Local Modeling and Robust Estimation for High-Resolution Direction Finding[1]

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Abstract

Many signal processing applications are concerned with estimating signal parameters with high accuracy from sensor array measurements. To obtain accurate location estimates, knowledge of the array response is required. Most estimation techniques are sensitive to perturbations in the array model and array calibration is often necessary. This paper discusses the issue of generating the array response parametrization from a finite number of noise contaminated calibration vectors. An estimation technique based on local parametric modeling of the array response is proposed. The potential improvement using the suggested scheme rather than an ideal array model is demonstrated on real data collected from a full-scale hydro-acoustic array.

1 Introduction

High-resolution signal parameter estimation is a problem of significance in many signal processing applications. These include emitter location estimation, jammer suppression, system identification, and time series analysis. Sensor arrays play an important role in areas such as radar, sonar, and electronic surveillance, where the objective is to detect and estimate incoming signals. Signal parameters of interest can include azimuth, elevation, frequency, etc. Though not necessary, we shall herein assume that only the azimuthal angle, termed the *direction-of-arrival* (DOA) is of interest.

Much of the recent work in array processing has focussed on methods for high-resolution DOA estimation, see e.g. [1, 2]. To yield reliable estimates, accurate knowledge of the array parametrization is necessary. Thus, it is assumed that the array response is known for all DOAs of interest. The collection of all array response vectors over the parameter range is often termed the *array manifold*, [1]. See e.g., [3, 4] for an analysis of the MUSIC algorithm in the presence of modeling errors. For certain types of errors, it is possible to estimate a set of array response parameters and the DOAs simultaneously. Such *auto-calibration* techniques are reported in, e.g. [5, 6]. Apart from their limited applicability, these methods suffer from high computational demands and degraded performance, due to the increased number of parameters. Alternative, so-called *robust estimation* techniques are proposed in [7, 8]. However, these require information of the (structure of the) array response uncertainty and will in general also require some sort of array calibration.

In many applications, *array calibration* using sources at known positions appears to be necessary for high-performance DOA estimation. A set of calibration data is then collected in the presence of a single emitter, whos DOA is varied in a known way. This paper addresses some of the practical issues that arise in generating the array response at arbitrary DOAs from the given calibration data set. For high-resolution signal parameter estimation techniques to be successful, the interpolated array manifold has to satisfy certain smoothness conditions. Furthermore, since real-terrain measurements are time consuming and expensive it is clearly desirable to use as few calibration points as possible. To still achieve accurate calibration, a model based approach appears the most practical. The array response model should be flexible enough to accommodate sensor and geometry imperfections without increasing the computational burden too much. Herein, a so-called *local model* is introduced. The array response is modeled as a known nominal response and a multiplicative unknown deviation. The nominal model may be obtained, e.g. from knowledge of the array geometry and the deviations are modeled by a polynomial expansion. For each given DOA, the model is linear in the polynomial coefficients and by appropriate formulation of the optimization criterion, a closed form solution is obtained.

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2 Problem Formulation

Assume an array of m sensors at which d narrow band plane waves are arriving. The center frequency of the signals is denoted ω. Let the parameter θ_i denote the DOA associated with the i^{th} signal, $s_i(t)$. The time-delay of a signal arriving at sensor k relative to the same signal at sensor 1 (the reference sensor) is given by $\mathbf{r}_k \cdot \mathbf{k}(\theta_i)$, where $\mathbf{r}_k$ is the position vector describing the location of the k^{th} sensor and $\mathbf{k}(\theta_i)$ is the wave-vector of the i^{th} signal. Let the response of the k^{th} receiver at frequency ω and in the direction θ_i be $\rho_k(\theta_i)$. The functional form of $\rho_k(\theta_i)$ as well as the sensor positions, $\mathbf{r}_k$ are in general uncertain. We shall assume that a nominal array response model is available and introduce a parametrized model of the deviation from this model. The response of the k^{th} sensor to a wavefront from θ_i is denoted $a_k(\theta_i, \mathbf{c}) = g_k(\theta_i, \mathbf{c}) f_k(\theta_i)$, where $f_k(\theta_i)$ is known and the unknown array parameters are denoted $\mathbf{c}$. Consider a p^{th} order model of $g_k(\theta_i, \mathbf{c})$. Locally around a point θ, the response of the k^{th} sensor is

