HEREDITARY DIFFERENTIAL SYSTEMS

DEFINED ON A COMPACT TIME INTERVAL

by

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Abstract

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This research is concerned with the study of hereditary differential systems with initial data which are not necessarily continuous. Let N be a positive integer and a, t_0, t_1, $\{\theta_i\}_{i=0}^{N}$ be real numbers such that

$$a > 0, \quad t_1 - t_0 \geq a, \qquad 0 = \theta_0 > \theta_1 > \ldots > \theta_N = -a.$$

Let E be a Banach space, and let $\mathfrak{F}$ be a vector space of Lebesgue measurable maps defined on [-a,0] with values in E. The hereditary differential systems considered here take their values in E and are of the form:

$$(E_1) \qquad \frac{dx(t)}{dt} = f(t, x(t+\theta_N), \ldots, x(t+\theta_1), x(t))$$

$$(E_2) \qquad \frac{dx(t)}{dt} = f(t, x(t), x_t)$$

$$(E_3) \qquad \frac{dx(t)}{dt} = f(t, x(t+\theta_N), \ldots, x(t+\theta_1), x(t), x_t)$$

for almost all (with respect to the Lebesgue measure on $[t_0, t_1]$)

t in $[t_0,t_1]$ and with initial data $x(s) = h(s-t_0)$ $(s \in [t_0-a,t_0]$, $h \in \mathfrak{F}$).

For such systems we study the Cauchy problem. In order to obtain global existence (and uniqueness) theorems as well as a result which shows that the solution is continuous with respect to the initial data we introduce two function spaces: one $(M^p(-a,0;E))$ in which the space of initial data will be considered and another $(AC^p(t_0,t_1;E))$ in which solutions will be sought.

We study further affine hereditary systems. For such systems the main results are i) a representation theorem for the map f, ii) a representation of solutions in terms of the fundamental matrix, and iii) a theory of adjoint systems. The above research opens the way to various applications, one of which is the analog of the classical optimal control problem for an affine differential system with a quadratic cost.

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1. Introduction.

This thesis is concerned with the study of hereditary differential systems with initial data which are not necessarily continuous. Although hereditary differential systems with initial data in the space of continuous functions have been extensively treated in the literature, a study on the lines of this thesis has apparently not been made.

We now indicate the contents of this thesis. The discussion in this chapter will be somewhat loose since our main objective is to give a summary description of the results and their motivation.

Let N be a positive integer and $a, t_0, t_1, \{\theta_i\}_{i=0}^{N}$ be real numbers such that

$$a > 0, \quad t_1 - t_0 \geq a, \quad 0 = > \theta_0 > \theta_1 \quad \ldots > \theta_N = -a.$$

Let m denote the Lebesgue measure on the compact interval $[t_0, t_1]$, let E be a Banach space, and let $\mathfrak{F}$ be a vector space of Lebesgue measurable maps defined on $[-a, 0]$ with values in E. Consider the following differential equations which are to be satisfied for almost all (m) t in $[t_0, t_1]$:

$$(E_1) \qquad \frac{dx(t)}{dt} = f(t,x(t+\theta_N),\ldots,x(t+\theta_1),x(t)), \tag{1.1}$$

$$(E_2) \qquad \frac{dx(t)}{dt} = f(t,x(t),x_t), \tag{1.2}$$

and

$$(E_3) \qquad \frac{dx(t)}{dt} = f(t,x(t+\theta_N),\ldots,x(t+\theta_1),x(t),x_t) \tag{1.3}$$

with initial data

$$(D) \qquad x(s) = h(s-t_0), \quad s \in [t_0 -a,t_0], \ h \in \mathfrak{F}$$

where $x(t) \in E$, $x_t \in \mathfrak{F}$

$$x_t(\theta) = x(t+\theta), \ t \in [t_0,t_1], \theta \in [-a,0].$$

Here f is a map defined on $[t_0,t_1] \times \mathcal{V}$ with values in E, and $\mathcal{V}$ is equal to E^{N+1}, $E \times \mathfrak{F}$ or $E^{N+1} \times \mathfrak{F}$ for (E_1), (E_2) and (E_3), respectively (the precise topological framework will be given in Chapter 3).

If $t \in [t_0,t_1]$ is interpreted as time, the right hand sides of the differential equations (E_i), i = 1,2,3 depend on t and the past history of x corresponding to the time interval [t-a,t]. This explains the terminology - <u>hereditary differential systems</u>. The three cases considered here do not exhaust the family of such systems. However the techniques and notions developed here are also applicable to other members of the family of problems. For ordinary differential equations, where f depends solely on t and $x(t)$, the space E is simultaneously the space of initial data, the space in which the differential equation takes its values at each time t and the space to which the "information" to be used at time t belongs. This is not true of hereditary systems where in general three different spaces $\mathfrak{F}$, E and Ψ are needed.

Solutions to hereditary differential systems (E_i) will be sought in the space of <u>absolutely continuous maps</u> defined on $[t_0,t_1]$ with values in E. The <u>Cauchy problem</u> for hereditary systems (E_i) with initial data (D) consists of finding an absolutely continuous map $\tilde{x}$ defined on $[t_0,t_1]$ with values in E for which $\tilde{x}(t_0) = h(0)$ and such that the map x defined on $[t_0-a,t_1]$, by

$$x(s) = \begin{cases} h(s-t_0) & s \in [t_0-a,t_0[\,, \\ \tilde{x}(s) & s \in [t_0,t_1], \end{cases} \tag{1.4}$$

satisfies equation (E_i) for almost all t in $[t_0,t_1]$.

The Cauchy problem for (E_1) has been extensively studied in the literature. An account of this may be found in R. Bellman and K.L. Cooke [1] and C. Corduneanu [2]. In both references the space of data $\mathfrak{F}$ is the space $C(-a,0;E)$ of continuous functions defined on $[-a,0]$ with values in E. The study of the same problem for (E_2) also provided a more general and elegant technique [3,4] for dealing with problem (E_1). Indeed, the map

$$x_t \mapsto (x_t(\theta_N),\ldots,x_t(\theta_0)) = (x(t+\theta_N),\ldots,x(t+\theta_0)) \quad (1.5)$$

defined on $C(-a,0;E)$ with values in E^{N+1} is continuous and linear. Therefore when $\mathcal{F}$ is equal to $C(-a,0;E)$ the Cauchy problem for (E_1) is a special case of the Cauchy problem for (E_2) and it is clear that only the latter problem need be studied. We refer the reader to J.K. Hale and C. Imaz [5], and C. Corduneanu [6] for a study of the case when f is a continuous function of its arguments. The Cauchy problem for (E_3) is a combination of (E_1) and (E_2) and as such the preceding remarks also apply to the latter.

Our objective is two-fold. First the space of initial data $\mathcal{F}$ will be enlarged from $C(-a,0;E)$ to $\mathcal{L}^p$ spaces of p-integrable functions defined on $[-a,0]$ $(1 \leq p < \infty)$ (not to be confused with L^p the space of equivalence classes of such maps). In doing this the Cauchy problem for (E_1) is no longer a particular case of the corresponding Cauchy problem for (E_2). Secondly, the hypothesis which requires that f be continuous in its arguments will be relaxed in favor of hypotheses of the Carathéodory type, namely, measurability with respect to t and continuity with respect to the other arguments.

In order to obtain global existence (and uniqueness) theorems as well as a result which shows that the solution is continuous with respect to the initial data (Chapter 3, Theorems 3.3 and 3.4) we introduce two function spaces: one ($M^p(-a,0;E)$) in which the space of initial datawill be considered and another ($AC^p(t_0,t_1;E)$) in which solutions will be sought.

The M^p-spaces are obtained when one considers a partition on $\mathcal{L}^p$-spaces which is different from that used to obtain L^p-spaces.

Let us remark that the pointwise character of the initial datum h is only used to obtain h(0) which fixes the value of x at time t_0. The remaining part of h is treated as an element of L^p since f need only be defined almost everywhere for integration with respect to t. This very naturally leads to the following equivalence relation $\sim$ among the elements of $\mathcal{L}^p$:

$$h \sim k \iff h(0) = k(0) \text{ and } h = k \text{ a.e. } [-a,0[. \tag{1.6}$$

It will be shown (Chapter 2) that the quotient spaces of $\mathcal{L}^p$ ($1 \leq p < \infty$) with respect to the above equivalence relation $\sim$ is a Banach space when endowed with the norm

$$\| h \| = [\, |h(0)|_E^p + \int_{-a}^{0} |h(\theta)|_E^p d\theta \,]^{1/p}.$$

This space will be denoted by M^p. It turns out that $(x(t), x_t)$ in (E_2) and (E_3) can be interpreted as an element of M^p since

$x(t) = x_t(0)$. In view of this it will be more convenient to have f defined on $\mathscr{V}$ equal to M^p and $E^N \times M^p$ for (E_2) and (E_3), respectively (see Chapters 2 and 3).

In Chapter 4 we study affine hereditary systems, an affine system being one where the map f: $[t_0,t_1] \times \mathscr{V} \longrightarrow H$, H is a Hilbert space, is affine for fixed t. The main results in this chapter are i) a representation theorem for the map f, ii) representation of solutions in terms of the fundamental matrix, and iii) a theory of adjoint systems.

The motivation for this thesis (not included here) is an application to the following problem of optimal control. Consider an affine system of the form

$$\frac{dx(t)}{dt} = \tilde{A}_0(t)x(t) + \tilde{A}_1(t)x_t + B(t)v(t), \text{ a.e. } [t_0,t_1] \tag{1.7}$$

or

$$\frac{dx(t)}{dt} = A_0(t)x(t) + \sum_{i=1}^{N} A_i(t)x(t+\theta_i) + B(t)v(t), \text{ a.e. } [t_0,t_1] \tag{1.8}$$

with initial data

$$x(s) = h(s-t_0), \quad s \in [t_0-a,t_0] \tag{1.9}$$

and control v in $\mathscr{L}^2(t_0,t_1;R^m)$ the space of square integrable functions defined on $[t_0,t_1]$ with values in $R^m (m \leq n)$. (The precise hypotheses on $\tilde{A}_0, \tilde{A}_1, B$, $A_0,\ldots,A_n$ and h are not given here.

The problem consists of finding a control u in $\mathcal{L}^2(t_0,t_1;R^m)$ which minimizes a given cost function of the form

$$J(v) = \int_{t_0}^{t_1} [(x_v(s),Q(s)x_v(s)) + (v(s),N(s)v(s))]ds \qquad (1.10)$$

($x_v(s)$ is the solution of (1.7) or (1.8) corresponding to h and v). In addition, it can be shown that the optimal control u can be synthesized at each time t directly from the "state" $(x(t),x_t)$ of the system as follows:

$$u(t) = P(x(t),x_t) + r(t), \quad t \in [t_0,t_1]. \qquad (1.11)$$

In the above P is a linear map defined on $R^n \times \mathcal{F}$ into R^m and r is an element of $\mathcal{L}^2(t_0,t_1;R^m)$.

A problem of this type was partially studied by N.N. Krasovskiǐ [7,8] as early as 1962. Based on Krasovskiǐ's techniques, attempts were made to find differential equations of Riccati type for the feedback map P and the function r. Results (incomplete) of this type have been reported by D.W. Ross and I. Flügge-Lotz [9], D.H. Eller, J.K. Aggarwal and H.T. Banks [10], and H.J. Kushner and D.I. Barnea [11]. The approach adopted is different. It is "the direct method", first introduced by J.L. Lions [12] in the context of parabolic partial differential equations. The adaptation of the latter method of hereditary differential systems was made possible by the introduction of the M^p spaces and the spaces AC^p of absolutely continuous map studied in Chapter 2. In summary, the direct method consists in establishing the existence of the feedback law (1.11), studying the properties of P and r, and showing that these satisfy natural differential equations exhibiting a Riccati feature. In this way it is not necessary to study the Riccati differntial equation in order to establish the existence of the feedback law, a task which is extremely difficult for the problem considered here.

Notations and Terminology.

R will be the set of real numbers.

Given a topological vector space E and an integer $n \geq 1$, E^n denotes the product topological vector space of n copies of E.

$B \setminus A$ will denote the complement $\{x \in B \mid x \notin A\}$ of a set A with respect to a set B.

Let B be a Banach space and let B^* denote its topological dual space. We define the symbol $\langle x,x^* \rangle_B$ by $\langle x,x^* \rangle_B = x^*(x)$, where the right hand side is the value of the linear form x^* at the point x. The map $(x,x^*) \mapsto \langle x,x^* \rangle_B$ is a bilinear form on $B \times B^*$.

$\mathcal{L}(X,Y)$ denotes the space of continuous linear map from a real Banach space X into another real Banach space Y. For $S \in \mathcal{L}(X,Y)$, $S^* \in \mathcal{L}(Y^*,X^*)$ is the transpose of the linear map S.

Let $f : X \to Y$ be a map between two real topological vector spaces X and Y. The map f is an isomorphism if it is bijective, linear, and bicontinuous; it is an embedding if the map $x \mapsto \bar{f}(x) = f(x) : X \to f(X)$ ($f(X)$, the image of X under f in Y with the relative topology) is an isomorphism.

Let $[a,b]$ be a compact interval $(b > a)$ in R, E a Banach space, and $f : [a,b] \to E$ a map. When $t \in]a,b]$ (resp. $t \in [a,b[$) the left (resp. right) derivative at t (if it exists) is defined as follows:

$$\lim_{\substack{y \to t \\ y \in [a,t[}} \frac{f(y) - f(t)}{y - t} \quad \left(\text{resp. } \lim_{\substack{y \to t \\ y \in]t,b]}} \frac{f(t) - f(y)}{t - y} \right).$$

When the left and right derivatives at t are equal we say that f is differentiable at t and denote as $\frac{df(t)}{dt}$ the derivative of f at t.

2. Function Spaces for the Study of Hereditary Differential Systems.

This chapter contains the basic material relevant to the study of hereditary differential systems. The notations and terminology here will be consistently used in the following chapters.

2.1 Preliminary Definitions.

2.1.1 Lang's theory of $L^p(\mu,E)$ and $\mathcal{L}^p(\mu,E)$ spaces.

We first summarize the development of Integration Theory as given in Lang [13, Chapter X and XI]. Let X be a non empty set, $\mathfrak{M}$ a σ-algebra on X, [13, p.222] and $\mu: \mathfrak{M} \to [0, \infty]$ a positive measure on $\mathfrak{M}$ [13, p.229]. $(X, \mathfrak{M}, \mu)$ is said to be a measured space [13,p.229]. A map $f: X \to E$ is said to be measurable if the inverse image of each open set in E is measurable in X [13, p.224]. A map $f: X \to E$ is said to be a simple map if it takes only a finite number of values, and if, for each $v \in E$ the inverse image $f^{-1}(v)$ is measurable [13, p.227]. By a partition of A ($\subset$ X) we mean a finite sequence $\{A_i\}$ $(i = 1,\dots, r)$ of measurable sets which are disjoint and such that $A = \bigcup_{i=1}^{r} A_i$. A map $f: X \to E$ is called a step map with respect to such a partition if f is equal to 0 outside A and $f(A_i)$ has one element for each i. A map $f: X \to E$ is said to be a step map if it is a step map with respect to some partition of some set of finite measure.

We denote the set of all step maps by $St(\mu, E)$ [13, p.231]. We define a map to be μ-measurable if it is a pointwise limit of a sequence of step maps almost everywhere [13, p.232].

If A is a measurable set of finite measure in X, and f is a step map with respect to a partition $\{A_i\}$ $(i = 1,\ldots,r)$ of A, then we define its <u>integral</u> to be

$$\int_X f\, d\mu = \sum_{i=1}^{r} \mu(A_i)\, f(A_i). \tag{2.1}$$

The <u>L^1-seminorm</u> on $St(\mu, E)$ is then defined as

$$\|f\|_1 = \int_X |f|_E\, d\mu \tag{2.2}$$

[13, p.235-237]. Let $\mathcal{N}(St)$ be the subspace of all step maps which are zero except perhaps on a subset of X of measure zero. The L^1-seminorm now becomes a norm on $St(\mu, E)/\mathcal{N}(St)$. Denote by $L^1(\mu, E)$ the completion of the latter space with respect to the L^1-norm. The integral is then naturally defined on $St(\mu, E)/\mathcal{N}(St)$ and extended to $L^1(\mu, E)$ by density as indicated in the diagram below:

$$\begin{tikzcd}
L^1(\mu, E) \arrow[rd, dashed, "\int_X"] & \\
St(\mu, E)/\mathcal{N}(St) \arrow[u, "i"] \arrow[r, "\int_X"'] & E
\end{tikzcd}$$

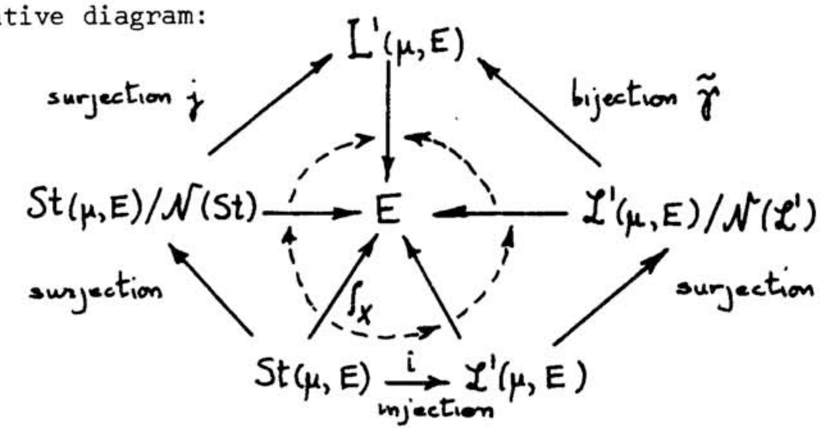
We define $\mathcal{L}^1(\mu, E)$ to be the set of all mappings $f\colon X \to E$ such that there exists an L^1-Cauchy sequence of step mappings converging almost everywhere to f. The integral and the L^1-seminorm are readily extended to $\mathcal{L}^1(\mu, E)$. Let $\mathcal{N}(\mathcal{L}^1)$ be the set of all elements of $\mathcal{L}^1(\mu, E)$ which are zero on X except perhaps on a subset of measure zero. If i denotes the injection of $St(\mu, E)$ into $\mathcal{L}^1(\mu, E)$ the diagrams below summarize the situation

$$\begin{tikzcd}
\mathcal{L}^1(\mu, E) \arrow[rd, dashed, "\int_X"] & \\
St(\mu, E) \arrow[u, "i"] \arrow[r, "\int_X"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{L}^1(\mu, E) \arrow[rd, dashed, "\|\ \|_1"] & \\
St(\mu, E) \arrow[u, "i"] \arrow[r, "\|\ \|_1"'] & \mathbb{R}
\end{tikzcd} .$$

To each f in $\mathcal{L}^1(\mu,E)$ we can associate a unique element $\tilde{f}$ in $L^1(\mu,E)$ determined by an L^1-Cauchy sequence of step maps converging a.e. to f. Let $\tilde{\gamma}: \mathcal{L}^1(\mu,E) \to L^1(\mu,E)$ be such a map. It is isometric, surjective [1, Lemmas 1 and 2, p.238-239] and the integrals coincide

$$\int_X f\,d\mu = \int_X \tilde{\gamma}(f)\,d\mu . \tag{2.3}$$

So $L^1(\mu,E)$ is isometrically isomorphic to $\mathcal{L}^1(\mu,E)/\mathcal{N}(\mathcal{L}^1)$ [13, Cor. 1, p.247] in such a way that the integrals on the respective spaces coincide. The situation is summarized in the following commutative diagram:



The curved arrows in the above diagram indicate liftings of the map $\int_X$. A similar diagram can be drawn with R in the center and the seminorm $\| \ \|_1$ in place of $\int_X$.

Likewise for $1 < p < \infty$, we define $\mathcal{L}^p(\mu,E)$ to be the vector space of all μ-measurable map f defined on X with values in E for which

$$\|f\|_p = \left[\int_X |f|^p d\mu \right]^{1/p} < \infty . \tag{2.4}$$

$\| \ \|_p$ is called the L^p-seminorm and $L^p(\mu,E)$ will denote the quotient space of $\mathcal{L}^p(\mu,E)$ by its subspace of all elements which are zero almost everywhere in X. It can be shown that the step maps are dense in $L^p(\mu, E)$ $(1 \le p < \infty)$.

When $p = \infty$, we define $\mathcal{L}^\infty(\mu, E)$ to be the vector space of maps f such that there exists a bounded μ-measurable g equal to f almost everywhere. We define the essential sup to be

$$\| f \|_\infty = \inf_g \| g \|$$

where $\| \;\|$ is the sup norm and the inf is taken over all bounded μ-measurable maps g equal to f almost everywhere. This defines a seminorm on $\mathcal{L}^\infty(\mu, E)$. It is clear that we have $\| f \|_\infty = 0$ if and only if f is equal to 0 almost everywhere. Denote by $L^\infty(\mu, E)$ the space of equivalence classes of elements of $\mathcal{L}^\infty(\mu, E)$.

If we identify $L^1(\mu, E)$ with the quotient space $\mathcal{L}^1(\mu, E)/\mathcal{N}(\mathcal{L}^1)$, we shall denote by

$$\gamma : \mathcal{L}^p(\mu, E) \longrightarrow L^p(\mu, E)$$

the <u>canonical surjection</u> for all p, $1 \le p \le \infty$.

2.1.2 <u>Some Function Spaces on $[\alpha, \beta]$</u>.

Let $\alpha < \beta$ be real numbers and E a real Banach space. Let $C(\alpha,\beta; E)$ denote the Banach space of continuous functions defined on $[\alpha,\beta]$ with values in E endowed with the norm

$$||f||_C = \max_{[\alpha,\beta]} |f(t)|, \tag{2.5}$$

where $| \;|$ is the norm in E.

In terms of the notation of section 2.1.1 let $X = [\alpha, \beta]$ and let $\mu = m$ be the (complete) Lebesgue measure on X. We now denote the space $\mathcal{L}^p(\mu, E)$ (resp. $L^p(\mu, E)$) by $\mathcal{L}^p(\alpha, \beta; E)$ (resp. $L^p(\alpha, \beta; E)$).

A map $f: [\alpha, \beta] \to E$ is <u>m-measurable</u> if it is a pointwise limit of a sequence of step maps almost everywhere.

2.2 M^p and N^p spaces.

We now introduce function spaces M^p and N^p. The purpose is to enlarge the space of initial functions for hereditary differential systems from continuous functions to $\mathcal{L}^p$-functions. This is done by introducing a special seminorm on $\mathcal{L}^p$-spaces which is different from the partition used to obtain L^p-spaces.

Let a be a positive real number. Consider the linear subspace S^p of $\mathcal{L}^p(-a,0; E)$ defined by

$$S^p = \{ f \in \mathcal{L}^p(-a, 0; E) \mid f(0) = 0, \ f(\theta) = 0 \text{ a.e. on } [-a, 0[\} \tag{2.6}$$

Let $M^p(-a, 0; E)$ denote the quotient space of $\mathcal{L}^p(-a, 0; E)$ by the subspace S^p. Let $[f]_{M^p}$ denote the equivalence class of $f \in \mathcal{L}^p(-a, 0; E)$ in $M^p(-a, 0; E)$.

When $[-a, 0]$ is replaced by $[0, a]$, we define the linear subspace T^p of $\mathcal{L}^p(0, a; E)$ by

$$T^p = \{ f \in \mathcal{L}^p(0, a; E) \mid f(0) = 0 , \ f(\theta) = 0 \text{ a.e. on }]0,a] \} \tag{2.7}$$

and denote by $N^p(0, a; E)$ the quotient of $\mathcal{L}^p(0, a; E)$ by the subspace T^p.

We see that there is a symmetry between $M^p(-a, 0; E)$ and $N^p(0 , a; E)$; for this reason only the space $M^p(-a, 0; E)$ will be studied.

Wherever there is no ambiguity we shall write simply M^p and N^p instead of $M^p(-a, 0; E)$ and $N^p(0, a ; E)$.

2.2.1 Seminorms and Norms on $\mathcal{L}^p$ and M^p.

We now define an appropriate seminorm on $\mathcal{L}^p$ and a norm on M^p.

Let

$$\alpha_p(f) = \begin{cases} (|f(0)|^p + \|f\|_p^p)^{1/p}, & 1 \le p < \infty \\ \max\{|f(0)|, \|f\|_\infty\}, & p = \infty \end{cases} \tag{2.8}$$

<u>Proposition 2.1</u>

(i) The functional

$$f \longmapsto \alpha_p(f) : \mathcal{L}^p(-a, 0; E) \longrightarrow \mathbb{R}$$

is a seminorm on $\mathcal{L}^p(-a, 0; E)$.

(ii) This seminorm α_p defines a norm $\bar{\alpha}_p$ on $M^p(-a, 0; E)$:

$$\bar{\alpha}_p([f]) = \alpha_p(f).$$

<u>Proof</u>: (i) By construction, α_p clearly satisfies the axioms of a seminorm . (ii) To prove that $\bar{\alpha}_p$ is a norm, we first show that $\bar{\alpha}_p$ is a well defined map from $M^p(-a, 0; E)$ into R. Indeed, let $f_1, f_2 \in \mathcal{L}^p(-a, 0; E)$ such that $[f_1] = [f_2]$. By definition

$$f_1(0) = f_2(0) \text{ and } f_1(\theta) = f_2(\theta) \text{ a.e. in } [-a, 0]$$

and hence

$$\bar{\alpha}_p([f_1]) = \alpha_p(f_1) = \alpha_p(f_2) = \bar{\alpha}_p([f_2]).$$

$\bar{\alpha}_p$ clearly satisfies the axioms of a seminorm since α_p does. To show that a seminorm is actually a norm, pick any $f \in \mathcal{L}^p(-a, 0; E)$ such that $\alpha_p(f) = 0$. Then $\|f\|_p = 0$ and $|f(0)| = 0$ and hence $f(\theta) = 0$ a.e. on $[-a, 0]$ and $f(0) = 0$. Therefore $f \in S^p$ and $[f] = 0$. Hence

$$\bar{\alpha}_p([f]) = 0 \quad \Longrightarrow \quad [f] = 0$$

proving that $\bar{\alpha}_p$ is a norm.

2.2.2 Some Elementary Results for M^p-spaces.

Let $E \times L^p(-a, 0; E)$ be endowed with the norm

$$\|(x,f)\|_{E\times L^p} = \begin{cases} (|x|^p + \|f\|_p^p)^{1/p}, & 1 \le p < \infty \\ \max(|x|, \|f\|_\infty), & p = \infty \end{cases} \tag{2.11}$$

Theorem 2.2

The map

$$f \mapsto K(f) = (f(0), [f]_{L^p}) : M^p(-a,0;E) \to E \times L^p(-a,0;E) \tag{2.12}$$

is a norm preserving isomorphism.

