QUALITATIVE PROPERTIES OF A CLASS OF INFINITE DIMENSIONAL SYSTEMS

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ABSTRACT

This research is concerned with the theory of Infinite Dimensional Systems and their Qualitative properties: controllability, observability, stabilizability and other related notions. Such properties may be "unstable under approximation", in which case there may exist a "property-gap". Considerable use is made of functional analysis, especially spectral theory in deriving several criteria for checking these system properties. As an illustration of the general theory linear systems with hereditary dependence (the most general type known to be useful) are considered. Several new results on controllability and stabilizability are presented.

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Dedicated to Beatriz, Karin and Kathia.

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NOMENCLATURE

This table lists the symbols used, with their meaning. This meaning in a specific context may differ, but then it will be explicitly stated.

A, B, C, D, F, G, L, T, ... $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, ...	Linear operators
A*, B*, ...	Adjoint of A, B,...
P, P_1, $\tilde{P}$,...	Projections
C_π, C_Λ	Specific subspaces (chapter IV)
H_x, H_y, H_u ...	Hilbert spaces
X, Y, Z,...	Banach spaces (generally)
X*, Y*,...	Dual Banach space of X, Y,...
x, y, z, u,...	Elements of a normed vectorspace
M, N, M_1,...E_1,E_2,...W_1,W,..V	Subspaces (generally)
$\mathcal{N}$, $\mathcal{N}$(A) , N(A)	Nulspace, Nulspace of A
Ω	Subset in R^n
t, t_1, ... T	Real numbers (usually stands for variable "time")
a, b, c,... α, β, γ,..τ..	Scalars
δ, ε	Small positive numbers
r, h	Scalars, occasionally vectors (chapter IV
m_i, y_i, or m_{λ_i}, y_{λ_i}	Algebraic multiplicity of eigenvalue λ_i

k^T	Transpose of vector k
$D_o(A)$	Domain of A
$G(A)$	Graph of A
$R(A)$	Range of A
$A \restriction_M$	The restriction of A to subspace M
$R(\lambda, A)$	Resolvent of A
$\rho(A)$	Resolvent set of A
$\sigma(A)$, $P\sigma(A)$, $C\sigma(A)$	Spectrum of A; point spectrum of A; continuous spectrum of A
$\mathfrak{Z}_i$, $\mathfrak{Z}_{\lambda_i}$	Eigenspace corresponding to eigenvalue λ_i
$\mathfrak{M}_i$, $\mathfrak{M}_{\lambda_i}$, $\mathfrak{M}_{\lambda_i}(A)$; M_i	Rootspace corresponding to eigenvalue λ_i of
Φ_i or Φ_{λ_i}	Matrix whose columns are basisvectors for $\mathfrak{M}_{\lambda_i}(A)$
Ψ_i, Ψ_{λ_i}	Matrix whose rows (usually are basis-vectors for $\mathfrak{M}_{\lambda_i}(A^*)$
ϕ, T, $\{\phi(t)\}_{t>0}$,$\{T(t)\}_{t>0}$	One-parameter semigroup of linear bounded operators
$\mathcal{L}(X,Y)$	Space of linear bounded operators from X to Y
$\mathcal{L}^2([0,T]; H_x)$	Space of (equivalence classes of) Lebesque square integrable functions on $[0,T]$ with values in H_x
$C^\infty(\Omega)$	Space of infinitely differentiable functions on Ω
$C([-\tau,0]; R^n)$, C	Space of continuous functions on $[-\tau, 0] \subset R$ with values in R^n

x_t, ϕ	Elements of $C([-\tau, 0]; R^n)$
y^t, ψ	Elements of $C([0,\tau]; R^{n^*})$
d_Λ in R^{d_Λ}	A positive integer indicating "dimension"
Λ, $\tilde{\Lambda}$	Finite symmetric set of eigenvalues $\{\lambda, \ldots \lambda_N\}$
R^+; I^+	Positive real line; positive integers
Σ_α	A sector in the complex plane (App A. def 7)
E_λ	Resolution of identy (App. B. def 5)
$N(\varepsilon, x)$, N_ε	ε-neighborhood of x
Γ	Positive real number standing for "gapwidth" (chapters II, III)
Q	Set of analytic vectors of semigroup (chapters II, III)
Q_i	See Chapter III, section 8
Q(T)	See Chapter IV, section 11
$X_o(\theta)$, X_o	See Chapter IV, section 3
A_n^i, B_n^i, $\tilde{A}_n^i$ ...	Coefficients in the Laurent expansion of $R(\lambda, A)$ in a small neighborhood of $\lambda=\lambda_i \in \sigma(A)$
$A/\mathcal{B}$	Range of $[B \mid AB \mid \ldots \mid A^{n-1}B]$ if $A : R^n \to R^n$.
$\triangleq$, $\equiv$	"Is defined to be"
F.D.S.	Finite Dimensional (linear) System
I.D.S	Infinite Dimensional (linear) System
D.P.S.	(Linear) Distributed Parameter System
$\mathcal{L}^{-1}$	Inverse Laplace transform
$\mathcal{F}^{-1}$	Inverse Fourier transform.

TABLE OF CONTENTS

page

CHAPTER I

INTRODUCTION.

We are interested in studying qualitative properties of linear infinite dimensional systems whose dynamics can be described by an operator differential eqaution of the general form

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \qquad y = Cx,$$

where $x(t)$, $y(t)$, $u(t)$ for fixed t are elements of spaces H_x, H_y, H_u. These spaces in general are Hilbertspaces unless specidied otherwise; x, y, u, as functions of t belong to specified spaces X, Y, U. One calls "state " the solution $x(t)$ to the above equation when appropriate initial conditions are specified; x is called the "trajectory, y the "output" or "observation" and u the "input" or "control". The above formulation includes as special cases linear distributed parameter systems, linear finite dimensional and infinite dimensional systems and finally a very general class of systems with hereditary dependence.

Operator differential equations as presented above are considered in some detail in Appendices A, B, and C. In these appendices some concepts from the realm of modern analysis are presented in order to establish a basis for the understanding of the main body of the thesis. These results are mostly well known although a major effort is required to gather the material relevant to control theory. A conscious effort was made to refer to refernces for proofs, retaining only those strictly necessary for the coherence and understanding of the material presented. It should be made clear from the outset that it is not our intention to contribute to the theory of partial differential equations, rather we are trying to apply existing theory to physical systems considered relevant to control-oriented applications. The main references for appendices A, B, and C are Hille-Phillips [H5] , Dunford-Schwartz [D2] , Goldstein [G3] and references [B1, B4, K1] .

In Appendix A operator differntial equations of the type $\frac{dx(t)}{dt} = Ax(t) + h(t)$ are discussed for existence and uniqueness of solutions and more specifically for the properties of the solution x(t). The study of these properties after all is the major topic of this thesis. The solutions will be characterized in terms of certain types of semigroups like analytic semigroups or C^o-unitary groups... These characterizations play a major role in chapters II and III on controllability respectively observability. Conditions on a dissipative operator A will be stated for which a C^o-contraction semigroup is being being generated. Stabilizability, one of the topics in Chapters III and IV, will be the systemproperty indicating the possibility for transforming a non-dissipative into a dissipative operator, thus changing the character of the solution.

In Appendix B some notions on spectral decompositions are discussed in a Banach space setting (used in Chapter IV). The theory is then specialized to Hilbert spaces. A case of major importance is that of a "normal spectrum and special attention will be given to this case.

In Appendix C important types of 1st and 2nd order operator differential equations, respectively parabolic and 2nd order hyperbolic partial differential equations, are discussed as illustration of the semigroup theory and special attention is given to the kind of semigroup generated.

Chapters II and III focus on controllability, observability and the new notion of gaps. Some attention is given to specifying methods for finding the gap. Observers and design procedures are touched upon in III. Finally Chapter IV is devoted exclusively to delay-systems. Specifically controllability, stabilizability of state and output by state-feedback are considered.

CHAPTER II

CONTROLLABILITY AND CONTROLLABILITY-GAPS FOR A CLASS OF INFINITE DIMENSIONAL SYSTEMS

2.1 Introduction

A great deal of theoretical work in the field of control theory for Distributed Parameter Systems, and in general for Infinite Dimensional Systems, has been centered on extending results known for finite dimensional systems. In a major portion of the work oriented towards solving practical problems approximation methods are used based on reducing the system-dimension to a tractable finite level. However, it is not a priori clear that a property which holds for any finite dimensional approximation to the system carries through in the limit. Such property will be termed unstable under approximation. The systems considered will be linear, time-invariant, with certain restrictions on the system operators in order to insure well-posedness. For these systems, the most outstanding such "unstable" property discussed here is related to the notion of complete controllability. It may happen that such systems are completely controllable in the sense that for every state x_d there exists a finite time $t(x_d) > 0$ and an admissible control u driving the system from the origin at time $t = 0$ to a point in the ε-neighborhood of x_d at time t, for any $\varepsilon > 0$. If time $t = T$ were fixed it might be that some states would not be controllable in the above sense.

