Massachusetts Institute of Technology

Physics Department

Physics 8.20 **IAP 2008**
Introduction to Special Relativity

Problem Set #1 Solutions

Note: All problems marked **(RH)** *are taken from Resnick and Halliday, Basic Concepts in Relativity (MacMillan, New York, 1992).*

1. **Speeds** (4 points)

   What fraction of the speed of light does each of the following speeds represent? (If any calculation is required, use Newtonian mechanics; ignore any relativistic effects. In cases where calculation — as opposed to unit conversion — is required, comment on whether your Newtonian results are good approximations to the correct speed.)

   (a) A billiard ball moving at 1 m/sec.

   $$\beta = \frac{1\,m/s}{3\times 10^8\,m/s} = 3.33\times 10^{-9}$$

   (b) A Kurt Schilling fast-ball, crossing the plate at 90 miles/hr.

   $$\beta = \left(90\,\frac{miles}{hr}\times 1609\,\frac{m}{mile}\times\frac{1}{3600}\,\frac{hr}{s}\right)/(3\times 10^8\,m/s) = 1.34\times 10^{-7}$$

   (c) A satellite orbiting the Earth in low-Earth orbit. The radius of the Earth is 6.4×10^6 m.
   `The Earth's gravitational force on the satellite causes its centripetal acceleration.`

   $$\frac{mv^2}{R_e} = mg$$
   $$\Rightarrow v = \sqrt{gR_e} \approx 7.9\times 10^3 m/s$$
   $$\Rightarrow \beta \approx 2.6\times 10^{-5}$$

   `Since` $\beta \ll 1$`, the relativistic corrections are not significant and the Newtonian calculation gives a good approximation to the true speed.`

   (d) A proton dropped onto the surface of a white dwarf star from rest at a great distance. A white dwarf star is a compact star with a mass of about 1.4 times the mass of the sun and a radius of about 5,000 km. The mass of the sun is 2.0×10^{30} kg. Assume that the proton starts at rest infinitely far from the star. Calculate its kinetic energy when it crashes into the neutron star surface, and then calculate its velocity.
   `Using conservation of mechanical energy,`

   $$\frac{1}{2}mv^2 - GM_w m/R_w = 0$$

$$\Rightarrow v = \sqrt{\frac{2GM_w}{R_w}} \approx 8.6 \times 10^6 m/s$$
$$\Rightarrow \beta \approx 0.03.$$

`Newtonian approximation holds good in this case as well.`

(e) A spaceship, starting from rest, accelerated at 1m/sec^2 for 50 years. Note this is about 1/10th the acceleration due to gravity at the surface of the earth.

$$\begin{aligned} v &= a\,t \\ &= 1\frac{m}{s^2} \times 50\,yr \times (365 \times 24 \times 3600)\,\frac{s}{yr} \\ &\approx 1.6 \times 10^9 m/s \\ \Rightarrow \beta &\approx 5. \end{aligned}$$

`The Newtonian calculation is inaccurate; it gives a speed about 5 times larger than the speed of light.`

(f) An electron in the circular LEP accelerator at CERN in Geneva. This is the highest energy accelerator in the world, and accelerates electrons to a final energy of 1.01×10^{11} electron-Volts.
`Using conservation of energy,`

$$\frac{1}{2}mv^2 = qV$$

$$v = \sqrt{\frac{2qV}{m}}$$

$$v = \sqrt{\frac{2 \times 1.01 \times 10^{11}(eV) \times 1.60 \times 10^{-19}(J/eV)}{9.11 \times 10^{-31}(kg)}} = 1.88 \times 10^{11} m/s \approx 600c$$

$$\beta \approx 600$$

`The Newtonian calculation is inaccurate.`

2. Dropping a ball on a train (4 points)

A train moves at constant speed 20 m/s in the x direction as measured by ground observers. A ball on the train is released from rest at a height of 5 m. Let S denote the ground frame of reference and S' the train's rest frame.

(a) Describe the motion of the ball as seen by an observer on the train. Write equations describing the ball's motion in the frame S' by specifying x', y', z' as functions of t', with the initial condition that ball is released at the position $x' = y' = 0$, $z' = 5$ m at time $t' = 0$.

`In the frame` S' `(train's rest frame) :`

$$\vec{a} = -g\hat{z}$$

$$\vec{r'}_0 = +5\hat{z}$$

$$\vec{v'}_0 = 0$$

Note: Since two frames have the same orientation: $\hat{z} = \hat{z}'$, etc.
For a constant acceleration we have:

$$\vec{r'} = \frac{1}{2}\vec{a}t'^2 + \vec{v'}_0 t' + \vec{r'}_0$$

Write the components of the above vector equation:

$$\begin{aligned} x' &= 0 \\ y' &= 0 \\ z' &= -\frac{1}{2}gt'^2 + 5\,. \end{aligned}$$

The ball is moving down in the z direction.

(b) Use the Galilean transformation to write equations that describe the ball's motion in the frame S. Sketch the trajectory in this frame and state what curve it corresponds to.

Galilean Transformation (assume at $t' = 0$; $t = 0$ and Origins of the two frames coincide) is :

$$x = x' + u_x t'$$

$$y = y' + u_y t'$$

$$z = z' + u_z t'$$

$$t = t'.$$

Where $\vec{u}$ is relative velocity of the S' with respect to S which for this problem is $\vec{u} = +20\hat{x}$. Then we get:

$$x = 0 + 20t' = 20t$$

$$y = 0$$

$$z = -\frac{1}{2}gt'^2 + 5 = -\frac{1}{2}gt^2 + 5$$

We should get rid of t for obtaining the trajectory in the z-x plane:

$$t = x/20$$

$$\Rightarrow z = -\frac{g}{800}x^2 + 5$$

which is a parabola.

(c) By differentiating the expressions derived in parts (a) and (b), find the three components of the ball's velocity and acceleration in each frame. Verify that Newton's second law is satisfied in each frame.