$$a_k(\theta_i, \mathbf{c}) = (c_{k1} + c_{k2}\Delta\theta_i + \cdots + c_{kp+1}\Delta\theta_i^p) f_k(\theta_i) \ , \tag{1}$$

where $\Delta\theta_i = \theta - \theta_i$. If the geometry of the array is known, the nominal model could for example be $f_k(\theta_i) = e^{-j\omega \mathbf{r}_k \cdot \mathbf{k}(\theta_i)}$. The individual sensor responses are collected in the m-vector $\mathbf{a}(\theta_i, \mathbf{c})$. Using the first sensor as a reference we can write

$$\mathbf{a}(\theta_i, \mathbf{c}) = (1,\ \mathbf{c}^* \boldsymbol{\phi}_i^*)^* \ , \tag{2}$$

where $(\cdot)^*$ denotes complex conjugate transpose and

$$\begin{aligned} \boldsymbol{\phi}_i &= [\mathbf{F}_i, \Delta\theta_i \mathbf{F}_i, \ldots, \Delta\theta_i^p \mathbf{F}_i] && (3) \\ \mathbf{F}_i &= \mathrm{diag}[f_2(\theta_i), \ldots, f_m(\theta_i)] && (4) \\ \mathbf{c} &= \mathrm{vec}(\mathbf{C}) \ . && (5) \end{aligned}$$

Here $\mathbf{C}$ is the matrix whose elements are c_{kl} for $k = 2, \ldots, m$, $l = 1, \ldots, p+1$ and the $(m-1)(p+1)$-vector $\mathbf{c}$ is obtained by stacking the columns of $\mathbf{C}$.

The output of the the k^{th} sensor is modeled by

$$x_k(t) = \sum_{i=1}^{d} a_k(\theta_i, \mathbf{c}) s_i(t) + n_k(t) \ , \tag{6}$$

where $n_k(t)$ is additive noise. The individual sensor outputs, $x_k(t)$, are collected in the vector $\mathbf{x}(t)$ resulting in

$$\mathbf{x}(t) = \sum_{i=1}^{d} \mathbf{a}(\theta_i, \mathbf{c}) s_i(t) + \mathbf{n}(t) = \mathbf{A}(\boldsymbol{\theta}, \mathbf{c})\mathbf{s}(t) + \mathbf{n}(t) \tag{7}$$

The problem formulation is now stated as follows: Given an *estimation* data set $\mathbf{x}(t)$ for $t \in \{t_1, \ldots, t_N\}$ the major objective is to estimate the parameter vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d)^T$. Since the array propagation vector also depends on $\mathbf{c}$, which cannot in general be estimated from $\mathbf{x}(t)$ alone, a *calibration* set is used. In the l^{th} calibration set, a single emitter at the known DOA η_l is present. The signal model is thus

$$\mathbf{z}_l(t) = \mathbf{a}(\eta_l, \mathbf{c}) \mathbf{s}_l(t) + \mathbf{n}_l(t) \ . \tag{8}$$

Observations of $\mathbf{z}_l(t)$ for $l = 1, \ldots, M$ and $t \in \{t_1, \ldots, t_{N_l}\}$ are assumed available.

3 Array Calibration

In this section the problem of estimating the array parameters and the unknown DOAs is considered. A maximum likelihood approach is used, which is later approximated to yield a closed-form solution.

3.1 Maximum Likelihood Approach

To apply the ML technique, a stochastic framework for the involved signals and noise must be formulated. Following [2], the emitter signals and the noise are modeled as temporally white, independent, zero-mean Gaussian random processes. In addition, the noise is assumed to be spatially white, i.e., the spatial noise covariance is proportional to the identity. Thus we have for the calibration data

$$\begin{aligned} \mathbf{z}_l(t) &\in \mathrm{N}(\mathbf{0}, \mathbf{R}_l) && (9) \\ \mathbf{R}_l &= \pi_l \mathbf{a}(\eta_l, \mathbf{c}) \mathbf{a}^*(\eta_l, \mathbf{c}) + \sigma_l^2 \mathbf{I} && (10) \end{aligned}$$

where η_l is known and $\pi_l, \mathbf{c}$ and σ_l^2 are all unknown. The distribution of the estimation data is

$$\begin{aligned} \mathbf{x}(t) &\in \mathrm{N}(\mathbf{0}, \mathbf{R}) && (11) \\ \mathbf{R} &= \mathbf{A}(\boldsymbol{\theta}, \mathbf{c}) \mathbf{P} \mathbf{A}^*(\boldsymbol{\theta}, \mathbf{c}) + \sigma^2 \mathbf{I} && (12) \end{aligned}$$

where now $\boldsymbol{\theta}, \mathbf{c}, \mathbf{P}$ and σ^2 are unknown.