Assume the system is completely controllable for $t = T_1$, and let $\Gamma = \inf T_1$ for which this is true.

If $\Gamma > 0$ we will say that a controllability-gap exists and is of width Γ. Thus a controllability-gap occurs if the system is completely controllable but only so after a finite time has elapsed. The existence of this phenomenon for infinite dimensional systems has been recognized by at least one author. Russell [R2, R3] discussed the occurrence of a gap for oscillator-type systems (vibrating strings and beams, lossless transmission lines.). A class of systems for which the gap-case is obvious is the class of delay systems $\dot{x}(t) = L(x_t,t)+B(t)u(t)$, $x_{t_o} = \phi$; x_t, x_{t_o}, $\phi \in C([-\tau,0];R^n)$. These systems are treated in Chapter IV.

Observe that the systems Russell considers have discrete, simple spectra, lying on the imaginary axis in the complex plane, and that the delay systems have a discrete spectrum, with finite multiplicities of the eigenvalues and lying in a left half plane $\{\lambda \in \mathbb{C} \mid \text{Re } \lambda \leq a, \text{ some real } a\}$.

Diffusive systems cannot have a gap as will be demonstrated. A subclass of these are systems governed by a self-adjoint system operator with discrete spectrum with finite multiplicities. The spectrum lies on the real line. Another subclass is the class governed by perturbed self-adjoint operators (see App C-2). For the spectral properties we refer to Appendix B.3.

The three "types" discussed have one thing in common: a normal spectrum. The geometrical configuration of the spectrum over the complex plane is different.

It would be of interest to isolate configurations or other conditions related to the spectrum excluding or confirming the possibility of occurrence of a gap. This theory could be made to apply to a more general case then that of a normal spectrum including the case of continuous or residual spectra. Clearly operator B will play a key-role in any configuration.

It would be equally desirable to devise ways to calculate the width Γ of the gap. This could in fact be used as a criterion for complete controllability. The system is completely controllable if $\Gamma < \infty$; it has no gap if $\Gamma = 0$.

The aim of this chapter is then:

1) To specify conditions on the system operators under which a controllability gap may or may not occur.

2) If a gap exists, indicate ways to obtain an analytical expression for the width of the gap.

3) Assuming one could manipulate or choose the operator through which the control is applied to the system, answer the question as to whether it is possible to choose the operator to make the gap disappear.

2.2 Systems Under Consideration

Consider the abstract linear system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(t_o) &= x_o \end{aligned} \qquad (1)$$

where $x(t) \in H_x$

$u(t) \in H_u$

$A : D_o(A) \subset H_x \to H_x$

$B : H_u \to H_x$

A and B are linear operators, B is bounded. H_x and H_u are given Hilbert spaces, respectively the state space and the input space. The function u will in general be assumed to be piecewise continuous, taking values in H_u in case one wants strong solutions, or just measurable maps from $R^+ \to H_u$ in case one wants to consider solutions in a weaker sense.

A minimal assumption on A which will not be restated is that the Cauchy problem

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) \\ x(t_o) &= x_o \end{aligned} \tag{2}$$

be wellposed. For the nonhomogeneous system (2) we will in addition assume, for strong solutions, that $Bu(t) \in C(R^+ D_o(A))$ and $ABu(t) \in C(R^+, H_x)$ or $Bu(t) \in C^1(R^+, H_x)$ in which case there exists a unique solution $x(.) \in C^1(R^+, H_x)$ of the form

$$x(t) = \phi(t)x_o + \int_0^t \phi(t-\sigma)Bu(\sigma)d\sigma \tag{3}$$

The details are given in the appendices. Systems that can be treated this way include finite dimensional systems, distributed systems (e.g. of parabolic type of hyperbolic type), a large class of continuous fixed time-lag systems (Chapter IV).

2.3 Definition: Controllability, Controllability-Gap

When solving an optimization problem one will have to require often the system to have certain qualities, the most important of which is its responsiveness to the available controls and sensors. For systems whose evolution is governed by the variation of constants formula one can define complete controllability (henceforth written as c.c.) as follows:

Definition 1.

A dynamical control system of type (1) will be called completely controllable from the origin if one can drive the system from the origin to an ε-neighborhood of any desired state with an admissible control during the time interval $[0,t]$ for some real $t \geq 0$, $\varepsilon > 0$.

Let the space of admissible controls be denoted by U. Let the operator $F_t : U \to H_x$ be defined by

$$F_t \triangleq \int_0^t \phi(t-\sigma)B.d\sigma$$

A mere reformulation of definition 1 using (3) gives then that the system is c.c. iff

$$\overline{\bigcup_{t\geq 0} R(F_t)} = H_x$$

or equivalently

$$\bigcup_{t\geq 0} R(F_t)^{\perp} = \{0\}$$

Even if the set of admissible controls is allowed to be the total space U it is conceivable that the density of $\bigcup_{t\geq 0} R(F_t)$ in H_x is completed only after a certain finite time T_1 has elapsed i.e.

$$\overline{\bigcup_{0\leq t\leq T_1} R(F_t)} = H_x \quad \text{but} \quad \overline{\bigcup_{0\leq t\leq T_2} R(F_t)} \underset{\text{strictly}}{\subset} H_x, \; T_2 < T_1$$

For linear time invariant systems is it easily demonstrated that

$$\bigcup_{0\leq t\leq T_1} R(F_t) = R(F_{T_1})$$

<u>Definition 2.</u>

A system of type (1) will be said to have a controllability-gap if it is c.c. but only after a finite time T_1 has elapsed. The g.l.b. on the time interval $[0,T_1]$ in which c.c. is achieved determines the "width" of the gap.

Alternatively the width Γ would be defined as

$$\Gamma = \sup_{x_d \varepsilon H_x} \; \inf_{u \varepsilon U} \Big\{ t \; \varepsilon \; R^+ \,|\, x(t,0) = x_1, \text{ for some}$$

$$x_1 \; \varepsilon \; N_\varepsilon(x_d) \text{ and for arbitrary small } \varepsilon > 0 \Big\}.$$

The well known controllability test for continuous, linear, time invariant, lumped systems (in R^n) namely

$$\text{rank } [B \,\vdots\, AB \,\vdots\, \ldots \, A^{n-1}B] = n$$

should be recognized as expressing an "instantaneous" property i.e. if $R(F_t) = R^n$ for some $t > 0$ then $R(F_\varepsilon) = R^n$, $\varepsilon > 0$. Moreover the criterion expressed above states that this property can be recognized at $t = 0$. It is then clear that in finite dimensional systems controllability-gaps are excluded.

In the next paragraph we will derive the infinite dimensional version of the above criterion, obtaining at the same time conditions on the systems excluding the possibility of occurrence of controllability-gaps. The conditions involve the set of analytic vectors for the semigroup ϕ. Subsequently it will be shown that, when ϕ is analytic existence of gaps is equally excluded (although the controllability test may not hold).

2.4 Controllability when ϕ is an Analytic Semigroup

Scholium 1

If A generates an analytic C^o-semigroup of type α, $0 < \alpha < \pi/2$ the following set equality holds

$$\overline{R(F_{t_1})} = \overline{R(F_{t_2})} \quad \text{for all } t_1, t_2 > 0 .$$

Proof :

Let $x \in R(F_{t_1})^{\perp}$. Then $B^*\phi^*(s)x=0$, $0 \leq s \leq t_1$.
Because ϕ^* is analytic if ϕ is we conclude that

$$B^*\phi^*(s)x = 0 \quad , \quad 0 \leq s \leq t_2 \quad , \quad t_2 > t_1$$

Therefore $R(F_{t_1})^{\perp} \subset R(F_{t_2})^{\perp}$. But $R(F_t)$ is a monotone increasing set with t for the systems under consideration. Therefore

$$R(F_{t_1}) \subset R(F_{t_2}) \quad \text{or} \quad R(F_{t_1})^{\perp} \supset R(F_{t_2})^{\perp}$$

Altogether then

$$R(F_{t_1})^{\perp} = R(F_{t_2})^{\perp} \quad \text{or} \quad \overline{R(F_{t_1})} = \overline{R(F_{t_2})} \ . \quad \blacksquare$$

The scholium states in fact that if ϕ is analytic, the set obtained as the closure of the set of attainable states from the origin is instantly established. Gaps are therefore excluded. Examples of cases where ϕ is analytic and therefore no gaps can occur were given in chapter I.