From expression for $x'(t')$, $y'(t')$, $z'(t')$ we have in frame S' : (We will write the equations in the vetor form, it's just a compact way of writing 3 equations)

$$\vec{v'} = \frac{d\vec{r'}}{dt'} = -gt'\hat{z}$$

$$\vec{a'} = \frac{d\vec{v'}}{dt'} = -g\hat{z}\,.$$

In the frame S from the expressions for $x(t)$, $y(t)$, $z(t)$ we have:

$$\vec{v} = \frac{d\vec{r}}{dt} = 20\hat{x} - gt\hat{z}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -g\hat{z}$$

Which results in:

$$\vec{v} = \vec{v'} + \vec{u}$$

$$\vec{a} = \vec{a'}$$

Newton's second law is obviously satisfied in each frame because

$$m\vec{a} = \vec{F} = -mg\hat{z} = \vec{F}' = m\vec{a}'$$

3. The Galilean transformation generalized (RH) (4 points)

Write the Galilean coordinate transformation equations (see Resnick Eqs. 1-1a and 1-1b) for the case of an arbitrary direction for the relative velocity $\vec{v}$ of one frame with respect to the other. Assume that the corresponding axes of the two frames remain parallel. (Hint: let $\vec{v}$ have components v_x, v_y, v_z.)

The laws of physics (and how to transform from one frame to another) are in the vector form since Nature doesn't have any preferance for any direction. Assume the two coordinates have the same orientation at all times. (If they don't have the same orientation even when there is no relative velocity $x \neq x'$, etc. They will be related by a rotation which at this moment is not of interest to us.) The vector form of Galilean transformation takes the form:

$$\vec{r'} = \vec{r} - \vec{v}t$$

$$t' = t$$

Write this in the component form:

$$x' = x - v_x t$$

$$y' = y - v_y t$$

$$z' = z - v_z t$$

$$t' = t$$

4. Frame independence of momentum conservation (RH) (4 points)

(a) Let the two particles of masses m_1 and m_2 have pre-collision velocities of $\vec{u}_1$ and $\vec{u}_2$ respectively. Let the post-collision velocities be $\vec{v}_1$ and $\vec{v}_2$ respectively. Then, by conservation of momentum,

$$\begin{aligned} \vec{P}_i &= \vec{P}_f\,, \\ \Rightarrow m_1\vec{u}_1 + m_2\vec{u}_2 &= m_1\vec{v}_1 + m_2\vec{v}_2\,. \end{aligned}$$

Now the collision is observed from a train moving with a velocity $\vec{v}$ relative to the first observer. The total momentum before the collision in the train's frame is

$$\vec{P}_i' = m_1\vec{u}_1' + m_2\vec{u}_2' = m_1(\vec{u}_1 - \vec{v}) + m_2(\vec{u}_2 - \vec{v}) = \vec{P}_i - (m_1 + m_2)\vec{v}\,.$$

Similarly, the final momentum is

$$\vec{P}_f' = \vec{P}_f - (m_1 + m_2)\vec{v}\,.$$

Therefore,

$$\begin{aligned} \vec{P}_f' &= \vec{P}_f - (m_1 + m_2)\vec{v} \\ &= \vec{P}_i - (m_1 + m_2)\vec{v} \text{ (using } \vec{P} \text{ conservation in unprimed frame)} \\ &= \vec{P}_i'\,. \end{aligned}$$

Hence, if momentum is conserved in one inertial reference frame, then it is conserved in all inertial reference frames.

(b) Now a transfer of mass takes place and the initial masses are m_1 and m_2 while the final masses are m_1' and m_2'. Proceeding exactly as in part (a), we find that

$$\begin{aligned} \vec{P}_i' &= \vec{P}_i - (m_1 + m_2)\vec{v}\,, \\ \vec{P}_f' &= \vec{P}_f - (m_1' + m_2')\vec{v} \end{aligned}$$

Now if

$$\begin{aligned} \vec{P}_f' &= \vec{P}_i' \\ \Rightarrow \vec{P}_f - (m_1' + m_2')\vec{v} &= \vec{P}_i - (m_1 + m_2)\vec{v} \\ \Rightarrow m_1 + m_2 &= m_1' + m_2' \end{aligned}$$

i.e. momentum in ground frame is conserved only if the total mass is conserved $(m_1 + m_2 = m_1' + m_2')$.

5. The invariance of elastic collisions (RH) (4 points)

A collision between two particles in which kinetic energy is conserved is defined to be *elastic*. Show, using the Galilean velocity transformation equations, that if a collision is elastic in one intertial reference frame, it will also be elastic in all other such frames. Could this result have been predicted from the principle of conservation of energy?

For elastic collision in the frame S we have:

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \left(\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2\right) = 0 \tag{1}$$

The question is do we have

$$\frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2 - \left(\frac{1}{2}m_1 u_1'^2 + \frac{1}{2}m_2 u_2'^2\right) = 0\,?$$

From Galilean transformation we have:

$$v = v' + w$$

Where w is the relative velocity of the frame S' with respect to frame S. Put this into equation (1) we have:

$$\frac{1}{2}m_1(v_1' + w)^2 + \frac{1}{2}m_2(v_2' + w)^2 - \left(\frac{1}{2}m_1(u_1' + w)^2 + \frac{1}{2}m_2(u_2' + w)^2\right) = 0$$

$$\Rightarrow \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2 - \left(\frac{1}{2}m_1 u_1'^2 + \frac{1}{2}m_2 u_2'^2\right) + w(P_f' - P_i') = 0$$

We know that the last term is zero from problem 4(a)

$$\frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2 - \left(\frac{1}{2}m_1 u_1'^2 + \frac{1}{2}m_2 u_2'^2\right) = 0$$

Energy conservation is a consequance of Newton's equations and Galilean transformations don't change the Newton's equations. So energy has to be conserved after doing Galilean transformation.

6. Work in two inertial frames, I (5 points)

Observer G is on the ground and observer T is on a train moving with uniform velocity $\vec{u}$ with respect to the ground. Each observes that a particle of mass m, initially at rest with respect to the train, is acted on by a constant force $\vec{F}$ applied to it in the forward direction for a time t and a is the common acceleration of the particle.

(a) w.r.t train frame T:

$$\Delta\vec{r}_T = \frac{1}{2}\vec{a}t^2 + 0 \times t = \frac{1}{2}\vec{a}t^2$$

Work done

$$W_T = \vec{F}\cdot\Delta\vec{r}_T = m\vec{a}\cdot\frac{1}{2}\vec{a}t^2 = \frac{1}{2}ma^2t^2$$

Change in kinetic energy

$$K_f^T - K_i^T = \frac{1}{2}m(at+0)^2 - 0 = \frac{1}{2}ma^2t^2 = W_T$$

w.r.t. ground frame G:

$$\Delta\vec{r}_G = \frac{1}{2}\vec{a}t^2 + \vec{u}t$$

Work done

$$W_G = \vec{F} \cdot \Delta \vec{r}_G = \frac{1}{2} m a^2 t^2 + m u a t$$

Change in kinetic energy

$$K_f^G - K_i^G = \frac{1}{2} m (at + u)^2 - \frac{1}{2} m u^2 = \frac{1}{2} m a^2 t^2 + m u a t = W_G$$

So work-energy theorem holds in both frames.