Proof: K is well defined and linear. To prove that K is injective suppose $K(f_1) = K(f_2)$. Consider $\bar{f}_1$ and $\bar{f}_2$ in $\mathcal{L}^p(-a,0;E)$ such that $[\bar{f}_i]_{M^p} = f_i$, $i = 1,2$. From (2.12) $f_1(0) = f_2(0)$ and $f_1(\theta) = f_2(\theta)$ a.e. $[-a.0]$. In particular $f_1 - f_2 \in S^p$ and $f_1 = [\bar{f}_1]_{M^p} = [\bar{f}_2]_{M^p} = f_2$. To prove K is surjective, for any $(x,g) \in E \times L^p(-a,0;E)$ let $\bar{g} \in \mathcal{L}^p(-a,0;E)$ be such that $[\bar{g}]_{L^p} = g$. Define $f: [-a,0] \to E$ as follows:

$$f(0) = x$$
$$f(\theta) = \bar{g}(\theta), \quad \theta \in [-a,0[.$$

Then f being a modification of $\bar{g}$ on a set of measure zero belongs to $\mathcal{L}^p(-a,0;E)$. Now

$$K([f]_{M^p}) = (f(0), [f]_{L^p}) = (x, [\bar{g}]_{L^p}) = (x, g).$$

Finally from the definition of $\bar{\alpha}_p$ and the choice of norm on $E \times L^p(-a,0;E)$, K is clearly isometric.

Corollary 2.3

$M^p(-a,0;E)$ is a Banach space isometrically isomorphic to $E \times L^p(-a,0;E)$.

Corollary 2.4

Consider the maps

$$\lambda : x \mapsto (x,0) : E \rightarrow E \times L^p(-a,0;E), \tag{2.13}$$

and

$$\mu : f \mapsto (0,f) : L^p(-a,0;E) \rightarrow E \times L^p(-a,0;E). \tag{2.14}$$

Then the composition map $\kappa^{-1} \circ \lambda$ (resp. $\kappa^{-1} \circ \mu$) isometrically embeds E (resp. $L^p(-a,0;E)$ into $M^p(-a,0;E)$.

2.2.3 A Density Theorem.

It is known that $C(-a,0;E)$ is dense in $L^p(-a,0;E)$ for $1 \leq p < \infty$ [13, p.336, Thm 6]. A similar result for M^p spaces is given in

Theorem 2.5

$C(-a,0;E)$ is dense in $M^p(-a,0;E)$ for $1 \leq p < \infty$.

Corollary 2.6

The subspace

$$\left\{ (f(0),\ [f]_{L^p}) \;\middle|\; f \in C(-a,0;E) \right\} \tag{2.15}$$

of $E \times L^p(-a,0;E)$ is dense.

Proof of Theorem 2.5: We claim that given f in M^p there exists a sequence $\{g_n\}$ of maps in $C = C(-a,0;E)$ for which

$$[g_n]_{M^p} \rightarrow f \quad \text{in } M^p.$$

Let $\bar{f}$ in $\mathcal{L}^p$ be such that $[\bar{f}]_{M^p} = f$. By the density of C in L^p [13, p.336, Thm 6] there exists a sequence $\{h_n\}$ of maps in C for which

$$[h_n]_{L^p} \rightarrow [\bar{f}]_{L^p} \text{ in } L^p$$

and

$$h_n(\theta) \rightarrow \bar{f}(\theta) \text{ a.e. in } [-a,0].$$

Now consider the approximations $(n \geq 1)$,

$$g_n(\theta) = \begin{cases} h_n(\theta) + [f(0) - h_n(\theta)] \dfrac{n\theta + a}{a} & , \ -\dfrac{a}{n} \leq \theta \leq 0, \\ h_n(\theta) & , \ -a \leq \theta < -\dfrac{a}{n} . \end{cases}$$

For all n, $g_n \in C$ and $g_n(0) = f(0) = \bar{f}(0)$. Also

$$\bar{\alpha}_p(g_n - f) = \left[|g_n(0) - \bar{f}(0)|^p + \|g_n - \bar{f}\|_p^p \right]^{1/p}$$

$$= \|g_n - \bar{f}\|_p \leq \|g_n - h_n\|_p + \|h_n - \bar{f}\|_p .$$

Consider the expression

$$\|g_n - f_n\|_p = \left[\int_{-a}^{0} |g_n(\theta - h_n(\theta)|^p d\theta \right]^{1/p}$$

$$= \left[\int_{-a/n}^{0} |f(0) - h_n(\theta)|^p \left(\frac{n\theta + a}{a}\right)^p d\theta \right]^{1/p}$$

$$\leq \left[\int_{-a/n}^{0} |f(0) - \bar{f}(\theta)|^p \left(\frac{n\theta + a}{a}\right)^p d\theta \right]^{1/p}$$

$$+ \left[\int_{-a/n}^{0} |\bar{f}(\theta) - h_n(\theta)|^p \left(\frac{n\theta + a}{a}\right)^p d\theta \right]^{1/p}$$

$$\leq \left[\int_{-a/n}^{0} |f(0) - \bar{f}(\theta)|^p d\theta \right]^{1/p} + \left[\int_{-a}^{0} |\bar{f}(\theta) - h_n(\theta)|^p d\theta \right]^{1/p}$$

since $0 \leq \dfrac{n\theta + a}{a} \leq 1$ when $\theta \in [-a/n, 0]$. But for arbitrary $\varepsilon > 0$, there exists $N_1 > 0$ such that

$$\|\bar{f} - h_n\|_p \leq \varepsilon/3 \quad , \quad \forall \ n \geq N_1$$

and there exists $N_2 > 0$ such that

$$\left[\int_{-a/n}^{0} |f(0) - \bar{f}(\theta)|^p d\theta \right]^{1/p} < \varepsilon/3 \ , \ \forall \ n \geq N_2 .$$

Finally for $n \geq N = \max\{N_1, N_2\}$

$$\bar{\alpha}_p(g_n - f) \leq 2\|h_n - \bar{f}\|_p + \left[\int_{-a/n}^{0} |f(0) - \bar{f}(\theta)|^p d\theta \right]^{1/p} \leq \varepsilon .$$

This proves the theorem.

2.2.4 Duality and Representation of Functionals on M^p.

Because of the particular structure of the M^p spaces, the well known duality and representation results on L^p spaces are also true for M^p spaces.

If B is a Banach space and B^* its topological dual, the pairing between B and B^* will be denoted $\langle x,x^* \rangle_B$ for $x^* \in B^*$ and $x \in B$.

Theorem 2.7

Let E be a reflexive Banach space.

(i) $M^p(-a,0;E)^*$ is isometrically isomorphic to $E^* \times L^p(-a,0;E)^*$, and each continuous linear functional Λ on M^p has the following representation in terms of a unique element (y,g^*) in $E^* \times L^p(-a,0;E)^*$:

$$\Lambda f = \langle f(0), y \rangle_E + \langle f, g^* \rangle_{L^p}, \quad \forall f \in M^p. \tag{2.16}$$

(ii) for $1 \le p < \infty$, $M^p(-a,0;E)^*$ is isometrically isomorphic to $M^q(-a,0;E^*)$ $(q^{-1} + p^{-1} = 1)$, and each continuous linear functional Λ on $M^p(-a,0;E)$ has the following representation in terms of a unique element g in $M^q(-a,0;E^*)$:

$$\Lambda f = \langle f(0),g(0) \rangle_E + \int_{-a}^{0} \langle f(\theta),g(\theta) \rangle_E d\theta, \quad \forall f \in M^p. \tag{2.17}$$

Proof: Let κ be the map given by (2.12). Corresponding to κ there exists an isometric isomorphism

$$\kappa^* : (E \times L^p)^* \longrightarrow (M^p)^*$$

defined in the natural way

$$\langle f, \kappa^* g^* \rangle_{M^p} = \langle \kappa f, g^* \rangle_{E \times L^p}, \quad \forall f \in M^p.$$

We also define the canonical map

$$\mathcal{J} : E^* \times (L^p)^* \longrightarrow (E \times L^p)^*$$

in the usual way: for each $(x^*, y^*) \in E^* \times (L^p)^*$

$$<(x,y), \mathcal{J}(x^*, y^*)>_{E \times L^p} = <x, x^*>_E + <y, y^*>_{L^p}$$

for all (x,y) in $E \times L^p$. $\mathcal{J}$ is an isometry if the norm on $E^* \times (L^p)^*$ is defined as

$$\|(x^*, y^*)\| = \begin{cases} [|x^*|^q + \|y^*\|_q^q]^{1/q}, \quad (q^{-1}+p^{-1}=1), & 1 < p \le \infty \\ \max\{|x^*\|, \|y^*\|_\infty\}, & p = 1; \end{cases}$$

it is surjective by a proposition in Horváth [14, p.267, Prop.2]. The composite map $K^* \circ \mathcal{J}$ is then the isometric isomorphism which establishes the first part of the theorem. The second part of the theorem follows from the fact that the Lebesgue measure is δ-finite and E is a reflexive Banach space [28, p.607 Thm 8.20.5, p.590 Thm 8.18.3].

Corollary 2.8

E^* (resp. $(L^p)^*$) is isometrically embedded into $(M^p)^*$.

Remark The corollary says that any continuous linear functional on E (resp. L^p) can be extended to a continuous linear functional on M^p.

All the results we have presented for M^p-spaces have their obvious counterparts for N^p-spaces.

As is commonly done for R^n an element f of M^p or N^p will be written

(f^0, f^1) or (f_0, f_1), $f^0, f_0 \in E$, f^1, $f_1 \in L^p$

instead of

$$\kappa^{-1}(f^0,f^1) \text{ or } \kappa^{-1}(f_0,f_1),$$

wherever there is no ambiguity. The notation (f^0,f^1) has definite advantages in computations.

2.3 The AC^p spaces.

Let α, β be as defined in section 2.1.2, and let $1 \leq p \leq \infty$. Further assume that E is a real Banach space. Let $AC^p(\alpha,\beta;E)$ be the vector space of all maps $f : [\alpha,\beta] \longrightarrow E$ which are differentiable almost everywhere on $[\alpha,\beta]$ with derivative $\frac{df}{dt}$ in $L^p(\alpha,\beta;E)$ and are such that

$$f(t) = f(\alpha) + \int_\alpha^t \frac{df(s)}{ds}\, ds, \quad t \in [\alpha,\beta]. \tag{2.18}$$

In other words $AC^p(\alpha,\beta;E)$ is the space of absolutely continuous maps defined on $[\alpha,\beta]$ with values in E which have a derivative in $L^p(\alpha,\beta;E)$. Such a space naturally arises in the study of differential equations since it is precisely the space in which solutions will be sought.

2.3.1 Norms on $AC^p(\alpha,\beta;E)$.

We first choose an appropriate norm on $AC^p(\alpha,\beta;E)$.

Proposition 2.9

(i) The functional

$$f \longmapsto n_p(f) = \begin{cases} \left[|f(\alpha)|^p + \left\|\frac{df}{dt}\right\|_p^p\right]^{1/p}, & 1 \leq p < \infty \\ \max\left\{|f(\alpha)|, \left\|\frac{df}{dt}\right\|_\infty\right\}, & p = \infty \end{cases}$$

$$: AC^p(\alpha,\beta;E) \longrightarrow R \tag{2.19}$$

is a norm on $AC^p(\alpha,\beta;E)$.

(ii) The functionals

$$f \mapsto \max\left\{ \|f\|_C \,,\, \left\|\frac{df}{dt}\right\|_p \right\} \tag{2.20}$$

and

$$f \mapsto \max\left\{ |f(\alpha)| \,,\, \left\|\frac{df}{dt}\right\|_p \right\} \tag{2.21}$$

also define norms on $AC^p(\alpha, \beta; E)$ which are equivalent to n_p.

Proof: (i) The axioms of a seminorm are clearly satisfied. So let $n_p(f) = 0$. Then $|f(\alpha)| = 0$ and $\left\|\frac{df}{dt}\right\|_p = 0$ and hence $f(t) = 0$ in $[\alpha, \beta]$.

(ii) The equivalence of the norm defined by (2.21) with n_p is obvious. The functional (2.20) also defines a norm on AC^p and by definition

$$n_p(f) \leq 2 \max\left\{ \|f\|_C \,,\, \left\|\frac{df}{dt}\right\|_p \right\}.$$

In the other direction it is sufficient to show the existence of some constant $b > 0$ for which

$$\|f\|_C \leq b \,.\, n_p(f) \quad :$$

$$|f(t)| = \left| f(\alpha) + \int_\alpha^t \frac{df}{dt}(s)\,ds \right| \leq |f(\alpha)| + c_1(\alpha, \beta, p) \left\|\frac{df}{dt}\right\|_p$$
$$\leq c_2(\alpha, \beta, p) \; n_p(f),$$

where $c_1(\alpha, \beta, p) = \begin{cases} (\beta - \alpha)^{1-p^{-1}} & , \; 1 \leq p < \infty \\ \beta - \alpha & , \; p = \infty \end{cases}$

and

$$c_2(\alpha, \beta, p) = \max\left\{ 1, \, c_1(\alpha, \beta, p) \right\}.$$

From now on we shall assume that AC^p is endowed with the norm n_p.

2.3.2 Properties of the AC^p spaces, Duality, and Representation of functionals.

It turns out that the AC^p spaces are structurally identical to the M^p spaces in the sense that they are isometrically isomorphic to the product space $E \times L^p$ endowed with the norm (2.11).

Proposition 2.10

The map

$$f \mapsto \nu(f) = (f(\alpha), \frac{df}{dt}) : AC^p(\alpha, \beta; E) \longrightarrow E \times L^p(\alpha, \beta; E) \tag{2.22}$$

is an isometric ismorphism.

Corollary 2.11

$AC^p(\alpha, \beta; E)$ is a Banach space isometrically isomorphic to $E \times L^p(\alpha, \beta; E)$.

Remark A particular case of the above proposition can be found in Dunford and Schwartz [15, p.242, p.338 Thm 3].

Proof of Proposition 2.10: The map ν is clearly an isometry by the definition of the norm n_p on AC^p and the particular norm chosen for $E \times L^p$. It is surjective by definition of AC^p. Corresponding to each $(x,g) \in E \times L^p$, we can construct an element f of AC^p as follows:

$$\begin{cases} f(\alpha) = x \\ f(t) = x + \int_\alpha^t g(s)\, ds \quad . \; t \in \,]\alpha, \beta]. \end{cases}$$

Trivially $\nu(f) = (x,g)$.

With the help of this key proposition we have a density result similar to Theorem 2.5. Let $AC(\alpha,\beta;E)$ be the vector space of all maps $f:[\alpha,\beta] \to E$ with a derivative $\frac{df}{dt}$ in $C(\alpha,\beta;E)$ and such that identity (2.18) is satisfied: at the points α and β $\frac{df(\alpha)}{dt}$ is defined as the right hand side derivative and $\frac{df(\beta)}{dt}$ as the left hand side derivative. The right norm for $AC(\alpha,\beta;E)$ is

$$n(f) = \max\left\{ \left|f(\alpha)\right| , \left\| \frac{df}{dt} \right\|_C \right\}.$$

<u>Theorem 2.12</u>

$AC(\alpha,\beta;E)$ is dense in $AC^p(\alpha,\beta;E)$ $(1 \leq p < \infty)$.

<u>Proof</u>: By Corollary 2.6 the subspace

$$[C] = \left\{ (f(\alpha),f) \;\middle|\; f \in C(\alpha,\beta;E)\right\}$$

of $E \times L^p$ is dense in $E \times L^p$. But $\nu(AC)$ the image of AC under ν is equal to $[C]$. Hence the density of AC in AC^p by the properties of ν.

The duality and representation results are also corollaries to Proposition 2.10.

<u>Theorem 2.13</u>

Let E be a Banach space.

(i) $AC^p(\alpha,\beta;E)^*$ is isometrically isomorphic to $E^* x L^p(\alpha,\beta;E)^*$ and each continuous linear functional Λ on $AC^p(\alpha,\beta;E)$ has the following representation in terms of a unique element (x,g) of $E^* \times L^p(\alpha,\beta;E)^*$:

$$\Lambda f = \langle f(\alpha), x \rangle_E + \langle \frac{df}{dt}, g \rangle_{L^p} , \quad \forall f \in AC^p.$$

(ii) For $1 \leq p < \infty$, $AC^p(\alpha,\beta;E)^*$ is isometrically isomorphic to $AC^q(\alpha,\beta;E^*)$ $(q^{-1}+p^{-1} = 1)$ and each continuous linear functional Λ on $AC^p(\alpha,\beta;E)$ has the following representation in terms of a unique element h in $AC^q(\alpha,\beta;E^*)$:

$$\Lambda f = \langle f(\alpha), h(\alpha) \rangle_E + \int_\alpha^\beta \langle \frac{df(t)}{dt}, \frac{dh(t)}{dt} \rangle_E \, dt, \quad \forall f \in AC^p.$$

Proof: The result is clear from Proposition 2.10.

3. The Cauchy Problem for Hereditary Differential Systems.

Let N be a positive integer, and a, t_0, t_1, $\{\theta_i\}_{i=1}^{N}$ be real numbers such that

$$a > 0, \; t_1 - t_0 \geq a, 0 = \theta_0 > \theta_1 > \dots > \theta_N = -a. \tag{3.1}$$

In this chapter we prove a global existence and uniqueness theorem and a global existence theorem for hereditary differential systems defined on the compact interval $[t_0, t_1]$. We also prove that a solution is continuous with respect to the initial data.

3.1 Problem Formulation and Main Theorems.

Let E be a fixed Banach space with norm $|\ |$. Consider the Banach spaces B_i^p, i = 1,2,3, constructed from E,

$$\left.\begin{array}{l} \left\{\begin{array}{l} B_1^p = E^{N+1} \\ \|y\|_{B_1^p} = \begin{cases} \left[\sum\limits_{j=0}^{N} |y_j|^p\right]^{1/p}, & 1 \leq p < \infty \\ \max\limits_{j=0,\dots,N} |y_j|, & p = \infty, \end{cases} \end{array}\right. \\ \\ \left\{\begin{array}{l} B_2^p = M^p(-a,0;E) \\ \|y\|_{B_2^p} = \|y\|_{M^p}, \end{array}\right. \end{array}\right\} \tag{3.2}$$

$$\left.\begin{cases} B_3^p = E^N \times M^p(-a,0;E) \\ \|y\|_{B_3^p} = \begin{cases} \left\{ \sum_{j=1}^{N} |y_j|^p + \|y_0\|_p^p \right\}^{1/p}, & 1 \le p < \infty, \\ \max\left\{ \max_{j=1,\dots,N} |y_j|, \|y_0\|_\infty \right\}, & p = \infty, \end{cases} \end{cases}\right\} \quad (3.2)$$

where $y = (y_N, \dots, y_1, y_0) \in B_3^p$.

The topology of the space B_1^p is not dependent on p, but we keep the superscript p since it specifies the particular norm chosen for B_1^p. When i = 1 or 3 the norm on B_i^p generates the product topology induced by the factors of B_i^p.

Because of the introduction of the M^p space it is more convenient to work with a map

$$\tilde{f} : [t_0, t_1] \times B_i^p \to E$$

rather than a map

$$f : [t_0, t_1] \times \mathcal{V}_i^p \to E,$$

where

$$\left.\begin{aligned} \mathcal{V}_1^p &= B_1^p \\ \mathcal{V}_2^p &= E \times L^p(-a,0;E), \text{ and} \\ \mathcal{V}_3^p &= E^{N+1} \times L^p(-a,0;E). \end{aligned}\right\} \quad (3.3)$$

From the isometric isomorphism $\kappa : M^p \to E \times L^p$ it is easy to construct another isometric isomorphism

$$\nu : B_i^p \to \mathcal{V}_i^p.$$

For all $t \in [t_0,t_1]$ $\tilde{f}$ can be constructed from f by

$$\tilde{f}(t,z) = f(t,\nu(z)), \qquad \forall\ z \in B_i^p,$$

and conversely f can be constructed from $\tilde{f}$

$$f(t,u) = \tilde{f}(t,\ \nu^{-1}(u)), \qquad \forall\ u \in \mathcal{V}_i^p,$$

since ν is a bijection. Hence we have a one to one correspondence between the maps $\tilde{f}$ and f. Since ν is an isometric isomorphism, the topological properties are not altered when one goes from $\tilde{f}$ to f or vice-versa.

Consider the product space $\mathfrak{F} \times \mathfrak{G}$, where $\mathfrak{F}$ is any one of the spaces $C(-a,0;E)$, $\mathcal{L}^p(-a,0;E)$ or $M^p(-a,0;E)$, and $\mathfrak{G}$ is taken to be either $C(t_0,t_1;E)$ or $AC^p(t_0,t_1;E)$. $\mathfrak{F} \circ \mathfrak{G}$ will denote the closed subspace (with respect to the product topology) of all (h,x) in $\mathfrak{F} \times \mathfrak{G}$ for which $x(t_0) = h(0)$.

<u>Definition 3.1</u>

The map

$$t \mapsto x_t(h) : [t_0,t_1] \to \mathfrak{F},$$

where

$$x_t(h)(\theta) = \begin{cases} h(t - t_0 + \theta), & -a \le \theta < -(t - t_0) \\ x(t + \theta), & -(t - t_0) \le \theta \le 0 \end{cases} \tag{3.4}$$

is called the <u>memory map</u> of (h,x) in $\mathfrak{F} \circ \mathfrak{G}$ and denoted by $x_\bullet(h)$.

The equations (E_i) $(i=1,2,3)$ can now be rewritten in the form

$$(E_1)\quad \frac{dx(t)}{dt} = f(t,x_t(h)(\theta_N),\ldots,x_t(h)(\theta_1),x(t)), \tag{3.5}$$

$$(E_2)\quad \frac{dx(t)}{dt} = f(t,x_t(h)), \tag{3.6}$$

$$(E_3)\quad \frac{dx(t)}{dt} = f(t,x_t(h)(\theta_N),\ldots,x_t(h)(\theta_1),\ x_t(h)), \tag{3.7}$$

for almost all t in $[t_0,t_1]$. As we mentioned earlier this formulation is entirely equivalent to the formulation of the Introduction. It however has definite technical advantages.

We can now give a precise definition of the Cauchy problem for such equations.

Definition 3.2

The global Cauchy problem on $[t_0,t_1]$ for the hereditary differential system (E_i) (i =i,2, or 3) with initial datum h in $M^p(-a,0;E)$ at time $t = t_0$ consists of finding an element x in $AC^1(t_0,t_1;E)$ for which $x(t_0) = h(0)$ and the equation (E_i) (i =1,2, or 3) is satisfied almost everywhere on $[t_0,t_1]$ for the memory map $x_\bullet(h)$ of (h,x). Such a map will be termed a global solution to the Cauchy problem on $[t_0,t_1]$ with initial data h at time $t = t_0$.

The local Cauchy problem for the hereditary differential system (E_i) (i =1,2,3) with initial datum h in $M^p(-a,0;E)$ at time $t = t_0$ consists of finding a real number α $(0<\alpha\le t_1-t_0)$ for which the global Cauchy problem on $[t_0,t_0+\alpha]$ has a global solution. The global solution on $[t_0,t_0+\alpha]$ is called a local solution to the Cauchy problem on $[t_0,t_1]$ with initial data h at time $t = t_0$.

Our main results in this chapter are given in the next two theorems:

Theorem 3.3

Let the map $f: [t_0,t_1] \times B_i^p \to E$ (i =1,2, or 3) have the following properties:

(CAR-1) the map $t \mapsto f(t,z) : [t_0,t_1] \to E$ is m-measurable for all $z \in B_i^p$;

(LIP) there exists a non negative function n in $\mathcal{L}^q(t_0,t_1;R)$ $(p^{-1}+ q^{-1} = 1)$ such that for all z_1 and z_2 in B_i^p

$$|f(t,z_1) - f(t,z_2)| \leq n(t) \| z_1 - z_2 \|_{B_i^p} , \text{ a.e.} [t_0,t_1] \tag{3.8}$$

and

(BC) the map $t \mapsto f(t,0) : [t_0,t_1] \to E$ is an element of $\mathcal{L}^1(t_0,t_1;E)$.

Then there exists a unique global solution x(h) in $AC^1(t_0,t_1;E)$ to the Cauchy problem on $[t_0,t_1]$ with initial datum at time $t = t_0$ for the hereditary differential system (E_i). Moreover the map

$$h \mapsto x(h) : M^p(-a,0;E) \to AC^1(t_0,t_1;E) \tag{3.9}$$

is Lipschitz continuous.

Theorem 3.4

Let $h \in M^p(-a,0;E)$ and let the map $f : [t_0,t_1] \times B_i^p \to E$ ($i = 1,2$, or 3) satisfy the following properties:

(CAR-1) the map $t \mapsto f(t,z) : [t_0,t_1] \to E$ is m-measurable for all $z \in B_i^p$;

(CAR-2) the map $z \mapsto f(t,z) : B_i^p \to E$ is continuous for almost all t in $[t_0,t_1]$.

(CAR-3) Let V be a non empty closed convex subset in $C(t_0,t_1;E)$, and assume there exists a non negative map m (possibly dependent on h) in $\mathcal{L}^1(t_0,t_1;R)$ such that

a) the set

$$V_m = \left\{ x \in C(t_0,t_1;E) \;\middle|\; x(t_0) = h(0),\ \max_{[t_0,t]} |x(s) - h(0)| \le \int_{t_0}^{t} m(s)\,ds,\ \forall\, t \in [t_0,t_1] \right\}, \quad (3.10)$$

is a subset of V,

b) and for all $x \in V_0 = \left\{ x \in V \;\middle|\; x(t_0) = h(0) \right\}$,

$$|f(t, \alpha_i(h,x)(t)| \le m(t), \text{ a.e. in } [t_0,t_1]. \quad (3.11)$$

Then there exists at least one global solution in $AC^1(t_0,t_1;E)$ to the Cauchy problem with initial datum h at time $t = t_0$.

Remarks 1) The set V can be defined pointwise. Let $\{ V(t) \}_{t \in [t_0,t_1]}$ be a family of closed convex subsets of E. The set $V = \{ x \in C(t_0,t_1;E) \mid x(t) \in V(t) \}$ is closed and convex in $C(t_0,t_1;E)$. The converse is not true since the image of an arbitrary closed set V in $C(t_0,t_1;E)$ under the map $x \mapsto x(t) : C(t_0,t_1;E) \to E$ is convex but not necessarily closed.

2) This alternative method of defining V was originally introduced by Carathéodory [16] in the context of ordinary differential equations. He chose $V(t) = \{ z \in R^n \} \mid z - x_0 \mid \leq b\}$ for some positive non zero constant b and $x_0 \in R^n$.

3) The local Cauchy problem arises when the set $V_m \not\subset V$. In such circumstances we seek an α , $t_1 - t_0 \geq \alpha > 0$,for which

$$\pi_\alpha (V_m) \subset \pi_\alpha (V)$$

where the map

$$\pi_\alpha : C(t_0, t_1; E) \longrightarrow C(t_0, t_0 + \alpha ; E)$$

is the restriction of the elements of $C(t_0, t_1; E)$ to the interval $[t_0, t_0 + \alpha]$.

4) The introduction of the set V_m is due to C. Corduneanu [6].

The hypotheses (CAR-1),(CAR-2) and (CAR-3) are the classical Carathéodory hypotheses [16]; (LIP) is the Lipschitz hypothesis for uniqueness; and (BC) is the hypothesis first introduced by A. Bielecki [17] and C. Corduneanu [6] in the context of global differential systems for continuous maps f.