An optimal approach requires simultaneous estimation of all unknown parameters. However, it is natural to base the estimate of the array parameters, $\mathbf{c}$, on the calibration data only. This does not significantly change the properties of the estimates if $\sum_{l=1}^{M} N_l \gg N$ and/or if the signal-to-noise ratio (SNR) is much higher in the calibration data. Thus, one can conceptually think of this as two separate steps

1. Estimate $\mathbf{c}$ (and π_l, σ_l^2) from observations of $\mathbf{z}_l(t)$ only.
2. Given $\hat{\mathbf{c}}$, estimate $\boldsymbol{\theta}$ (and $\mathbf{P}, \sigma^2$) using measurements of $\mathbf{x}(t)$.

It is possible to approach this problem in different ways. If the parameter range is small, the manifold model may be a sufficiently accurate *global* model and a fixed set of array parameters can be estimated. Herein, the model is viewed as *locally* valid, i.e., the parameters, $\mathbf{c}$, are functions of θ. This suggests a two step procedure; for every θ estimate $\mathbf{c}$ and then evaluate a cost function (e.g. the MUSIC spatial spectrum) for that θ. Later, we show that with appropriate approximations, $\mathbf{c}$ can be solved for explicitly without loosing accuracy.

The negative log likelihood function (ignoring parameter-independent terms) of the calibration data can be expressed as

$$L(\mathbf{c}, p_l, \sigma_l^2) = \sum_{l=1}^{M} N_l \left(\log|\mathbf{R}_l| + \mathrm{Tr}\{\mathbf{R}_l^{-1}\hat{\mathbf{R}}_l\}\right) \qquad (13)$$

where $\hat{\mathbf{R}}_l$ is the sample covariance

$$\hat{\mathbf{R}}_l = \frac{1}{N_l}\sum_{k=1}^{N_l} \mathbf{z}_l(t_k)\mathbf{z}_l^*(t_k) \qquad (14)$$

The ML estimates of $\mathbf{c}, \pi_l, \sigma_l^2$, are the minimizing arguments of (13). Using the same technique as in [2], the above can be minimized explicitly with respect to π_l and σ_l^2 resulting in the concentrated criterion

$$\sum_{l=1}^{M} N_l \log\left| \hat{\pi}_l(\mathbf{c})\mathbf{a}(\eta_l, \mathbf{c})\mathbf{a}^*(\eta_l, \mathbf{c}) + \hat{\sigma}_l^2(\mathbf{c})\,\mathbf{I}\,\right| \qquad (15)$$

where $\hat{\pi}_l(\mathbf{c})$ and $\hat{\sigma}_l^2(\mathbf{c})$ are fairly complicated functions of $\mathbf{c}$ whose precise form is irrelevant.

3.2 Least Squares Approximation

Since for a given θ, $\mathbf{a}(\mathbf{c})$ is linear in $\mathbf{c}$, see (2), one may hope for a closed form solution for $\hat{\mathbf{c}}(\theta)$. However, to achieve this the criterion function (15) must be simplified. Applying the results of [9, 10], shows that each term in the sum (15) can be replaced by the expression

$$N_l \frac{(\hat{\lambda}_l - \hat{\sigma}_l^2)^2}{\hat{\sigma}_l^2\hat{\lambda}_l}|\hat{\mathbf{e}}_l - \mathbf{a}(\eta_l, \mathbf{c})t_l|^2 \qquad (16)$$

without affecting the asymptotic (for large N_l) properties of the estimate. In (16), $\hat{\lambda}_l$ is the largest eigenvalue of $\hat{\mathbf{R}}_l$, $\hat{\mathbf{e}}_l$ is the corresponding unit-norm eigenvector, $\hat{\sigma}_l^2$ is the average of the $m-1$ smallest eigenvalues, and t_l is a free "scaling" parameter. Inserting (16) in (15) yields the cost function