(b) From part (a) we see that $\Delta \vec{r}_T \neq \Delta \vec{r}_G$ but the forces (for two inertial frames) are always the same. Displacement in the ground frame is more than the displacement in the train frame, therefore, despite the forces are same work done is different in the two frames.

In reference frame attached to train if one wishes to stop the particle one can draw the work from the particle until it comes to a stop. For example letting the particle work against a spring attached to train until it stops relative to train. The amount of work drawn via this process is equal to the kinetic energy of the particle in the reference frame of the train. With respect to the ground particle is still moving and still has kinetic energy. One can repeat the same experiment but this time with an spring attached to ground to bring it to rest relative to ground. In this process more work can be drawn from the particle by the ground observer and hence ground observer measuring higher kinetic energy than train observer. However notice that in the second process train observer will observe that spring attached to the ground did work on the particle eventually accelerating in the other direction.

7. Work in two inertial frames, II (5 points)

(a) In the train frame work done by friction is

$$W_f^T = f \Delta r_T = \frac{1}{2} f a t^2.$$

In the ground frame work done by friction is (frictional force $\times$ relative displacement *w.r.t.* train surface)

$$W_f^G = f(\Delta r_G - vt) = \frac{1}{2} f a t^2$$

Alternatively one can think as (work done by friction = work done by friction on particle - work done by friction on train), since frictional force acting on the train is in the other direction.

$$\Rightarrow W_f^G = f \Delta r_G - f v t = \frac{1}{2} f a t^2.$$

Either way heat produced due to friction is measured to be the same in either frame of reference.

(b) The only external force acting on the train is reaction of friction from the particle in forward direction. Since in the time t train moves distance ut work done on train by friction is fut. Notice that this is also the gain in the kinetic energy by the train. For the observer on the train, train surface does not move so there is no equivalent performance by the observer on the train.

From ground frame, applied force on particle is $(ma+f)$ and distance travelled is $(\frac{1}{2}at^2+ut)$. So work done on the particle is $(ma+f)(\frac{1}{2}at^2+ut)$. From previous problem, gain in kinetic energy by the particle is $ma(\frac{1}{2}at^2+ut)$. However in this problem kinetic energy gained by train is fut. Subtracting total gain in kinetic energy from total work done by the applied force we get the heat loss due to friction $\frac{1}{2}fat^2$ which is independent of the frame of reference. Alternatively, this amounts to saying that no matter what frame of reference we are using heat loss due to friction only depends on the relative surface displacement.

Note: In problems like this, it is useful to break down the system into simpler components and make free body diagrams for each object separately.

8. Binomial expansion (4 points)

We will be using the "binomial expansion" often in 8.20. It reads:

$$\begin{aligned}(1+\epsilon)^a &= 1+\frac{a}{1}\epsilon+\frac{a(a-1)}{2\cdot 1}\epsilon^2+\frac{a(a-1)(a-2)}{3\cdot 2\cdot 1}\epsilon^3 \\ &+ \frac{a(a-1)(a-2)(a-3)}{4\cdot 3\cdot 2\cdot 1}\epsilon^4+\dots \end{aligned} \tag{2}$$

This expansion converges when $|\epsilon|<1$.

(a) For what values of a does the expansion terminate with a finite number of terms?
If 'a' is a non-negative integer, then the Binomial expansion terminates with a finite number of terms. The number of terms is then $a+1$.

(b) Use eq. (2) to derive an expansion for $(a+b)^c$ when $|b|<|a|$.

$$\begin{aligned}(a+b)^c &= a^c\left(1+\frac{b}{a}\right)^c \quad \text{(where } \left|\tfrac{b}{a}\right|<1\text{)} \\ &= a^c\left(1+\frac{c}{1}\frac{b}{a}+\frac{c(c-1)}{2\cdot 1}\left(\frac{b}{a}\right)^2+\dots\right)\end{aligned}$$

(c) Consider $1/\sqrt{1-v^2/c^2}$. Write this in the form $(1+\epsilon)^a$ and expand it using the binomial expansion and give the first four terms.

$$\begin{aligned}1/\sqrt{1-v^2/c^2} &= \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \\ &= 1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\left(\frac{v^2}{c^2}\right)^2+\frac{5}{16}\left(\frac{v^2}{c^2}\right)^3+\dots\end{aligned}$$

(d) If $v/c=0.3$ how large an error would you make by keeping only the first two terms in part (c)? the first three terms?
For $\frac{v}{c}=0.3$, the above series becomes

$$1+0.045+0.0030375+\dots$$

and the true answer (to 6 dec. pl.) is 1.048285.
Keeping two terms, we get 1.045 which gives an error of 0.003285 or about

0.3% of the true answer.
Keeping three terms, we get 1.0480375 which gives an error of 0.000247 or about 0.02% of the true answer.

9. **Numbers for Michelson-Morley** (4 points)

In the Michelson-Morley experiment of 1887, the length, ℓ of each arm of the interferometer was 11 meters, and sodium light of wavelength 5.9×10^{-7} meters was used. Suppose that the experiment would have revealed any shift larger than 0.005 fringe. What upper limit does this place on the speed of the Earth through the supposed aether? How does this compare with the speed of the Earth around the Sun?

From eq. (1-8) in Resnick and Halliday,

$$\begin{aligned} \Delta N &= \frac{2l\beta^2}{\lambda} \\ \Rightarrow \beta &= \sqrt{\frac{\lambda \Delta N}{2l}} \\ \Rightarrow \beta &< \sqrt{\frac{5.9 \times 10^{-7} \times 0.005}{2 \times 11}} \\ \Rightarrow \beta &< 1.2 \times 10^{-5} \\ \Rightarrow v &< 3.6\, km/s\,. \end{aligned}$$

The earth's orbital speed is about $30\, km/s$ which is 12 times greater than the upper limit predicted above.