The proof of the two theorems (section 3.4) will proceed via several lemmas and propositions (sections 3.2 and 3.3).

3.2 Properties of the Memory Maps.

Consider the following product spaces and their respective norms:

1) $C(-a,0;E) \times C(t_0,t_1;E)$

$$\|(h,x)\|_{C \times C} = \max\left\{ \|h\|_C, \|x\|_C \right\}; \tag{3.12}$$

2) $M^p(-a,0;E) \times C(t_0,t_1;E)$

$$\|(h,x)\|_{M^p \times C} = \max\left\{ \|h\|_p, \quad |h(0)|, \|x\|_C \right\}, \tag{3.13}$$

where $\|\ \|_p$ denotes the L^p norm.

It is not too difficult to see that these norms generate the product topologies on the corresponding product spaces $\mathfrak{F} \times \mathfrak{C}$. Moreover on the subspace $\mathfrak{F} \circ \mathfrak{C}$ the above norms reduce to $\max\left\{\|h\|_C, \|x\|_C\right\}$ and $\max\left\{\|h\|_p, \|x\|_C\right\}$, respectively.

Proposition 3.5

For $1 \le p < \infty$,

(i) $x_\bullet(h) \in C(t_0,t_1;M^p)$ for all $(h,x) \in M^p \circ C$,

(ii) we have the inequalities

$$\max_{s \in [t_o,t]} \|x_s(h)\|_{M^p} \le \|h\|_p + c(p)\|x\|_{C(t_0,t;E)} \tag{3.14}$$

where $c(p) = \max\left\{1, a^{1/p}\right\}$, and

$$\max\left\{\|h\|_p, \|x\|_{C(t_0,t;E)}\right\} \le \max_{[t_0,t]} \|x_s(h)\|_{M^p}, \tag{3.15}$$

(iii) and the map

$$(h,x) \mapsto x_\bullet(h): M^p \circ C \to C(t_0,t_1;M^p) \tag{3.16}$$

is an isomorphism.

Proof: We use the density of $C(-a,0;E)$ in $M^p(-a,0;E)$ (Theorem 2.5). Part (i) of the proposition is first proved with $C(-a,0;E)$ in place of $M^p(-a,0;E)$. Corresponding to each $(h,x) \in C \circ C$ we define the map

$$s \mapsto \bar{x}(s) = \begin{Bmatrix} h(s-t_0), & t_0-a \le s < t_0 \\ x(s), & t_0 \le s \le t_1 \end{Bmatrix} : [t_0-a,t_1] \to E.$$

Since $x(t_0)=h(0)$, clearly $\bar{x} \in C(t_0-a,t_1;E)$.

For all u,v in $[t_0,t_1]$

$$\|x_u(h)-x_v(h)\|_C = \max_{\theta \in [-a,0]} |x_u(h)(\theta)-x_v(h)(\theta)|$$

$$= \max_{\theta \in [-a,0]} |\bar{x}(u+\theta)-\bar{x}(v+\theta)| .$$

Hence by the uniform continuity of $\bar{x}$ on $[t_0-a,t]$, the memory map $x_\bullet(h) \in C(t_0,t_1; C(-a,0:E))$.

Now pick any (h,x) in $M^p(-a,0;E) \circ C(t_0,t_1;E)$. There exists a sequence $\{h_n\}$ in $C(-a,0;E)$ for which $h_n(0) = h(0)$ and $h_n \to h$ in $L^p(-a,0;E)$. Such a sequence was constructed in the proof of Theorem 2.5. Notice that $x_\bullet(h_n)$ is in $C(t_0,t_1;C(-a,0;E))$ for all n. By definition

$$\| x_t(h)-x_t(h_n)\|_{M^p} = [\,|h(0)-h_n(0)|^p+\|h-h_n\|_p^p]^{1/p}$$
$$= \| h-h_n \|_p .$$

The continuity of the map $t \mapsto x_t(h)$ is now a consequence of the following sequence of inequalities: for any t and t' in $[t_0,t_1]$

$$\| x_t(h)-x_{t'}(h)\|_{M^p} \le \| x_t(h)-x_t(h_n) \|_{M^p} + \| x_t(h_n)-x_{t'}(h_n) \|_{M^p}$$
$$+ \| x_{t'}(h)-x_{t'}(h_n) \|_{M^p}$$
$$\le 2 \| h-h_n \|_p$$
$$+ [\,|x(t)-x(t')|^p+\|x_t(h_n)-x_{t'}(h_n)\|_p^p]^{1/p}$$
$$\le 2 \| h-h_n\|_p+\max\left\{1,a^{1/p}\right\}\| x_t(h_n)-x_{t'}(h_n)\|_C .$$

This establishes (i).

The inequalities (3.14) and (3.15) in (ii) follow directly from the definition of the memory map and the inequalities

$$\| x_t(h)\|_{M^p} = \left[|x(t)|^p+\begin{cases} \left[\int_{-a}^{-(t-t_0)} |h(t-t_0+\theta)|^p d\theta + \int_{-(t-t_0)}^{0} |x(t+\theta)|^p d\theta\right], & t-t_0<a \\ \left[\int_{-a}^{0} |x(t+\theta)|^p d\theta\right]^{1/p}, & t-t_0 \ge a \end{cases}\right]^{1/p}$$

$$\le \| h \|_p + |x(t)| + a^{1/p} \max_{[t_0,t]} | x(s) |$$

$$\leq \| h \|_p + \max\left\{1, a^{1/p}\right\} \max_{[t_0,t]} | x(s) |$$

and

$$\| h \|_p = \| x_{t_0}(h) \|_p \leq \| x_{t_0}(h) \|_{M^p}$$

$$| x(t) | = | x_t(h)(0) | \leq \| x_t(h) \|_{M^p} .$$

Parts (i) and (ii) of the proposition establish that the map (3.19) is a topological isomorphism. It is clearly linear and bijective since

$$x_{\bullet}(h) = y_{\bullet}(k) \Rightarrow h = x_{t_0}(h) = y_{t_0}(k) = k$$

$$\text{and } x(t) = x_t(h)(0) = y_t(k)(0) = y(t)$$

$$\text{for all } \; t \in [t_0, t_1],$$

and for any $z \in C(t_0, t_1; M^p)$, $x_{\bullet}(h) = z$ where

$$x(t) = (z(t))(0) \quad , \quad t \in [t_0, t_1]$$

$$h = z(t_0).$$

Corollary 3.6

(i) $x_{\bullet}(h) \in C(t_0, t_1; C(-a, 0; E))$, $\forall (h, x) \in C \circ C$,

(ii) $$\max_{[t_0,t]} \| x_s(h) \|_C = \max\left[\max_{[t_0,t]} | x(s) | , \| h \|_C \right], \qquad (3.17)$$

(iii) and the map

$$(h,x) \mapsto x_{\bullet}(h) : C \circ C \longrightarrow C(t_0,t_1;C(-a,0;E)) \tag{3.18}$$

is an isometric isomorphism.

Proposition 3.7

Let p $(1 \leq p \leq \infty)$ and $\theta \in [-a,0]$.

(i) Given $(h,x) \in M^p \circ C$, the map $t \mapsto x_t(h)(\theta):[t_0,t_1] \longrightarrow E$ is an element of $L^p(t_0,t_1;E)$.

(ii) For all $t \in [t_0,t_1]$

$$\left[\int_{t_0}^{t} |x_s(h)(\theta)|^{1/p} ds\right]^{1/p} \leq \|h\|_p + (t_1-t_0)^{1/p} \max_{s\in[t_0,t]} |x(s)| \quad (1 \leq p < \infty) \tag{3.19}$$

and

$$\operatorname*{ess\,sup}_{s\in[t_0,t]} |x_s(h)(\theta)| \leq \|h\|_\infty + \max_{[t_0,t]} |x(s)| \quad (p=\infty) \tag{3.20}$$

(iii) Denote by $x_{\bullet}(h,\theta)$ the map $t \mapsto x_t(h)(\theta)$; the map

$$(h,x) \mapsto x\,(h,\theta) : M^p \circ C \longrightarrow L^p(t_0,t_1;E) \tag{3.21}$$

is linear, injective and continuous.

Proof: By definition of $x_\bullet(h)$

$$x_t(h,\theta) = \begin{cases} h(t-t_0+\theta) , & t_0 \le t < t_0-\theta \\ x(t+\theta) , & t_0-\theta \le t \le t_1 \end{cases}$$

and $x_\bullet(h,\theta)$ is clearly m-measurable.

Moreover

$$\left[\int_{t_0}^{t} |x_s(h,\theta)|^p ds \right]^{1/p} = \begin{cases} \left[\int_{t_0}^{t} |h(s-t_0+\theta)|^p ds \right]^{1/p}, & t \le t_0-\theta, \\ \left[\int_{t_0}^{t_0-\theta} |h(s-t_0+\theta)|^p ds \right]^{1/p} + \left[\int_{t_0-\theta}^{t} |x(s+\theta)|^p ds \right]^{1/p}, & t > t_0-\theta, \end{cases}$$

$$\le \left[\int_{-a}^{0} |h(\xi)|^p d\xi \right]^{1/p} + (t_1-t_0)^{1/p} \|x\|_{C(t_0,t;E)}$$

$$\le \|h\|_p + (t_1-t_0)^{1/p} \|x\|_{C(t_0,t;E)}$$

and similarly for $p = \infty$. Hence $x_\bullet(h,\theta)$ is in $L^p(t_0,t_1;E)$.

This establishes the theorem since the linearity and the injective property of the map (3.21) are obvious.

Counterexample to Proposition 3.4 for M^∞.

For the sake of completeness we give an example of a pair $(h,x) \in M^\infty \circ C$ for which the map $t \mapsto x_t(h) : [t_0,t_1] \to M^\infty$ is not continuous. Let $t_0=0$, $t_1=2$, $a=2$, $x=0$ and

$$h(\theta) = \begin{cases} e, & -2 \le \theta < -1 \\ 0, & -1 \le \theta \le 0 \end{cases} \tag{3.22}$$

where $e \in E$ has norm equal to 1.

Then

$$x_t(h)(\theta) = \begin{cases} e, & -2 \leq \theta < -(1+t) \\ 0, & -(1+t) \leq \theta \leq 0. \end{cases} \tag{3.23}$$

Hence $x_1(h) = 0$ and

$$\| x_t(h) - x_1(h) \|_{M^\infty} = \| x_t(h) \|_{M^\infty} = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & 1 \leq t \leq 2. \end{cases} \tag{3.24}$$

It is not too difficult to show that the maximal subspace of $M^\infty(-a,0;E)$ for which the map $t \mapsto x_t(h)$ is continuous is $C(-a,0;E)$. It is suspected that in general the map $t \mapsto x_t(h):[t_0,t_1] \to M^\infty$ belongs to $L^\infty(t_0,t_1;M^\infty)$, but we shall not attempt to verify this conjecture (the crutial part is the m-measurability of the map).

3.3 Auxiliary Results.

In this section we complete the groundwork which will allow us to prove Theorems 3.3 and 3.4. via the elegant techniques developed by A. Bielecki [17] and C. Corduneanu [6].

Proposition 3.7

Let p, $1 \leq p < \infty$, and i, $i = 1,2,3$, be given.

(i) The maps

$$\alpha_i : M^p(-a,0;E) \circ C(t_0,t_1;E) \to L^p(t_0,t_1;B_i^p), \tag{3.25}$$

where for $(h,x) \in M^p \circ C$

$$t \mapsto \alpha_1(h,x)(t)=(x_t(h,\theta_N),\ldots,x_t(h,\theta_0)):[t_0,t_1] \to B_1^p$$

$$t \mapsto \alpha_2(h,x)(t)=x_t(h):[t_0,t_1] \to B_2^p \tag{3.26}$$

$$t \mapsto \alpha_3(h,x)(t)=(x_t(h,\theta_N),\ldots,x_t(h,\theta_1),x_t(h)):[t_0,t_1] \to B_3^p$$

are linear and continuous when the B_i^p's are endowed with the product topology.

(ii) If the product norm on B_i^p is chosen as defined in the relations (3.2), then for all

$t \in [t_0,t_1]$

$$\left[\int_{t_0}^{t} \|\alpha_i(h,x)(s)\|_{B_i^p}^p \, ds\right]^{1/p} \leq k(p,a,t_1-t_0) \max\left\{\|h\|_p, \|x\|_{C(t_0,t;E)}\right\} \tag{3.27}$$

for some $k(p,a,t_1-t_0) > 0$.

(iii) Let

$$g_x = \kappa^{-1}(x(t_0),g) \tag{3.28}$$

($\kappa : M^p \to E \times L^p$ as in Theorem 2.2) for all $(x,g) \in C(t_0,t_1;E) \times L^p(-a,0;E)$. Then for fixed $g \in L^p(-a,0;E)$ and almost all

$t \in [t_0,t_1]$

$$\|\alpha_i(g_x,x)(t) - \alpha_i(g_y,y)(t)\|_{B_i^p} \leq c'(p) \|x-y\|_{C(t_0,t;E)} \tag{3.29}$$

for all x and y in $C(t_0,t_1;E)$ $c'(p) = (N+1)^{1/p}$ $(i = 1)$, $(1 + a)^{1/p}$ $(i = 2)$, $(N + 1 + a)^{1/p}$ $(i = 3)$.

Proof: (i) The map α_i is an element of $L^p(t_0,t_1;B_i^p)$ by Propositions 3.4 and 3.6 and the fact that $C(t_0,t_1;M^p)$ is contained in $L^p(t_0,t_1;M^p)$. The linearity of the map α_i is clear by its very construction from (h,x). The continuity of the map α_i follows from the continuity of the restrictions of α_i t_0 each factor of B_i^p and a Theorem in Lang [13, p.245, Theorem 3].

(ii) From the definition of the $\alpha_i(h,x)(t)$ the inequalities

$$\left[\int_{t_0}^{t}|x_s(h,\theta_i)|^p ds\right]^{1/p} \le \|h\|_p + (t_1-t_0)^{1/p}\|x\|_{C(t_0,t;E)}$$

$$\left[\int_{t_0}^{t}\|x_s(h)\|^p_{M^p} ds\right]^{1/p} \le (t_1-t_0)^{1/p}\left[\|h\|_p + c(p)\|x\|_{C(t_0,t;E)}\right]$$

establish the existence of a constant $k(p,a,t_1-t_0) > 0$ for which the inequality (3.27) is true.

(iii) Again by definition

$$|x_t(g_x,\theta_i)-y_t(g_y,\theta_i)| = \begin{cases} 0 & , \ t_0 \le t < t_0-\theta_i \\ |x(t)-y(t)| & , \ t_0-\theta_i \le t \le t_1 \end{cases}$$

and

$$\|x_t(g_x)-y_t(g_y)\|^p_{M^p} = \|(x-y)_t\ (g_x-g_y)\|^p_{M^p}$$

$$\le \quad \|g_x-g_y\|^p_p + (1+a)\max_{[t_0,t]}|x(s)-y(s)|^p$$

(by linearity of $x_t(h)$ in (h,x) and the inequality (3.16)); but $\|g_x-g_y\|_p = 0$ by construction and hence the inequality (3.29).

To summarize, note that the three hereditary differential systems (E_1), (E_2) and (E_3) now have the same structure:

1) a map $f : [t_0,t_1] \times B_i^p \to E \quad (1 \leq p < \infty)$

2) and a map $\alpha_i : M^p(-a,0;E) \circ C(t_0,t_1;E) \to \mathcal{L}^p(t_0,t_1;B_i^p)$ with the properties

$$\left[\int_{t_0}^{t} \|\alpha_i(h,x)(s)\|_{B_i^p}^p \, ds\right]^{1/p} \leq k(p) \max\left\{\|h\|_p , \|x\|_{C(t_0,t;E)}\right\}$$

for all $(h,x) \in M^p \circ C$ and some constant $k(p) > 0$,

and

$$\|\alpha_i(g_x,x)(t) - \alpha_i(g_y,y)(t)\|_{B_i^p} \leq c^{\circ}(p) \|x-y\|_{C(t_0,t;E)}$$

for all $g \in L^p(-a,0;E)$, all $x,y \in C(t_0,t_1;E)$, almost all $t \in [t_0,t_1]$, and some constant $c(p) > 0$ independent of g,x,y and t. All the structural properties of the system are contained in the couple (B_i^p, α_i). The map α_i bears a certain similarity to the lag function introduced by G.S. Jones [18], but there is a fundamental difference. Jones' lag function was defined on the time variable to generate a hereditary time set:

$$\alpha : \mathbb{R} \to \Omega$$

(Ω denotes the set of all closed subsets of R which are bounded above); for (E_1) we would have

$$\alpha(t) = \left\{ t+\theta_N, \ldots, t+\theta_1, t \right\},$$

for (E_2)

$$\alpha(t) = [t-a,t],$$

and for (E_3)

$$\alpha(t) = \left\{ t+\theta_N, \ldots, t+\theta_1, t , [t-a,t] \right\}.$$

Our map $\alpha_i (i=1,2,3)$ acts on the "information $x_t(h)$ stored in a memory at time t" and samples what the system needs at time t precisely $\alpha_i(h,x)(t)$.

Lemma 3.8 (Carathéodory)

Let B and E be Banach spaces with their respective σ-algebra of all Borel sets. Assume that the map $f:[t_0,t_1] \times B \to E$ satisfy the first two Carathéodory hypotheses:

(CAR-1) $t \mapsto f(t,z)$ is m-measurable for fixed z;

(CAR-2) and $z \mapsto f(t,z)$ is continuous on B for almost all t in $[t_0,t_1]$.

Then for any $y \in \mathcal{L}^1(t_0,t_1;B)$ the map

$$t \mapsto f_y(t) = f(t,y(t)) : [t_0,t_1] \to E \tag{3.30}$$

is m-measurable.

Proof: There exists an L^1-Cauchy sequence of step maps $\{s_n\}$ converging almost everywhere to y. It is sufficient to show that

1) f_{s_n} is a sequence of m-measurable map, and

2) $\{f_{s_n}\}$ converges almost everywhere to f_y.

By the hypothesis (CAR-2)

$$f_{s_n}(t)=f(t,s_n(t)) \to f(t,y(t)) = f_y(t)$$

for almost all t since $s_n(t) \to y(t)$ for almost all t. Let s(t) be an arbitrary step map defined on $[t_0,t_1]$. Its most general form is

$$s(t) = \sum_{i=1}^{r} a_i \chi_{A_i}(t)$$

where $r \geq 1$ is a finite integer and the A_i's are Lebesgue measurable disjoint subsets of $[t_0,t_1]$ the union $A = \bigcup_{i=1}^{r} A_i$ of which has finite measure. Then

$$f_s(t) = f(t, s(t)) = f(t,0)\left[1 - \chi_A(t)\right] + \sum_{i=1}^{N} f(t, a_i)\chi_{A_i}(t)$$

is clearly the sum of N+1 m-measurable maps by the hypothesis (CAR-1).

<u>Remark</u> The proof of the above Lemma is essentially Carathéodory's original proof [16, p.665].

The proof of Theorem 3.3 will also make use of the Banach fixed point Theorem [19, p.305] and some techniques borrowed from A. Bielecki [17] and C. Corduneanu [6].

<u>Lemma 3.9</u> (A. Bielecki, C. Corduneanu)

Let $n \in \mathcal{L}^1(t_0,t_1;R)$ be a non negative function, and α, $0 < \alpha < 1$, be given. The inequality

$$\int_{t_0}^{t} n(s)\, g(s)\, ds \leq \alpha\, g(t) \quad , \quad t \in [t_0,t_1] \qquad (3.31)$$

has a solution in $C(t_0,t_1;R)$ which is strictly positive and non-decreasing. In particular

$$g_\alpha(t) = \exp\left\{\alpha^{-1}\int_{t_0}^{t}\eta(s)\,ds\right\}, \quad t \in [t_0,t_1] \tag{3.32}$$

is such a solution.

Proof: By direct substitution.

Remark The introduction of the function g_α in the context of "global differential equations" is due to A. Bielecki [17]. Thereafter this idea was successfully used by C. Corduneanu [6,2] in the global case.

Definition 3.10

Let α, $0 < \alpha < 1$, and g_α be given by (3.32).

$C_\alpha(t_0,t_1;E)$ will denote the space of all continuous maps defined on $[t_0,t_1]$ with values in E, where E is endowed with the norm

$$\|x\|_\alpha = \max_{s\in[t_0,t_1]}\left\{|x(s)|/g_\alpha(s)\right\}.$$

Remark $C_\alpha(t_0,t_1;E)$ and $C(t_0,t_1;E)$ are equal as sets and equivalent as topological vector spaces, that is their respective norms generate the same topology.

Lemma 3.11 (C. Corduneanu)

For all α, $0 < \alpha < 1$, $x \in C(t_0,t_1;E)$

and $t \in [t_0,t_1]$

$$\max_{[t_0,t]} \left\{ |x(s)| / g_\alpha(s) \right\} = \max_{[t_0,t]} \left\{ \|x\|_{C(t_0,s;E)} / g_\alpha(s) \right\}. \tag{3.33}$$

Proof: See reference [6].

3.4 Proofs of Theorems 3.3 and 3.4.

The spaces $C(t_0,t_1;E)$ and $M^p(-a,0;E)$ will be abbreviated as C and M^p.

3.4.1 Proof of Theorem 3.3.

1) For arbitrary $(h,x) \in M^p \circ C$ consider the map

$$t \mapsto \tilde{f}(h,x)(t) = f(t, \alpha_i(h,x)(t)) : [t_0,t_1] \longrightarrow E . \tag{3.34}$$

We claim $\tilde{f}(h,x) \in \mathcal{L}^1(t_0,t_1;E)$. By the hypothesis (LIP) the map f is continuous in z for almost all $t \in [t_0,t_1]$; hence hypotheses (CAR-1) and (CAR-2) are satisfied. Also $\alpha_i(h,x) \in \mathcal{L}^p(t_0,t_1;B_i^p)$.
By Lemma 3.8 $\tilde{f}(h,x)$ is m-measurable. Also

$$\int_{t_0}^{t_1} |\tilde{f}(h,x)(t)|\,dt \le \int_{t_0}^{t_1} |\tilde{f}(h,x)(t) - f(t,0)|\,dt + \int_{t_0}^{t_1} |f(t,0)|\,dt. \tag{3.35}$$

By the hypothesis (BC), the last term on the right hand side of (3.35) is in $\mathcal{L}^1(t_0,t_1;E)$. By hypothesis (LIP)

$$\int_{t_0}^{t_1} |\tilde{f}(h,x)(t) - f(t,0)|\, dt \le \int_{t_0}^{t_1} n(t)\, \|\alpha_i(h,x)(t)\|_{B_i^q}\, dt$$
$$\le \|n\|_q \quad \|\alpha_i(h,x)\|_p$$

and by Proposition 3.7

$$\le \|n\|_q\, k(p) \max\{\|h\|_p, \|x\|_C\}.$$

This shows that $\tilde{f}(h,x) \in \mathcal{L}^1(t_0,t_1;E)$.

2) From the following inequalities it follows that the map $(h,x) \mapsto \tilde{f}(h,x) : M^p \circ C \to L^1(t_0,t_1;E)$ is Lipschitz continuous:

$$\|\tilde{f}(h,x) - \tilde{f}(k,y)\|_{L^1(t_0,t;E)} \le \int_{t_0}^{t} n(s)\, \|\alpha_i(h,x)(s) - \alpha(k,y)(s)\|_{B_i^p}\, ds$$
$$\le \int_{t_0}^{t} n(s)\, \|\alpha_i(h-k, x-y)(s)\|_{B_i^p}\, ds \tag{3.36}$$
$$\le \|n\|_q\, k(p) \max\{\|h-k\|_p, \|x-y\|_{C(t_0,t;E)}\},$$

for all $t \in [t_0,t_1]$. Hence the continuity of $\tilde{f}$.

3) Fix $\tilde{h} \in M^p$ and let $(\tilde{h}^0, \tilde{h}^1) = \kappa(\tilde{h})$ (κ being the isometric isomorphism between M^p and $E \times L^p$).

Consider the map

$$x \mapsto \tilde{h}^1_x = \kappa^{-1}(x(t_0), \tilde{h}^1) : C(t_0,t_1;E) \to M^p.$$

By construction $(\tilde{h}^1_x, x) \in M^p \circ C$ for all $x \in C$.

We define the map

$$t \mapsto \tilde{U}_f(x)(t) = \tilde{h}^0 + \int_{t_0}^{t} \hat{f}(\tilde{h}^1_x, x)(s)\, ds : [t_0,t_1] \to E.$$

From part 1) we conclude that $\tilde{U}_f \in AC^1(t_0,t_1;E)$ which is a subset of $C(t_0,t_1;E)$. So for an arbitrary α $(0 < \alpha < 1)$ to be chosen later we have a map

$$\widetilde{\mathcal{U}}_f : C_\alpha(t_0,t_1;E) \longrightarrow C_\alpha(t_0,t_1;E) \tag{3.37}$$

since C_α and C are equal as sets and have equivalent norms. We show that for some $\alpha = \tilde{\alpha}$ such a map is a contraction mapping. For arbitrary x and y in $C_\alpha(t_0,t_1;E)$ and $t \in [t_0,t_1]$,

$$\begin{aligned} |\widetilde{\mathcal{U}}_f(x)(t) - \widetilde{\mathcal{U}}_f(y)(t)| &\le \int_{t_0}^{t} n(s)\, \| \alpha_i(\tilde{h}^1_x, x)(s) - \alpha_i(\tilde{h}^1_y, y)(s) \|_{B^p_i}\, ds \\ &\le \int_{t_0}^{t} n(s)\, c'(p) \left[\max_{r\in[t_0,s]} |x(r) - y(r)| \right] ds. \end{aligned} \tag{3.38}$$

(Proposition 3.7). Pick $\tilde{\alpha}$ $(0 < \tilde{\alpha} < 1)$ such that $0 < \tilde{\alpha}\, c'(p) < 1$.

Thus

$$\begin{aligned} &\int_{t_0}^{t} n(s) \left[\max_{r\in[t_0,s]} |x(r) - y(r)| \right] ds \\ &\le \left[\int_{t_0}^{t} n(s) g_\alpha(s)\, ds \right] \max_{s\in[t_0,t]} \left\{ \max_{r\in[t_0,s]} |x(r)-y(r)| / g_\alpha(s) \right\} \\ &\le \alpha\, g_\alpha(t)\, \| x - y \|_{C_\alpha(t_0,t;E)} \end{aligned}$$

(by Lemmas 3.9 and 3.11), and

$$\| \widetilde{\mathcal{U}}_f(x) - \widetilde{\mathcal{U}}_f(y) \|_{C_\alpha} \le c'(p)\, \alpha\, \| x - y \|_{C_\alpha}, \tag{3.39}$$

where $0 < c'(p)\alpha < 1$ (by choice of $\tilde{\alpha}$). By the Banach fixed point Theorem [19, p.305], $\widetilde{\mathcal{U}}_f$ has a unique fixed point which is necessarily in $AC^1(t_0,t_1;E)$. But a fixed point of $\widetilde{\mathcal{U}}_f$ is necessarily a solution of the differential equation (E_i), and conversely an absolutely continuous solution of (E_i) is a fixed point of $\widetilde{\mathcal{U}}_f$ by the definition of $AC^1(t_0,t_1;E)$.