$$\sum_{i=1}^{M} N_l \frac{(\hat{\lambda}_l - \hat{\sigma}_l^2)^2}{\hat{\sigma}_l^2\hat{\lambda}_l}|\hat{\mathbf{e}}_l - \mathbf{a}(\eta_l, \mathbf{c})t_l|^2 \qquad (17)$$

which is to be minimized with respect to $\mathbf{c}$ and t_l. Making use of the convention $\mathbf{a}_1(\eta_l, \mathbf{c}) = 1$, we see that a consistent (as $N_l \to \infty$) estimate of t_l is simply $\hat{t}_l = \hat{\mathbf{e}}_{l1}$, the first component of $\hat{\mathbf{e}}_l$. Inserting this into (17) and using (2) gives the final criterion

$$\begin{aligned} V(\mathbf{c}) &= \sum_{l=1}^{M} w_l|\hat{\mathbf{e}}_l - \mathbf{a}(\eta_l, \mathbf{c})\hat{\mathbf{e}}_{i1}|^2 \\ &= \sum_{l=1}^{M} w_l|\hat{\mathbf{u}}_l - \hat{\mathbf{e}}_{i1}\boldsymbol{\phi}_l\mathbf{c}|^2 \end{aligned} \qquad (18)$$

where

$$w_l = N_l\frac{(\hat{\lambda}_l - \hat{\sigma}_l^2)^2}{\hat{\sigma}_l^2\hat{\lambda}_l} \qquad (19)$$

and where $\hat{\mathbf{u}}_l$ contains the $m-1$ last elements of $\hat{\mathbf{e}}_l$. Clearly (18) is quadratic in $\mathbf{c}$ and $\hat{\mathbf{c}}(\theta)$ is obtained as the least squares solution to (18)

$$\hat{\mathbf{c}}(\theta) = \boldsymbol{\Phi}^\dagger\mathbf{u} = (\boldsymbol{\Phi}^*\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^*\mathbf{u} \qquad (20)$$

where we have used the notation

$$\boldsymbol{\Phi} = \begin{bmatrix} \sqrt{w_1}\hat{\mathbf{e}}_{11}\boldsymbol{\phi}_1 \\ \vdots \\ \sqrt{w_M}\hat{\mathbf{e}}_{M1}\boldsymbol{\phi}_M \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \sqrt{w_1}\mathbf{u}_1 \\ \vdots \\ \sqrt{w_M}\mathbf{u}_M \end{bmatrix}$$

Let us again stress that the solution, (20), depends on the DOA of interest (observe that $\boldsymbol{\Phi}$ depends on θ through the $\boldsymbol{\phi}_l$:s). Hence, the principal eigenvector of each calibration data sample covariance matrix must be stored along with the corresponding eigenvalue and the estimated noise variance. To use (15) directly requires storage of the complete sample covariance matrices and in addition a closed form expression for $\hat{\mathbf{c}}$ is unavailable.

3.3 Modeling Errors

The philosophy behind using (2) with θ-dependent coefficients is that a global model (i.e., valid for all DOAs) would be impractically complicated due to imperfect receiver characteristics, mutual coupling effects, etc. However, the present weighting in equation (20) only accounts for finite sample effects. It is useful to apply a further weighting to reduce the bias due to modeling errors. When estimating the array parameters at θ, the calibration points that are close to θ should be given larger weights than those that are further away. Thus the block rows of $\boldsymbol{\Phi}$ and $\mathbf{u}$ are replaced by $\sqrt{q_l w_l}\hat{\mathbf{e}}_{l1}\boldsymbol{\phi}_l$ and $\sqrt{q_l w_l}\mathbf{u}_l$ respectively. A natural choice for q_l is

$$q_l = \mu^{(\Delta\theta_l)^2} = \mu^{(\theta-\eta_l)^2} \qquad (21)$$

for some constant μ, $0 < \mu < 1$.

4 DOA Estimation

The focus in this paper is on the calibration problem. However, we shall briefly discuss integration of the calibration procedure with a technique for generating DOA estimates.