10. **Michelson-Morley for a real wind (RH)** (4 points)

A pilot plans to fly due east from A to B and back again. If u is her airspeed and if ℓ is the distance between A and B, it is clear that her round-trip time t_0 — if there is no wind — will be $2\ell/u$.

(a) Suppose, however, that a steady wind of speed v blows from the west. What will the round trip travel time now be, expressed in terms of t_0, u, and v?

If a steady wind of speed v blows from west to east, then the speed of the airplane relative to the earth is $u+v$ toward the east and $u-v$ toward the west. So,

$$\begin{aligned} t_1 &= \frac{l}{u+v} + \frac{l}{u-v} \\ &= \frac{2lu}{u^2 - v^2} \\ &= \frac{t_0}{1-(v/u)^2} \end{aligned}$$

where $t_0 = \frac{2l}{u}$.

(b) If the wind is from the south, find the expected round-trip travel time, again as a function of t_0, u, and v.

If the wind is from the south, the plane will have to be directed at an angle θ south of east in going from A to B so that the southward speed relative to the earth is $u\sin\theta - v$. This will be zero when $\theta = \sin^{-1}\frac{v}{u}$. So the eastward travel time will be $\frac{l}{u\cos\theta}$. The westward travel time is exactly the same. So,

$$\begin{aligned} t_2 &= \frac{2l}{u\cos\theta} \\ &= \frac{2l}{u\sqrt{1-\sin^2\theta}} \\ &= \frac{t_0}{\sqrt{1-(v/u)^2}} \end{aligned}$$

(c) Note that these two travel times are not equal. Should they be? Did you make a mistake?

The two travel times must be unequal, as velocity of the wind is a vector. The only way these two times can be the same is if $v = 0$. Hence given a non-zero wind velocity, the result is correct.

(d) In the Michelson-Morley experiment, however, the experiment seems to show that (for arms of equal length) the travel times for light *are* equal; otherwise these experimenters would have found a fringe shift when they rotated their interferometer. What is the essential difference between these two situations?

The essential difference is that light does not propagate in an ether wind and the speed of light is the same in all inertial frames of reference. In other words something that propagates in a medium gets dragged by the medium in a frame of reference in which medium is moving. The result of Michelson-Morley experiment shows that light does not propagate in a medium.

11. Michelson-Morley generalized (French - Ch.2, problem 8, p 60) (5 points)

From Figure 1 schematics of Michelson-Morley experiment for general orientation should be evident. At time $t = 0$ light rays leave the beam splitter for mirrors M_1 and M_2. The light ray that left for mirror M_1 hits the mirror at $t = t_1$ and at $t = t_1 + t_2$ it comes back to beam splitter. The light ray that left for mirror M_2 hits the mirror at $t = t_3$ and at $t = t_3 + t_4$ it comes back to beam splitter.In order to evaluate $\Delta t = (t_3 + t_4) - (t_1 + t_2)$ we wish to evaluate time intervals $t_1 + t_2$ and $t_3 + t_4$.

Before we get down to calculation remind yourself of this identity for triangles. Consider a triangle ABC with sides AB, BC, CA and angle at C (say α), then we know that following relation holds

$$(AB)^2 = (BC)^2 + (CA)^2 - 2(BC)(CA)\cos\alpha. \tag{3}$$

Now consider triangle OAM_1. We know that angle $\angle OAM_1 = 90° + \theta$. Therefore using the triangle identity above we have

$$\begin{aligned} (ct_1)^2 &= (vt_1)^2 + l^2 - 2lvt_1\cos(90° + \theta) \\ &= (vt_1)^2 + l^2 + 2lvt_1\sin\theta. \end{aligned} \tag{4}$$

Similarly from triangle BAM_1 ($\angle BAM_1 = 90° - \theta$) we get

$$(ct_2)^2 = (vt_2)^2 + l^2 - 2lvt_2\sin\theta. \tag{5}$$

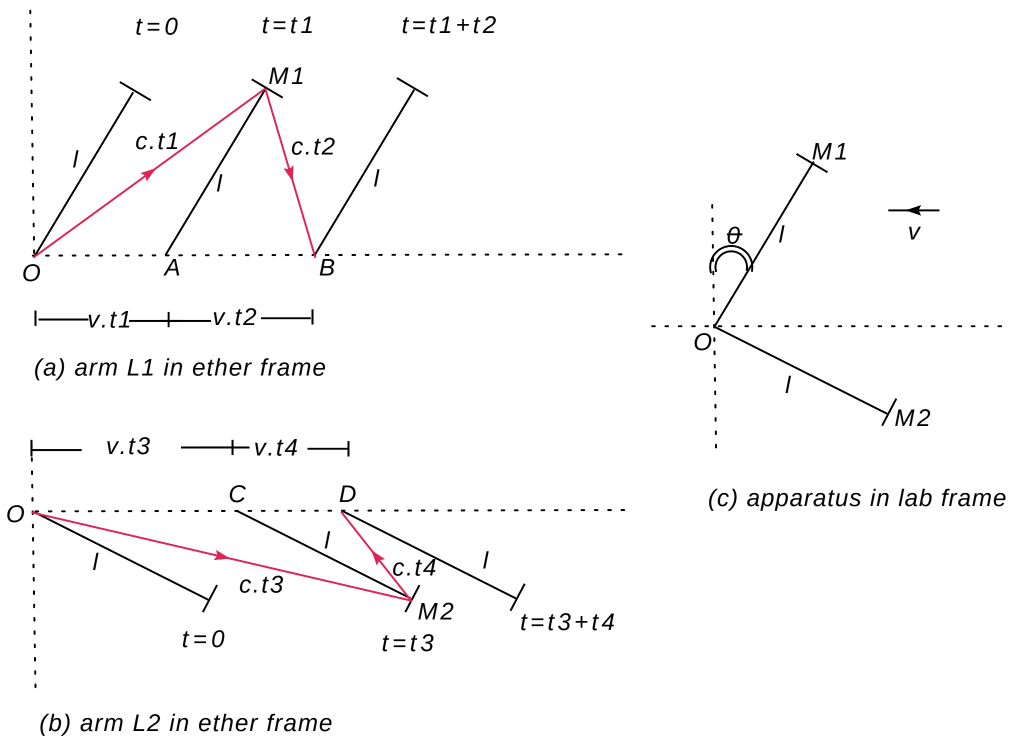

Figure 1: Schematics of Michelson-Morley experiment for general orientation *w.r.t* ether wind. Paths of light rays from beam splitter to mirrors and back are indicated in red.