2.5 A Controllability-Criterion using the Notion of Analytic Vectors of Semigroup ϕ

2.5.1 Analytic vectors of ϕ

For $t > 0$ and $x \in D_o(A^n)$ one can write the Taylor formula for semigroup ϕ :

$$\phi(t)x = \sum_{k=0}^{n-1} \frac{t^k}{k!} A^k x + \frac{1}{n-1!} \int_0^t (t-s)^{n-1} \phi(s) A^n x ds$$

It will be important to us to know conditions under which

$$\phi(t)x = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k x$$

One clearly need that $x \in D_\infty(A) \triangleq \bigcap_{k=1}^{\infty} D_o(A^k)$ and that the series converges.

Scholium 2.

If $x \in D_\infty(A)$ and the series converges it converges to $\phi(t)x$ absolutely in a neighborhood of the origin.

Proof : If the series converges we have

$$\lim_{n\to\infty} \frac{t^n}{n!} ||A^n x|| = 0$$

and hence

$$||\phi(t)x - \sum_{k=0}^{n-1} \frac{t^k}{k!} A^k x|| \leq \frac{1}{n-1!} \int_0^t (t-s)^{n-1} ||\phi(s)|| \; ||A^n x|| ds$$
$$\leq Me^{\omega t} \frac{t^n}{n!} ||A^n x|| \qquad (4)$$

since $||\phi(s)|| \leq Me^{\omega s}$ for some M, ω (see generation theorems for C^o-semigroups). The right hand side of expression (4) clearly goes to zero for $n \to \infty$. If $0 \leq s \leq t$, then $\sum_{k=0}^{\infty} \frac{s^k}{k!} ||A^k x|| < \infty$. This follows easily since

$$\frac{t^n}{n!} ||A^n x|| \to 0$$

implies $\frac{t^n}{n!} ||A^n x|| \leq C$ for all n and a real constant C. Then

$$\sum_{k=0}^{\infty} \frac{s^k}{k!} ||A^k x|| \leq C \sum_{0}^{\infty} \left(\frac{s}{t} \right)^k < \infty. \blacksquare$$

Motivated by the theory of ordinary analytic functions we will call x an analytic vector in case the Taylor series converges for some $t > 0$.

Definition 3.

A vector $x \in H_x$ is called an analytic vector for $\{\phi(t)\}_{t \geq 0}$ if

1) $x \in D_\infty(A)$ [i.e. x is a C_∞ vector]

2) $\sum_{k=0}^{\infty} \frac{t^k}{k!} ||A^k x|| < \infty$ for some $t(x) \geq 0$.

One important property of analytic vectors is illustrated by the following scholium.

Scholium 3.

Suppose x is an analytic vector for ϕ with radius $\varepsilon(x)$, and

$\langle A^n x, x^* \rangle = 0$ for n=1,2,3,... arbitrary $x^* \in H_x^*$

then $\langle \phi(t)x, x^* \rangle = 0$ for all $t \geq 0$.

Proof : For $0 \leq t < \varepsilon$ $(\varepsilon = \varepsilon(x) > 0)$ we have

$$\langle \phi(t)x, x^* \rangle = \sum_0^\infty \frac{t^n}{n!} \langle A^n x, x^* \rangle = 0$$

So by continuity at $t = \varepsilon$, $\langle \phi(t)x, x^* \rangle = 0$ for $0 \leq t \leq \varepsilon$

By the same argument[1] we have

$$\langle \phi(t)A^m x, x^* \rangle = 0 \text{ for } 0 \leq t \leq \varepsilon \qquad m = 0,1,2,...$$

Hence, if $0 \leq t \leq \varepsilon$ we have

$$\langle \phi(t+\varepsilon)A^m x, x^* \rangle = \sum_0^\infty \frac{t^n}{n!} \langle \phi(\varepsilon)A^{n+m} x, x^* \rangle = 0$$

So $\langle \phi(t)A^m x, x^* \rangle = 0$ for all $0 \leq t \leq 2\varepsilon$, $m = 0,1,2,...$

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(1) Here we made use of the fact that an absolutely convergent power series has the same radius of convergence. Therefore

$$\sum_{n=0}^\infty \frac{t^n}{n!} ||A^{n+m}x|| < \infty \quad \text{for } 0 \leq t < \varepsilon \qquad m = 0,1,2,...$$

A repetition of the argument gives

$$\langle\phi(t)A^m x, x^*\rangle = 0 \text{ for } 0 < t \leqslant 4\varepsilon, \quad m = 0,1,2,\ldots$$

and by induction for $0 \leqslant t$, $m = 0,1,2,\ldots$ For $t = t_1+s \quad 0 \leqslant s \leqslant \varepsilon$

$$\begin{aligned}\langle\phi(t)x,x^*\rangle &= \langle\phi(s)\phi(t_1)x,x^*\rangle \\ &= \sum_{k=1}^{\infty} \frac{s^k}{k!} \langle\phi(t_1)A^n x,x^*\rangle \\ &= 0 \qquad t \geqslant 0 .\end{aligned}$$

It would be nice if a semigroup always had "enough" analytic vectors, but unfortunately this is not the case. It can happen that ϕ has no nonzero analytic vectors (see Nelson [N1]). However, in a few important cases, one can show the existence of a dense set of analytic vectors.

Examples

(1) A is bounded (equivalently: the semigroup is uniformly continuous). Then $\sum \frac{t^n}{n!} ||A^n x|| \leqslant e^{t||A||}||x||$. So that every vector x is an analytic vector for ϕ.

(2) (-A) is also an infinite-simal generator so that $t \to \phi(t)$ can be extended to a group on $(-\infty,+\infty)$. In this case, let $\sigma > 0$ and define

$$x_\sigma \triangleq \frac{1}{\sqrt{4\pi\sigma}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{4\sigma}} \phi(t)x\,dt$$

Then one can show (*) that x_σ is an analytic vector for ϕ and

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(*) This method of constructing analytic vectors was used by Gelfand (see Nelson [N1] for references).

$x_\sigma \to x$ for $\sigma \to 0$.

In particular ϕ has a dense set of analytic vectors. This method works e.g. if A is skew-symmetric ($A^* = -A$). In this case $t \to \phi(t) = e^{tA}$ is a group of unitary operators (see Appendix A), and one could also construct analytic vectors by means of spectral analysis.

(3) Construction of Analytic Vectors Using Spectral Theory

Let H_x be a Hilbert space, $A = A^*$ and $\langle Ax,x\rangle \leq c\langle x,x\rangle$ for all $x \in D_o(A)$ (i.e., A is self adjoint and semibounded from above).

By the spectral theorem for self adjoint operators we have

$$A = \int_{-\infty}^{+\infty} \lambda E(d\lambda) \qquad \text{or} \quad = \int_{-\infty}^{+\infty} \lambda dE_\lambda$$

Let $[a,b] \subseteq \mathbb{R}$ be an interval and $y \in H_x$. If $x = E([a,b])y$ we have

$$||A^n x||^2 = \int_{-\infty}^{+\infty} |\lambda|^{2n} \langle dE_\lambda x, x\rangle = \int_a^b |\lambda|^{2n} \langle dE_\lambda y, y\rangle$$

$$\leq [\max(|a|,|b|)]^{2n} ||y||^2$$

$$\leq c^n \text{ for some constant } c$$

and that $\Sigma \frac{t^n}{n!} ||A^n x||$

converges absolutely. We have shown earlier that it then converges to $\phi(t)x$. So x is an analytic vector for ϕ.

Since $E([a,b])y \to y$ as $a \to -\infty$, $b \to +\infty$ the set of analytic vectors is dense.

2.5.2 Criterion for Complete Controllability

Let Q denote the set of analytic vectors of φ. Define

$$\mathcal{a} = \overline{\bigcup_{t\geq 0} R(F_t)} .$$

Theorem 1.

Let $B : H_u \to H_x$ be such that $R(B) \subset D_o(A^n)$ for $n \in I^+$. Then the following inclusions hold:

$$\sum_0^\infty R(A^nB)^\perp \supset R(F_t)^\perp \supset \bigcup_{t\geq 0} R(F_t)^\perp = \mathcal{a}^\perp$$

The reverse inclusions also hold if $\overline{B^{-1}(R(B) \cap Q)} = H_u$.