Following Section 3.1, the ML estimate of $\boldsymbol{\theta}$ based on measurements of $\mathbf{x}(t)$ is easily formulated. However, to illustrate the joint calibration and DOA estimation we will instead generate "MUSIC spectra", [1],

$$P_{MU}(\theta) = \frac{\mathbf{a}^*(\theta, \hat{\mathbf{c}}(\theta))\mathbf{a}(\theta, \hat{\mathbf{c}}(\theta))}{\mathbf{a}^*(\theta, \hat{\mathbf{c}}(\theta))\hat{\mathbf{E}}_n \hat{\mathbf{E}}_n^* \mathbf{a}(\theta, \hat{\mathbf{c}}(\theta))} \qquad (22)$$

where the matrix $\hat{\mathbf{E}}_n$ contains the $m-d$ "smallest" eigenvectors of the sample covariance. The manifold vectors have the following form

$$\mathbf{a}(\theta, \hat{\mathbf{c}}(\theta)) = (1,\ \mathbf{u}^* \boldsymbol{\Phi}^{\dagger *} \boldsymbol{\phi}^*)^* \ . \qquad (23)$$

When devising a decent algorithm, knowledge of the derivatives of $\mathbf{a}$ are required and can be derived from (23). Further details of such an approach will be published elsewhere. Let us only mention that the main computational requirement in general is forming $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ since M may be quite large compared to m. The bulk of this operation can however be done off-line. To see this examine the case where $p=2$

$$\begin{aligned} \boldsymbol{\phi}_l &= [1\ \Delta\theta_l\ \Delta\theta_l^2] \otimes \mathbf{F}_l \\ &= [1\ \theta_l\ \theta_l^2] \begin{bmatrix} 1 & \theta & \theta^2 \\ 0 & -1 & -2\theta \\ 0 & 0 & 1 \end{bmatrix} \otimes \mathbf{F}_l \qquad (24) \end{aligned}$$

where it is important to note that only the 3×3 matrix depends on θ. Using the "Mixed Product Rule" $(A \otimes B)(C \otimes D) = AC \otimes BD$ on $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ the computational savings become evident.

5 Experimental Results

Below, preliminary results from a project aimed at examining the viability of accurate source localization methods on hydro acoustic sensor array data are presented. The data is collected by a seven-element equispaced linear array with an aperture of approximately one wavelength. The array is mounted horizontally and consists of sensors which are designed to ideally be omni-directional in the horizontal plane. A far field narrow band source is placed, in turn, in 18 different locations around array broadside. The purpose is to estimate the azimuthal angle to the emitter. The noise is assumed spatially white, since no measurements of the noise covariance is available. Cleary, this produces noise modeling errors in addition to the array response errors.

In the experiments to be presented, data from one source location is considered the estimation set, whereas the other data sets are used for calibration.

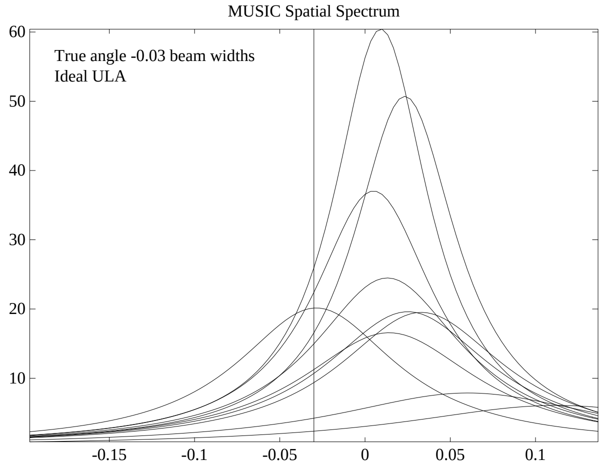

Figure 1: *Ideal uniform linear array manifold, experiment 1. True emitter location indicated by solid line.*