`Solving eqn. (4) and eqn. (5) we get`

$$
\begin{aligned}
t_1 &= \frac{2lv\sin\theta + \sqrt{4l^2v^2\sin^2\theta + 4l^2(c^2-v^2)}}{2(c^2-v^2)} \\
t_2 &= \frac{-2lv\sin\theta + \sqrt{4l^2v^2\sin^2\theta + 4l^2(c^2-v^2)}}{2(c^2-v^2)} \\
\Rightarrow \qquad t_1+t_2 &= \frac{2\sqrt{4l^2v^2\sin^2\theta + 4l^2(c^2-v^2)}}{2(c^2-v^2)} \\
&= \frac{2l\sqrt{c^2-v^2\cos^2\theta}}{(c^2-v^2)}.
\end{aligned}
$$

`Now for triangle` OCM_2 `(with` $\angle OCM_2 = 180° - \theta$`) and triangle` DCM_2 `(with` $\angle DCM_2 = \theta$`)`

$$
\begin{aligned}
(ct_3)^2 &= (vt_3)^2 + l^2 + 2lvt_3\cos\theta \\
(ct_4)^2 &= (vt_4)^2 + l^2 - 2lvt_4\cos\theta
\end{aligned}
$$

`respectively. Solving these equations we get`

$$
t_3 = \frac{2lv\cos\theta + \sqrt{4l^2v^2\cos^2\theta + 4l^2(c^2-v^2)}}{2(c^2-v^2)}
$$

$$
\begin{aligned}
t_4 &= \frac{-2lv\cos\theta + \sqrt{4l^2v^2\cos^2\theta + 4l^2(c^2-v^2)}}{2(c^2-v^2)} \\
\Rightarrow \qquad t_3 + t_4 &= \frac{2\sqrt{4l^2v^2\cos^2\theta + 4l^2(c^2-v^2)}}{2(c^2-v^2)} \\
&= \frac{2l\sqrt{c^2 - v^2\sin^2\theta}}{(c^2-v^2)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Delta t &= (t_3+t_4) - (t_1+t_2) \\
&= \frac{2l}{c^2-v^2}\left(\sqrt{c^2 - v^2\sin^2\theta} - \sqrt{c^2 - v^2\cos^2\theta}\right) \\
&= \frac{2l}{c^2}\left(1 + \frac{v^2}{c^2}\right)\left\{c\left(1 - \frac{v^2\sin^2\theta}{2c^2}\right) - c\left(1 - \frac{v^2\cos^2\theta}{2c^2}\right)\right\} \\
&= \frac{2l}{c}\left(1+\frac{v^2}{c^2}\right)\left(\frac{v^2}{2c^2}\cos(2\theta) + \mathcal{O}(\frac{v^4}{c^4})\right) \\
&\approx \frac{lv^2}{c^3}\cos(2\theta).
\end{aligned}
$$

12. Michelson-Morley for sound waves (French - Ch.2, problem 9, p 61) (5 points)

(a) From previous problem we know that the time difference between pulse reflected from M_1 and pulse reflected from M_2 is given

$$\Delta t = \frac{lv^2}{c^3}\cos(2\theta)$$

where θ is angle between wind and one of the arms. Microphone receives the pulse reflected from one of the arms and then a pulse reflected from another arm because of this time lag. In most orientations $\cos(2\theta)$ is a number of order 1, therefore Δt will be small but finite. Only when $\cos(2\theta) = 0$ there will be no time lag.

(b) Time delay is maximum when $\cos(2\theta) = \pm 1$. This happens at $\theta = 0,\ \pi/2,\ \pi,\ 3\pi/2$ *i.e.* in four orientations. In other words when wind is parallel to one of the arms.

(c) We have

$$(\Delta t)_{\texttt{max}} = \frac{lv^2}{c^3}$$

$$
\begin{aligned}
\Rightarrow \qquad v &= \sqrt{\frac{c^3(\Delta t)_{\texttt{max}}}{l}} \\
&= \sqrt{\frac{(300)^3 \times 10^{-5}}{3}}\ m/s \\
&= 9.49\ \ m/s.
\end{aligned}
$$

13. Ehrenfest's thought experiment (RH) (5 points)

Paul Ehrenfest (1880-1933) proposed the following thought experiment to illustrate the different behavior expected for light under the ether wind hypothesis and under Einstein's second postulate:

Imagine yourself seated at the center of a spherical shell of radius 3×10^8 meters, the inner surface being diffusely reflecting. A source at the center of the sphere emits a sharp pulse of light, which travels outward through the darkness with uniform intensity in all directions. *Explain* what you would expect to see during the three second interval following the pulse under the assumptions that,

(a) there is a steady ether wind blowing through the sphere at 100 km/sec,
`The light will radiate outwards, hitting first the wall in the direction the ether wind is blowing, in a time slightly less than one second. Light will bounce off of the walls, and slightly more than two seconds later you will see the first reflected light. You will be illuminated from reflected light from that time forward.`

(b) there is no ether and Einstein's second postulate holds.
`The light will radiate outwards, hitting all points on the spherical shell at the same time. Light will bounce off the walls, and exactly two seconds later you will see the first reflected light. You will be illuminated from reflected light from that time forward.`

(c) Discuss the relationship of this thought experiment to the Michelson-Morley Experiment.
`Both this thought experiment and the Michelson-Morley experiment set up situations in which the ether wind hypothesis and Einstein's second relativity postulate make observably different predictions.`

14. Weighing the sun (RH) (4 points)

(a) Show that M, the mass of the sun, is related to the aberration constant, $\alpha = \tan^{-1} \frac{v}{c}$, by

$$M = \frac{\alpha^2 c^2 R}{G}$$

in which R is the radius of the Earth's orbit (which we assume to be circular) and G is Newton's constant ($G = 6.67 \times 10^{-11}$N m/kg^2. Hint: apply Newton's second law to the Earth's motion around the sun.