4) The Lipschitz continuity of the map

$$h \mapsto x(h) : M^p \to AC^1(t_0,t_1;E)$$

is also obtained via the C_α -spaces. If x and y are solutions corresponding to h and k, respectively,

$$|x(t)-y(t)| \le \left|\int_{t_0}^{t} (\tilde{f}(\tilde{h}'_x,x)(s)-\tilde{f}(\tilde{k}'_y,y)(s))\,ds\right| + |h^0-k^0|$$

and

$$\int_{t_0}^{t} |\tilde{f}(\tilde{h}'_x,x)(s)-\tilde{f}(\tilde{k}'_y,y)(s)|\,ds$$

$$\le \int_{t_0}^{t} |\tilde{f}(\tilde{h}'_x,x)(s)-\tilde{f}(\tilde{h}'_y,y)(s)|\,ds + \int_{t_0}^{t} |\tilde{f}(\tilde{h}'_y,y)(s)-\tilde{f}(\tilde{k}'_y,y)(s)|\,ds$$

$$\le \int_{t_0}^{t} n(s)\,c'(p) \max_{[t_0,s]} |x(r)-y(r)|\,ds + \|n\|_q\, k(p)\, \|h-k\|_p$$

$$\le \alpha\, c'(p)\, g(t)\, \|x-y\|_{C_\alpha(t_0,t;E)} + \|n\|_q\, k(p)\, \|h-k\|_p .$$

Thus

$$\|x-y\|_{C_\alpha} \le \alpha\, c'(p)\, \|x-y\|_{C_\alpha} + d_1(p)\|h-k\|_{M^p},$$

where $d_1(p) = \max_{t\in[t_0,t_1]} \left(\frac{1}{g_\alpha(t)}\right) \begin{cases} [1+\|n\|_q^q k(p)^q]^{1/q}, & 1<p\le\infty \\ \max\{1, \|n\|_\infty k(1)\}, & p=1 \end{cases}$,

Hence

$$\|x-y\|_{C_{\tilde{\alpha}}} \le d_2(p) = \frac{d_1(p)}{1-\tilde{\alpha}\, c'(p)} \|h-k\|_{M^p}$$

for some $d_2(p)>0$. But x and y are in AC^1 and

$$\|x-y\|_{AC^1} \le d_3 \|x-y\|_C \le d_4(\tilde{\alpha}) \|x-y\|_{C_{\tilde{\alpha}}}$$

($d_4 > 0$) by Proposition 2.9. Finally for some $d_5 > 0$

$$\|x-y\|_{AC^1} \le d_5 \|h-k\|_{M^p} .$$

3.4.2 Proof of Theorem 3.4.

1) V_0 is clearly closed and convex by definition.

We show that for all $x \in V_0$, the map

$$t \mapsto f(t, \alpha_i(h,x)(t)) : [t_0,t_1] \to E \tag{3.40}$$

is in $\mathcal{L}^1(t_0,t_1;E)$. For each $x \in V_0$ the map

$$t \mapsto \alpha_i(h,x)(t) : [t_0,t_1] \to B_i^p$$

is in $\mathcal{L}^p(t_0,t_1;B_i^p)$ by Proposition 3.7 since $(h,x) \in M^p \circ C$. Hence hypotheses (CAR-1), (CAR-2) and Lemma 3.8 establish the m-measurability of the map (3.40). The assertion is now true by part b) of the hypothesis (CAR-3).

2) The map

$$x \mapsto U_f(x) : V_0 \to C(t_0,t_1;E),$$

where

$$(U_f(x))(t)=h(0)+\int_{t_0}^{t} f(s, \alpha_i(h,x)(s))ds, \forall t \in [t_0,t_1],$$

now makes sense. Moreover

$$|U_f(x)(t)-h(0)| = \left|\int_{t_0}^{t} f(s, \alpha_i(h,x)(s))\, ds\right|$$

$$\leq \int_{t_0}^{t} m(s)\, ds \ , \quad t \in [t_0,t_1],$$

and $U_f(x)(t_0) = h(0)$. Thus $U_f(x) \in V_m \subset V$.

But $V_m = V_m \cap V_0 \subset V \cap V_0 = V_0$. So the image of the map U_f is contained in V_0.

3) Let

$$M(s) = \int_{t_0}^{s} m(v)\, d(v) \quad , \; s \in [t_0,t_1].$$

M is uniformly continuous and monotonically increasing on $[t_0,t_1]$. For arbitrary s and t in $[t_0,t_1]$ and any $x \in V_0$

$$|U_f(x)(t) - U_f(x)(s)| \leq \left| \int_{t_0}^{t} f(v, \alpha_i(h,x)(v)\, dv - \int_{t_0}^{s} f(v, \alpha_i(h,x)(v) dv \right| \leq |M(t) - M(s)|.$$

The family $U_f(V_0)$ is equicontinuous. Also

$$|U_f(x)(t)| = \left| h(0) + \int_{t_0}^{t} f(v, \alpha_i(h,x)(v))\, dv \right| \leq |h(0)| + \int_{t_0}^{t} m(v)\, dv \leq |h(0)| + M(t_1)$$

and $U_f(V_0)$ is an equicontinuous and uniformly bounded family, hence a relatively compact subset of $C(t_0,t_1;E)$ by Ascoli's lemma [13, p.211].

4) Finally we show U_f is continuous on V_0. If $x \in V_0$ is an isolated point there is nothing to prove; if not consider an arbitrary Cauchy sequence $\{x_n\}$ of points in V_0 converging to x. Let $g_n(t) = f(t, \alpha_i(h,x_n)(t))$, $g(t) = f(t, \alpha_i(h,x)(t))$. By hypothesis (CAR-2) and the continuity of the map α_i

$$\alpha_i(h,x_n) \rightarrow \alpha_i(h,x) \text{ in } \mathcal{L}^p(t_0,t_1;B_i^p)$$

which implies that

$$\alpha_i(h,x_n)(t) \rightarrow \alpha_i(h,x)(t) \quad \text{a.e. in } [t_0,t_1]$$

and

$$g_n(t) \rightarrow g(t) \quad , \text{ a.e. in } [t_0,t_1].$$

By part b) of the hypothesis (CAR-3), the Lebesgue dominated convergence theorem can be applied and

$$g_n \rightarrow g \quad \text{in } \mathcal{L}^1(t_0, t_1; E).$$

5) The map

$$U_f : V_0 \rightarrow V_0$$

is continuous and $U_f(V_0)$ is relatively compact. The theorem now is true by Schauder's theorem [19, p.415].

4. Affine Hereditary Systems.

Let the notations and definitions of Chapter 3 hold, and assume that $E = H$ is a Hilbert space. A hereditary differential system characterized by the map $f:[t_0,t_1] \times B_i^p \to H$ is said to be affine if for all $t \in [t_0,t_1]$ the maps

$$z \mapsto f(t,z) : B_i^p \to H \tag{4.1}$$

are affine.

The Chapter is divided in three sections.

1) The first section studies the representation problem for the maps f which define an affine system and satisfy the hypotheses (CAR-1),(LIP) and (BC).Further a more precise version of Theorem 3.3 is given for the questions of continuity with respect to the data.

2) For a certain class of affine differential systems, the "fundamental matrix solution" is introduced in a way which generalizes the well known results for affine ordinary differential systems.

3) Finally the "adjoint problem" is solved for linear hereditary differential systems belonging to the special class mentioned in 2).

4.1 Representation of the maps f and Specialization of Theorem 3.3.

When f characterizes an affine differential system the hypotheses (CAR-1), (LIP) and (BC) of Theorem 3.3 reduce to

Hypotheses 4.1. (Affine systems)

Consider an affine differential system corresponding to the map $f:[t_0,t_1] \times B_i^p \rightarrow H$ affine on B_i^p for all fixed $t \in [t_0,t_1]$ $(1 \leq p < \infty)$.

- There exists two maps g and ℓ

$$g : [t_0,t_1] \rightarrow H, \ \ell : [t_0,t_1] \times B_i^p \rightarrow H \tag{4.2}$$

for which

$$f(t,z) = \ell(t,z) + g(t) \ , \ t \in [t_0,t_1], \ z \in B_i^p;$$

- the map g is in $\mathcal{L}^1(t_0,t_1;H)$ and the map ℓ has the following properties:

(i) the map $z \mapsto \ell(t,z) : B_i^p \rightarrow H$ is linear for all t in $[t_0,t_1]$,

(ii) the map $t \mapsto \ell(t,z) : [t_0,t_1] \rightarrow H$ is m-measurable for all $z \in B_i^p$,

(iii) and there exists $n \in \mathcal{L}^q(t_0,t_1;R)$ $(q^{-1}+p^{-1}=1)$ such that

$$|\ell(t,z)| \leq n(t) \ \|z\|_{B_i^p} \ \text{a.e. in } [t_0,t_1] \tag{4.3}$$

for all z in B_i^p.

In the remainder of this Chapter, it will be assumed that an affine hereditary system satisfies Hypotheses 4.1. Technically the affine version of equation (E_3) exhibits a combination of the features of the affine versions of (E_1) and (E_2). Thus in order to keep the computations as simple as possible only cases (E_1) and (E_2) will be studied. However the results are readily extendable to the case (E_3) due to the linearity of $\ell(t,z)$ in its second argument.

Notation Given two real topological vector spaces X and Y, $\mathcal{L}(X;Y)$ denotes the real vector space of all continuous linear maps defined on X with values in Y. If X and Y are normed spaces, $\mathcal{L}(X;Y)$ is endowed with the norm

$$\|A\| = \sup_{\|x\|_X=1} \|Ax\|_Y, \ \forall A \in \mathcal{L}(X;Y). \tag{4.4}$$

If in addition Y is complete, then $\mathcal{L}(X;Y)$ is a Banach space [15, p.61, Lemma 8].

4.1.1 Main Results.

Our first result is concerned with the possible representation of the maps $(t,z) \longmapsto \ell(t,z)$ satisfying the Hypotheses 4.1 in terms of time dependent elements of $\mathcal{L}(B_i^p;H)$. Consider the map

$$z \mapsto A(t)\, z = \ell(t,z) \; ; \; B_i^p \to H. \tag{4.5}$$

By Hypotheses 4.1 (i) and (iii) $A(t) \in \mathcal{L}(B_i^p, H)$

and by Hypotheses 4.1 (ii) and (iii) the map

$$t \mapsto \| A(t) \|_{\mathcal{L}} \; : [t_0, t_1] \to R \text{ is in } L^q(t_0, t_1; R).$$

If instead of Hypothesis 4.1 (ii) we assume that

$$\text{(M)} \qquad \boxed{\begin{array}{l} \text{the map} \\ \qquad t \mapsto A(t) \; : \; [t_0, t_1] \to \mathcal{L}(B_i^p, H) \\ \text{is m-measurable,} \end{array}} \tag{4.6}$$

then $A \in L^q(t_0, t_1 \; ; \mathcal{L}(B_i^p, H))$. It is clear that hypothesis (M) implies Hypothesis 4.1 (ii) (Carathéodory's hypothesis (CAR-2)), but the converse might in general not be true.

When $i = 1$ (that is $B_1^p = H^{N+1}$)

consider the maps

$$\left.\begin{array}{l} z_0 \mapsto A_0(t) z_0 = A(t)\,(z_0, 0, \ldots, 0) \; : \; H \to H \\ \ldots \\ z_N \mapsto A_N(t) z_N = A(t)\,(0, \ldots, 0, z_N) \; : \; H \to H \end{array}\right\} \tag{4.7}$$

It is clear that by linearity for each t

$$A(t) z = \sum_{j=0}^{N} A_j(t) z_j \; , \; \forall \; z = (z_N, \ldots, z_0) \in B_1^p \tag{4.8}$$

and $A_j \in L^q(t_0, t_1; \mathcal{L}(H,H))$, $j = 0, \ldots, N$.

When $i = 2$ (that is $B_2^p = M^p(-a, 0; H)$)

consider the maps

$$\left.\begin{aligned} z_0 &\mapsto A_0(t)z_0 = A(t)\kappa^{-1}(z_0,0) \; : H \to H \\ z_1 &\mapsto \tilde{A}_1(t)z_1 = A(t)\kappa^{-1}(0,z_1) \; : L^p(-a,0;H) \to H. \end{aligned}\right\} \quad (4.9)$$

where $\kappa : M^p(-a,0;H) \to H \times L^p(-a,0;H)$ is the isometric isomorphism of Theorem 2.2.

Again by linearity for each t

$$A(t)z = A_0(t)z_0 + \tilde{A}_1(t)z_1, \; \forall \; z = \kappa^{-1}(z_0,z_1) \in B_2^p \quad (4.10)$$

and $\quad A_0 \in L^q(t_0,t_1;\mathcal{L}(H,H))$ and

$\tilde{A}_1 \in L^q(t_0,t_1 \; ; \mathcal{L}(L^p(-a,0;H), H))$.

In the latter case one would like to have an integral representation for the elements of $L^q(t_0,t_1;\mathcal{L}(L^p(-a,0;H), H))$. Given $\tilde{A}_1$ in the latter space can we find an $A_1 \in L^q([t_0,t_1] \times [-a,0]; \mathcal{L}(H,H))$ such that for all $f \in L^p(-a,0;H)$

$$\tilde{A}_1(t) \; f = \int_{-a}^{0} A_1(t,\theta) \; f(\theta) \; d\theta \;, \quad \text{a.e. } [t_0,t_1], \quad (4.11)$$

where $L^q([t_0,t_1] \times [-a,0] \; ; \mathcal{L}(H,H)) = L^q(m_2,\mathcal{L}(H;H))$ and m_2 is the complete Lebesgue measure on $[t_0,t_1] \times [-a,0]$. When $H = R^n$ the answer is positive, but it does not seem the result is true for infinite dimensional Hilbert spaces.

All the results are summarized in

Theorem 4.2

(i) case $B_1^p = H^{N+1}$.

Any ordered family $(A_0,\dots,A_N)$ of elements of $L^q(t_0,t_1;\mathcal{L}(H,H))$ defines a unique map ℓ,

$$\ell(t,z) = \sum_{j=0}^{N} A_j(t)z_j,\ \forall\, z=(z_N,\dots,z_0) \in B_1^p,\quad t \in [t_0,t_1], \tag{4.12}$$

which satisfies Hypotheses 4.1 and hypothesis (M). Conversely for each map ℓ satisfying Hypotheses 4.1 (i), 4.1 (iii), and hypothesis (M) there exists $A_j \in L^q(t_0,t_1;\mathcal{L}(H,H))$, $j = 0,\dots,N$, for which identity (4.12) is true.

Hypotheses 4.1 (i), 4.1 (iii) and (M) imply Hypothesis 4.1 (ii). When $H = R^n$, Hypotheses 4.1 imply hypothesis (M).

(ii) case $B_2^p = M^p(-a,0;H)$.

Any pair (A_0,A_1), $A_0 \in L^q(t_0,t_1;\mathcal{L}(H,H))$ and $A_1 \in L^q([t_0,t_1] \times [-a,0];\mathcal{L}(H,H))$, defines a unique map ℓ,

$$\ell(t,z) = A_0(t)z_0 + \int_{-a}^{0} A_1(t,\theta)z_1(\theta)\, d\theta ,$$
$$\forall\, z = \kappa^{-1}(z_0,z_1) \in B_2^p, \forall t \in [t_0,t_1], \tag{4.13}$$

which satisfies Hypotheses 4.1 and (M).

When

- either $p = 1$ and hypothesis (M) is satisfied,
- or H is finite dimensional and the map

$$t \mapsto \ell(t,\kappa^{-1}(0,\cdot)) : [t_0,t_1] \to \mathcal{L}(L^p(-a,0;H),H)$$

is m-measurable,

there exists unique $A_0 \in L^q(t_0,t_1;\mathcal{L}(H,H))$ and $A_1 \in L^q([t_0,t_1] \times [-a,0];\mathcal{L}(H,H))$ for which identity (4.13) is verified.

The proof of Theorem 4.2 will be given in section 4.1.2.

<u>Corollary 4.3</u>

The map

$$(t,\theta) \mapsto A(t,\theta)\, f(\theta) : [t_0,t_1] \times [-a,0] \to H \tag{4.15}$$

is in $L^1([t_0,t_1] \times [-a,0];H)$ for all A in $L^q([t_0,t_1] \times [-a,0];\mathcal{L}(H,H))$ and f in $L^p(-a,0;H)$.

The proof of the corollary is obvious.

Theorem 4.2 is not completely satisfactory. For the case B_1^p, one would like to construct an example of a map ℓ satisfying Hypotheses 4.1 but violating hypothesis (M). Similarly in the case B_2^p, one would like to exhibit a map ℓ which satisfies Hypotheses 4.1 but for which there is no integral representation of the form (4.13). Such examples would make Theorem 4.2 completely precise.

The second set of results is a specialization of Theorem 3.3.

Theorem 4.4

Let the map ℓ satisfy Hypotheses 4.1. Denote by $x(h,g)$ the solution of the differential equation associated with the map $f_g : [t_0,t_1] \times B_i^p \to H$, where

$$f_g(t,z) = \ell(t,z) + g(t), \forall z \in B_i^p, \forall t \in [t_0,t_1]. \tag{4.16}$$

The map

$$(h,g) \mapsto x(h,g)$$
$$: M^p(-a,0;H) \times L^1(t_0,t_1;H) \to AC^1(t_0,t_1;H) \tag{4.17}$$

is linear and continuous.

Corollary 4.5

Let the hypotheses of Theorem 4.4 hold. Assume in addition that

$$g \in \mathcal{L}^p(t_0,t_1;H) \text{ and } n \in \mathcal{L}^\infty(t_0,t_1;R). \tag{4.18}$$

The solution $x(h,g)$ is in $AC^p(t_0,t_1;H)$ and the map

$$(h,g) \mapsto x(h,g) \tag{4.19}$$
$$: M^p(-a,0;H) \times L^p(t_0,t_1;H) \to AC^p(t_0,t_1;H)$$

is linear and continuous.

Remark In the case $B_2^p = M^p$, the memory map $t \mapsto x_t(h)$ is continuous. This means that the solution $x(h,g)$ is in $AC^q(t_0,t_1;H)$ for all $g \in \mathcal{L}^q(t_0,t_1;H)$ and $n \in \mathcal{L}^q(t_0,t_1;R)$.

Corollary 4.6

Let the hypotheses of Theorem 4.4 hold. Denote by $x(\cdot;s,h,g)$ the solution in $[s,t_1]$ of the differential equation associated with the restriction of the map f_g to $[s,t_1]$ and initial data h at time $t = s$ for some s in $[t_0,t_1]$. The map

$$(s,t) \mapsto x(t;s,h,g) \; : \{(s,t) \in [t_0,t_1]\times[t_0,t_1]\} \; s \leq t\} \to H \quad (4.20)$$

is continuous.

The proof of the above Theorem and Corollaries will be given in subsection 4.1.3.

4.1.2 Representation of the maps f.

We give the proof of Theorem 4.2 announced in section 4.1.1. We use some results proven in Appendices A and B.

Proof of Theorem 4.2

(i) Given an ordered family $(A_0,\ldots,A_N)$ of elements of $L^q(t_0,t_1;\mathcal{L}(H,H))$, the corresponding map ℓ defined as

$$\ell(t,z)=\sum_{j=0}^{N} A_j(t)z_j, \; \forall \; z=(z_N,\ldots,z_0) \in B_1^p$$

is clearly linear in z for fixed t, m-measurable in $[t_0,t_1]$ for fixed z, and

$$|\ell(t,z)| \leq \sum_{j=0}^{N} \|A_j(t)\| \, |z_j|$$
$$\leq n(t) \, |z|_{B_1^p}$$

where

$$n(t) = \begin{cases} \left[\sum_{j=0}^{N} \|A_j(t)\|^q\right]^{1/q} & , \ 1 \leq q < \infty \\ \max_{j=0,\dots,N} \|A_j(t)\| & , \ q = \infty \end{cases} \tag{4.21}$$

and

$$|z|_{B_1^p} = \begin{cases} \left[\sum_{j=0}^{N} |z_j|^p\right]^{1/p} & , \ 1 \leq p < \infty \\ \max_{j=0,\dots,N} |z_j| & , \ p = \infty . \end{cases} \tag{4.22}$$

Notice that $n \in L^q(t_0,t_1;R)$, and ℓ satisfies Hypotheses 4.1. Moreover by definition the map ℓ also satisfies hypothesis (M). The converse is true in the light of the remarks preceding Theorem 4.2. By using Lemma A.1 (ii), Lemma A.3, and Lemma A.1 (iii), it is not difficult to see that if $H = R^n$ Hypothesis 4.1 (ii) implies hypothesis (M).

(ii) Let $A_0 \in L^q(t_0,t_1;\mathcal{L}(H,H))$ and $A_1 \in L^q([t_0,t_1] \times [-a,0];\mathcal{L}(H,H))$. Consider the map $\ell: [t_0,t_1] \times B_2^p \to H$ defined as

$$\ell(t,z) = A_0(t) \, z_0 + \int_{-a}^{0} A_1(t,\theta) \, z_1(\theta) \, d\theta \, ,$$
$$\forall \; z = K^{-1}(z_0,z_1) \in B_2^p, \forall t \in [t_0,t_1].$$

By construction ℓ is linear in z for fixed t and m-measurable in $[t_0, t_1]$ for fixed z.

Moreover

$$\begin{aligned} |\ell(t,z)| &\leq \|A_0(t)\| \, |z_0| + \|A_1(t,\cdot)\|_q \, \|z_1\|_p \\ &\leq n(t) \; |z|_{B_2^p} \end{aligned}$$

where

$$n(t) = \begin{cases} \left[\|A_0(t)\|^q + \|A_1(t,\cdot)\|_q^q \right]^{1/q}, & 1 \leq q < \infty \\ \max\left\{ \|A_0(t)\|, \; \|A_1(t,\cdot)\|_\infty \right\}, & q = \infty \end{cases} \tag{4.23}$$

and

$$\begin{aligned} |z|_{B_2^p} &= \begin{cases} \left[|z_0|^p + \|z_1\|_p^p \right]^{1/p}, & 1 \leq p < \infty \\ \max\left\{ |z_0|, \; \|z_1\|_\infty \right\}, & p = \infty \end{cases} \\ &= \|K^{-1}(z_0,z_1)\|_{M^p} = \|z\|_{M^p} . \end{aligned} \tag{4.24}$$

Notice that $n \in L^q(t_0,t_1;R)$ and ℓ satisfies Hypotheses 4.1. Moreover by definition the map ℓ also satisfies hypothesis (M).

When hypothesis (M) is satisfied we can define from the map ℓ the maps (4.9) and (4.10) and show that A_0 (map (4.9)) and $\tilde{A}_1$(map (4.9)) are respectively in $L^q(t_0,t_1;\mathcal{L}(H,H))$ and $L^q(t_0,t_1; \mathcal{L}(L^p(-a,0;H),H))$. When p = 1 there is a norm preserving isomorphism between $\mathcal{L}(L^1(-a,0;H),H)$ and $L^\infty(-a,0; \mathcal{L}(H,H))$ (Theorem A.6). This induces a norm preserving isomorphism between $L^\infty(t_0,t_1; \mathcal{L}(L^1(-a,0;H),H))$ and $L^\infty(t_0,t_1; L^\infty(-a,0;\mathcal{L}(H,H)))$. Finally there is another norm preserving isomorphism between $L^\infty(t_0,t_1; L^\infty(-a,0;\mathcal{L}(H,H)))$

and $L^{\infty}([t_0,t_1] \times [-a,0]; \mathcal{L}(H,H))$ (Corollary B.4). This shows that when $p = 1$ and hypothesis (M) is satisfied, the map ℓ has a representation given by identity (4.13).

When H is finite dimensional we can again define the maps A_0 (equation (4.9)) and $\tilde{A}_1$ (equation (4.9)) from the map ℓ . By using Lemma A.1 (ii), Lemma A.3, and Lemma A.1 (iii), A_0 belongs to $L^q(t_0,t_1; \mathcal{L}(H,H))$. Because of the additional hypothesis "$t \mapsto \ell(t, \kappa^{-1}(0,\bullet)) : [t_0,t_1] \to \mathcal{L}(L^p(-a,0;H),H)$ is m-measurable" $\tilde{A}_1 \in L^q(t_0,t_1; \mathcal{L}(L^p(-a,0;H),H))$. But $\mathcal{L}(L^p(-a,0;H),H))$ and $L^q(-a,0; \mathcal{L}(H,H)))$ are isomorphic by Theorem A.6 since H is finite dimensional. This induces an isomorphism between $L^q(t_0,t_1; \mathcal{L}(L^p(-a,0;H),H))$ and $L^q(t_0,t_1; L^q(-a,0; \mathcal{L}(H,H)))$. Finally Corollary B.4 gives us an isomorphism between the latter space and $L^q([t_0,t_1] \times [-a,0]; \mathcal{L}(H,H))$. This shows that under such hypotheses the map ℓ has a representation of the form given by identity (4.13).

4.1.3 Specialization of Theorem 3.3.

The existence and uniqueness of a solution to the hereditary differential system defined by f_g (equation (4.16)) for each $(h,g) \in M^p(t_0,t_1;H) \times L^1(t_0,t_1;H)$ has been proven in Theorem 3.3 since f_g satisfies all of the hypotheses.

Our task is reduced to study the properties of the map (4.17). The linearity is clear. Consider the following inequalities:

$$\|x(h,g)\|_{AC'} = \|x(h,0)+x(0,g)\|_{AC'} \le \|x(h,0)\|_{AC'} + \|x(0,g)\|_{AC'}, \quad (4.25)$$

$$\|x(h,0)\|_{AC'} = \|x(h,0)-x(0,0)\|_{AC'} \le k(p)\ \|h\|_{M^p}\ \text{(Theorem 3.3)}. \quad (4.26)$$

As for the term x(0,g) we must revert to the techniques of Theorem 3.3 and the C_α spaces.

$$\begin{aligned}|x(0,g)(t)| &= \left|\int_{t_0}^{t} \ell(s,\alpha_i(0,x(0,g))(s))\,ds + \int_{t_0}^{t} g(s)\,ds\right| \\ &\le \int_{t_0}^{t} n(s)\,\|\alpha_i(0,x(0,g))(s)\|_{B_i^p}\,ds + \|g\|_1 \\ &\le \int_{t_0}^{t} n(s)\,c'(p) \max_{u\in[t_0,s]} |x(0,g)(u)|\,ds + \|g\|_1 \\ &\le \tilde\alpha\, c'(p)\ g_{\tilde\alpha}(t)\ \|x(0,g)\|_{C_{\tilde\alpha}} + \|g\|_1\end{aligned}$$

and

$$\|x(0,g)\|_{C_{\tilde\alpha}} \le \frac{1}{1-\tilde\alpha c'(p)} \max_{t\in[t_0,t_1]}\left[\frac{1}{g_{\tilde\alpha}(t)}\right]\|g\|_1 = k\ \|g\|_1,$$

where $k>0$ since $0<\tilde\alpha\ c'(p)<1$ and $g_{\tilde\alpha}(t)>0$.