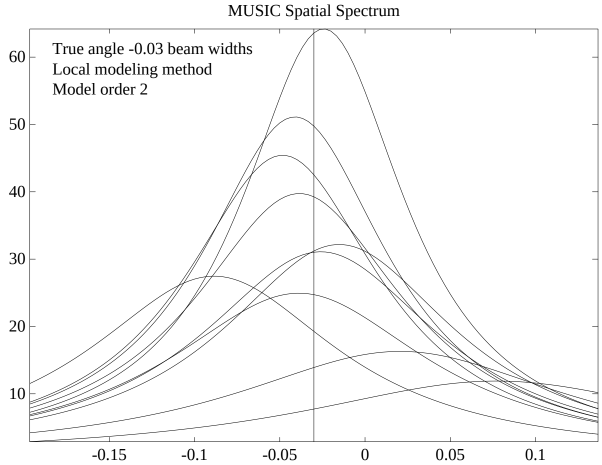

Figure 2: *Locally modeled array manifold, experiment 1. True emitter location indicated by solid line.*

The estimation data is divided into 10 segments of .32 seconds each and the corresponding MUSIC spectra are displayed. The x-axis is normalized in terms of the 3 dB array beam width at broadside, (zero degrees) and the SNR is approximately 6dB. The vertical line indicates the true emitter location. In Figures 1 and 3 the array manifold is assumed to be that of an ideal ULA and in Figures 2 and 4 the local modeling technique is used. No θ-dependent weighting (q_i) is employed.

Note the reduced bias in the peak locations for the local modeling technique. We expect that the performance differences will be even more pronounced when estimating several signal parameters simultaneously by a multi-dimensional technique, e.g. ML. The "variance" seen in the location estimates is due to measurement noise and time varying conditions.

It should be noted that in the examples above, the

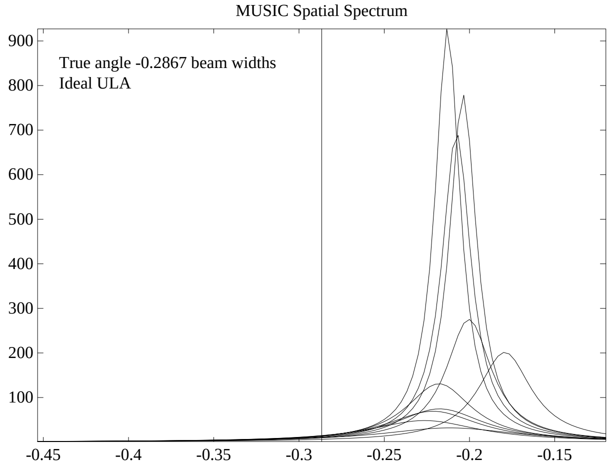

Figure 3: *Ideal uniform linear array manifold, experiment 2. True emitter location indicated by solid line.*

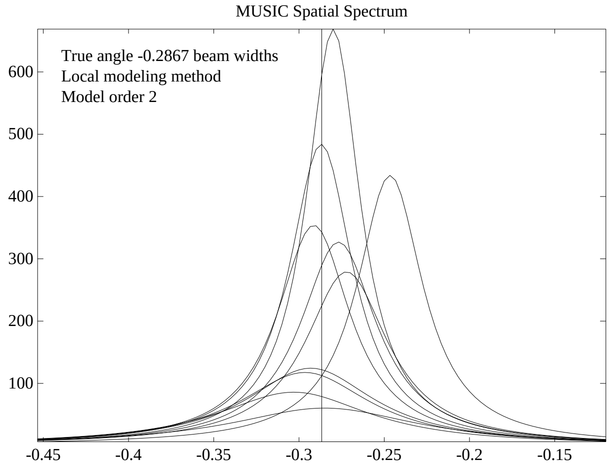

Figure 4: *Locally modeled array manifold, experiment 2. True emitter location indicated by solid line.*

estimated DOA is in the center region of the calibration set. When applying the local modeling technique in regions close to the boundary of the calibration set, the performance degrades and is comparable to that of the ideal ULA manifold.

6 Conclusions

High-resolution signal parameter estimation techniques require accurate knowledge of the array characteristics. It is often impossible to obtain an analytical description of the array response and one must resort to measurements, i.e., calibration of the array. This paper addresses the problem of generating the array manifold from a finite number of noisy calibration points in the parameter space. A model based approach motivated by local modeling of the array response is proposed. The ML estimator is formulated and a large sample approximation is described leading to a closed form solution of the parameters in the array model. By applying this technique together with a standard DOA estimation procedure, a significant reduction in estimation bias can be obtained. This is demonstrated on data collected from a large scale hydro acoustic array.

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