`The Earths orbital motion around the Sun causes the Sun to have an aberration angle` α `when viewed through a telescope. Since the distance between the Earth and the Sun is much larger than the diameter of the Earth, the motion of the telescope may be considered to be perpendicular to the direction of the sun. So the aberration is given by`

$$\alpha = \tan^{-1} \frac{v}{c} \approx \frac{v}{c}$$

`for small orbital velocity` v`.(NOTE:` α `for the above relation is in radians). Now` v `may be computed by considering the centripetal acceleration of the earth caused by the gravitational force due to the sun:`

$$\frac{GM}{R^2} = \frac{v^2}{R}$$

$$\Rightarrow v = \sqrt{\frac{GM}{R}}$$
$$\Rightarrow \alpha = \sqrt{\frac{GM}{Rc^2}}$$
$$\Rightarrow M = \frac{\alpha^2 c^2 R}{G}$$

(b) Calcuate M given that $\alpha = 20.5''$ and $R = 1.50 \times 10^{11}$m.

`Angel unit conversions:` $''$ `denotes arc seconds which is` $1/60$ `of arc minute` $(')$ `and arc minute itself is` $1/60$ `of a degree` $(^\circ)$. 180° `is` π `radians.`

Plugging in $\alpha = 20.5'' = 9.94 \times 10^{-5}$ rad., $R = 1.50 \times 10^{11} m$, $G = 6.67 \times 10^{-11} m^3/(kgs^2)$ and $c = 3 \times 10^8 m/s$, we get

$$M = 2.00 \times 10^{30} kg\,.$$

This agrees with the known mass of the sun.

15. "Emission theories" (6 points)

One of the early responses to the realization that electromagnetism predicts that the speed of light is a constant took the form of "emission theories". These theories assume that the speed of light is a constant, c, with respect to the source that emits it. They then have to deal with what happens when the emitted light reflects off a mirror — what velocity counts then? Consider three versions of emission theories that differ in their predictions of what the speed of light will be upon reflection from a mirror:

- The "original source" theory: the speed remains c relative to the original source.
- The "ballistic" theory: the speed is originally c relative to the original source, but upon reflection from the mirror it becomes c relative to the mirror.
- The "new source" theory: the speed is originally c relative to the original source, but becomes c relative to *the mirror image of the source* upon reflection.

Now suppose that a source of light, S, and a mirror M are moving away from one another. To be explicit, assume that the source is moving to the left with speed u in the laboratory, and a mirror, M is originally moving to the right with speed v. What is the speed of a light beam (as measured in the laboratory) originally emitted by the source after it has been reflected from the mirror according to each of the three "theories"?

Choose the left direction to be negative and the right positive. The speed of the mirror is v and that of the source is $-u$ in the lab frame. Consider the situation before the reflection from the mirrior of the beam of light. In all three emission theories the speed of light in the rest frame of the source is c, in the rest frame of the mirror is $c-u-v$, and in the lab frame is $c-u$.

After reflection from the mirror, the velocity of light as measured in the lab frame:

- in the ''original source'' theory is $-(c+u)$: In the rest frame of the source, the light is reflected back from the mirror with velocity $-c$. To transform back to the lab frame, we must add the velocity $-u$, which gives $-(c+u)$.

- in the ''ballistic'' theory is $-(c-v)$: This time, the light is refelcted back with velocity $-c$ in the rest frame of the mirror. So to go back to the lab frame, we must add v, which gives $-c+v$.
- in the ''new source'' theory is $-(c-u-2v)$: In the rest frame of the mirror, the source and its image move away from each other with equal and opposite velocites and the velocity of the image is $u+v$. The light is reflected with velocity $-c$ in the rest frame of the image, in which the mirror now moves with velocity $-(u+v)$. To transform back to the lab frame, we need to add $(u+v)+v$, resulting in $-c+u+2v$.

According to Einstein's second postulate what would be the the measured speed of a light pulse (either before or after reflection from the mirror) as viewed from i) the lab frame; ii) the rest frame of the mirror; iii) the rest frame of the source.

The lab frame and the rest frames of the source and the mirror, all three are inertial frames. So, according to Einstein's second postulate the speed of light is c, in all of them, both before and after the reflection from the mirror.

16. Invariance of the wave equation (3 points)

As discussed in lecture, starting from Maxwell's equations, it is possible to derive a *wave equation* whose solutions represent electromagnetic waves. The equation for the electric field, E, may be written

$$\frac{\partial^2}{\partial x^2}E(x,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(x,t) = 0 \tag{6}$$

where c is the speed of light.

(a) In lecture, we showed that $E(x,t) = f_0(x-ct)$ is a solution to this equation that travels to the right with speed c. *for any function,* f_0. Is $f_0(x+ct)$ also a solution?

Let $w_+ = x + ct$. Then,

$$\begin{aligned}
\frac{\partial}{\partial x}f(w_+) &= \frac{df(w_+)}{dw_+}\frac{\partial w_+}{\partial x} \\
&= \frac{df(w_+)}{dw_+} \\
\Rightarrow \frac{\partial^2}{\partial x^2}f(w_+) &= \frac{d^2}{dw_+^2}f(w_+)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}f(w_+) &= \frac{df(w_+)}{dw_+}\frac{\partial w_+}{\partial t} \\
&= +c\frac{\partial}{\partial w_+}f(w_+) \\
\Rightarrow \frac{\partial^2}{\partial t^2}f(w_+) &= c^2\frac{d^2}{dw_+^2}f(w_+)\,.
\end{aligned}$$

So,

$$\begin{aligned}\frac{\partial^2}{\partial x^2} f(w_+) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(w_+) &= \frac{d^2}{dw_+^2} f(w_+) - \frac{1}{c^2} c^2 \frac{d^2}{dw_+^2} f(w_+) \\ &= 0\end{aligned}$$

and f is a solution to the wave equation.

If so, describe its motion.

To track which way a wave is going we can focus at the particular height (or in the case of electromagnetic wave at some intensity) and follow its motion. For our solution this correspons to fixing the argument of f at x_0 then,

$$x + ct = x_0 \Rightarrow x = x_0 - ct.$$

This shows that the point that we have focused on is moving to the left direction. This solution corresponds to a wave which is moving to the left.

(b) Show that eq. (6) *is not invariant* under the Galilean transformation $x' = x - vt$, $t' = t$.