Also

$$\begin{aligned}\left\|\frac{dx}{dt}(0,g)\right\|_1 &= \int_{t_0}^{t_1} |\ell(t,\alpha_i(0,x(0,g))(t)) + g(t)|\,dt \\ &\le \int_{t_0}^{t_1} n(t)\ c'(p)\ \|x(0,g)\|_C + \|g\|_1 \\ &\le k'\ \|x(0,g)\|_C + \|g\|_1 .\end{aligned}$$

But there exists constants $k_1(\alpha)$ and $k_2(\alpha)$ for which

$$\| \ \|_C \le k_1(\alpha) \| \ \|_{C_\alpha} , \quad \| \ \|_{C_\alpha} \le k_2(\alpha) \| \ \|_C ,$$

and

$$\|x(0,g)\|_{AC'} \le k'' \max\left\{ \|x(0,g)\|_C , \left\| \frac{dx}{dt}(0,g) \right\|_1 \right\}$$

$$\le k'' \max\left\{ k_2(\alpha) k , k' k_1(\alpha) k + 1 \right\} \|g\|_1 . \tag{4.27}$$

Finally the substitution of (4.26) and (4.27) in (4.25) yields

$$\|x(h,g)\|_{AC'} \le c.\max\left\{ \|h\|_{M^p} , \|g\|_1 \right\} \tag{4.28}$$

for some constant $c > 0$.

Proof of Corollary 4.5

It suffices to show $\frac{dx}{dt}(h,g)$ is in $L^p(t_0,t_1;E)$.

$$\left\| \frac{dx}{dt}(h,g) \right\|_p = \left[\int_{t_0}^{t_1} | \ell(t, \alpha_i(h, x(h,g))(t)) + g(t) |^p dt \right]^{1/p}$$

$$\le \left[\int_{t_0}^{t_1} m(t)^p \|\alpha_i(h, x(h,g))(t)\|^p_{B_i^p} dt \right]^{1/p} + \|g\|_p$$

$$\le \|m\|_\infty \|\alpha_i(h, x(h,g))\|_p + \|g\|_p$$

$$\le \|m\|_\infty k \max\left\{ \|h\|_p , \|x(h,g)\|_C \right\} + \|g\|_p < \infty$$

(by Proposition 3.7).

Proof of Corollary 4.6

Let $t_0 \le s_1 \le s \le t_1$. Denote by $\bar{x}$ and x the respective solutions on $[s_1,t_1]$ and $[s,t_1]$. On the time interval $[s,t_1]$

$\bar{x}\,(t;s_1,h,g) = x(t;s,\bar{x}_s(h),g), t \in [s,t_1]$,

where $\bar{x}_s(h)$ is the memory map of $(h,\bar{x})$ at time s,

$$\bar{x}_s(h)(\theta) = \begin{cases} h(s-s_1+\theta), & -a \le \theta < -(s-s_1) \\ \bar{x}(s+\theta), & -(s-s_1) \le \theta \le 0. \end{cases}$$

By Theorem 4.4

$$\|x(\cdot;s,h,g) - x(\cdot;s,\bar{x}_s(h),g)\|_{AC^1(s,t_1;E)}$$
$$\le k \;\|h - \bar{x}_s(h)\|_{M^p}$$

and for $t \in [s,t_1]$ (by changing the norm on AC^1, Prop. 2.9)

$$|x(t;s,h,g)-\bar{x}(t;s_1,h,g)| \le k' \,\|\bar{x}_{s_1}(h)-\bar{x}_s(h)\|_{M^p}.$$

Similarly when $t_0 \le s \le s_1 \le t_1$ we obtain for all $t \in [s_1,t_1]$

$$|x(t;s,h,g)-\bar{x}(t;s_1,h,g)| \le k'' \,\|x_s(h)-x_{s_1}(h)\|_{M^p}.$$

But the memory maps

$u \mapsto x_u(h):[s_1,t_1] \to M^p$ and $u \mapsto x_u(h):[s,t_1] \to M^p$

are continuous maps. For all $t \in [t_0,t_1]$ the map

$$s \mapsto x(t;s,h,g) : [t_0,t_1] \to H$$

is continuous from the right and from the left.

Thus we obtain the continuity of the map (4.20) at (s,t).

4.2 The Fundamental Matrix Solution.

Let $A_j \in L^q(t_0,t_1;\ \mathcal{L}(H,H))$, $j = 0,\ldots,N$, in the case $B_1^p = H^{N+1}$; for B_2^p let $A_0 \in L^q(t_0,t_1;\ \mathcal{L}(H,H))$ and $A_1 \in L^q([t_0,t_1] \times [-a,0];\ \mathcal{L}(H,H))$. They define (Theorem 4.2) the following hereditary differential systems ($1 \leq p < \infty$):

$$(E_1') \begin{cases} \dfrac{dx}{dt}(t) = A_0(t)x(t) + \displaystyle\sum_{i=1}^{N} A_i(t)\,x(t+\theta_i) + g(t), \text{ a.e. } [t_0,t_1], \\ \\ x(t) = h(t-t_0), \quad t \in [t_0-a,t_0], \end{cases} \tag{4.29}$$

$h \in \mathcal{L}^p(-a,0;H)$, $g \in \mathcal{L}^1(t_0,t_1;H)$, $A_i \in \mathcal{L}^q(t_0,t_1;\mathcal{L}(H,H))$,

$i = 0,\ldots,N$, and $p^{-1}+q^{-1} = 1$;

$$(E_2') \begin{cases} \dfrac{dx}{dt}(t) = A_0(t)x(t) + \displaystyle\int_{-a}^{0} A_1(t,\theta)\,x(t+\theta)\,d\theta + g(t), \text{ a.e. } [t_0,t_1], \\ \\ x(t) = h(t-t_0), \quad t \in [t_0-a,t_0], \end{cases} \tag{4.30}$$

$h \in \mathcal{L}^p(-a,0;H)$, $g \in \mathcal{L}^1(t_0,t_1;H)$, $A_0 \in \mathcal{L}^q(t_0,t_1;\mathcal{L}(H,H))$,
$A_1 \in \mathcal{L}^q([t_0,t_1]\times[-a,0];\mathcal{L}(H,H))$ where $p^{-1}+q^{-1} = 1$.

Notice that the above formulation is the one of the Introduction. This formulation turns out to be more convenient whenever computations are necessary. The reader will be careful not to confuse the map x defined on $[t_0-a,t_1]$ and the solution x defined on $[t_0,t_1]$. This apparently confusing notation has the advantage of being simple and does not lead to any confusion. We shall also go from the formulations (E_1') and (E_2') to the formulations of Chapter 3 whenever necessary. Such operations have been justified in section 3.1.

We now state our main theorem. Its proof will be broken into several propositions.

Theorem 4.7

Let the hypotheses of Theorem 4.4 hold for the systems (E_1') and (E_2'). The solution $x(h,g)$ has a representation for all $(h,g) \in M^p(-a,0;H) \times L^1(t_0,t_1;H)$. At each time $t \in [t_0,t_1]$

$$x(h,g)(t) = \Phi^0(t,t_0)h(0) + \int_{-a}^{0} \Phi^1(t,t_0,\eta)h(\eta)\,d\eta + \int_{t_0}^{t} \Phi^0(t,s)g(s)\,ds, \tag{4.31}$$

where $\Phi^0(t,s)$ and $\Phi^1(t,t_0,\eta) \in \mathcal{L}(H,H)$ are defined as follows.

1) $\Phi^0(t,s)$, $t \geq s$.

For all $s \in [t_0,t_1[$, the map $t \mapsto \Phi^0(t,s)$ $:[s,t_1] \to \mathcal{L}(H,H)$ is the unique solution of the matrix equation:

(i) case (E_1')

$$\left\{\begin{array}{l} \dfrac{\partial \Phi^0}{\partial t}(t,s) = A_0(t)\,\Phi^0(t,s) + \displaystyle\sum_{\substack{i=1 \\ t+\theta_i \geq s}}^{N} A_i(t)\,\Phi^0(t+\theta_i,s), \text{ a.e. in } [t_0,t_1] \\ \\ \Phi^0(s,s) = I \quad (\text{identity in } \mathcal{L}(H,H)) \end{array}\right. \tag{4.32}$$

(ii) case (E_2')

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi^0(t,s) = A_0(t)\,\Phi^0(t,s) + \begin{cases} \displaystyle\int_{-(t-s)}^{0} A_1(t,\theta)\,\Phi^0(t+\theta,s)\,d\theta, & t-s<a \\ \displaystyle\int_{-a}^{0} \quad '' \quad\quad '' \quad\quad '' & , t-s \geq a, \end{cases} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. in } [s,t_1] \\ \Phi^0(s,s) = I \quad (\text{identity in } \mathcal{L}(H,H)). \end{cases} \tag{4.33}$$

Moreover the map $(t,s) \mapsto \Phi^0(t,s)$

$:\{(t,s) \in [t_0,t_1]\times[t_0,t_1] \mid t \geq s\} \to \mathcal{L}(H,H)$ is continuous.

2) $\Phi^1(t,t_0,\eta)$, $t \geq t_0$, $\eta \in [-a,0]$.

(i) case (E_1')

$$\Phi^1(t,t_0,\eta) = \begin{cases} \displaystyle\sum_{\substack{i=1 \\ -a \leq \theta_i < \eta}}^{N} \Phi^0(t,t_0+\eta-\theta_i)\,A_i(t_0+\eta-\theta_i), & -a \leq \eta < t-t_0-a \\ \displaystyle\sum_{\substack{i=1 \\ \eta-(t-t_0) \leq \theta_i < \eta}}^{N} \quad '' \quad\quad\quad '' & , t-t_0-a \leq \eta \leq 0, \end{cases} \tag{4.34}$$

(ii) case (E_2')

$$\Phi^1(t,t_0,\eta) = \begin{cases} \displaystyle\int_{-a}^{\eta} \Phi^0(t,t_0+\eta-\alpha)\,A_1(t_0+\eta-\alpha,\alpha)\,d\alpha, & -a \leq \eta < t-t_0-a \\ \displaystyle\int_{\eta-(t-t_0)}^{\eta} \quad '' \quad\quad\quad '' \quad\quad '' & , t-t_0-a \leq \eta \leq 0. \end{cases} \tag{4.35}$$

The proof of Theorem 4.7 will be given via a sequence of propositions.

Remark (i) Notice that $\Phi^1(t_0,t_0,\eta) = 0$, $\eta \in [-a,0]$, as is to be expected since $x(h,g)(t_0) = h(0) = \Phi^0(t_0,t_0)h(0)$.

(ii) As a function of η , $\Phi'(t,t_0,\eta)$ needs only be defined almost everywhere on [-a,0]. This explains a certain element of choice in the definition of $\Phi'(t,t_0,\eta)$.

Proposition 4.8

Let the hypotheses of Theorem 4.7 hold. The solution $x(t;t_0,h,g)$ at time t of the differential equation (E_1') or (E_2') for the data $(h,g) \in M^p(-a,0;H) \times L^1(t_0,t_1;H)$ has the representation

$$x(t;t_0,h,g) = \Phi(t,t_0)\, h + G(t,t_0)g \tag{4.36}$$

for all $t \in [t_0,t_1]$, where

$\Phi(t,t_0) \in \mathcal{L}(M^p;H)$ and $G(t,t_0) \in \mathcal{L}(L^1(t_0,t_1;H),H)$ are obtained in the following way:

$$\Phi(t,t_0)h = x(t;t_0,h,0) \tag{4.37}$$

and

$$G(t,t_0)g = x(t;t_0,0,g). \tag{4.38}$$

Proof: By theorem 4.4 and the properties of the evaluation map

$$x \mapsto x(t) : AC^1(t_0,t_1;H) \to H,$$

that is linearity and continuity.

We now study separately the maps $\Phi(t,t_0)$ and $G(t,t_0)$.

Proposition 4.9

Again let the hypotheses of Theorem 4.7 hold, and let $s \in [t_0, t_1]$. The solution $x(t;s,h,0)$ at time $t \in [s,t_1]$ of the differential equation (E_1') or (E_2') with initial data h at time $t = s$ has the representation

$$x(t;s,h,0) = \Phi(t,s)h, \tag{4.39}$$

for $\Phi(t,s) \in \mathcal{L}(M^p(-a,0;H),H)$.

Because of the isometric isomorphism κ between M^p and $H \times L^p(-a,0;H)$, we also have the representation

$$\Phi(t,s)h = \Phi^0(t,s)h^0 + \Phi^1(t,s)h^1 \ , \forall h \in M^p, \tag{4.40}$$

where $\Phi^0(t,s) \in \mathcal{L}(H,H)$, $\Phi^1(t,s) \in \mathcal{L}(L^p,H)$ and $(h^0,h^1) = \kappa(h)$ (κ as in Theorem 2.2).

Moreover

$$\Phi^0(t,s)h^0 = x(t;s,\kappa^{-1}(h^0,0),0) \ , \forall \ h^0 \in H \tag{4.41}$$

$$\Phi^1(t,s)h^1 = x(t;s,\kappa^{-1}(0,h^1),0) \ , \forall \ h^1 \in L^p. \tag{4.42}$$

Proof: Direct consequence of Proposition 4.8, Corollary 4.6 and Theorem 2.2.

Proposition 4.10

Let $\Phi^0(t,s)$, $t_1 \geq t \geq s \geq t_0$ be as defined in Proposition 4.9. For each $s \in [t_0,t_1[$ the map $t \mapsto \Phi^0(t,s) : [s,t_1] \to \mathcal{L}(H,H)$ is the unique solution of equation (4.32) (resp. (4.33)) for the system (E_1') (resp. (E_2')). Moreover the map

$$(t,s) \mapsto \Phi^0(t,s) : \{(t,s) \in [t_0,t_1]\times[t_0,t_1] \mid t \geq s\} \to \mathcal{L}(H,H) \tag{4.43}$$

is continuous.

Proof: We must first check that equations (4.32) and (4.33) make sense. They do if we look at them as equations with values in $\mathcal{L}_H = \mathcal{L}(H,H)$. Corresponding to $A_i \in \mathcal{L}^q(t_0,t_1;\mathcal{L}_H)$ we can define an $\tilde{A}_i \in \mathcal{L}^q(t_0,t_1;\ \mathcal{L}(\mathcal{L}_H,\mathcal{L}_H))$ as follows $(\tilde{A}_i(t)X)x = A_i(t)\ (Xx),\ \forall\, x \in H,\ \forall\, X \in \mathcal{L}_H$.

Moreover

$$\|A_i(t)\| = \sup_{|x|\leq 1} |A_i(t)x| = \sup_{|x|\leq 1} |(\tilde{A}_i(t)I)x| \leq \sup_{|x|\leq 1} \sup_{\|X\|\leq 1} |(\hat{A}_i(t)X)x|$$
$$\leq \sup_{\|X\|\leq 1} \|(\tilde{A}_i(t)X\| = \|\tilde{A}_i(t)\|$$

and

$$\|\tilde{A}_i(t)\| = \sup_{\|X\|\leq 1} \|(\tilde{A}_i(t)X)\| = \sup_{\|X\|\leq 1} \sup_{|x|\leq 1} |(\tilde{A}_i(t)X)x|$$
$$= \sup_{|x|\leq 1} \sup_{\|X\|\leq 1} |A_i(t)(Xx)| \leq \sup_{|x|\leq 1} |A_i(t)x| = \|A_i(t)\|$$

implies that $\|\tilde{A}_i(t)\| = \|A_i(t)\|$ for all $t \in [t_0,t_1]$. Similarly for $A_1 \in \mathcal{L}^q([t_0,t_1]\times[-a,0];\ \mathcal{L}_H)$ we can define an $\tilde{A}_1 \in \mathcal{L}^q([t_0,t_1]\times[-a,0];\ \mathcal{L}(\mathcal{L}_H,\mathcal{L}_H))$ as follows $(\tilde{A}_1(t,\theta)X)x = A_1(t,\theta)\ (Xx),\ \forall\, x\in H,\ \forall\, X \in \mathcal{L}_H$; we also have $\|\tilde{A}_1(t,\theta)\| = \|A_1(t,\theta)\|$ for all $(t,\theta) \in [t_0,t_1]\times[-a,0]$.

Thus equation (4.32) (resp. (4.33)) is an equation of the type (E_1') (resp. (E_2')) but with values in the Banach space $\mathcal{L}_H$ satisfying the hypotheses of Theorem 4.4. As for the initial data, we only need it at time $t = s$ since in both equations (4.32) and (4.33) we do not require any information on the time interval $[s-a,s[$ (or the initial data can be considered as zero on that interval). By Theorem 4.4 both systems respectively have a unique solution $t \mapsto X(t,s) : [s,t_1] \to \mathcal{L}_H$ in $[s,t_1]$; by Corollary 4.6 the map $(t,s) \mapsto X(t,s) : \{(t,s) \in [t_0,t_1] \times [t_0,t_1] \mid t \geq s\} \to \mathcal{L}_H$ is continuous.

To show $X(t,s)$ is the $\Phi^0(t,s)$ of Proposition 4.9, pick $h^0 \in H$ and "multiply" both side of equation (4.32) (resp. (4.33)) by h^0 on the right. This shows that the map

$$t \mapsto \tilde{x}(t) = X(t,s)h^0 \ : [s,t_1] \to H$$

is a solution of equation (E_1') (resp. (E_2')) with initial data $x(s) = X(s,s)h^0 = h^0$. In other words $\tilde{x}$ is a solution of (E_1') (resp. (E_2')) with initial data $\kappa^{-1}(h^0,0) \in M^p$ and $g = 0$. By uniqueness $\Phi^0(t,s)h^0 = x(t;s,\kappa^{-1}(h^0,0),0) = \tilde{x}(t) = X(t,s)h^0$, $t \in [s,t_1]$. Hence $X(t,s)$ obtained as a solution of equation (4.32) or (4.33) is equal to the corresponding $\Phi^0(t,s)$ of Proposition 4.9.

Remark If $(u_i)_{i\in I}$ is a maximal orthonormal family of H, $\Phi^0(t,s)$ can be constructed from the respective solutions of (E_1') and (E_2') on $[s,t_1]$ with initial data

$$h^i(\theta) = \begin{cases} e_i & ,\theta = 0 \\ 0 & ,\theta \in [-a,0[\end{cases} \quad , i \in I.$$

Proposition 4.11

Let the hypotheses and notations of Propositions 4.8, 4.9 and 4.10 hold. Then

$$x(t;t_0,0,g) = G(t,t_0)g = \int_{t_0}^{t} \Phi^0(t,u)\, g(u)\,du \tag{4.44}$$

for all $t \in [t_0,t_1]$ and $g \in L^1(t_0,t_1;H)$.

Proof: We only give a proof for the case (E_2'). By definition

$$\tilde{x}(t) = G(t,t_0)g = \int_{t_0}^{t} \Phi^0(t,s)g(s)ds, \forall t \in [t_0,t_1], \forall g \in L^1(t_0,t_1;H). \tag{4.45}$$

We differentiate with respect to t, substitute in equation (4.30), and show $G(t,t_0)g$ is indeed a solution. Because of uniqueness this is sufficient to establish the proposition.

Let $\Delta > 0$ such that t and $t+\Delta \in [t_0,t_1]$.

Consider the relation

$$\frac{1}{\Delta}\left[\tilde{x}(t+\Delta) - \tilde{x}(t)\right] = \frac{1}{\Delta}\int_{t}^{t+\Delta} \Phi^0(t+\Delta,s)g(s)ds$$

$$+ \int_{t_0}^{t} \frac{1}{\Delta}\left[\Phi^0(t+\Delta,s) - \Phi^0(t,s)\right]g(s)ds.$$

Now $\Phi^0(t,s)$ is differentiable with respect to t and

$$\frac{d\tilde{x}}{dt}(t) = \Phi^0(t,t)g(t) + \int_{t_0}^{t} \frac{\partial \Phi^0}{\partial t}(t,s)\, g(s)\, ds. \tag{4.46}$$

But $\frac{\partial \Phi^0}{\partial t}(t,s)$ can be computed from equation (4.33):

$$\int_{t_0}^{t} \frac{\partial \Phi^0}{\partial t}(t,s) g(s) ds = A_0(t)\int_{t_0}^{t} \Phi^0(t,s) g(s) ds + \int_{t_0}^{t} ds \begin{cases} \int_{-(t-s)}^{0} d\theta\, A_1(t,\theta)\Phi^0(t+\theta,s)g(s), & t-s<a, \\ \int_{-a}^{0} d\theta \quad \text{"} \quad \text{"} \quad \text{"}, & t-s \geq a. \end{cases} \tag{4.47}$$

But the integrand of the last term in equation (4.47),

$$(s,\theta) \mapsto \left\{ \begin{array}{ll} 0 & , -a \leq \theta \leq -(t-s) \\ A_1(t,\theta)\Phi^0(t+\theta,s)g(s) & , \text{otherwise} \end{array} \right\} : [t_0,t_1] \times [-a,0] \to H,$$

is in $\mathcal{L}^1([t_0,t] \times [-a,0];H)$ for almost all t in $[t_0,t_1]$.

When $1 < p < \infty$ (the case p = 1 is similar)

$$① = \begin{cases} \int_{-(t-s)}^{0} d\theta\, |A_1(t,\theta)\Phi^0(t+\theta,s)g(s)| \\ \int_{-a}^{0} d\theta \quad \text{"} \quad \text{"} \quad \text{"} \end{cases} \leq \begin{cases} \left[\int_{-(t-s)}^{0} |A_1(t,\theta)|^q d\theta\right]^{1/q} \cdot \left[\int_{-(t-s)}^{0} |\Phi^0(t+\theta,s)g(s)|^p d\theta\right]^{1/p} \\ \left[\int_{-a}^{0} \text{"}\right]^{1/q} \cdot \left[\int_{-a}^{0} \text{"}\right]^{1/p} \end{cases}$$

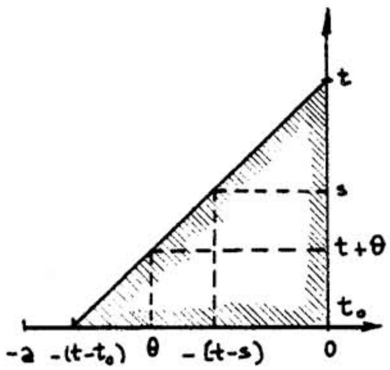
$$\leq \|A_1(t,\cdot)\|_q \max_{u\in[s,t]}\{\|\Phi^0(u,s)\|\, a^{1/p}\, |g(s)|\}$$

$$\int_{t_0}^{t} ①\, ds \leq \|A_1(t,\cdot)\|_q\, a^{1/p} \int_{t_0}^{t} \max_{u\in[s,t]} \|\Phi^0(u,s)\|\, |g(s)|\, ds$$

$$\leq \|A_1(t,\cdot)\|_q\, a^{1/p} \max_{(u,s)\in \mathcal{R}} \|\Phi^0(u,s)\|\, \|g\|_1 < \infty$$

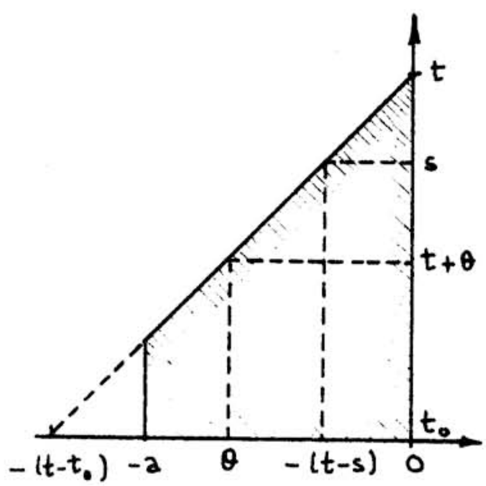
where $\mathcal{R} = \{(u,s) \in [t_0,t]\times[t_0,t] \mid u \geq s\}$. Thus Fubini's theorem can be applied to change the order of integration for almost all $t \in [t_0,t_1]$.

When $0 \le t-t_0 < a$, we integrate over the shaded region drawn below



$$\int_{t_0}^{t} ds \int_{-(t-s)}^{0} d\theta = \int_{-(t-t_0)}^{0} d\theta \int_{t_0}^{t+\theta} ds \;;$$

when $t-t_0 \ge a$, the region of integration is again drawn below



$$\int_{t_0}^{t} ds \begin{cases} \displaystyle\int_{-(t-s)}^{0} d\theta \,, & t-s<a \\ \displaystyle\int_{-a}^{0} d\theta \,, & t-s\ge a \end{cases} = \int_{-a}^{0} d\theta \int_{t_0}^{t+\theta} ds .$$

The second term on the right hand side of the identity (4.47) can now be rewritten as

$$\begin{cases} \displaystyle\int_{-(t-t_0)}^{0} d\theta \; A_1(t,\theta) \int_{t_0}^{t+\theta} ds \; \Phi^0(t+\theta,s) g(s), & t-t_0 < a \\ \displaystyle\int_{-a}^{0} d\theta \; A_1(t,\theta) \int_{t_0}^{t+\theta} ds \; \Phi^0(t+\theta,s) g(s), & t-t_0 \ge a \end{cases}$$

$$= \begin{cases} \displaystyle\int_{-(t-t_0)}^{0} d\theta \; A_1(t,\theta)\, \tilde{x}(t+\theta) & , \; t-t_0 < a \\ \displaystyle\int_{-a}^{0} d\theta \quad " \qquad " & , \; t-t_0 \geq a \, . \end{cases} \tag{4.48}$$

Finally

$$\frac{d\tilde{x}}{dt}(t) = g(t) + A_0(t)\,\tilde{x}(t) + \begin{cases} \displaystyle\int_{-(t-t_0)}^{0} d\theta \;\; A_1(t,\theta)\,\tilde{x}(t+\theta) \, , & t-t_0 < a \\ \displaystyle\int_{-a}^{0} d\theta \quad " \qquad " & , \; t-t_0 \geq a \end{cases}$$

In other words $\int_{t_0}^{t} \Phi^0(t,u)g(u)\, du$ is indeed equal to the solution $x(t;t_0,0,g)$ of equation (E_2') by uniqueness.

The last result makes it possible to give a more specific expression for $\Phi^1(t,s)$. Given $h^1 \in L^p(-a,0;H)$, define

(i) <u>case (E_1')</u>

$$\tilde{g}(t) = \sum_{\substack{i=1 \\ t-t_0+\theta_i < 0}}^{N} A_i(t)\; h^1(t-t_0+\theta_i)$$

$$= \begin{cases} \displaystyle\sum_{\substack{i=1 \\ -a \leq \theta_i < -(t-t_0)}}^{N} A_i(t)\, h^1(t-t_0+\theta_i) \, , & t-t_0 < a \\[2em] 0 & , \; t-t_0 \geq a \end{cases} \tag{4.49}$$

(ii) case (E_2')

$$\tilde{g}(t) = \begin{cases} \int_{-a}^{-(t-t_0)} A_1(t,\theta)\, h^1(t-t_0+\theta)\,d\theta & , t-t_0 < a, \\ 0 & , t-t_0 \geq a. \end{cases} \tag{4.50}$$

Notice that in both cases $\tilde{g} \in \mathcal{L}^1(t_0,t_1;H)$.