Proof :

The second inclusion $R(F_t)^\perp \supset \mathcal{a}^\perp$ is obvious. To prove the first inclusion observe that

$$x \in R(F_t)^\perp \quad \text{implies } B^*\phi^*(s)x = 0, \quad 0 \leq s \leq t .$$

Then, for any $z \in H_u$, $\langle\phi(s)Bz,x\rangle = 0 \quad 0 \leq s \leq t$. Since $R(B) \subset D_o(A^n)$ it follows that $\langle A^nBz,x\rangle = 0$ for all $z \in H_u$ and $n \in I^+$. This expresses that

$$x \in \sum_{n=1}^\infty R(A^nB)^\perp .$$

Under the additional assumption that

$$\overline{B^{-1}(R(B) \cap Q)} = H_o \quad \text{let } x \in \bigcup R(A^nB)^\perp$$

Then for $z_o \varepsilon B^{-1}(R(B) \cap Q)$ one has $<A^nBz_o,x> = 0$. Since $Bz_o \varepsilon Q$ this implies (Scholium 3) that

$$<\phi(s)Bz_o,x> = 0 \text{ for } 0 \leq s \leq \varepsilon(x), \text{ implying}$$

$$<\phi(s)Bz_o,x> = 0 \text{ for } 0 \leq s$$

But if $z \varepsilon H_u$ is arbitrary, let $z_n \varepsilon B^{-1}(R(B) \cap Q)$ be such that $z_n \to z$. Then $<\phi(s)Bz,x> = \lim_n <\phi(s)Bz_n,x> = 0$, $s \geq 0$, by boundedness of ϕ and B.

Therefore $< \int_0^t \phi(t-sBu(s)ds,x> = 0 \quad t \geq 0$ or $x \varepsilon \mathcal{O}\!\!\mathcal{L}^{\perp}$

We conclude that under the re-inforced condition

$$\sum_0^{\infty} R(A^nB)^{\perp} = R(F_t)^{\perp} = \mathcal{O}\!\!\mathcal{L}^{\perp}. \qquad \blacksquare$$

<u>Theorem 2</u>.

Under the combined conditions of theorem 1 a system of type (1) is c.c. iff $\overline{\sum_0^{\infty} R(A^nB)} = H_x$. $\blacksquare$

<u>Remark</u>. The condition $R(B) \subset Q$ is a stronger condition than the one used in theorem 1 and 2 and implies both $R(B) \subset D_{\infty}(A)$ and $\overline{B^{-1}(R(B) \cap Q)} = H_u$. In some cases this stronger condition may be unduly restrictive (for this reason we preferred the condition as stated in theorem 1), in other cases it may be automatically satisfied, like in the case that A is bounded or for finite dimensional systems; in the latter case our theorem 2 reduces to the usual Kalman test.

<u>Example 1</u> If $H_x = H_u$, B is bounded and $B : D_o(A) \to D_o(A)$, and $AB = BA$ then

$$||\Sigma \frac{t^nA^n}{n!} Bz|| \leq ||B|| \; ||\Sigma \frac{t^nA^n}{n!} z||$$

If $z \in Q$ then $Bz \in Q$ in this case, so $B^{-1}(R(B) \cap Q) \supset Q$. If Q is dense then $B^{-1}(R(B) \cap Q)$ dense in H_u is certainly satisfied. A and B commute for example if B is a convolution and A is a differential operator with constant coefficients (Goodman, [G7]).

Example 2 Let $H_x = H_u = L^2(\Omega)$, Ω a compact interval in R. Consider the case $Bu = g(z)u(t)$, $g(z) \in L^2(\Omega)$. Let A be a self adjoint positive elliptic operator and consider the system

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ Bu(t) \end{bmatrix} \triangleq \mathcal{A} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \tilde{B}u(t)$$

Then $\tilde{B} : R \rightarrow Z = D_o(\sqrt{A}) \times H_x$ and $R(B)$ is one-dimensional. Let $g(z) = \sum_\lambda \gamma_\lambda \phi_\lambda$, where $\lambda \in \sigma(A)$. ϕ_λ is the corresponding eigenfunction. Then g is analytic if it is a finite linear combination of eigenfunctions ϕ_λ (in which case the system cannot be controllable, see 2.7) or when $\sum_{\lambda \in \sigma(A)} \gamma_\lambda^2 e^{2\sqrt{\lambda}t} < \infty$. To see this observe that (Chapter I, (15))

$$e^{\mathcal{A}t}\tilde{B} = \begin{bmatrix} A^{-1}\sinh t\sqrt{A}.B \\ \cosh t\sqrt{A}.B \end{bmatrix} = \begin{bmatrix} \sum_\lambda \sqrt{\lambda}^{-1} \sin t\sqrt{\lambda}.\gamma_\lambda \phi_\lambda \\ \sum_\lambda \cosh t\sqrt{\lambda}.\gamma_\lambda \phi_\lambda \end{bmatrix} \triangleq \begin{bmatrix} a \\ b \end{bmatrix}$$

with norm

$$||e^{\mathcal{A}t}\tilde{B}||^2_{D_o(\sqrt{A}) \times H_x} = |\sqrt{A}\ a|^2_{H_x} + |b|_{H_x} \leq \sum_\lambda \gamma_\lambda^2 e^{2\sqrt{\lambda}t}$$

$R(B) \subset Q$ would then be satisfied if $\Sigma\gamma_\lambda^2 e^{2\sqrt{\lambda}t} < \infty$ for arbitrary small $t>0$.

If A is negative self adjoint, then $\mathcal{A}$ is skew adjoint in the right topology and $e^{\mathcal{A}t}$ is a C^o-unitary group given by I.(17). In the topology indicated there $||e^{\mathcal{A}t}\tilde{B}|| = \sum_\lambda \gamma_\lambda^2$ and $\sum_\lambda \gamma_\lambda^2 < \infty$ is satisfied since $g \in L^2(\Omega)$. However, for g to be analytic, $g \in D_\infty(\mathcal{A})$ requires

$\sum_{\lambda}\lambda^N\gamma_\lambda^2 < \infty$, for arbitrary big N.

We conclude with reference to a specific example in the literature where the occurrence of a gap is demonstrated.

Example 3. (Occurrence of a gap; [R2, R3, R4])

Russell considered the system

$$\rho(x)\,\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x)\,\frac{\partial v}{\partial x}\right) = g(x)f(t)$$

with $g(x) \in L^2[0,1]$, $f(t) \in L^2[0,T]$. $\rho(x)$ and $p(x)$ are positive and twice differentiable on $[0,1]$. It is assumed that $u(x,t)$ satisfies the boundary conditions

$$A_o v(0,t) + B_o \frac{\partial v}{\partial x}(0,t) \equiv 0$$

$$A_1 v(1,t) + B_1 \frac{\partial v}{\partial x}(1,t) \equiv 0$$

where A_i, B_i for $i = 1,2$, are real constants with $A_o^2+B_o^2 \neq 0$, $A_1^2+B_1^2 \neq 0$. The initial conditions are $v(x,0) = v_{o1}(x)$, $\frac{\partial v}{\partial t}(x,0) \in v_{o2}(x)$ such that the boundary conditions are satisfied at $t = 0$ and $\frac{\partial^2 v_{o1}}{\partial x^2}, \frac{\partial^2 v_{o2}}{\partial x^2} \in L^2[0,1]$. Let

$$\ell \triangleq \int_0^1 \sqrt{\frac{\rho(x)}{p(x)}}\,dx$$

then Russell claims that the system cannot be controllable in $[0,T]$ for $T < 2\ell$. If, in general, an infinite dimensional oscillator has eigenvalues $\{j\omega_k\}_{k=-\infty}^{-\infty}$ and if δ denotes the asymptotic gap then $p = \frac{2\pi}{\delta}$ is the controllability gap-width.

The case of boundary control, where $g(x)v(t)$ is of the form

$\delta(z)\ u_1(t) + \delta(z-1)u_2(t)$ and therefore $g(x) \notin L^2[0,1]$ is considered in [R3,R4].

Example 4. (Occurrence of a gap; linear systems with hereditary dependence)

Let $x_t(\theta)$ or in short x_t, $\theta \in [-\tau,0]$, $\tau \in R^+$, be defined by

$$x_t(\theta) \triangleq x(t+\theta)$$

Let $x_t \in C([-\tau,0];R^n)$ and let $C([-\tau,0];R^n)$ be equipped with the norm of uniform convergence (to make it a Banach space).

