric problem is however not very satisfactory, as was explained in section 2.

In the discrete time case the situation is more complicated. We then get a system of nonlinear equations in the $f_i$. So if we consider the $h_i$ as given functions and the $f_i$ as variables we may write

$$\begin{aligned} h_0(f_1, \ldots, f_n) &= h_1(x_1, \ldots, x_n) \\ &\vdots \\ h_{n-1}(f_1, \ldots, f_n) &= h_n(x_1, \ldots, x_n) \end{aligned} \tag{7}$$

The general solubility of this system is not as simple as that of its continuous time correpondence.

**Theorem 2.4** *A nonsingular cubic hypersurface of dimension $\geq 2$ is unirational.*

**Proof.** See [21, page 52]. $\square$

**Example 2.1** The curve $x_1^2 - x_2^2 - x_2^3 = 0$ is rational. It can be parametrized as

$$x_1 \;=\; t^2 - 1, \quad x_2 \;=\; t\,(t^2 - 1) \tag{2}$$

$\square$

**Example 2.2** The curve $x_1^n + x_2^n - 1 = 0$ is not rational for any $n > 2$. For a proof of this see [26, page 7]. $\square$

It should be noted that a parameterization need not be surjective, even if it is proper:

**Example 2.3** The circle $x_1^2 + x_2^2 = 1$ is properly parametrized as

$$x_1 \;=\; \frac{1 - t^2}{1 + t^2}, \quad x_2 \;=\; \frac{2t}{1 + t^2}$$

However, the point $(-1, 0)$ on the circle does not correspond to any $t \in \mathbb{C}$. $\square$

For more comments on this, see [1, Lecture 28].

# 3 Rational Realizations

To give some insight in the rational realization problem we initially consider systems without inputs. The main result here is that a continuous time system without input given on scalar form has a rational explicit state space realization if and only if the associated hypersurface is unirational.

Let us thus consider state space realizations of the type

$$\dot{x}_1 = f_1(x), \; \ldots, \; \dot{x}_n = f_n(x), \quad y = h(x) \tag{3}$$

where $f_1, \ldots, f_n, h \in k(x)$ using the abbreviation $x = (x_1, \ldots, x_n)$.

The following lemma is of great importance; unfortunately it is remarkably difficult to find the lemma in any of the major modern textbooks on commutative algebra and algebraic geometry.

**Lemma 3.1** *The $n$ polynomials $f_1, \ldots, f_n$ in $k[x_1, \ldots, x_n]$ are algebraically dependent iff the jacobian matrix*

$$\Big(\frac{\partial f_i}{\partial x_j}\Big)_{ij}$$

*is singular.*

**Definition 2.2** An algebraic set $X$ is *unirational* if there is a rational mapping $\varphi : \mathbf{A}^n \to X$ such that $\varphi(\mathbf{A}^n)$ is dense in $X$. □

Thus $X$ is unirational if it can be parametrized with $n$ rational functions. Of course every rational variety is unirational. The difference between rationality and unirationality is one of injectivity:

**Definition 2.3** The parametrization of a unirational variety is *proper* if it is a birational equivalence. □

It is a highly non-trivial problem to determine if a given variety is rational or not; it is unsolved in the general case. Some results can be found in e.g. [26, 23, 16]. A necessary and sufficient condition for a plane curve to be rational is that its *genus* is zero. In Semple and Roth [24, Chapter XIII, §9.7] a necessary and sufficient condition is given for a *surface* to be rational. Algorithms for determining whether an algebraic *curve* is rational and in this case parametrize it are given in [25]. These algorithms have also been implemented in Maple [5] by Gebauer et al. [13].

Let us make some brief comments on a classical question, known as the *Lüroth problem*:

"When is a unirational variety rational?" Equivalently: "When is there a proper parametrization of a rationally parametrized variety?"

The following celebrated theorem gives the answer for curves:

**Theorem 2.1 (Lüroth's theorem)**
*Let $t$ be transcendental over $k$. If $L$ is a subfield of $k(t)$ then there is a $g \in k(t)$ such that $L \simeq k(g)$.*

**Proof.** [29, page 222] or [24, page 62]. □

In the language of geometry Lüroth's theorem says: a curve is unirational iff it is rational. For surfaces the same thing holds as long as $k$ is algebraically closed; this is known as Castelnuovo's theorem [4], see also [26, page 174]. However, the Lüroth problem has a negative solution in dimension $\geq 3$; there are varieties of dimension $n \geq 3$ that are unirational but not rational [2].