By the structure of the differential equations (E_1') and (E_2')

$$x(t;t_0,0,\tilde{g}) = x(t;t_0,\kappa^{-1}(0,h^1),0) \tag{4.51}$$

and hence

$$\Phi^1(t,s)h^1 = \int_{t_0}^{t} \Phi^0(t,s)\tilde{g}(s)\,ds \quad ,\forall\, h^1 \in L^p. \tag{4.52}$$

We shall show that the operator $\Phi^1(t_0,t) \in \mathcal{L}(L^p(-a,0;H),H)$ $(1 \leq p < \infty)$ has an integral representation of the form

$$\Phi^1(t,t_0)\,h^1 = \int_{-a}^{0} \Phi^1(t,t_0,\eta)\,h^1(\eta)\,d\eta \ , \ \forall\, h^1 \in L^p(-a,0;H),$$

$(\Phi^1(t,t_0,\eta) \in \mathcal{L}(H,H))$.

<u>Proposition 4.12</u>

Let the hypotheses and notations of Propositions 4.8 to 4.11 hold. $\Phi^1(t,t_0)$ has a representation of the form

$$\Phi^1(t,t_0)\,h^1 = \int_{-a}^{0} \Phi^1(t,t_0,\eta)\,h^1(\eta)\,d\eta \ , \quad \forall\, h^1 \in L^p, \tag{4.53}$$

where $\Phi^1(t,t_0,\eta) \in \mathcal{L}(H,H)$ is given by the identity (4.34) (resp. (4.35)) for the case (E_1') (resp. (E_2')) for all $t \in [t_0,t_1]$ and $\theta \in [-a,0]$.

Proof: We again only verify identity (4.35) corresponding to the case (E_2'). With $\tilde{g}$ as defined in (4.50) the right hand side of identity (4.52) becomes

$$\int_{t_0}^{t} \Phi^0(t,s)\,\tilde{q}(s)\,ds = \int_{t_0}^{t} \Phi^0(t,s) \begin{cases} \int_{-a}^{-(s-t_0)} A_1(s,\theta)\,h^1(s-t_0+\theta)\,d\theta\,, & s-t_0<a \\ 0\,, & s-t_0 \geq a. \end{cases} \tag{4.54}$$

Change the variable θ to $\eta = s-t_0+\theta$ and then the variable s to $\alpha = s-t_0-a$. The right hand side of identity (4.54) becomes:

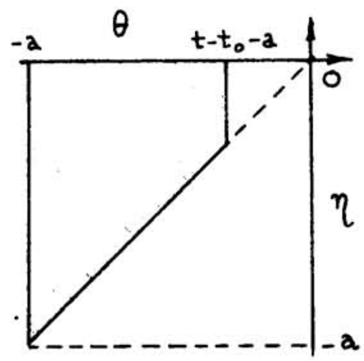
$$\int_{-a}^{t-t_0-a} d\alpha\, \Phi^0(t,t_0+a+\alpha) \begin{cases} \int_{\alpha}^{0} d\eta\, A_1(t_0+a+\alpha,\eta-\alpha-a)\,h^1(\eta)\,, & \alpha<0 \\ 0\,, & \alpha \geq 0. \end{cases} \tag{4.55}$$

But the integrand

$(\alpha,\eta) \mapsto \Phi^0(t,t_0+a+\alpha)\, A_1(s+a+\alpha,\eta-\alpha-a)\, h^1(\eta)$

is in $\mathcal{L}^1([-a,0]\times[-a,0];H)$ (Corollary 4.3 and Proposition 4.10).

By Fubini's theorem the order of integration can be changed over the region drawn below:



$$\int_{-a}^{t-t_0-a} d\alpha \begin{cases} \int_{\alpha}^{0} d\eta\,, & \alpha<0 \\ 0\,, & \alpha\geq 0 \end{cases} = \int_{-a}^{0} d\eta \begin{cases} \int_{-a}^{\eta} d\alpha\,, & -a \leq \eta < t-t_0-a \\ \int_{-a}^{t-t_0-a} d\alpha\,, & t-t_0-a \leq \eta \leq 0. \end{cases}$$

Finally after a last change of the variable α to $\theta = \eta - \alpha - a$, we obtain identity (4.35).

Remark It is important to note the structural implications of Proposition 4.12. The "dynamics" of a hereditary system of the type (E_1') or (E_2') are completely characterized provided the behavior of the system is known for initial data of the form $h = K^{-1}(c,0)$, $c \in H$.

Remark The "fundamental matrix solution $\Phi^0(t,t_0)$" is generally not invertible. Consider the following example. For the system (E_1') let $t_1 > t_0+a$, $H = R^n$, $N = 1$, $g = 0$, $0 < \varepsilon < t_1-(t_0+a)$

$$A_0(t) = \begin{cases} \tilde{A}_0(t) & , t_0 \le t \le t_0+a \\ 0 & , t_0+a < t \le t_1 \end{cases}$$

$$A_1(t) = \begin{cases} I & , t_0 \le t \le t_0+a \\ -\frac{1}{\varepsilon}\tilde{\Phi}(t_0+a, s-a) & , t_0+a < t \le t_0+a+\varepsilon \\ 0 & , t_0+a+\varepsilon < t \le t_1 , \end{cases}$$

where $\tilde{A}_0$ is arbitrary in $L^1(t_0,t_0+a;\mathcal{L}(R^n,R^n))$ and $\tilde{\Phi}(t,u)$ $(t_0+a \ge t \ge u \ge t_0)$ is the fundamental matrix solution of the differential equation

$$\frac{dx(t)}{dt} = \tilde{A}_0(t)x(t) \quad , \quad t \in [t_0,t_0+a].$$

It is easy to compute the fundamental matrix solution $\Phi^0(t,t_0)$ of the differential equation $\frac{dx}{dt}(t)=A_0(t)x(t) + A_1(t)x(t-a)$, $t \in [t_0,t_1]$:

$$\Phi(t,t_0) = \begin{cases} \tilde{\Phi}(t,t_0), & t_0 \leq t \leq t_0+a \\ \tilde{\Phi}(t_0+a,t_0)\left[\dfrac{t_0+a+\varepsilon-t}{\varepsilon}\right], & t_0+a < t \leq t_0+a+\varepsilon \\ 0, & t_0+a+\varepsilon < t \leq t_1. \end{cases}$$

Notice that $\Phi(t,t_0)= 0$ on the interval $]t_0+a+\varepsilon, t_1]$.

4.3 The Adjoint Problem for Linear Hereditary Systems.

For research related to the adjoint problem we refer the reader to references [3,20,21,22,23,12]. The adjoint problem for a linear hereditary system presents some unusual structural features. The first question is the very definition of the adjoint. For a linear ordinary differential equation,

$$\begin{cases} \dfrac{dx}{dt}(t) = A(t)\,x(t), & \text{a.e. } [t_0,t_1] \\ x(t_0) = x_0, \end{cases}$$

the adjoint solution p is defined as an element of $AC^1(t_0,t_1;H)$ (H, Hilbert with inner product $(.|.)$) for which

$$(p(t)\,|\,x(t)) = \text{constant}, \quad \forall\, t \in [t_0,t_1].$$

For hereditary differential systems this definition requires modifications. In the first subsection we give an appropriate definition of an adjoint solution and give the specific form of the adjoint hereditary differential system. The second subsection contains the proof of the main lemma.

4.3.1 The A-product and the Adjoint System.

Before we can introduce the right definition for the adjoint solution, some preliminary definitions and a key lemma are necessary.

Consider the product space $\tilde{\mathfrak{F}} \times \mathcal{C}$, where $\tilde{\mathfrak{F}}$ denotes any of the spaces $C(0,a;H)$, $\mathcal{L}^p(0,a;H)$ and $N^p(0,a;H)$, and $\mathcal{C}$ is chosen among the spaces $C(t_0,t_1;H)$ or $AC^p(t_0,t_1;H)$. $\tilde{\mathfrak{F}} \circ \mathcal{C}$ will denote the closed subspace (with respect to the product topology) of all $(\tilde{h},p)$ in $\tilde{\mathfrak{F}} \times \mathcal{C}$ for which $p(t_1)=\hat{h}(0)$.

Definition 4.13

The map

$$t \mapsto p_t(\tilde{h}) \; : \; [t_0,t_1] \longrightarrow \tilde{\mathfrak{F}} \tag{4.56}$$

where

$$p_t(\tilde{h})(\alpha) = \begin{cases} p(t+\alpha), & 0 \le \alpha \le t_1-t \\ \hat{h}(t-t_1+\alpha), & t_1-t < \alpha \le a \end{cases} \tag{4.57}$$

will be referred to as the <u>adjoint memory map</u> for $(\tilde{h},p)\in \tilde{\mathcal{F}}\circ\mathcal{C}$ and denoted as $p_\bullet(\tilde{h})$.

<u>Remark</u> All the results of section 3.2 on memory maps have obvious equivalents for the adjoint memory map.

<u>Definition 4.14</u> <u>(The A-product)</u>

Consider (E_1') or (E_2') with $g = 0$.

The A-product between $N^P(0,a;H)$ and $M^P(-a,0;H)$ is defined at each time $t \in [t_0,t_1]$ as follows:

$$(\nu,\mu) \longmapsto \langle \nu,\mu\rangle_t : N^P \times M^P \to R \tag{4.58}$$

where for $\nu = \tilde{K}(\nu_0,\nu_1)$ and $\mu = K(\mu_0,\mu_1)$ ($\tilde{K}$ and K as in Theorem 2.2) $\langle\nu,\mu\rangle_t$ is equal to

(i) <u>case (E_1')</u>

$$(\nu_0|\mu_0) + \sum_{i=1}^{N} \int_{\theta_i}^{0} (\nu_1(\alpha-\theta_i)\,|\,A_i(t+\alpha-\theta_i)\mu_1(\alpha))\,d\alpha \tag{4.59}$$

(ii) <u>case (E_2')</u>

$$(\nu_0|\mu_0) + \int_{-a}^{0} d\theta \int_{\theta}^{0} d\alpha\, (\nu_1(\alpha-\theta)\,|\,A_1(t+\alpha-\theta,\theta)\mu_1(\alpha)). \tag{4.60}$$

<u>Remark</u> To the author's knowledge, a group of terms of which the A-product is a generalization was introduced as a convenient entity by A. Halanay [20] and J.K. Hale [3] for "time invariant" functional equations with continuous initial data.

Lemma 4.15

Let $(h,x)\in M^p(-a,0;H)\circ AC^1(t_0,t_1;H)$ and $(\tilde{h},p)\in N^p(0,a;H)\circ AC^1(t_0,t_1;H)$. The following relations are true for all t in $[t_0,t_1]$:

case (E_1')

$$<p_t(\tilde{h}),x_t(h)>_t - <p_{t_0}(\hat{h}),x_{t_0}(h)>_{t_0}$$

$$= \int_{t_0}^{t} (p(s) \,|\, \frac{dx}{ds}(s) - \sum_{i=0}^{N} A_i(s)x_s(h)(\theta_i))\,ds \ldots$$

$$\ldots + \int_{t_0}^{t} (\frac{dp}{ds}(s) + \sum_{i=0}^{N} A_i^*(s)p_s(\hat{h})(-\theta_i) \,|\, x(s))\,ds \quad , \qquad (4.61)$$

where

$$A_i^*(s) = \begin{cases} A_i^T(s-\theta_i) & , -(t_1-s) \le \theta_i \le 0 \\ 0 & , -a \le \theta_i < -(t_1-s)\ ; \end{cases} \qquad (4.62)$$

case (E_2')

$$<p_t(\tilde{h}),x_t(h)>_t - <p_{t_0}(\tilde{h}),x_{t_0}(h)>_{t_0}$$

$$= \int_{t_0}^{t} (p(s) \,|\, \frac{dx}{ds}(s) - A(s)x_s(h))\,ds \ldots$$

$$\ldots + \int_{t_0}^{t} (\frac{dp}{ds}(s) + A^*(s)p_s(\tilde{h}) \,|\, x(s))\,ds \quad , \qquad (4.63)$$

where the maps $\mu \mapsto A(s)\mu : M^p \to H$ and $\nu \mapsto A^*(s)\nu : N^p \to H$ are defined as follows:

$$A(s)\mu = A_o(s)\mu_o + \int_{-a}^{0} A_1(s,\theta)\mu_1(\theta)\,d\theta \tag{4.64}$$

and

$$A^*(s)\nu = A_o^T(s)\nu_o + \int_{-a}^{0} \begin{cases} A_1^T(s-\theta,\theta)\nu_1(-\theta)d\theta, & -(t_1-s) \le \theta \le 0 \\ 0 & , -a \le \theta < -(t_1-s). \end{cases} \tag{4.65}$$

The proof of the lemma will be given in section 4.3.2.

Remark It is extremely important to note that the identities (4.61) and (4.63) are only dependent on $\tilde{h}_0$ and not on $\tilde{h}_1$ ($\kappa(\tilde{h}) = (\tilde{h}_0,\tilde{h}_1)$). This shows that the space of final data is $H \approx N^p(0,a;H) / L^p(0, a ;H)$ and not $N^p(0,a;H)$.

Notation: In the case (E_2') it will be convenient to define

$$A_o^*(s) = A_o^T(s) \tag{4.66}$$

$$A_1^*(s,\theta) = \begin{cases} A_1^T(s-\theta,\theta), & -(t_1-s) \le \theta \le 0 \\ 0 & , -a \le \theta < -(t_1-s) \end{cases} \tag{4.67}$$

Convention: For future development, it will be very convenient to adopt the following conventions:

<u>case (E_1')</u>

$$A_i(t) = 0 \ , \ t > t_1 \ , \ i = 1,\ldots,N \tag{4.68}$$

<u>case (E_2')</u>

$$A_1(t,\theta) = 0 \ , \ t > t_1 \ , \forall \theta \in [-a,0]. \tag{4.69}$$

With such a convention we now have for all $s \in [t_0,t_1]$,

$$A_i^*(s) = A_i^T(s-\theta_i) \ , \ i = 0,\ldots,N \tag{4.70}$$

$$A_1^*(s,\theta) = A_1^T(s-\theta,\theta) \ , \ \forall \theta \in [-a,0]. \tag{4.71}$$

The identities (4.61) and (4.63) almost give the right definition for the adjoint system corresponding to a linear hereditary differential system. This stresses the importance of the A-product a particular case of which is the inner product in H. It takes into account the way the delayed information is processed through its dependence on $\{A_i(s)\}_{i=1}^N$ and $A_1(s,\theta)$.

<u>Definition 4.16</u>

Given the linear hereditary system (E_1') (resp. (E_2')) that is $g = 0$) with initial data $h \in M^p(-a,0;H)$, the <u>adjoint solution with final data $\tilde{h}_0 \in H$</u> is a map p in $AC^1(t_0,t_1;H)$ for which

$$< p_t(\hat{\kappa}^{-1}(\tilde{h}_0,0)), x_t(h)>_t = (\tilde{h}_0 \mid x(t_1)) \ , \ \forall t \in [t_0,t_1], \tag{4.72}$$

where x is the solution of (E_1') (resp. (E_2')) on $[t_0,t_1]$ with initial data h, and $\tilde{\kappa}$ is the isometric isomorphism between $N^p(0,a;H)$ and $H \times L^p(0,a;H)$ (as in Theorem 2.2).

With this definition in hand and Lemma 4.15 the following theorem is obvious.

<u>Theorem 4.17</u>

Given $\tilde{h}_0 \in H$ and the linear hereditary system (E_1') or (E_2') (with g = 0) the adjoint solution with final data $\tilde{h}_0 \in$ H is the unique solution of the following linear hereditary system:

<u>case (E_1')</u>

$$(a_1) \quad \begin{cases} \frac{dp}{dt}(t) + \sum_{i=0}^{N} A_i^*(t)\, p(t-\theta_i) = 0\,, \text{ a.e. in } [t_0,t_1] \\ p(t_1) = \tilde{h}_0 \end{cases} \tag{4.73}$$

<u>case (E_2')</u>

$$(a_2) \quad \begin{cases} \frac{dp}{dt}(t) + A_0^*(t)p(t) + \int_{-a}^{0} A_1^*(t,\theta)\, p(t-\theta)\, d\theta = 0\,, \text{ a.e. in } [t_0,t_1] \\ p(t_1) = \tilde{h}_0\,. \end{cases} \tag{4.74}$$

4.3.2 Proof of Lemma 4.17.

Again a proof will only be provided for the case (E_2') (which is the more difficult case!). For simplicity of notation we shall write x_s instead of $x_s(h)$ and $\dot{x}(s)$ instead of $\frac{dx(s)}{ds}$.

Consider the expression

$$\int_{t_0}^{t} (p(s) \mid \dot{x}(s) - A(s)\, x_s)\, ds$$

$$= (p(t) \mid x(t)) - (p(t_0) \mid x(t_0)) - \int_{t_0}^{t} (\dot{p}(s) \mid x(s))\, ds \ \ldots$$

$$\ldots - \int_{t_0}^{t} (p(s) \mid A(s)\, x_s)\, ds \tag{4.75}$$

$$= (p(t) \mid x(t)) - (p(t_0) \mid x(t_0)) - \int_{t_0}^{t} (\dot{p}(s) + A_0^T(s)\, p(s) \mid x(s))\, ds \ \ldots$$

$$\ldots - \int_{t_0}^{t} ds\, (p(s) \mid \int_{-a}^{0} d\theta \ \ A_1(s,\theta)\, x_s(\theta)), \tag{4.76}$$

where integration by parts was made possible because p and x are in $AC^1(t_0, t_1; H)$.

The last term in equation (4.76) can be decomposed as follows:

$$\int_{t_0}^{t} ds \begin{cases} \displaystyle\int_{-a}^{-(s-t_0)} d\theta\ (p(s) \mid A_1(s,\theta)\, h(s-t_0+\theta)) + \int_{-(s-t_0)}^{0} d\theta\ (p(s) \mid A_1(s,\theta)\, x(s+\theta)), & s - t_0 < a, \\ \displaystyle\int_{-a}^{0} d\theta\ \ (p(s) \mid A_1(s,\theta)\, x(s+\theta)), & s - t_0 \geq a. \end{cases} \tag{4.77}$$

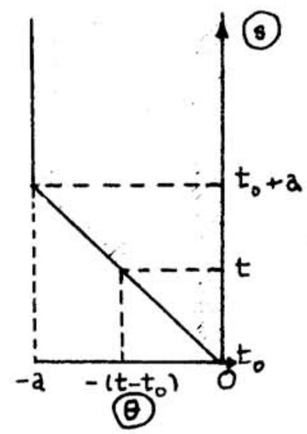
The above expression can be broken in two: ① + ②

$$① = \int_{t_0}^{t} ds \begin{cases} \int_{-a}^{-(s-t_0)} d\theta \, (p(s) \,|\, A_1(s,\theta)\, h(s-t_0+\theta)), & s-t_0 < a \\ 0 & , s-t_0 \geq a \end{cases} \tag{4.78}$$

$$② = \int_{t_0}^{t} ds \begin{cases} \int_{-(s-t_0)}^{0} d\theta \quad (p(s) \,|\, A_1(s,\theta)\, x(s+\theta)), & s-t_0 < a \\ \int_{-a}^{0} d\theta \quad '' \quad '' \quad '' & , s-t_0 \geq a. \end{cases} \tag{4.79}$$

Consider term ② where we change the order of integration

(Corollary 4.3 and Fubini's theorem)

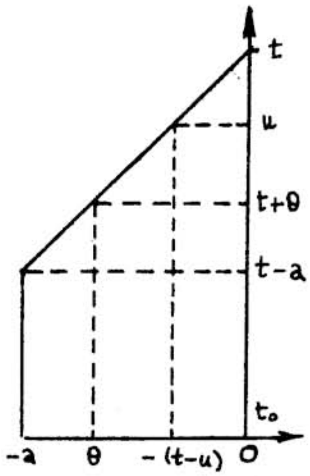


$$\int_{-(t-t_0)}^{0} ds \begin{cases} \int_{-(s-t_0)}^{0} d\theta, & s-t_0 < a \\ \int_{-a}^{0} d\theta, & s-t_0 \geq a \end{cases} = \begin{cases} \int_{-(t-t_0)}^{0} d\theta \int_{t_0-\theta}^{t} ds, & t-t_0 < a \\ \int_{-a}^{0} d\theta \int_{t_0-\theta}^{t} ds, & t-t_0 \geq a. \end{cases}$$

Change the variable s to u = s + θ

$$② = \begin{cases} \int_{-(t-t_0)}^{0} d\theta \int_{t_0}^{t+\theta} du \, (p(u-\theta) \,|\, A_1(u-\theta,\theta)\, x(u)), & t-t_0 < a \\ \int_{-a}^{0} d\theta \int_{t_0}^{t+\theta} du \quad '' \quad '' \quad '' & , t-t_0 \geq a. \end{cases} \tag{4.80}$$

Again the order of integration is changed.



$$\begin{cases} \displaystyle\int_{-(t-t_0)}^{0} d\theta \int_{t_0}^{t+\theta} du \;, & t-t_0 < a \\[2ex] \displaystyle\int_{-a}^{0} d\theta \int_{t_0}^{t+\theta} du \;, & t-t_0 \geq a \end{cases} = \int_{t_0}^{t} du \begin{cases} \displaystyle\int_{-(t-u)}^{0} d\theta \;, & t-u < a \\[2ex] \displaystyle\int_{-a}^{0} d\theta , & t-u \geq a \;. \end{cases}$$

We introduce here $p_u(\tilde{h})$ (see Definition 4.13) which will be abbreviated p_u. The term ② becomes

$$② = \int_{t_0}^{t} du \int_{-a}^{0} d\theta \; (p_u(-\theta) | \, A_1(u-\theta, \theta) x(u)) - \begin{cases} \displaystyle\int_{-a}^{-(t-u)} d\theta \, (p_u(-\theta) | A_1(u-\theta, \theta) x(u)), \\ \qquad\qquad\qquad t-u < a, \\ 0 \qquad\qquad\quad , t-u \geq a, \end{cases} \tag{4.81}$$

where $A_1(u-\theta, \theta) = 0 \, , \; u-\theta > t_1$ (see convention).

Change the variable u to s = u in the first term of ② :

$$② = \int_{t_0}^{t_1} ds \; (A_1^{*}(s, \theta) \, p_s(-\theta) \, | \, x(s)) \; - \; ③ \tag{4.82}$$

Change the variable u to $\alpha = -(t-u)$ in ③

$$③ = \int_{-(t-t_0)}^{0} d\alpha \begin{cases} \displaystyle\int_{-a}^{\alpha} d\theta \, (p_t(\alpha-\theta) \, | \, A_1(t+\alpha-\theta, \theta) \, x_t(\alpha)) \, , & -\alpha < a \\[2ex] 0 & , -\alpha \geq a \, . \end{cases} \tag{4.83}$$

Integration for $(t-t_0) \geq a$ can be disregarded. Also the convention $A_1(t,\theta) = 0, t > t_1$ makes it possible to rewrite ③ in the form

$$③ = \int_{-a}^{0} d\alpha \int_{-a}^{\alpha} d\theta \, (p_t(\alpha-\theta) \,|\, A_1(t+\alpha-\theta, \theta)\, x_t(\alpha))$$

$$= \int_{-a}^{0} d\theta \int_{\theta}^{0} d\alpha \, (p_t(\alpha-\theta) \,|\, A_1(t+\alpha-\theta, \theta)\, x_t(\alpha)) \tag{4.84}$$

after the order of integration has once more been changed.

Finally

$$② = \int_{t_0}^{t} ds \, \Big(\int_{-a}^{0} d\theta \, A_1^*(s,\theta)\, p_s(-\theta) \,\Big|\, x(s) \Big)$$
$$- \int_{-a}^{0} d\theta \int_{\theta}^{0} d\alpha \big(p_t(\alpha-\theta) \,|\, A_1(t+\alpha-\theta, \theta)\, x_t(\alpha)\big). \tag{4.85}$$

The last term that we must take care of is ① . Again with the convention $A_1(t,\theta) = 0$, $t > t_1$, ① can be rewritten in the form

$$① = \int_{t_0}^{t_0+a} ds \int_{-a}^{-(s-t_0)} d\theta \, \big(p_{t_0}(s-t_0) \,|\, A_1(t_0+s-t_0, \theta)\, x_{t_0}(s-t_0+\theta)\big). \tag{4.86}$$

Change the variable s to $\eta = -(s-t_0)$

$$① = \int_{-a}^{0} d\eta \int_{-a}^{\eta} d\theta \, \big(p_{t_0}(-\eta) \,|\, A_1(t_0-\eta, \theta)\, x_{t_0}(\theta-\eta)\big). \tag{4.87}$$

Change the order of integration once more

$$① = \int_{-a}^{0} d\theta \int_{\theta}^{0} d\eta \; (\quad " \qquad " \qquad " \quad). \tag{4.88}$$

Change the variable η to $\alpha = \theta - \eta$

$$① = \int_{-a}^{0} d\theta \int_{\theta}^{0} d\alpha \left(p_{t_0}(\alpha-\theta) \mid A_1(t_0+\alpha-\theta, \theta) x_{t_0}(\alpha) \right). \tag{4.89}$$

Substitute back (4.89) and (4.85) in (4.76) via (4.77), (4.78) and (4.79) to obtain

$$\int_{t_0}^{t} (p(s) \mid \dot{x}(s) - A(s)x_s)\, ds + \int_{t_0}^{t} (\dot{p}(s) + A^*(s)p_s \mid x(s))\, ds$$

$$= (p(t) \mid x(t)) + \int_{-a}^{0} d\theta \int_{\theta}^{0} d\alpha \left(p_t(\alpha-\theta) \mid A_1(t+\alpha-\theta, \theta) x_t(\alpha) \right)$$

$$- \left[(p(t_0) \mid x(t_0)) + \int_{-a}^{0} d\theta \int_{\theta}^{0} d\alpha \left(p_{t_0}(\alpha-\theta) \mid A_1(t_0+\alpha-\theta, \theta) x_{t_0}(\alpha) \right) \right]. \tag{4.90}$$

This and Definition 4.14 yield the identity (4.63).

5. Concluding Remarks.

This thesis has obvious extensions. The first one is to look at hereditary differential equations on $[t_0,t_1[$, where $t_0 < t_1 \leq +\infty$. To do this it suffices to go to L^p-spaces of locally p-integrable maps ($1 \leq p < \infty$) or locally essentially bounded maps ($p = \infty$). Such problems bring up the question of the behaviour of the solutions as the time t goes to $+\infty$. Secondly, existing stability theory can be adapted to hereditary differential systems with initial data in a M^p space.

As far as the Optimal Control Theory is concerned, this thesis contains the basic material for a treatment of the "hereditary version" of the "classical problems". An example of such a problem has been mentioned in the Introduction. In fact the optimization of a quadratic cost for a linear hereditary differential system has been the central motivation for the study of hereditary differential systems in this particular framework.

Appendix A. Integral Representation of continuous linear operators defined on $L^p(\mu,E)$ with values in H, when $1 \leq p < \infty$, E is a reflexive Banach space and H is a Hilbert space.

Lemma A.1

Let X be a measured space with measure μ, and E and F be Banach spaces.