Consider $\dot{x}(t) = L(x_t,t) + B(t)u(t),$ (a)

$u(t) \in U$ where U is a closed set in R^m, L is a linear bounded mapping from $C([-\tau,0];R^n) \times R \to R^n$, B is an nxm-matrix with initial condition $x_o = \phi \in C([-\tau,0];R^n)$. The "state-space-representation" of (a) is (see Chapter IV)

$$\begin{cases} \dot{x}_t = Ax_t + \chi_o B(t)u(t) \\ x_{t_o} = \phi \\ \chi_o(\theta) \triangleq I \quad \theta = 0 \\ \qquad\quad \triangleq 0 \quad \text{otherwise} \end{cases} \qquad \text{(b)}$$

Since for $t < \tau$ $x_t(\theta) = \phi(\theta+t)$, $\theta + t \leq 0$, part of the state is still equal to the part of the initial function and hence uncontrollable. It is then clear that the gap width Γ satisfies $\Gamma \geq \tau$.

Conclusion:

As part of theorem 1 it was shown that $\overline{R(F_t)} = \overline{\bigcup_{t\geq 0} R(F_t)}$

under the appropriate conditions namely $R(B) \subset D_\infty(A)$ and $\overline{B^{-1}(R(B) \cap Q)} = H_u$ or under the more severe condition $R(B) \subset Q$. If the system is c.c. in this case (i.e. $\overline{\bigcup_{t\geq 0} R(F_t)} = H_x$) it must be so instantly; in other words existence of a gap is excluded. Our attention has been focused on smoothness of the vector Bu with respect to ϕ. If ϕ itself is an analytic C^o-semigroup one can make a similar conclusion, as was seen in II.4.

As demonstrated above $R(B) \subset Q$ may be satisfied for ϕ being a C^o-group. In general groups are not analytic. Thus the two criteria for excluding gaps may apply to non-overlapping cases. In case ϕ is analytic and assuming it is possible to manipulate the expression for B we may satisfy the condition $R(B) \subset Q$ as well, by taking the new operator to be $\tilde{B} = \phi(\varepsilon)B$. Clearly $\tilde{B}u$ for arbitrary u is an analytic vector for ϕ. (Observe that $R(\phi(\varepsilon)) \subset \bigcap_{n=1}^{\infty} D(A^n)$ is satisfied by analycity of ϕ.)

2.6 A Controllability-Criterion Using the Resolvent of A.

In this section a controllability criterion for linear time invariant systems of type (1) is derived using frequency-domain techniques. It is well known [H5] that the resolvent $R(\lambda,A) : H_x \to D_o(A)$ is the Laplace transform of semigroup ϕ generated by A, for all λ such that $\mathrm{Re}\,\lambda > \omega_o \triangleq \lim_{t\to 0} \frac{\log ||\phi(t)||}{t}$. Also, in a Hilbert space setting, it is known that, if A generates ϕ then A^* generates ϕ^*, with Laplace transform $R(\lambda,A^*)$.

<u>Lemma</u> [Fattorini, F3]

$$x \in \bigcup_{t\geq 0} R(F_t)^{\perp} \quad \text{iff} \quad \mathcal{J}^*(\lambda)x = 0 \quad \text{for Re } \lambda > \omega_o$$

Therein denotes $\mathcal{J}^*$ the Laplace transform of $B^*\phi^*(s)$, $s \geq 0$.

<u>Proof:</u> We have shown before that

$$x \in \bigcup_{t\geq 0} R(F_t)^{\perp} \Leftrightarrow B^*\phi^*(s)x = 0 \quad s \geq 0$$

Then $\mathcal{J}^*(\lambda)$ is well defined for Re $\lambda > \omega_o$ and $\mathcal{J}^*(\lambda)x = 0$

Conversely:

if $\mathcal{F}^*(\lambda)x = 0$ for Re $\lambda > \omega_o$ then, for arbitrary $\tilde{u} \in H_u$

$$\langle \tilde{u}, B^*R(\lambda, A^*)x\rangle = 0$$

Then

$$\int_0^{\infty} e^{-\lambda s} \langle \tilde{u}, B^*\phi^*(s)x\rangle ds = 0$$

for arbitrary $\tilde{u}$, or $B^*\phi^*(s)\, x = 0$, $s \geq 0$. ∎

<u>Remark:</u> The condition Re $\lambda > \omega_o$ in the lemma could be replaced by $\lambda \in \rho_o(A) \triangleq$ {the connected part of $\rho(A)$ containing the half plane Re $\lambda > \omega_o$}.

<u>Theorem 3</u>

$$x \in \bigcup_{t\geq 0} R(F_t)^{\perp} \quad \text{iff } x \in \sum_{n\in I^+} R(R(\lambda_o, A)^n B)^{\perp} \quad \text{for some}$$

$\lambda_o \in \rho_o(A)$.

<u>Proof:</u> From the lemma it follows that $x \in \bigcup_{t\geq 0} R(F_t)^{\perp} \Rightarrow$ for $\lambda_o \in \rho_o(A)$ $B^*R(\lambda_o, A^*)x = 0$ or for any $\tilde{u}$ of H_u, $\langle x, R(\lambda_o, A)B\tilde{u}\rangle = 0$.

Using

$$B^*R(\lambda, A^*)^n x = \int_0^{\infty} e^{-\lambda s} \frac{s^{n-1}}{n-1!} B^*\phi^*(s)x ds \quad n=1,2,3\ldots$$

and $B^*\phi^*(s)x = 0$, $s \geq 0$ it is clear that

$$\langle x, R(\lambda_o,A)^n B\tilde{u}\rangle = 0 \quad n=1,2,3\ldots, \text{ using } R(\lambda,A^*) = R(\bar{\lambda},A)^* .$$

Therefore, $x \perp \Sigma\, R[R(\lambda_o,A)^n B]$ for all $\lambda_o\, \varepsilon \rho_o(A)$. Conversely if this last statement is true for some $\lambda_o \varepsilon\ \rho_o(A)$ then $B^*R(\lambda_o,A^*)^n x=0$ $n=0,1,2,\ldots$ implies $B^*R(\lambda,A^*)x = 0$ for $\lambda\ \varepsilon\ \rho_o(A)$. Indeed

$$B^*R(\lambda,A^*) = B^*R(\lambda_o,A^*)\,\frac{1}{1-(\lambda_o-\lambda)R(\lambda_o,A^*)}$$

For $||R(\lambda_o,A^*)|| < \frac{1}{|\lambda-\lambda_o|}$ the series $\overset{\infty}{\Sigma}(\lambda_o-\lambda)^{n-1}B^*R(\lambda_o,A^*)^n$ converges and then clearly

$$B^*R(\lambda,A^*)x = \sum_{n=1}^{\infty} (\lambda_o-\lambda)^{n-1}B^*R(\lambda_o,A^*)^n x = 0$$

The lemma then states that $x\ \varepsilon \bigcup_{t\geq 0} R(F_t)^{\perp}$. ∎

Theorem 4

A necessary and sufficient condition for complete controllability is that

$$\sum_{n=0}^{\infty} R(R(\lambda_o,A)^n B)^{\perp} = \{0\} \quad \text{for some } \lambda_o\ \varepsilon\ \rho_o(A).$$

Proof: It was shown earlier that complete ε-controllability implies and is implied by $\bigcup_{t\geq 0} R(F_t)^{\perp} = \{0\}$. By Theorem 3 an equivalent expression is

$$\overline{\sum_{n=0}^{\infty} R(R(\lambda_o,A)^n B)} = H_x \quad \text{or} \quad \sum_{n=0}^{\infty} R(R(\lambda_o,A)^n B)^{\perp} = \{0\}\ .\ ∎$$

Remark: It follows trivially from Theorem 3 that the orthogonal spaces to the controllability spaces of the two following systems coincide

$$\dot{x} = Ax + Bu$$

$$\dot{x} = R(\lambda_o, A)x + Bu \qquad \lambda_o \in \rho_o(A) .$$

i.e.,

$$\bigcup_{t \geq 0} R(F_t)^{\perp} = \sum_{n=0}^{\infty} R(R(\lambda_o, A)^n B)^{\perp} .$$

The latter system however, does not involve unbounded operators.

2.7 On the Calculation of the Gap-Width for the Case $\sigma(A)$ is a Normal Spectrum.

The method developed in this section will make use of a special criterion for controllability for systems isomorphic to a countable direct sum of finite dimensional subsystems. Also, a slightly different definition of controllability is needed. However, for normal spectra equivalence with our former definition will be demonstrated.

2.7.1 Controllability when A has normal spectrum.

For finite dimensional systems the controllable set C is defined as the set of states for which it is possible to specify an admissible control to drive from x_1 at t_o to x_2 at some finite $t > t_o$, where x_1 and x_2 are arbitrary states in C.

For infinite dimensional systems there are in the literature two or three definitions for a controllable set. We will now write out these definitions, give the sets different names and make some comments as to how these sets compare.