Under a Galilean transformation, $E(x,t) \to E(x',t')$.

$$\begin{aligned}\frac{\partial}{\partial x} E(x,t) &= \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial x} \\ &= \frac{\partial E}{\partial x'} \cdot 1 + \frac{\partial E}{\partial t'} \cdot 0 \\ &= \frac{\partial E}{\partial x'} \\ \Rightarrow \frac{\partial^2}{\partial x^2} E(x,t) &= \frac{\partial^2 E}{\partial (x')^2}\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial}{\partial t} E(x,t) &= \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial t} \\ &= -v\frac{\partial E}{\partial x'} + \frac{\partial E}{\partial t'} \\ \Rightarrow \frac{\partial^2}{\partial t^2} E(x,t) &= \left(\frac{\partial^2}{\partial (t')^2} + v^2\frac{\partial^2}{\partial (x')^2} - 2v\frac{\partial^2}{\partial x' \partial t'}\right) E(x',t')\end{aligned}$$

So,

$$\frac{\partial^2}{\partial x^2} E(x,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} E(x,t) \neq \frac{\partial^2}{\partial (x')^2} E(x',t') - \frac{1}{c^2}\frac{\partial^2}{\partial (t')^2} E(x',t')$$

and the wave equation is not Galilean-invariant.

(c) Show however that eq. (6) *is invariant* under the Lorentz transformation $z' = a(z - vt)$, $t' = a(t - vz/c^2)$, where $a = 1/\sqrt{1 - v^2/c^2}$.

Plug in the expressions for the primed coordinates and simply to get the same wave equation in primed coordinates.

$$\begin{aligned}
\frac{\partial}{\partial x}E(x,t) &= \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial x} \\
&= \frac{\partial E}{\partial x'}\cdot\gamma + \frac{\partial E}{\partial t'}\cdot\frac{-\gamma v}{c^2} \\
\Rightarrow \frac{\partial^2}{\partial x^2}E(x,t) &= \gamma^2\left(\frac{\partial^2}{\partial (x')^2} + \frac{v^2}{c^4}\frac{\partial^2}{\partial (t')^2} - \frac{2v}{c^2}\frac{\partial^2}{\partial t'\partial x'}\right)E
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial t}E(x,t) &= \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial t} \\
&= \frac{\partial E}{\partial x'}\cdot(-\gamma v) + \frac{\partial E}{\partial t'}\cdot\gamma \\
\Rightarrow \frac{\partial^2}{\partial t^2}E(x,t) &= \gamma^2\left(\frac{\partial^2}{\partial (t')^2} + v^2\frac{\partial^2}{\partial (x')^2} - 2v\frac{\partial^2}{\partial x'\partial t'}\right)E
\end{aligned}$$

So,

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}E(x,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(x,t) &= \gamma^2\left(1-\frac{v^2}{c^2}\right)\left(\frac{\partial^2}{\partial (x')^2}E(x',t') - \frac{1}{c^2}\frac{\partial^2}{\partial (t')^2}E(x',t')\right) \\
&= \frac{\partial^2}{\partial (x')^2}E(x',t') - \frac{1}{c^2}\frac{\partial^2}{\partial (t')^2}E(x',t')
\end{aligned}$$

therefore the wave equation is Lorentz-invariant.

17. Aberration, before and after Einstein (RH) (5 points)

Show that, according to special relativity, the classical aberration equation,

$$\tan\alpha_{\mathrm{c}} = \frac{v}{c} \quad \text{classical theory}$$

must be replaced by

$$\sin\alpha_{\mathrm{r}} = \frac{v}{c} \quad \text{relativity theory.}$$

According to the special theory of relativity, the speed of light is a constant in all inertial frames, and so the speed along the telescope axis is

$$c\,.$$

According to relativity, in a time Δt, light travels a distance $c\Delta t$ along the telescope and the telescope itself travels a distance $v\Delta t$. So,

$$\sin\alpha_r = \frac{v\Delta t}{c\Delta t} = \frac{v}{c}\,.$$

According to the ether hypothesis, the light is traveling with a speed c and an ether wind of speed v is perpendicular to it. So, the net speed of light along the telescope axis is

$$\sqrt{c^2+v^2}.$$

$$\sin\alpha_c = \frac{v\Delta t}{\sqrt{c^2+v^2}\Delta t} = \frac{v}{\sqrt{c^2+v^2}} \Rightarrow \tan\alpha_c = \frac{v}{c}$$

Thus the ether theory and relativity make different predictions for the aberration of starlight. However the differences are very small. To see this, consider a realistic case. Assume that the Earth's orbital speed is 30 km/sec and take $c = 3.00 \times 10^8$ m/sec. Find the *fractional difference*,

$$f \equiv \frac{\alpha_c - \alpha_r}{\alpha_r}$$

Note the differences are so small that your calculator may fail to capture the significant figures. Instead use the series expansions:

$$\begin{aligned} \sin^{-1} x &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 \\ \tan^{-1} x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 \end{aligned}$$

The earth's orbital speed is $30km/s$ which corresponds to

$$\beta = 10^{-4} \ll 1\,.$$

Now,

$$\tan \alpha_c = \beta = \sin \alpha_r\,.$$

Since β is much smaller than 1, α_c and α_r may be expanded in a series in powers of β and a truncation after the first few terms will yield an accurate answer.

$$\begin{aligned} \alpha_r &= \sin^{-1}\beta \\ &= \beta + \frac{1}{6}\beta^3 + \dots\,, \\ \alpha_c &= \tan^{-1}\beta \\ &= \beta - \frac{1}{3}\beta^3 + \dots\,, \\ \Rightarrow \frac{\alpha_c - \alpha_r}{\alpha_r} &= -\frac{\frac{1}{2}\beta^3}{\beta} + \mathrm{O}(\beta^4) \\ &\approx -5 \times 10^{-9}\,. \end{aligned}$$

So, stellar aberration is not very sensitive to the difference between the ether hypothesis and relativity.

18. Aberration due to the Earth's rotation (5 points)

In class we discussed the stellar aberration generated by the Earth's motion around the Sun. The rotation of the Earth about its axis also causes stellar aberration.

(a) Explain why the amount of stellar aberration generated by the Earth's rotation depends upon the latitude of the observer.