(i) Let $f: X \to E$ and $g: X \to F$ be μ-measurable maps. Then the map $t \mapsto (f(t), g(t)): X \to E \times F$ is μ-measurable.

(ii) Let $f_i: X \to E$, $i = 1, \ldots, n$ be a finite family of μ-measurable maps. Then the map $t \mapsto (f_1(t), \ldots, f_n(t)): X \to (E)^n$ is μ-measurable.

(iii) Let $f: X \to E$ be μ-measurable and $\Lambda: E \to F$ be linear and continuous. Then the map $t \mapsto \Lambda(f(t)): X \to F$ is μ-measurable.

Proof: 1) Since f(resp. g) is μ-measurable there exists a sequence $\{f_n\}$ (resp. $\{g_n\}$) of step maps such that $f_n \to f$ (resp. $g_n \to g$) pointwise almost everywhere. Assume f_n (resp. g_n) is of the form

$$f_n(t) = \sum_{i=1}^{M_n} a_i^n \chi_{S_i^n}(t) \quad \left(\text{resp. } g_n(t) = \sum_{j=1}^{N_n} b_i^n \chi_{T_i^n}(t)\right)$$

where $\{a_i^n\} \subset E$ (resp. $\{b_j^n\} \subset F$) and $\{S_i^n\}$ (resp. $\{T_j^n\}$) are disjoint measurable subsets of X with finite measures. For each n, define the map $h_n : X \to E \times F$:

$$h_n(t) = (f_n(t), g_n(t)) = \sum_{i=1}^{M_n} \sum_{j=1}^{N_n} (a_i^n \chi_{S_i^n}(t), b_j^n \chi_{T_j^n}(t))$$

$$= \sum_{i=1}^{M_n} \sum_{j=1}^{N_n} \left[(a_i^n, 0)\chi_{S_i^n \setminus T_j^n}(t) + (0, b_j^n)\chi_{T_j^n \setminus S_i^n}(t) + (a_i^n, b_j^n)\chi_{T_j^n \cap S_i^n}(t) \right],$$

where $A \setminus B = \{x \in A \mid x \notin B\}$. Thus h_n is a step map since the measurable sets $\{S_i^n \setminus T_j^n\}_{ij}$, $\{T_j^n \setminus S_i^n\}_{ij}$ and $\{T_j^n \cap S_i^n\}_{ij}$ are disjoint. Hence the map $t \mapsto (f(t), g(t)) : X \to E \times F$ is μ-measurable since it is the almost everywhere pointwise limit of step maps [13, p.235, M12].

2) This is a direct consequence of 1). Pick $F = E$ and the result is true for $n = 2$. Pick $F = E \times E$ and the results is true for $n = 3$, etc...

3) There exists a sequence f_n of step maps converging almost everywhere to f. Say $f_n(t) = \sum_{i=1}^{M_n} a_i^n \chi_{S_i^n}(t)$ for $\{a_i^n\} \subset E$ and $\{S_i^n\}$ disjoint measurable subsets of X with finite measure.

Let

$$g_n(t) = \sum_{i=1}^{M_n} \Lambda(a_i^n) \chi_{S_i^n}(t) ;$$

the map $t \mapsto g_n(t)$ are step maps from X to F.

By continuity of Λ for almost all $t \in X$

$$f_n(t) \to f(t) \Rightarrow g_n(t) = \Lambda(f_n(t)) \to \Lambda(f(t)).$$

Thus the map $t \mapsto \Lambda(f(t)) : X \to F$ is μ-measurable.

• In the proof of the following lemmas we shall need some results on Hilbert spaces and filters. For the definitions. and properties of filters we refer the reader to Horváth [14]; as for Hilbert spaces the necessary results may be found in Horváth and W. Rudin [25, Chapter 4].

Let I be a set. Denote by $\Phi(I)$ the set of all finite subsets of I. If $\Phi(I)$ is ordered by inclusion "$\subseteq$", it becomes a directed set. For each $J \in \Phi(I)$ let $S(J) = \{ K \in \Phi(I) \mid K \supseteq J \}$. The collection $\mathfrak{S} = \{ S(J) \mid J \in \Phi(I) \}$ is a filter basis on $\Phi(I)$. The filter generated by $\mathfrak{S}$ is called the <u>filter of sections</u> of $\Phi(I)$ [14, p.77, Ex. 5].

• Let $(x_i)_{i \in I}$ be a family of elements of a topological vector space E. Associate with each $J \in \Phi(I)$ the element $x_J = \sum_{i \in J} x_i$ of E. We say $(x_i)_{i \in I}$ is <u>summable</u> to an element $x \in E$ if the filter generated by the image of $\mathfrak{S}$ under the map $J \mapsto x_J$ converges to x. If this is the case, we say that x is the <u>sum</u> $(x_i)_{i \in I}$ and we write $x = \sum_{i \in I} x_i$ [14, p.127, Def. 2].

• Let $\ell^2(I)$ be the set of all families $(\xi_i)_{i\in I}$ of real numbers such that the family $(|\xi_i|^2)_{i\in I}$ is summable. $\ell^2(I)$ is a Hilbert space with the inner product

$$((\xi_i)|(\eta_i)) = \sum_{i\in I} \xi_i \eta_i \tag{A.1}$$

[14, p.35, Ex. 7].

• Every Hilbert space contains a maximal orthonormal family $(u_i)_{i\in I}$ [14, p.31, Prop. 7]. If $(.|.)$ denotes the inner product in H, for each $x \in H$, the real numbers $x_i = (x|u_i)$, $i \in I$, are called the Fourier coefficients with respect to $(u_i)_{i\in I}$ and $\sum_{i\in I} (x|u_i)\, u_i$ is called the Fourier expression of x [14, p.32, Def. 6]. The map $x \mapsto (x_i)_{i\in I} : H \to \ell^2(I)$ is an isometric isomorphism (top. and alg.) and

$$(x|y) = ((x_i)|(y_i)) \;, \forall x,y \in H$$

[14, p.35, Ex. 7]. In particular:

$$x \in H \iff (|x_i|^2)_{i\in I} \text{ is summable} \iff (x_i u_i)_{i\in I} \text{ is summable.}$$

Thus for each x, let $\mathcal{F}_x$ be the filter on H generated by the image of the filter basis $\mathcal{G}$ under the map

$$J \mapsto x_J = \sum_{i\in J} x_i u_i \; : \Phi(I) \to H. \tag{A.2}$$

By definition of summability $\mathcal{F}_x$ converges to x.

Lemma A.2

Let H and F be Hilbert and Banach spaces respectively. Assume $A \in \mathcal{L}(H;F)$. Denote by $(u_i)_{i\in I}$ a maximal orthonormal family in H. Define for each $J \in \Phi(I)$ the map $A_J : H \to F$ as

$$A_J x = \sum_{i \in J} x_i \, (Au_i) \, , \; \forall \, x \in H, \tag{A.3}$$

where $(x_i)_{i \in I}$ are the Fourier coefficients of x with respect to $(u_i)_{i \in I}$. Then $A_J \in \mathcal{L}(H;F)$ for all $J \in \Phi(I)$ and the filter generated by the image of $\mathfrak{S}$ under the map $J \mapsto A_J : \Phi(I) \to \mathcal{L}(H;F)$ converges to A.

Proof: By definition A_J is linear and

$$\sup_{|x| \le 1} |A_J x| = \sup_{|\sum_{i \in J} x_i u_i| \le 1} |A(\textstyle\sum_{i \in J} x_i u_i)| \le \sup_{|x| \le 1} |Ax| = \|A\| .$$

The family $(A_J)_{J \in \Phi(I)}$ is bounded in $\mathcal{L}(H;F)$.

1) For each x in H the family $(x_i u_i)_{i \in I}$ is summable. That is the filter $\mathfrak{F}_x$ on H generated by the image of the filter basis $\mathfrak{S}$ under the map

$$J \mapsto x_J = \sum_{i \in J} x_i u_i \; : \Phi(I) \to H$$

converges to x. By continuity of the map A the filter $\mathfrak{F}_{Ax}$ on F generated by the image of the filter basis $\mathfrak{S}$ under the map

$$J \mapsto A_J x = Ax_J \; : \Phi(I) \to F$$

converges to Ax.

2) Let $\mathfrak{F}_A$ be the filter on $\mathcal{Q} = \{ A_J \mid J \in \Phi(I) \}$ generated by the image of the filter basis $\mathfrak{S}$ under the map

$$J \mapsto A_J \; : \Phi(I) \to \mathcal{L}(H;F).$$

3) But the subset $\mathcal{A}$ of $\mathcal{L}(H;F)$ is equicontinuous since $\|A_J\| \leq \|A\|$ for all $J \in \Phi(I)$. It now follows from a version of the Banach-Steinhaus theorem [26, Chap. 3, p.25, Cor. to Prop. 5] that the filter $\mathcal{F}_A$ converges to A.

Lemma A.3

Let E be a Banach space, and $(u_i)_{i \in I}$ be a maximal orthonormal family in a Hilbert space H. For $J \in \Phi(I)$ let E^J be the space of all mappings b defined on the finite subset J of I with values in E endowed with the norm

$$\|b\| = \Big[\sum_{i \in J} |b(i)|^2\Big]^{1/2} \quad , \quad b \in E^J \quad , \tag{A.4}$$

where $|\;|$ is the norm in E.

Corresponding to each $b \in E^J$ we define the map $\beta(b) : H \to E$,

$$\beta(b)x = \sum_{i \in J} x_i b(i) \qquad , \quad x \in H \quad , \tag{A.5}$$

where $x_i = (x \mid u_i)$ is the Fourier coefficient of x in H and $(.|.)$ is the inner product in H. For all $b \in E^J$

$$\sup_{|x| \leq 1} |\beta(b)x| \leq \Big[\sum_{i \in J} |b(i)|^2\Big]^{1/2} \tag{A.6}$$

and the map $\beta : E^J \to \mathcal{L}(H;E)$ is a continuous linear injection.

Proof: β(b) is linear by definition. For all $x \in H$

$$|\beta(b)x| = |\sum_{i\in J} x_i\, b(i)| \leq \sum_{i\in J} |x_i|\;|b(i)|$$

$$\leq \Big[\sum_{i\in J} |x_i|^2\Big]^{1/2} \cdot \Big[\sum_{i\in J} |b(i)|^2\Big]^{1/2}.$$

Because of the isometry between H and $\ell^2(I)$,

$$\Big[\sum_{i\in J} |x_i|^2\Big]^{1/2} \leq \Big[\sum_{i\in I} |x_i|^2\Big]^{1/2} = |x|\,,$$

and

$$|\beta(b)x| \leq |x|\;\|b\|\;,\ \forall\, x \in H.$$

So β is indeed a continuous map by our last result. It is linear and injective by construction:

$$\beta(b) = \beta(b') \Rightarrow b(i) = \beta(b)\,u_i = \beta(b')u_i = b'(i)$$

for all i in J.

Proposition A.4

Let E be a reflexive Banach space and let X be a Banach space. Let γ(A) be the transpose of $A \in \mathcal{L}(X,E)$ defined as follows:

$$\langle x, \gamma(A)e^*\rangle_X = \langle Ax, e^*\rangle_E\ ,\ \forall x \in X, \forall e^* \in E^*. \qquad (A.7)$$

The map

$$\gamma : \mathcal{L}(X,E) \rightarrow \mathcal{L}(E^*, X^*) \qquad (A.8)$$

is a norm preserving isomorphism.

Proof: 1) Let $c: X \to X^{**}$ and $d: E \to E^{**}$ be the respective canonical isometric embeddings on X and E in their second dual X^{**} and E^{**} (d is an isomorphism). For each $A \in \mathcal{L}(X,E)$ define $\gamma(A) : E^{*} \to X^{*}$ as follows: for each $e^{*} \in E^{*}$ let

$$<Ax, e^{*}>_E = <x, \gamma(A)e^{*}>_X \ , \ \forall x \in X .$$

Clearly $\sup_{|e^{*}| \leq 1} |\gamma(A)e^{*}| \leq \|A\|_{\mathcal{L}(X,E)}$, $\gamma(A) \in \mathcal{L}(E^{*}, X^{*})$,

and γ is linear. Similarly we define for each $B \in \mathcal{L}(E^{*},X^{*})$ $\mathcal{J}_1(B) : X^{**} \to E^{**}$ as follows: for each $x^{**} \in X^{**}$ let

$$<Be^{*}, x^{**}>_{X^{*}} = <e^{*}, \mathcal{J}_1(B)x^{**}>_{E^{*}} \ , \ \forall e^{*} \in E^{*}.$$

Clearly $\mathcal{J}_1 : \mathcal{L}(E^{*},X^{*}) \to \mathcal{L}(X^{**},E^{**})$ is linear and $\|\mathcal{J}_1(B)\|_{\mathcal{L}(X^{**},E^{**})} \leq \|B\|_{\mathcal{L}(E^{*},X^{*})}$.

Finally let

$$D \mapsto \mathcal{J}_2(D) = d^{-1} \circ D \circ c : \mathcal{L}(X^{**},E^{**}) \to \mathcal{L}(X,E).$$

Since d and c are linear, so is $\mathcal{J}_2$. Moreover

$$\begin{aligned}
\|(\mathcal{J}_2(D))\, x\|_E &= \\
&= \|d^{-1}(D(c(x)))\|_E = \|D(c(x))\|_{E^{**}} \\
&\leq \|D\|_{\mathcal{L}(X^{**},E^{**})} \|c(x)\|_{X^{**}} \\
&\leq \|D\|_{\mathcal{L}(X^{**},E^{**})} \|x\|_X
\end{aligned}$$

and

$$\|\mathcal{J}_2(D)\|_{\mathcal{L}(X,E)} \leq \|D\|_{\mathcal{L}(X^{**},E^{**})}.$$

Let $\mathcal{J} = \mathcal{J}_2 \circ \mathcal{J}_1$.

We first show $\mathcal{J}(\gamma(A)) = A,\ \forall\, A \in \mathcal{L}(X,E)$.

Consider the following chain of equalities: $\forall\, x \in X, \forall\, e^* \in E^*$,

$$
\begin{aligned}
\langle (\mathcal{J}(\gamma(A)))x, e^* \rangle_E &= \langle (\mathcal{J}_2(\mathcal{J}_1(\gamma(A))))x, e^* \rangle_E \\
&= \langle d^{-1}(\mathcal{J}_1(\gamma(A)))\, c\, x, e^* \rangle_E \\
&= \langle e^*, (\mathcal{J}_1(\gamma(A)))\,(c\, x) \rangle_{E^*} \\
&= \langle \gamma(A)e^*, c\, x \rangle_{X^*} \\
&= \langle x, \gamma(A)e^* \rangle_X \\
&= \langle Ax, e^* \rangle_E .
\end{aligned}
$$

Clearly $\mathcal{J}(\gamma(A)) = A$ and $\|\mathcal{J}(\gamma(A))\| \leq \|\gamma(A)\| \leq \|A\|$.

Similarly we show $\gamma(\mathcal{J}(B)) = B,\ \forall\, B \in \mathcal{L}(E^*, X^*)$.

Consider the following chain of equalities: $\forall\, x \in X, \forall\, e^* \in E^*$,

$$
\begin{aligned}
\langle x, (\gamma(\mathcal{J}(B)))\, e^* \rangle_X &= \langle \mathcal{J}(B)x, e^* \rangle_E \\
&= \langle \mathcal{J}_2(\mathcal{J}_1(B))x, e^* \rangle_E \\
&= \langle d^{-1}(\mathcal{J}_1(B))(c\, x), e^* \rangle_E \\
&= \langle e^*, \mathcal{J}_1(B)(c\, x) \rangle_{E^*} \\
&= \langle Be^*, c\, x \rangle_{X^*} \\
&= \langle x, Be^* \rangle_X .
\end{aligned}
$$

Again $\gamma(\mathcal{J}(B)) = B$ and $\|\gamma(\mathcal{J}(B))\| \leq \|\mathcal{J}(B)\| \leq \|B\|$.

Thus γ is an isomorphism since it has a continuous linear inverse $\mathcal{J}$.

Moreover

$$\| \gamma(A) \| \leq \| A \|$$

and $$\| A \| = \| \mathfrak{J}(\gamma(A)) \| \leq \| \gamma(A) \|$$

makes γ an isometric isomorphism.

Proposition A.5

(i) Let E and F be Banach spaces, (X, μ) a measured space, and $1 \leq p \leq \infty$. The map

$$\pi : L^p(\mu, \mathcal{L}(F,E)) \longrightarrow \mathcal{L}(F, L^p(\mu,E)), \tag{A.9}$$

defined for each $A \in L^p(\mu, \mathcal{L}(F,E))$ as

$$(\pi(A)z)(x) = A(x)z, \forall z \in F, \forall x \in X, \tag{A.10}$$

is linear, injective, continuous, and

$$\| \pi(A) \|_{\mathcal{L}} \leq \| A \|_p, \forall A \in L^p(\mu, \mathcal{L}(F,E)). \tag{A.11}$$

(ii) When F is a Hilbert space the image of $L^p(\mu, \mathcal{L}(F,E))$ under π is dense in $\mathcal{L}(F, L^p(\mu,E))$; if in addition $p = \infty$, π is a norm preserving isomorphism.

(iii) When F is finite dimensional, π is an isomorphism.

Proof: (i) We construct a map π from $L^p(\mu, \mathcal{L}(F,E))$ to $\mathcal{L}(F, L^p(\mu,E))$. Let $A \in L^p(\mu, \mathcal{L}(F,E))$. For all $z \in F$ the map $B \mapsto z^*(B) = Bz : \mathcal{L}(F,E) \to E$ is linear and continuous. By Lemma A.1 (iii) the map

$$x \mapsto (\pi(A)z)(x) = A(x)z : X \to E$$

is μ-measurable as the composition of a continuous linear map z^* and a μ-measurable map A. Also

$$|(\pi(A)z)(x)| = |A(x)z| \leq \|A(x)\| \, |z|$$

which implies that

$$\|\pi(A)z\|_p \leq \|A\|_p \, |z|$$

and $\pi(A)z \in L^p(\mu,E)$. Thus $\pi(A)$ defines a linear map

$$z \longmapsto \pi(A)z \quad : F \longrightarrow L^p(\mu,E)$$

for which

$$\sup_{|z|\leq 1} \|\pi(A)z\|_p \leq \|A\|_p.$$

Hence the map π of equation (A.9) is a continuous linear map for which inequality (A.11) is true.

Pick A_1 and A_2 in $L^p(\mu, \mathcal{L}(F,E))$ for which $\pi(A_1) = \pi(A_2)$. Then

$$A_1(x)z = (\pi(A_1)z)(x) = (\pi(A_2)z)(x) = A_2(x)z$$

for all $z \in F$ and all $x \in X$, and clearly $A_1 = A_2$. This proves the injective property of π and the first part of the proposition.

(ii) When F is a Hilbert space any element B in $\mathcal{L}(F,L^p(\mu,E))$ can be approximated in the sense of Lemma A.2. If $(u_i)_{i \in I}$ is a maximal orthonormal family in F, we define for each $J \in \Phi(I)$ the element $B_J \in \mathcal{L}(F,L^p(\mu,E))$

$$B_J z = \sum_{i\in J} z_i \, (Bu_i), \; \forall z \in F.$$

The filter $\mathcal{F}_B$ on the subset $\mathcal{B} = \{ B_J \} \, J \in \Phi(I)\}$ of $\mathcal{L}(F,L^p(\mu,E))$ which is generated by the image of $\mathcal{G}$ under the map

$$J \mapsto B_J : \Phi(I) \to \mathcal{L}(F, L^p(\mu, E))$$

converges to B.

We now show for all $J \in \Phi(I)$ there exists $\tilde{B}_J \in L^p(\mu, \mathcal{L}(F,E))$ for which $\pi(\tilde{B}_J) = B_J$.

The map

$$x \mapsto ((B_J u_i)(x))_{i \in J} : X \to E^J$$

is μ-measurable by Lemma A.1. The image of $((B_J u_i)(x))_{i \in J}$ under the map β of Lemma A.3 will be denoted by $\tilde{B}_J(x)$. Since β is linear and continuous the map

$$x \mapsto \tilde{B}_J(x) = \beta(((B_J u_i)(x))_{i \in J}) : X \to \mathcal{L}(F,E)$$

is μ-measurable by Lemma A.1. Moreover by Lemma A.3

$$\|\tilde{B}_J(x)\| \leq \Big[\sum_{i \in J} |(B_J u_i)(x)|^2\Big]^{1/2}$$

and $\|\tilde{B}_J\|_p \leq \sum_{i \in J} \|B_J u_i\|_p < \infty$.

Therefore $\tilde{B}_J \in L^p(\mu, \mathcal{L}(F,E))$ and $\pi(\tilde{B}_J) = B_J$ for all $J \in \Phi(I)$.

This proves the density property of the map π. When in addition $p = \infty$,

$$\|\pi(A)\|_{\mathcal{L}} = \sup_{|z| \leq 1} \|\pi(A) z\|_\infty$$

$$= \sup_{|z| \leq 1} \Big(\operatorname*{ess\,sup}_{x \in X} |(\pi(A) z)(x)|\Big) = \sup_{|z| \leq 1} \Big(\operatorname*{ess\,sup}_{x \in X} |A(x) z|\Big)$$

$$= \operatorname*{ess\,sup}_{x \in X} \Big(\sup_{|z| \leq 1} |A(x) z|\Big) = \operatorname*{ess\,sup}_{x \in X} \|Ax\|_{\mathcal{L}(F,E)} = \|A\|_\infty .$$

Because of the isometry the image of $L^\infty(\mu, \mathcal{L}(F,E))$ under the map π is closed in $\mathcal{L}(F,L^{\infty}(\mu,E))$. By density the range of π is all of $\mathcal{L}(F,L^{\infty}(\mu,E))$. This proves the surjectivity of the map π when $p = \infty$.

(iii) When F is a finite n-dimensional Banach space it is algebraically and topologically isomorphic to R^n. The space R^n can be regarded as a Hilbert space endowed with the norm

$$|z| = \Big[\sum_{i=1}^{n} |z_i|^2\Big]^{1/2}, \quad \forall\, z = (z_1,\dots,z_n) \in R^n.$$

To prove (iii) it suffices to establish the result for F equal to the Hilbert space R^n. By (ii) the result is true when $p = \infty$. When $1 \leq p < \infty$ we show there exists a constant $d > 0$ such that

$$\|A\|_p \leq d\, \|\pi(A)\|_{\mathcal{L}}, \quad \forall\, A \in L^p(\mu, \mathcal{L}(F,E)).$$

This will show that the range of π is closed and dense (part (ii)) in $\mathcal{L}(F,L^p(\mu,E))$; this will be sufficient to establish the surjectivity and a fortiori the fact that π is an isomorphism. For any orthonormal basis $\{u_i\}_{i=1}^{n}$ in R^n $z = \sum_{i=1}^{n} z_i u_i$, where the z_i's are the Fourier coefficients of $z \in R^n$. In particular

$$\Big|\sum_{i=1}^{n} z_i (A(x)u_i)\Big| \leq \Big[\sum_{i=1}^{n} |z_i|^2\Big]^{1/2} \Big[\sum_{i=1}^{n} |A(x)u_i|^2\Big]^{1/2}$$
$$\leq |z| \Big[\sum_{i=1}^{n} |(\pi(A)u_i)(x)|^2\Big]^{1/2}.$$

The above calculations are now used in the following sequence of inequalities:

$$\|A\|_p = \left\{ \int_X \left[\sup_{|z| \le 1} |A(x) z| \right]^p d\mu \right\}^{1/p}$$

$$= \left\{ \int_X \left[\sup_{|z| \le 1} \left| A(x) \left(\sum_{i=1}^{n} z_i u_i \right) \right| \right]^p d\mu \right\}^{1/p}$$

$$= \left\{ \int_X \left[\sup_{|z| \le 1} \left| \sum_{i=1}^{n} z_i \; (A(x) u_i) \right| \right]^p d\mu \right\}^{1/p}$$

$$\le \left\{ \int_X \left[\sup_{|z| \le 1} \left(|z| \left[\sum_{i=1}^{n} |(\pi(A) u_i)(x)|^2 \right]^{1/2} \right) \right]^p d\mu \right\}^{1/p}$$

$$\le \left\{ \int_X \left[\sum_{i=1}^{n} |(\pi(A) u_i)(x)|^2 \right]^{p/2} d\mu \right\}^{1/p}$$

$$\le \sum_{i=1}^{n} \|\pi(A) u_i\|_p \le n \, \|\pi(A)\|_{\mathcal{L}} .$$

This completes the proof of Proposition A.5.

One would like to know precisely under what conditions on E, F, (X,μ) and p the map π is an isomorphism. We only know the result is true when

- $p = \infty$, F a Hilbert space, (X,μ) and E arbitrary

and

- F a finite dimensional space, p, (X,μ) and E being arbitrary.

In the other direction we know that when $1 < p < \infty$, $E = R^2$, $F = L^p(m, R^2)$ (m the complete Lebesgue measure on [0,1]) the result is not true. Pick $I \in \mathcal{L}(L^p(m,R^2), L^p(m,R^2))$ the identity operator and assume there exists

$\tilde{I} \in L^p(m, \mathcal{L}(L^p(m,R^2), R^2))$ such that

$$f(t) = (If)(t) = \tilde{I}(t)f \quad , \forall f \in L^p(m,R^2), \forall t \in [0,1].$$

By definition

$$\|\tilde{I}\|_p = \left[\int_X \left[\sup_{|f| \le 1} |f(t)|^p dm\right]^{1/p}.\right.$$

Consider for $n \ge 1$ and $1 \le m \le n$ the maps s_{nm} in $L^p(m,R^2)$

$$s_{nm}(t) = \begin{cases} n^{1/p}u, & t \in \left[\frac{m-1}{n}, \frac{m}{n}\right] \\ 0, & \text{otherwise} \end{cases}$$

where u is an element of norm 1 in R^2.

Clearly $\|s_{nm}\|_p = 1 \quad , \forall n, \forall m$, but

$$\sup_{\substack{n \ge 1 \\ 1 \le m \le n}} |s_{nm}(t)| = \sup_{n \ge 1} n^{1/p} = +\infty .$$

This shows that $\|\tilde{I}\|_p = +\infty$. This contradicts the fact that $\tilde{I} \in L^p(m, \mathcal{L}(L^p(m,R^2),R^2))$. In this particular case π is not an isomorphism.

Even if the result is generally not true, this does not mean it is not true when $F = E$. This case is of particular interest to us, but we cannot say whether under these circumstances π is an isomorphism or not.

Theorem A.6

Let E be a Banach space, F a reflexive Banach space, (μ, X) a measured space, and p,q such that $1 \leq p \leq \infty$ and $q^{-1} + p^{-1} = 1$.

(i) There is a continuous linear injective map

$$\rho : L^q(\mu, \mathcal{L}(E,F)) \to \mathcal{L}(L^p(\mu,E),F) \tag{A.12}$$

such that for all $A \in L^q(\mu, \mathcal{L}(E,F))$

$$\rho(A)f = \int_X A(x)f(x)\, d\mu \;,\; \forall f \in L^p(\mu,E) \tag{A.13}$$

and

$$\|\rho(A)\|_{\mathcal{L}} \leq \|A\|_q . \tag{A.14}$$

(ii) When E is reflexive, μ is regular and F is Hilbert the image of $L^q(\mu, \mathcal{L}(E,F))$ under the map ρ is dense in $\mathcal{L}(L^p(\mu,E),F)$.