Definition 4. (Set of attainability from the origin)

The set of controllable states is the set $\mathcal{A}$ of states x_d for which an admissible control exists driving the origin at time t_o to a point in the ε-neighborhood of x_d at time $t > t_o$. For the system under consideration

$$\mathcal{A} = \overline{\bigcup_{t \geq t_o} R(F_t)} \text{ where } F_t \triangleq \int_{t_o}^{t} \phi(t,\sigma)B.d\sigma .$$

Definition 5. (Reducible set)

The set of controllable states is the set D of states x_o such that some point in the ε-neighborhood of x_o can be driven, with an admissible control u, to 0 in some real time interval $[t_o,t]$ $t > t_o$. For the systems under consideration

$$D = \text{Closure of } \{x \in H_x / \phi(t,t_o)x + \int_{t_o}^{t} \phi(t,\sigma)Bu(\sigma)d\sigma = 0$$

$$\text{for some admissible } u \text{ and } t \geq t_o\}$$

Definition 6

The set of controllable states C is the set of states such that for an arbitrary pair x_1, $x_2 \in C$ there exists an admissible control u to drive from a point x_1' in the ε-neighborhood of x_1 to x_2' in the ε-neighborhood of x_2 in the finite time interval $[t_o,t]$, $t \geq t_o$.

These three definitions coincide obviously for finite dimensional systems and since the sets $\mathcal{A}$, D, C are then automatically closed (always under the assumption that the set of admissible controls is the whole space U) one can omit all "neighborhood" statements.

In general, if ϕ is invertible (e.g. when ϕ is a group, of which the finite dimensional case is a special case) the defined sets coincide.

It can be shown that $C \subset A \cap D$ and that C has some interesting invariance properties [V1]. The sets A and D are in general hard to compare. If the infinite dimensional system has normal spectrum then the restriction of semigroup ϕ to the rootspace τ_Λ of a finite symmetric set Λ of eigenvalues of A is a group from which it then follows that A_Λ, D_Λ, C_Λ for the subsystem

$$\dot{x} = A_\Lambda x + P_\Lambda Bu$$
$$x(0) = P_\Lambda x_o$$

where P_Λ is the appropriate Riesz integral, coincide. Then A, D, C coincide for the total system since they are obtainable as direct sums of components and the equality holds component-wise. In the sequel we may then use any of the definitions 4,5,6 (up to now only definition 4 was used.)

Next an easy lemma is stated on <u>complete</u> controllability.

<u>Lemma</u>: The system under consideration is completely controllable (def. 4) iff each spectral subsystem is completely controllable.

<u>Proof</u> : Consider the subsystems

$$\dot{x} = A_{\lambda_i} x + P_{\lambda_i} Bu$$

$$x(0) = P_{\lambda_i} x \ .$$

in space $P_{\lambda_i} H$ and with $\mathcal{Q}_{\lambda_i}$ being the set of attainable states from 0. The implication $\mathcal{A} = H_x \Rightarrow \mathcal{A}_{\lambda_i} = P_{\lambda_i} H_x$ is clear. Conversely, let $\mathcal{A}_{\lambda_i} = P_{\lambda_i} H_x$ for all i and let $y \in \mathcal{A}^{\perp}$. Then $P_{\lambda_i} y \in P_{\lambda_i}(\mathcal{A}^{\perp}) = (P_{\lambda_i}\mathcal{A})^{\perp}$. But $P_{\lambda_i}\mathcal{A} = \mathcal{A}_{\lambda_i} = P_{\lambda_i} H_x$. Hence $P_{\lambda_i} y = 0 \Rightarrow y = 0$. Hence $\mathcal{A} = H_x$. ∎

Let $P_{\lambda_i} B$ be denoted by B_i, $P_{\lambda_i} A \triangleq A_i$, τ_{λ_i} the rootspace of λ_i and $\mathfrak{Z}_{\lambda_i}$ the eigenspace of λ_i. Let $\mathfrak{Z}^*_{\lambda_i}$ denote the eigenspace of $\lambda_i \in \sigma(A^*) = \sigma(A)$. The following easy theorem is an extension of a result in ref. [S2].

Theorem 5. Let A have a normal spectrum.

Then the system under consideration is completely controllable iff $\mathcal{N}(B_i^*) \cap \mathfrak{Z}^*_{\lambda_i} = \{0\}$ for all $\lambda_i \in \sigma(A)$.

Proof : It suffices, according to the lemma, to concentrate on the condition for the subsystem in τ_{λ_i}.

Let $\tau_{\lambda_i} \overset{\text{isom.}}{=} R^n$. The Kalman test rank $[B_i \mid A_i B_i \mid \dots \mid A_i^{n-1} B_i] = n$ is equivalent to rank $[B_i \mid (\lambda_i I_i - A_i)B_i \mid \dots \mid (\lambda_i I_i - A_i)^{n-1} B_i] = n$ or

$$\mathcal{N}(B_i^*) \cap \dots \cap \mathcal{N}(B_i^*(\lambda_i I_i - A_i^*)^{n-1}) = \{0\} \qquad \text{(a)}$$

We want to show that the last statement is equivalent to

$$\mathcal{N}(B_i^*) \cap \mathcal{N}(\lambda_i I_i - A_i^*) = \{0\} \qquad \text{(b)}$$

If there exists an $x \, \{ {\in \tau_{\lambda_i} \atop \neq 0}$ such that $B_i^* x = 0$ and $(\lambda_i I_i - A_i^*)x = 0$ then obviously $B_i^*(\lambda_i I_i - A_i^*)^k x = 0$, $k \geq 0$, so that (a) $\Rightarrow$ (b). Conversely assume there exists an $x \, \{ {\in \tau_{\lambda_i} \atop \neq 0}$ such that $B^*(\lambda_i I_i - A_i^*)^k x = 0$ for $0 \leq k \leq n-1$. Since dim $\mathcal{N}(\lambda_i I_i - A_i^*)^p = n$ for some $p \leq n$, there is a $k_o \leq p$ such that $x \in \mathcal{N}(\lambda_i I_i - A_i^*)^{k_o}$ and $x \notin (\lambda_i I_i - A_i^*)^{k_o - 1}$. Let $z = (\lambda_i I_i - A_i^*)^{k_o - 1} x$, then $z \neq 0$ and $B_i^* z = 0$ and $(\lambda_i I_i - A_i^*) z = 0$. Therefore (b) $\Rightarrow$ (a). ∎

Summarizing the lemma and the theorem, one can again state that invertibility of $(P_{\lambda_i} B)^*$ on $\mathfrak{Z}_{\lambda_i}^*$ for each $\lambda_i \in \sigma(A)$ is a necessary and sufficient condition for controllability of the total system. Also, ~~we repeat that this~~ does not exclude the existence of a gap.

<u>Remark</u>: The condition $\mathcal{N}(B_i)^* \cap \mathfrak{Z}_{\lambda_i}^* = \{0\}$ for all i can also be written $\mathcal{N}(B^*) \cap \mathfrak{Z}_{\lambda_i}^* = \{0\}$ for all i or $B^*|_{\mathfrak{Z}_{\lambda_i}^*}$ is one-to-one for all i.

2.7.2 <u>On the calculation of the width of the controllability-gap.</u>

In this section an attempt is made to develop a method for explicitly specifying the control driving an initial condition (or a point in its ε-neighborhood) to 0 in finite time. Such control we know, does exist under the assumption of c.c. and the assumption of a normal spectrum. If a fixed interval $[0,T]$ is specified and the initial condition made arbitrary, the validity of the expression for u will tell if the gap is or is not "wider" than $[0,T]$. To check this validity it is necessary to have some idea of what minimal condition such u must satisfy.

Let $u(t)$, $t \in [0,T]$ be the control such that for any x_o' given, $x(T,0;x_o,u) = 0$ for $x_o \in N(\varepsilon, x_o')$. The trajectory $x(t)$ may be considered as belonging to $L^2(0,T)$ for $t \in [0,T]$, and $x(t) \equiv 0$

for $t \notin [0,T]$. This implies something about the Fourier transform $\tilde{x}(\lambda)$ of $x(t)$ namely (Paley-Wiener theorems) :

- $\tilde{x}(\lambda)$ is an entire function
- $\tilde{x}(\lambda)$ is a subsine function of type T
- $\tilde{x}(\lambda)$ has some special asymptotic behaviour and is square-integrable on certain lines. (Rudin, [R1])

We state a special version of the Paley-Wiener theorems using Laplace transforms:

<u>Statement 1</u> (Theorem by P.E. Pfeiffer, [P1])

Let $\tilde{x}(\lambda)$ denote the unilateral Laplace transform of a function $x(t)$. Then $x(t)$ vanishes for $t \notin [0,T]$ $T \geq 0$ iff

1) $\tilde{x}(\lambda)$ is entire
2) $|\tilde{x}(\lambda)| < Me^{|\mathrm{Re}\lambda|T}$
3) $\tilde{x}(\lambda) \to 0$ for $\lambda \to \infty$ along any path in $\lambda \geq \lambda_o$, for any $\lambda_o > -\infty$.