The linear velocity, v, at a latitude θ, owing to the Earth's rotation, is

$$v = \omega(R_e \cos\theta)$$

where the angular velocity of rotation is $\omega = 2\pi/(1\text{day})$ and the radius of the Earth is $R_e = 6400km$. So, the stellar aberration for a star in a direction perpendicular to the direction of v is given by

$$\begin{aligned} \tan\alpha &= \frac{v}{c} \\ &= \frac{\omega R_e \cos\theta}{c} \\ &\approx 1.55 \times 10^{-6} \cos\theta \\ \Rightarrow \alpha &\approx (1.55 \times 10^{-6} \cos\theta) \text{ rad. or } (0.32 \cos\theta)''. \end{aligned}$$

(b) For an observer at a given latitude explain why the amount of aberration depends on the *compass direction* of the star being observed. Compare, for example, the aberration of a star viewed on the eastern or western horizon with one on the northern horizon and with one directly overhead.

A star viewed on the eastern or western horizon has no aberration because the direction of motion of the telescope is along the direction of the starlight. However, if a star is viewed on the northern horizon then the telescope is moving perpendicular to the starlight and $\tan\alpha$ is given by the expression in part (a). Similarly, for a star directly overhead, the expression in part (a) holds.

(c) What is the largest aberration angle (the tilt of the telescope) due to the Earth's rotation alone for an observer a) at the North Pole, b) at the equator, and c) at latitutde 45° north.

The largest aberration angle is when the telescope moves in a direction perpendicular to the direction of the starlight and is given by

$$\alpha \approx (0.32 \cos\theta)''$$

as derived in part (a).

North Pole ($\theta = 90°$): $\alpha = 0$.
Equator ($\theta = 0°$): $\alpha \approx 0.32''$.
$\theta = 45°$: $\alpha \approx 0.238''$.

19. Relativity of simultaneity (5 points)

A plane flies overhead an observer on the Earth. Treat both the Earth and the plane as inertial frames for this problem. The speed of the plane is v. When the plane is overhead a light signal is emitted from the center of the plane. Subsequently it is detected by observer A in the front of the plane and observer B in the rear of the plane. Both observers measure their distance from the center of the plane to be d.

(a) Assume the speed of light is c as measured by the observers in the plane. Explain why observers A and B agree that the light signal reaches them simultaneously. How much time does the light take to reach them?
The time the light takes to reach each of A and B is $t = d/c$.

(b) Assuming that the speed of light is also c as measured by an observer on the Earth, explain why the Earth-bound observer would say that the arrival of the light signal at A and at B *were not simultaneous events*.
`A` is moving towards the light source, and hence sees the light signal at time t_A given by solving $ct_A = d - vt_A$. Point B is moving away from the light source, and hence sees the light signal at time t_B given by solving $ct_B = d + vt_B$. We find $t_A < t_B$ -- the events are not simultaneous in the frame of the observer at rest on the Earth.

20. A feeling for the Lorentz factor (5 points)

The "Lorentz factor", $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ determines the magnitude of many of the most unusual consequences of relativity (time dilation and length contraction, for example). What must an object's velocity be relative to you, the observer, for it's Lorentz factor to be: (a) 1.001, (b) 1.2, (c) 20, (d) 1000, (e) 10^9 ?

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow \beta = \frac{\sqrt{\gamma^2 - 1}}{\gamma}$$

(a)

$$\gamma = 1.001 \Rightarrow \beta = 0.0446879$$

(b)

$$\gamma = 1.2 \Rightarrow \beta = 0.5527708$$

(c)

$$\gamma = 20 \Rightarrow \beta = 0.9987492$$

(d)

$$\gamma = 1000 \Rightarrow \beta = 0.9999995$$

(e)

$$\gamma = 10^9 \Rightarrow \beta = 1.0000000$$

where the answers are given to 7 decimal places.

21. Inverse Lorentz transformation

Suppose two inertial frames are related by a Lorentz transformation:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - vx/c^2) \end{aligned}$$

Solve for x, y, z, t in terms of x', y', z', t' and show that the transformation is identical except for $v \to -v$.

This is straightforward algebra.

22. Lorentz transformation in an arbitrary direction (6 points)

Suppose two inertial frames, Σ and Σ' move such that their coordinate axes are parallel, their origins coincide at $t = t' = 0$, and the origin of Σ' is observed to move with velocity $\vec{v}$ in Σ. Starting from the form of the Lorentz transformation when the relative motion is along a coordinate axis, derive the Lorentz transformation relating x', y', z', t' to x, y, z, t. [Hint: it will be useful to decompose $\vec{x}$ into $\vec{x}_{\parallel} = \hat{v}(\hat{v} \cdot \vec{x})$ and $\vec{x}_{\perp} = \vec{x} - \vec{x}_{\parallel}$.]

The parallel component of $\vec{x}$ will transform and contribute to the the time transformation:

$$\begin{aligned} t' &= \gamma\left(t - \frac{v}{c^2} x_{\parallel}\right) \\ &= \gamma\left(t - \frac{v}{c^2}(\hat{v} \cdot \vec{x})\right) \\ &= \gamma\left(t - \frac{1}{c^2}(\vec{v} \cdot \vec{x})\right) . \\ x'_{\parallel} &= \gamma\left(x_{\parallel} - vt\right) . \end{aligned}$$

The perpendicular component of $\vec{x}$ will remain Lorentz-invariant:

$$\begin{aligned} \vec{x}'_{\perp} &= \vec{x}_{\perp} \\ \Rightarrow \vec{x}' - x'_{\parallel}\hat{v} &= \vec{x} - x_{\parallel}\hat{v} \\ \Rightarrow \vec{x}' &= \vec{x} + x'_{\parallel}\hat{v} - x_{\parallel}\hat{v} \\ &= \vec{x} + \gamma\left(x_{\parallel} - vt\right)\hat{v} - x_{\parallel}\hat{v} \\ &= \vec{x} + \frac{\gamma - 1}{v^2}(\vec{v} \cdot \vec{x})\vec{v} - \gamma\vec{v}t\,. \end{aligned}$$

Hence, the Lorentz transformation equations in an arbitrary direction are:

$$\begin{aligned} t' &= \gamma\left(t - \frac{1}{c^2}(\vec{v} \cdot \vec{x})\right) , \\ \vec{x}' &= \vec{x} + \frac{\gamma - 1}{v^2}(\vec{v} \cdot \vec{x})\vec{v} - \gamma\vec{v}t\,. \end{aligned}$$

It is easy to check that for $\vec{v} = v\hat{i}$, the above equations reduce to the usual Lorentz transformation equations when the relative motion is along a coordinate axis.