(iii) When E is reflexive, μ is regular, and

- either F is finite dimensional and $1 \leq p < \infty$,
- or F is Hilbert and $p = 1$,

ρ is an isomorphism (norm preserving when $p = 1$).

Proof: We construct four maps and study the circumstances under which each of them is a (norm preserving) isomorphism.

1) By hypothesis the space F is reflexive. The map $T \mapsto T^* : \mathcal{L}(E,F) \to \mathcal{L}(F^*,E^*)$, defined for each $z^* \in F^*$ as

$$\langle Ty, z^* \rangle_F = \langle y, T^* z^* \rangle_E \quad , \quad \forall y \in E, \forall z^* \in F^*,$$

is a norm preserving isomorphism by Proposition A.4. This induces an isometric isomorphism ($1 \leq q \leq \infty$)

$$\tau : L^q(\mu, \mathcal{L}(E,F)) \to L^q(\mu, \mathcal{L}(F^*, E^*))$$

defined for each $A \in L^q(\mu, \mathcal{L}(E,F))$ as

$$\tau(A)(x) = [A(x)]^* \quad , \quad \forall x \in X.$$

2) We denote by

$$\gamma : \mathcal{L}(L^p(\mu,E),F) \to \mathcal{L}(F^*, L^p(\mu,E)^*)$$

the norm preserving isomorphism defined for each $z^* \in F^*$ as

$$\langle \Lambda f, z^* \rangle_F = \langle f, \gamma(\Lambda) z^* \rangle_{L^p} \quad , \quad \forall f \in L^p(\mu, E)$$

(Proposition A.4, since the conjugate F^* of a reflexive Banach space is reflexive).

3) We denote by

$$c : L^q(\mu, E^*) \to L^p(\mu,E)^*$$

the map defined for each $g \in L^q(\mu, E^*)$ as

$$\langle f, c(g) \rangle_{L^p} = \int_X \langle f(x), g(x) \rangle_E \, d\mu$$

for all $f \in L^p(\mu,E)$. It is clearly linear, injective and

$$\| c(g) \|_{L^{p*}} \leq \| g \|_q .$$

A sufficient condition for c to be a norm preserving isomorphism is that E be reflexive, $1 \leq p < \infty$ and μ be regular. The map induces the linear injective map

$$\Gamma \mapsto \lambda(\Gamma) = c \circ \Gamma$$

$$: \mathcal{L}(F^*, L^q(\mu, E^*)) \longrightarrow \mathcal{L}(F^*, L^p(\mu, E)^*)$$

for which

$$\|\lambda(\Gamma)\| = \|c \circ \Gamma\| \leq \|\Gamma\|, \ \forall\, \Gamma \in \mathcal{L}(F^*, L^q(\mu, E^*)).$$

When c is a norm preserving isomorphism, so is λ .

4) Finally let

$$\pi : L^q(\mu, \mathcal{L}(F^*, E^*)) \longrightarrow \mathcal{L}(F^*, L^q(\mu, E^*))$$

be the map of Proposition A.5 for F^* and E^* instead of F and E: for each $A \in L^q(\mu, \mathcal{L}(F^*, E^*))$

$$(\pi(A)z^*)(x) = A(x)z^*, \ \forall\, z^* \in F^*, \ \forall\, x \in X.$$

It is linear injective and

$$\|\pi(A)\|_{\mathcal{L}} \leq \|A\|_q .$$

When F is Hilbert the range of π is dense in $\mathcal{L}(F^*, L^q(\mu, E^*))$.
When $p = \infty$ and F is Hilbert it is a norm preserving isomorphism; when F is finite dimensional it is an isomorphism.

(i) We define the map ρ as the composition map $\gamma^{-1} \circ \lambda \circ \pi \circ \tau$. From the properties of the maps γ^{-1} (isomorphism since F is reflexive), λ, π and τ , ρ is a linear injective map such that

$$\|\rho(A)\|_{\mathcal{L}} \leq \|A\|_q \ , \ \forall\, A \in L^q(\mu, \mathcal{L}(E, F)).$$

All we need to verify is the identity (A.13).
Let $f \in L^p(\mu, E)$, $z^* \in F^*$, $A \in L^q(\mu, \mathcal{L}(E, F))$:

$$<\int_X A(x)f(x)\,d\mu, z^*>_F \;=\; \int_X < A(x)f(x), z^*>_F \,d\mu$$

$$= \int_X < [\tau^{-1}(\tau(A))](x) f(x), z^*>_F d\mu = \int_X < f(x), [\tau(A)](x)\, z^* >_E d\mu$$

$$= \int_X < f(x), [(\pi(\tau(A)))z^*](x)>_E d\mu \;=\; < f, [\lambda(\pi(\tau(A)))] z^*>_{L^p}$$

$$= < [\gamma^{-1}(\lambda(\pi(\tau(A))))] f, z^* >_F \;=\; < [\rho(A)] f, z^* >_F \;.$$

Since the identity

$$< [\rho(A)] f, z^* >_F \;=\; < \int_X A(x)\, f(x)\, d\mu, z^* >_F$$

is true for all $z^* \in F^*$ (locally convex and Hausdorff) and the topological pairing $< \,,\, >$ separates points of F,

$$[\rho(A)]\, f = \int_X A(x)f(x)\, d\mu, \quad \forall\, f \in L^p(\mu,E).$$

(ii) and (iii) When E is reflexive, μ is regular and $1 \le p < \infty$, the map c becomes a norm preserving isomorphism [27, p.607, Thm. 8.20.5, p.590, Thm. 8.18.3]. This makes λ a norm preserving isomorphism. When F is Hilbert ρ has the density property since π does. When either F is finite dimensional, or p = 1 and H is Hilbert, the map π becomes an isomorphism. Under the above combined conditions ρ is the composition of four isomorphisms. This proves the last part of the theorem.

Remark (i) This theorem is a generalization of the classical integral representation theorem for continuous linear forms defined on $L^p(\mu,E)$ [27, p.607 Thm. 8.20.5 and p.590 Thm. 8.18.3].

(ii) A theory of the linear representation of linear operators defined on $L^p(\mu,E)$ with values in F, was developed by N. Dinculeanu and C. Foias [28], I. Singer [29, 30], N. Dinculeanu and C. Foias [31], A. and C. Ionescu Tulea [32], C. Ionescu Tulcea [33], and N. Dinculeanu [34]. When p = 1 the theorem may be considered as a corollary of the above representation theory. But when $1 < p < \infty$, μ is regular, and $F = R^n$ (finite dimensional) it shows that this class of continuous linear operators has an integral representation. This was not obvious since it was to be shown that for all $A \in \mathcal{L}(L^p(\mu,E),R^n)$

$$|||A||| = \sup \sum_{i=1}^{N} |A(b_i \chi_{B_i})|_{R^n} < \infty,$$

where the sup is taken over all step maps of the form $s = \sum_{i=1}^{N} b_i \chi_{B_i}$ with L^p-norm equal to 1.

To our knowledge, the density property (part (ii)) is also new.

(iii) Our proof of the theorem is different from that of the above mentioned authors and is constructive. It uses in a critical manner the integral representation results for the dual of $L^p(\mu,E)$. This gives the sufficient conditions on μ, p and E. The sufficient conditions on p and F arise from the map π of Proposition A.5. If any of the above two sets of conditions could be relaxed, the hypotheses of Theorem A.6 will also be relaxed.

Appendix B. Let B be a Banach space. Assume $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ are measured spaces [13, p.229] : X(resp. Y) denotes the space, $\mathfrak{M}$ (resp. $\mathfrak{N}$) a δ-algebra of subsets of X(resp. Y) and $\mu : \mathfrak{M} \to R$ (resp. $\nu : \mathfrak{N} \to R$) a positive measure on $\mathfrak{M}$ (resp. $\mathfrak{N}$) . $\mathfrak{M} \otimes \mathfrak{N}$ and $\mu \otimes \nu$ will denote the product δ -algebra in the product space X x Y and the product measure defined on $\mathfrak{M} \otimes \mathfrak{N}$ respectively.

When $(X, \mathfrak{M}, \mu)$ is a measured space, let $\overline{\mathfrak{M}}$ consist of all subsets Y of X which differ from an element of $\mathfrak{M}$ by a set contained in a set of measure 0. In other words, there exists a set A in $\mathfrak{M}$ such that $(Y \setminus A) \cup (A \setminus Y)$ is contained in a set of measure zero. If we define $\bar{\mu}(Y) = \mu(A)$ for Y, A as above, $(X, \overline{\mathfrak{M}}, \bar{\mu})$ is a measured space. $(X, \overline{\mathfrak{M}}, \bar{\mu})$ is the complete measure space determined by $(X, \mathfrak{M}, \mu)$ and $\bar{\mu}$ the completion of μ . [13, p.280]. If $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ are two measured spaces

$$\overline{\overline{\mathfrak{M}} \otimes \overline{\mathfrak{N}}} = \overline{\mathfrak{M} \otimes \mathfrak{N}} \quad \text{and} \quad \overline{\bar{\mu} \otimes \bar{\nu}} = \overline{\mu \otimes \nu} \tag{B.1}$$

[13, p.280].

Lemma B.1 Let $(X, \mathfrak{M}, \mu)$ be a measured space and $1 \leq p \leq \infty$.

There is a norm preserving isomorphism

$$\bar{\alpha} : L^p(\mu, B) \to L^p(\bar{\mu}, B), \tag{B.2}$$

such that for each $f \in \mathcal{L}^p(\bar{\mu}, B)$ there exists an element

$\tilde{f} \in \mathcal{L}^p(\mu, B)$ for which

$$\tilde{f}(x) = f(x) \ , \text{ a.e. } (\mu) \text{ in } X \tag{B.3}$$

and

$$\bar{\alpha}(\gamma(\tilde{f})) = \bar{\gamma}(f) , \tag{B.4}$$

where γ and $\bar{\gamma}$ are the respective canonical surjections (section 2.1.1) for $L^p(\mu, B)$ and $L^p(\bar{\mu}, B)$.

Proof: 1) $\forall f \in \mathcal{L}^p(\bar{\mu}, B)$, $\exists \ \tilde{f} \in \mathcal{L}^p(\mu, B)$ such that $\tilde{f} = f$ a.e. (μ) in X.

We first prove the result for characteristic functions. Let S be a $\bar{\mu}$-measurable subset of X for which $\bar{\mu}(S)$ is finite. By definition of the completion of $(X, \mathcal{M}, \mu)$ there exists a μ-measurable subset S of X for which the symmetric difference of S and $\tilde{S}$,

$$S \,\Delta\, S = (S \setminus \tilde{S}) \cup (\tilde{S} \setminus S),$$

is a subset of an element Z in $\mathcal{M}$ with μ-measure 0. In particular $\bar{\mu}(S) = \mu(\tilde{S})$ and the characteristic functions χ_S and $\chi_{\tilde{S}}$ of S and $\tilde{S}$ are equal except at most on a set of μ-measure 0. The integrals

$$\int_X \chi_S \, d\bar{\mu} = \bar{\mu}(S) = \mu(\tilde{S}) = \int_X \chi_{\tilde{S}} \, d\mu$$

and the L^p-seminorms

$$\|\chi_S\|_p \ (\bar{\mu}) = \|\chi_{\tilde{S}}\|_p \ (\mu)$$

also coincide. It is clear that the result is true for step maps and hence for $\bar{\mu}$-measurable maps. Finally $\tilde{f}$ is a μ-measurable map and

$$\|\hat{f}\|_p\ (\mu) = \|f\|_p\ (\bar{\mu}) < \infty .$$

This shows that $\tilde{f} \in \mathcal{L}^p(\mu,B)$.

2) The map $\bar{\alpha}$ and its properties.

Pick g_1 and g_2 in $\mathcal{L}^p(\mu,B)$ such that $g_1 = g_2$ a.e. (μ) in X. Clearly $g_1 = g_2$ a.e. $(\bar{\mu})$ in X,

$$\gamma(g_1) = \gamma(g_2) \Rightarrow \bar{\gamma}(g_1) = \bar{\gamma}(g_2), \text{ and}$$

$\|g_1\|_p\ (\mu) = \|g_1\|_p\ (\bar{\mu})$. This defines a norm preserving linear injective map

$$\bar{\alpha} : L^p(\mu,B) \to L^p(\bar{\mu},B)$$

as follows:

$$\bar{\alpha}(\gamma(g)) = \bar{\gamma}(g) \quad , \forall\, g \in \mathcal{L}^p(\mu,B).$$

As for the surjectivity it follows from part 1):

$\forall\, f \in \mathcal{L}^p(\bar{\mu},B)$, $\exists\ \tilde{f} \in \mathcal{L}^p(\mu,B)$ for which $\bar{\alpha}(\gamma(\tilde{f})) = \bar{\gamma}(f)$.

Theorem B.2 Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be two finite measured spaces and $1 \le p \le \infty$. There is a norm preserving isomorphism

$$\bar{\beta} : L^p(\mu\otimes\nu,B) \to L^p(\mu,L^p(\nu,B)) \tag{B.5}$$

such that for each $f \in \mathcal{L}^p(\mu\otimes\nu,B)$ there exists an $\tilde{F} \in \mathcal{L}^p(\mu, L^p(\nu,B))$ and $F(x) \in \mathcal{L}^p(\nu,B)$ a.e. (μ) in X for which

$$\gamma_Y(F(x)) = \tilde{F}(x) \text{ , a.e. } (\mu) \text{ in } X, \tag{B.6}$$

$$F(x)(y) = f(x,y) \text{ , a.e. } (\mu\otimes\nu) \text{ in } X \times Y, \tag{B.7}$$

and

$$\bar{\beta}(\gamma(f)) = \gamma_X(\tilde{F}), \tag{B.8}$$

where

$$\gamma_X : \mathcal{L}^p(\mu, L^p(\nu, B)) \longrightarrow L^p(\mu, L^p(\nu, B)), \tag{B.9}$$

$$\gamma_Y : \mathcal{L}^p(\nu, B) \longrightarrow L^p(\nu, B) \text{ and} \tag{B.10}$$

$$\gamma : \mathcal{L}^p(\mu\otimes\nu, B) \longrightarrow L^p(\mu\otimes\nu, B) \tag{B.11}$$

are the respective canonical surjections (section 2.1.1).

By symmetry similar results are true with $L^p(\nu, L^p(\mu, B))$ in place of $L^p(\mu, L^p(\nu, B))$.

Proof: 1) Construction of the map $\bar{\beta}$.

Let $f \in \mathcal{L}^p(\mu\otimes\nu, B)$. We define for each x in X the map $F(x) : Y \longrightarrow B$ as follows

$$F(x)(y) = f(x,y) \ , \ y \in Y.$$

We shall show that

- $F(x) \in \mathcal{L}^p(\nu, B)$,
- the map $x \mapsto \gamma_Y(F(x)) : X \longrightarrow L^p(\nu, B)$ is μ -measurable,
- $\forall f, f' \in \mathcal{L}^p(\mu\otimes\nu, B)$ such that $\gamma(f) = \gamma(f')$,

$\gamma_X(\gamma_Y(F)) = \gamma_X(\gamma_Y(F'))$ for F and F' constructed from f and f'.

This defines the map

$$\bar{f} \mapsto \bar{\beta}(\bar{f}) = \gamma_X(\gamma_Y(F)) : L^p(\mu\otimes\nu, B) \longrightarrow L^p(\mu, L^p(\nu, B))$$

where F is constructed from f and $\gamma(f) = \bar{f}$.

• $\mathcal{L}^p(\mu\otimes\nu, B) \subset \mathcal{L}^1(\mu\otimes\nu, B)$ since X x Y has finite $\mu\otimes\nu$ -measure. By Fubini's theorem [13, p.269 Thm. 9] $F(x) \in \mathcal{L}^1(\nu, B)$ a.e. (μ) in X. Hence F(x) is ν -measurable. Now when $1 \leq p < \infty$ the map

$$(x,y) \mapsto |f(x,y)|^p : X \times Y \to R$$

is in $\mathcal{L}^1(\mu\otimes\nu, B)$; therefore always by Fubini's theorem the map

$$y \mapsto |F(x)(y)|^p = |f(x,y)|^p : Y \to R$$

is in $\mathcal{L}^1(\nu, R)$. Hence $F(x) \in \mathcal{L}^p(\nu, B)$ a.e. (μ) in X. When $p = \infty$ clearly

$$\|F(x)\|_\infty = \operatorname*{ess\,sup}_{y\in Y} |f(x,y)| \leq \operatorname*{ess\,sup}_{(x,y)\in X\times Y} |f(x,y)| = \|f\|_\infty$$

and $F(x) \in \mathcal{L}^\infty(\nu, B)$ a.e. (μ) in X.

• Since X x Y has finite $\mu\otimes\nu$ -measure, the step maps defined on $\mathfrak{M}\otimes\mathfrak{N}$ are dense in $L^p(\mu\otimes\nu, B)$ even in the case $p = \infty$ [13, p.289 Thm. 4 (iii)].

The set of all step maps defined on the algebra

$$\mathfrak{A} = \{ S\times T \} \ S \in \mathfrak{M},\ T \in \mathfrak{N},\ \mu(S) < \infty,\ \nu(T) < \infty \}$$

is dense in $L^p(\mu\otimes\nu, B)$ (by extension of a result in Lang [13, p.257, Thm. 6]). Let $f \in \mathcal{L}^p(\mu\otimes\nu, B)$ and $\{f_n\}$ be an L^p-Cauchy sequence of step maps on $\mathfrak{A}$ converging a.e. ($\mu\otimes\nu$) to f. Each f_n is of the form

$$f_n(x,y) = \sum_{i=1}^{N_n} b_i^n \chi_{S_i^n \times T_i^n}(x,y) = \sum_{i=1}^{N_n} b_i^n \chi_{T_i^n}(y) \chi_{S_i^n}(x),$$

where $(x,y) \in X \times Y$. The corresponding F_n's are of the form

$$F_n(x) = \sum_{i=1}^{N_n} (b_i^n \chi_{B_i^n}) \chi_{A_i^n}(x), \text{ a.e. } (\mu) \text{ in } X,$$

that is they are step maps defined on X with values in $\mathcal{L}^p(\nu,B)$. Moreover when $p = \infty$

$$\begin{aligned} \| F_n(x) - F(x) \|_\infty &= \operatorname*{ess\,sup}_{y \in Y} | f_n(x,y) - f(x,y) | \\ &\leq \operatorname*{ess\,sup}_{X \times Y} | f_n(x,y) - f(x,y) | \\ &\leq \| f_n - f \|_1 , \text{ a.e. } (\mu) \text{ in } X, \end{aligned}$$

and when $1 \leq p < \infty$

$$\int_X d\mu \left\{ \int_Y d\nu \, | f_n(x,y) - f(x,y) |^p \right\} \to 0 \quad \text{as } n \to \infty$$

implies that for almost all (μ) x in X,

$$\| F_n(x) - F(x) \|_p^p = \int_Y d\nu \, | f_n(x,y) - f(x,y) |^p \to 0 \text{ as } n \to \infty.$$

We conclude that in all situations $F_n(x) \to F(x)$ a.e. (μ) in X. So F is a μ-measurable map.

Finally by the very definition of F,

$$\| F \|_p = \| f \|_p$$

and we can conclude that the map

$$x \mapsto \gamma_Y(F(x)) : X \to L^p(\nu,B)$$

is an element of $\mathcal{L}^p(\mu, L^p(\nu,B))$, and $\gamma_X(\gamma_Y(F))$ is in $L^p(\mu, L^p(\nu,B))$.

- We now check that given f and f' in $\mathcal{L}^p(\mu\otimes\nu, B)$

$$\gamma(f) = \gamma(f') \Rightarrow \gamma_X(\gamma_Y(F)) = \gamma_X(\gamma_Y(F'))$$

where F and F' are constructed from f and f' by the previous techniques. Clearly

$$\gamma(f) = \gamma(f') \Rightarrow f = f' \text{ a.e.}(\mu\otimes\nu) \text{ in } X \times Y$$

$$\Rightarrow \text{ for almost all } (\mu)\ x \text{ in } X$$

$$F(x)(y) = F'(x)(y), \text{ for a.a. } (\nu)\ y \text{ in } Y$$

$$\Rightarrow \gamma_Y(F(x)) = \gamma_Y(F'(x)), \text{ a.e. } (\mu) \text{ in } X$$

$$\Rightarrow \gamma_X(\gamma_Y(F)) = \gamma_X(\gamma_Y(F')).$$

2) The map $\bar{\beta}$ just constructed is clearly linear and norm preserving (hence injective). Only the surjectivity must be proved. Pick $G \in L^p(\mu, L^p(\nu, B))$. We can choose $g \in \mathcal{L}^p(\mu, L^p(\nu, B))$ for which $\gamma_X(g) = G$. There exists an L^p-Cauchy sequence of step maps $\{g_n\}$ converging a.e. (μ) to g. If the g_n's are of the form

$$g_n(x) = \sum_{i=1}^{N_n} g_i^n \chi_{A_i^n}(x), \quad x \in X,$$

we can pick $\tilde{g}_i^n \in \mathcal{L}^p(\nu, B)$ such that $\gamma_X(\tilde{g}_i^n) = g_i^n$ $(i = 1, \ldots, N_n)$ and define

$$\tilde{g}_n(x) = \sum_{i=1}^{N_n} \tilde{g}_i^n \chi_{A_i^n}(x), \quad x \in X$$

(in particular $\gamma_Y(\tilde{g}_n(x)) = g_n(x)$). We now naturally define

$$\bar{g}_n(x,y) = \hat{g}_n(x)(y) = \sum_{i=1}^{N_n} \tilde{g}_i^n(y) \chi_{A_i^n}(x), \quad (x,y) \in X \times Y.$$

By construction the maps $y \mapsto \tilde{g}_i^n(y)$ are ν -measurable, and then the map $(x,y) \mapsto \tilde{g}_i^n(y)\chi_{A_i^n}(x)$ and a fortiori the maps $\bar{g}_n$ are $\mu \otimes \nu$ -measurable. If we define the map

$$(x,y) \mapsto \bar{g}(x,y) = \lim_{n \to \infty} \bar{g}_n(x,y) : X \times Y \to B,$$

it is also $\mu \otimes \nu$-measurable [13, p.235, M 12].

Notice that by construction $\tilde{g}_i^n \in \mathcal{L}^p(\nu,B)$ and consequently $\|\bar{g}_n\|_p < \infty$ and $\bar{g}_n \in \mathcal{L}^p(\mu \otimes \nu, B)$.

Moreover, always by construction,

$$\|g_n - g_m\|_p = \|\gamma_Y(\tilde{g}_n - \tilde{g}_m)\|_p = \|\tilde{g}_n - \tilde{g}_m\|_p = \|\bar{g}_n - \bar{g}_m\|_p,$$

for any n and m. Since $\{g_n\}$ converges, $\{\bar{g}_n\}$ is necessarily Cauchy in $\mathcal{L}^p(\mu \otimes \nu, B)$ and converges to an element of $\mathcal{L}^p(\mu \otimes \nu, B)$ which is almost everywhere equal to $\bar{g}$. In particular

$\bar{g} \in \mathcal{L}^p(\mu \otimes \nu, B)$ and $\gamma(\bar{g}) = \lim_{n \to \infty} \gamma(\bar{g}_n)$.

Notice that (always by construction)

$$\bar{\beta}(\gamma(\bar{g}_n)) = \gamma_X(g_n).$$

By continuity of $\bar{\beta}$,

$$\lim_{n \to \infty} \bar{\beta}(\gamma(\bar{g}_n)) = \bar{\beta}(\gamma(\bar{g}))$$

and by continuity of γ_X

$$\lim_{n \to \infty} \gamma_X(g_n) = \gamma_X(g) = G.$$

In summary

$$\bar{\beta}(\gamma(\bar{g})) = \lim_{n \to \infty} \bar{\beta}(\gamma(\bar{g}_n)) = \lim_{n \to \infty} \gamma_X(g_n) = \gamma_X(g) = G.$$

This completes the proof of the theorem.

Corollary B.3

Let the notations and definitions of Theorem B.2 hold. Assume the measured spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are also complete. There is a norm preserving isomorphism

$$\tilde{\beta} : L^p(\overline{\mu \otimes \nu}, B) \longrightarrow L^p(\mu, L^p(\nu, B)) \tag{B.12}$$

such that for each $f \in \mathcal{L}^p(\overline{\mu \otimes \nu}, B)$ there exists an $\tilde{F} \in \mathcal{L}^p(\mu, L^p(\nu, B))$ and $F(x) \in \mathcal{L}^p(\nu, B)$ a.e. (μ) in X for which

$$\gamma_Y(F(x)) = \tilde{F}(x) \text{ , a.e. } (\mu) \text{ in } X, \tag{B.13}$$

$$F(x)(y) = f(x,y), \text{ a.e. } (\mu \otimes \nu) \text{ in } X \times Y \tag{B.14}$$

and

$$\tilde{\beta}(\bar{\gamma}(f)) = \gamma_X(\tilde{F}), \tag{B.15}$$

where $\bar{\gamma} : \mathcal{L}^p(\overline{\mu \otimes \nu}, B) \longrightarrow L^p(\overline{\mu \otimes \nu}, B)$ is the canonical surjection (section 2.1.1).

By symmetry similar results are true with $L^p(\nu, L^p(\mu, B))$ in place of $L^p(\mu, L^p(\nu, B))$.

Proof: Let $\bar{\alpha} : L^p(\mu \otimes \nu, B) \longrightarrow L^p(\overline{\mu \otimes \nu}, B)$ be the map of Lemma B.1. The composition map $\bar{\beta} \circ \bar{\alpha}^{-1}$ is precisely the map $\tilde{\beta}$ with the required properties.

Corollary B.4

Let the notations, definitions and hypotheses of Corollary B.3 hold. Assume $(X, \mathcal{M}, \mu) = ([t_0, t_1], \mathcal{M}, m)$,

$(Y, \mathcal{N}, \nu) = ([-a,0], \mathfrak{M}', m')$ and $([t_0,t_1]\times[-a,0], \mathfrak{M}_2, m_2)$ $(t_1 > t_0,\ a > 0)$ are complete Lebesgue measured spaces.

Then $\mathfrak{M}_2 = \overline{\mathfrak{M}\otimes\mathfrak{M}'}$, $m_2 = \overline{m\otimes m'}$ and the conclusions of Corollary B.3 are true with $L^P(m_2,B)$ in place of $L^P(\overline{m\otimes m'},B)$.

Proof: From Corollary B.3 and a proposition in W. Rudin [26, p.144, Thm. 7.11] : $([t_0,t_1] \times [-a,0], \mathfrak{M}_2, m_2)$ is the completion of $([t_0,t_1] \times [-a,0], \mathfrak{M}\otimes\mathfrak{M}', m\otimes m')$.

Remarks (i) Similar partial results can be found in Dunford and Schwartz [15, p.196 Lemma 16 and p.198 Lemma 17].

(ii) It seems that our results are readily extendable to σ-finite measured spaces.

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