As for the control function $u(t)$, we will assume always that such u exists i.e. that the system is c.c. For a system with normal spectrum this means that $B^*\restriction_{\mathcal{J}^*_{\lambda_i}}$ is invertible for all $\lambda_i \in \sigma(A)$. This invertibility condition obviously has provided us already with a necessary condition on the dimension of u. For example, if $n = \sup_i\{\nu_{\lambda_i}\}$, where ν_{λ_i} is the geometric multiplicity of eigenvalue λ_i then n is a lower bound on the dimension of u.

Consider the Laplace transform of the variation of constants formula

$$x(t) = \phi(t,0)x_o + \int_0^t \phi(t-\sigma)Bu(\sigma)d\sigma.$$

namely $x(s) = R(s,A)x_o + R(s,A)Bu(s)$ (5)

If $x(t) \equiv 0$ outside $[0,T]$ we have seen that $x(s)$ must be entire; the righthand side of expression (5) therefore must be entire by right choice of $u(t)$. In case of normal spectrum, $\sigma(A)$ contains no essential singularities and $R(s,A)$ has only poles. In a small enough neighborhood of an arbitrary $\lambda \in \sigma(A)$, the Laurent series expansion for $R(s,A)$, $s \in \rho(A)$ therefore has only a finite number of negative terms. (see Chapter I.2.3, expecially formulae (5), (6), (7), and (8)).

$$R(s,A) = \sum_{n=1}^{m_i} (s-\lambda_i)^{-n} B_n^i + \sum_{n=0}^{\infty} (s-\lambda_i)^n A_n^i \quad . \tag{6}$$

This expansion is valid on $G_i \triangleq \{s: |s-\lambda_i| = \delta\}$ where δ is chosen such that $G_i \cap [\sigma(A)-\{\lambda_i\}] = \phi$. A_n^i and B_n^i are bounded operators and satisfy the recurrence relations (8) of Chapter I.2.3.

We would like to obtain the general form of expansion

$$x(s) = \frac{X_{m_i}^i}{(s-\lambda_i)^{m_i}} + \dots \frac{X_1^i}{(s-\lambda_i)} + X_o^i + f^i(s) \tag{7}$$

where $f^i(s)$ is an entire function. In a similar way to (6) write

$$R(s,A)Bu(s) = \sum_{n=1}^{m_i} \frac{\tilde{B}_n^i}{(s-\lambda_i)^n} + \sum_{n=0}^{\infty} \frac{\tilde{A}_n^i}{(s-\lambda_i)^{-n}} \tag{8}$$

where

$$\tilde{B}^i_{m_i} = (s-\lambda_i)^{m_i} R(s,A)Bu(s)\Big|_{s=\lambda_i} = B^i_{m_i} Bu^{(0)}(\lambda_i)$$

$$\tilde{B}^i_{m_i-1} = \frac{d}{ds} (s-\lambda_i)^{m_i} R(s,A)Bu(s)\Big|_{s=\lambda_i} = B^i_{m_i} Bu^{(0)}(\lambda_i)+B^i_{m_i} Bu^{(1)}(\lambda_i).$$

. . . .

$$\tilde{B}^i_n = \frac{1}{m_i-n!} \frac{d^{m_i-n}}{ds^{m_i-n}} (s-\lambda_i)^{m_i} R(s,A)Bu(s)\Big|_{s=\lambda_i}$$

$$= \sum_{j=0}^{m_i-n} B^i_{m_i-j} Bu^{(m_i-n-j)}(\lambda_i) \tag{9}$$

. . . .

$$B^i_1 = \sum_{j=0}^{m_i-1} B_{m_i-j} Bu^{(m_i-j-1)}(\lambda_i)$$

Therefore

$$X^i_{m_i} = B^i_m x_o + B^i_{m_i} Bu^{(0)}(\lambda_i)$$

$$\vdots$$

$$X^i_n = B^i_n x_o + \sum_{j=0}^{m_i-n} B_{m_i-j} Bu^{(m_i-n-j)}(\lambda_i) \tag{10}$$

$$\vdots$$

$$X^i_1 = B^i_1 x_o + \sum_{j=0}^{m_i-1} B^i_{m-j} Bu^{(m_i-1-j)}(\lambda_i)$$

Entirety of $x(s)$ requires

$$X^i_n = 0 \text{ for } n=1,2,\ldots,m_i \text{ and all } \lambda_i \in \sigma(A). \tag{11}$$

It will be shown that the system of equations (11) can be solved for the coefficients $u^{(0)}(\lambda_i),\ldots,u^{(m-1)}(\lambda_i)$ if the system is completely controllable. One should realize that $\mathcal{T}_{\lambda_i}$ is finite dimensional. Each of the equations in (11) is a vector equation. We state a theorem and a couple of examples.

Theorem 6

Assume $H_u = R^p$, $p < \infty$. The system of equations (11) is solvable for the coefficients $\underline{u}^{(j)}(\lambda_i)$ for $j = 0,1,\ldots,m_i-1$ and for all i, $\lambda_i \varepsilon \sigma(A)$ if the system $\dot{x} = Ax + Bu$ is completely controllable.

Proof : The proof makes use of the results expressed in Theorem 5 and the remark following it. It was already observed before that the assumption of c.c. implies $p \leqslant \sup_i \nu_i$ where ν_i denotes the geometric multiplicity of $\lambda_i \varepsilon \sigma(A)$.

Consider the first equation of system (11). A sufficient condition for

$$B^i_{m_i} B u^{(0)}(\lambda_i) + B^i_{m_i} x_o = 0$$

to be solvable for $u^{(0)}(\lambda_i)$ would be

$$R(B^i_{m_i}) \subset R(B^i_{m_i} B)$$

Clearly $R(B^i_{m_i}) \supset R(B^i_{m_i} B)$ is always true; so that the condition in fact is

$$R(B^i_{m_i}) = R(B^i_{m_i} B) \quad \text{or} \quad \mathcal{N}[(B^i_{m_i} B)^*] \cap R(B^i_{m_i}) = \{0\}$$

Since $R(B^i_{m_i}) = (A-\lambda_i)^{m_i-1} P_{\lambda_i} H_x$ and $(A-\lambda_i I)^{m_i} P_{\lambda_i} = 0$ one has immediately that $R(B^i_{m_i}) = \mathcal{N}(A-\lambda_i I)$. It may then be checked that $\mathcal{N}(B^*B^{i*}_{m_i}) \cap \mathcal{N}(A-\lambda_i I) = \{0\}$ if $B^*|_{\mathcal{N}(A^*-\lambda_i I)}$ is one-to-one.

The last condition is equivalent to c.c. as expressed in Theorem 5. Consider next the equation for $u^{(1)}(\lambda_i)$:

$$B^i_{m_i-1}\, x_o + B^i_{m_i} Bu^{(1)}(\lambda_i) + B^i_{m_i-1} Bu^{(0)}(\lambda_i) = 0$$

or

$$B^i_{m_i-1} B(B^i_{m_i} B)^{-1} B^i_{m_i} x_o - B_{m_i-1} x_o = B^i_{m_i} Bu^{(1)}(\lambda_i)$$

Since by c.c. $B^i_{m_i} B$ is invertible on $R(B^i_{m_i}) = \mathcal{N}(A-\lambda_i I)$ it would suffice to prove that the lefthand side of the last equation is an element of $\mathcal{N}(A-\lambda_i I)$ or $(A-\lambda_i I)[B^i_{m_i-1} x_o - B^i_{m_i-1} B(B^i_{m_i} B)^{-1} B^i_{m_i} x_o] = 0$. But $(A-\lambda_i I)B^i_{m_i-1} = B^i_{m_i}$, which makes the equality obvious. The same recursive reasoning goes through for the other coefficients $\underline{u}^{(j)}(\lambda_i)$ for the index j.