Let us mention some special cases of varieties that are rational:

**Theorem 2.2** *Hypersurfaces defined by an equation in which one of the variables occurs linearly are rational.*

**Proof.** Choose the other variables as parameters. □

**Theorem 2.3** *An irreducible hypersurface defined by an equation of degree 2 is rational.*

**Proof.** See [26, page 27]. □

for an arbitrary set of differential equations describing a system; it is discussed in for instance [6, 15].

If we only consider systems where all nonlinearities are polynomial it seems that differential and commutative algebra are proper mathematical tools in systems theory; see [7, 8, 11, 15].

Parametrization is the inverse problem of implicitization. In [11] the state space to input-output conversion problem is seen to be equivalent to implicitization, so it is natural to try to tackle the state space realization problem as a parametrization problem. However this approach presents some additional difficulties that does not appear in the implicitization problem. Basically the difficulty is that we cannot adjoin the input variables to the ground field and expect everything to work as smoothly as if we had $\mathbb{Q}$ for ground field.

We will assume that the reader is familiar with basic algebraic geometry and commutative algebra; some standard references are [17] and [26]. Here we just recall some notations:

The affine $n$-space is denoted $\mathbf{A}^n$. If $T$ is a subset of $k[x_1, \ldots, x_n]$ the manifold of $T$ is denoted $Z(T)$:

$$Z(T) = \{ \eta \in \mathbf{A}^n \mid \forall p \in T : \quad p(\eta) = 0 \}$$

The coordinate ring of a variety $V$ is denoted $k[V]$ and its field of quotients, the ring of rational functions on $V$, is written $k(V)$.

Some abbreviations that will be used are:

$$y_i = \frac{d^i}{dt^i} y(t), \qquad h_i = \frac{d^i}{dt^i} h(t) \tag{1}$$

and $k^* = k \setminus 0$.

# 2 Some Facts about Rational Varieties

In this section we recall some well known facts from algebraic geometry, and especially problems related to unirationality and birationality of varieties.

Some classical problems in algebraic geometry concern the rationality of varieties. A variety is said to be unirational if it can be parametrized using rational functions, and rational if the parametrization is injective.

**Definition 2.1** Let $V$ be an $n$-dimensional affine variety. We say that $V$ is *rational* if the field of functions $k(V)$ is a purely transcendental extension of $k$. □

Another way to define rationality is to say that an $n$-dimensional affine variety $V$ is rational if it is birationally equivalent to $\mathbf{A}^n$. There is a slightly weaker concept called unirationality:

# On Rational State Space Realizations

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**Abstract.** It is investigated when a polynomial input-output differential equation can be realized in rational, explicit state space form. In the case where there are no inputs the problem is seen to be equivalent to a famous problem in algebraic geometry, which is solved only in various special cases. For systems with inputs the problem is more complicated, as is the discrete time case.

**Keywords:** Realization theory, polynomial systems, algebraic geometry, rational varieties

## 1 Introduction

The problem of state space realization of nonlinear systems is a classical one, recognized as being quite difficult. Depending on what kinds of realizations we allow, the problem becomes more or less difficult. The most conservative generalization of the Kalmanian state space theory for linear systems to nonlinear systems is the one where no derivatives of the input are allowed to occur, and where all $\dot{x}_i$ occur linearly. If we do not have both these requirements the realization problem is easier, and if we drop both it is trivial. In [8] and [9] reasons for being rather liberal concerning the definition of state space realization are given. It is argumented that derivatives of the input should be allowed, and in [10] some practical examples of systems with implicit equations are given.

In this paper we treat the special case of rational realizations, i.e. realizations in explicit state space form where all right hand sides are rational functions. (Even if we are of the opinion that implicit equations should be allowed it is interesting to know when an explicit realization is possible.) We will allow derivatives of the input.

Let us just give some references to earlier work on the realization problem. The nonlinear realization problem is treated in e.g. [8, 12, 14, 18, 20, 22, 27, 28] and the linear one in [3, 31]. *Modeling* consists in retrieving a state space realization