Since in this reasoning the index i was arbitrary we have shown that c.c. allows us to solve for all $\underline{u}^{(j)}(\lambda_i)$ $j=0,\ldots,m_i-1$ and all i, $\lambda_i \in \sigma(A)$. ∎

Example 1. Consider a system with geometric multiplicities of all $\lambda_i \in \sigma(A)$ equal to 1. Complete controllability implies then that the dimension of u be at least 1. Consider a system with $u(\lambda)$ a scalar function of λ, i.e. $H_u = R$. Then $B_{m_i} x_o + B_{m_i} bu^{(0)}(\lambda_i) = 0$ has a solution iff B_{m_i} and $B_{m_i} b$ are multiples of each other. It may be checked out that $B_{m_i} b$ is a scalar. Therefore

$$u^{(0)}(\lambda_i) = \frac{B_{m_i} x_o}{B_{m_i} b}$$

$$u^{(1)}(\lambda_i) = (B_{m_i} b)^{-1}(B_{m_i-1} x_o + B_{m_i-1} b (B_{m_i} b)^{-1} b x_o)$$

etc...

Thus the set $u^{(j)}(\lambda_i)$, $j=0,\ldots,m_i-1$ is well defined. Since equations (11) involve only the system in τ_{λ_i} we will write out a little numerical example.

$$\text{Let } P_{\lambda_i} A = A_1 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \text{ and } P_{\lambda_1} b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ with } b_3 \neq 0$$

$$\text{Then } B_3^1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2^1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } B_1^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The defining equations are:

$$\begin{bmatrix} x_{03} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} b_3 \\ 0 \\ 0 \end{bmatrix} u(\lambda_1) = 0$$

$$\begin{bmatrix} x_{02} \\ x_{03} \\ 0 \end{bmatrix} + \begin{bmatrix} b_2 \\ b_3 \\ 0 \end{bmatrix} U(\lambda_1) + \begin{bmatrix} b_3 \\ 0 \\ 0 \end{bmatrix} u^{(1)}(\lambda_1) = 0$$

$$\begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} + \begin{bmatrix} b_2 \\ b_3 \\ 0 \end{bmatrix} u^1(\lambda_1) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u(\lambda_1) + \begin{bmatrix} b_3 \\ 0 \\ 0 \end{bmatrix} u^{(2)}(\lambda_1) = 0$$

with solutions

$$u(\lambda_1) = -\frac{x_{03}}{b_3}$$

$$u^{(1)}(\lambda_1) = -\frac{x_{02} - \frac{b_2}{b_3} x_{03}}{b_3}$$

$$u^{(2)}(\lambda_1) = -\frac{x_{03} + b_2 u^{(1)}(\lambda_1) + b_1 u(\lambda_1)}{b_3}$$

The case of geometric multiplicity greater than 1 is a little bit more complicated. However the condition that $B^*_{\Gamma_{3\lambda_i}}$ is invertible allows us to solve for the coefficients $u^{(j)}(\lambda_i)$.

Example 2

Consider the example

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix}, \text{ with } b_3 b_6 - b_5 b_4 \neq 0$$

Observe that $B_1 = P_{\lambda_1} = I$, and $B_2 = (\lambda_1 I - A)$. Let $H_u = R^2$.

1) $B_2 \underline{x}_o + B_2 b\underline{u}(\lambda_1) = 0$ becomes $x_{20} + b_3 u_1 + b_4 u_2 = 0$ (12)

2) $B_1 \underline{x}_o + B_2 B\underline{u}^{(1)}(\lambda_1) + B_1 B\underline{u}^{(0)}(\lambda_1) = 0$ becomes

$$\begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} b_3 & b_4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = 0 \qquad (13)$$

As we see equation (12) is repeated in (13).

$$\begin{cases} \begin{bmatrix} u_1(\lambda_1) \\ u_2(\lambda_1) \end{bmatrix} = \begin{bmatrix} b_3 & b_4 \\ b_5 & b_6 \end{bmatrix}^{-1} \begin{bmatrix} x_{20} \\ x_{30} \end{bmatrix} \\ b_3 u_1^{(1)} + b_4 u_2^{(1)} + b_1 u_1 + b_2 u_2 + x_{10} = 0 \end{cases} \qquad (14)$$

Choose for example $u_2^{(1)}(\lambda_1) = 0$ if $b_3 \neq 0$ and $u_1^{(1)}(\lambda_1) = 0$ if $b_4 \neq 0$. (This is always possible since by virtue of controllability b_3 or b_4 is nonzero and solve equations (14). Observe that solutions are not necessarily unique. ▮

If the interval [0,T] is such that $Me^{T|\mathrm{Re}\lambda|} < u_{\lambda_i}^{(0)}$ for some i, then one has clearly that the gap-width is greater than T, since in this case $u(\lambda)$ cannot be a subsine function of type T. Herein $M = \int_0^T |u(t)|dt$. Unfortunately this condition is difficult to check if no such M is imposed a priori on the problem.

Similarly if λ_i and λ_j are close then $u_{\lambda_i}^{(0)}$ and $u_{\lambda_j}^{(0)}$ must be close. Indeed

$$u(\lambda_i) - u(\lambda_j) = \int_{\lambda_i}^{\lambda_j} u'(\lambda)d\lambda \qquad , u'(\lambda) \triangleq \frac{du}{d\lambda}$$

so that $|u(\lambda_i) - u(\lambda_j)| \leq |\lambda_i - \lambda_j| N$

where N is the maximum of u' along the straight line path joining λ_i and λ_j. On the other hand from the Cauchy integral formula

$$u'(z) = \frac{1}{2\pi i} \int_{|\xi|=R} \frac{u(\xi)}{(\xi - z)^2} d\xi$$

and putting $R = 2|z|$ let's say gives $u'(z) < \frac{2M}{|z|} e^{2T|z|}$ so that N can be evaluated. Similar expressions can be written for higher derivatives.

One has that $u(t) = \mathcal{L}^{-1}(\tilde{u}(\lambda))$, if one can find the explicit expressions for $\tilde{u}(\lambda)$ satisfying the conditions of Statement 1 in this paragraph and such that

$$\tilde{u}^{(0)}(\lambda)\big|_{\lambda=\lambda_i} \triangleq u_K$$

$$\tilde{u}^{(1)}(\lambda)\big|_{\lambda=\lambda_i} \triangleq u_K'$$

$$\vdots$$

$$\tilde{u}^{(m_k-1)}(\lambda)\big|_{\lambda=\lambda_i} \triangleq u_k^{m_k-1} \quad \text{for } k=1,2,\ldots$$

and $u_k,\ldots,u_k^{m_k-1}$ denote the solutions to system (11). Explicit reconstruction formulae for $u(\lambda)$ for normal spectra and especially when allowing $m_i > 1$ are hard to come by and are under investigation. Such formula would be extremely useful not only for solving the gap-width problem but in time-optimal, minimum energy problems and others. The case of time-optimal controls for normal spectrum restricted to the imaginary line was treated in [G6]. Such reconstruction formula has been found in case H_x is finite dimensional (which is of not too much interest in our present case since we know that then $\Gamma = 0$.) and in a case of an infinite dimensional oscillator. This case is presented next as an illustration of the method and confirms results obtained earlier by Russell [R2, R3].

Observe that in these references only simple spectra on the imaginary line of the complex plane are considered and a specific expression for A, B and the boundary conditions is assumed given. This makes it substantially less general and complicated, but allows direct specification of the gap-width Γ as a function of the minimum distance between the eigenvalues. [R2]

Example (Infinite dimensional oscillator)

Consider the system described by $\dfrac{\partial^2 x}{\partial t^2} = \dfrac{\partial^2 x}{\partial z^2}$

$$z \in [0,\pi] \qquad t \in [0,T] \tag{15}$$

with initial conditions $x(z,0) = x_0(z)$, $\dot{x}(z,0) = x_1(z)$, and boundary condition $x(\pi,t) = -u(t)$, $x(0,t) = 0$.

This is a system with normal spectrum and geometric and algebraic multiplicities equal to 1. One can obtain the subsystems using eigenfunction expansions. Let $\{\psi_k\}$ be the set of eigenfunctions of the adjoint system for the boundary conditions at 0 and π equal to zero. Then the system with homogeneous boundary conditions is in fact self adjoint as seen from .

$$\int_0^\pi \frac{\partial^2 x(z)}{\partial z^2}\, y(z)dz = \frac{\partial x}{\partial z}\, y \Big|_0^\pi - x\,\frac{\partial y}{\partial z}\Big|_0^\pi + \int_0^\pi x\,\frac{\partial^2 y}{\partial z^2}\, dz$$

Then $\{\psi_k(z)\} = \{\sin kz\}$ for $k=1,2,\ldots$ is the set of eigenfunctions and $\{\lambda_k\} = \{-k^2\}$ the eigenvalues.

Taking the inner product of both sides of (15) with the eigenfunctions $\psi_k(z)$ and defining $x_k(t) \triangleq \int_0^\pi x(t,z)\phi_k(z)dz$ one obtains

$$\ddot{x}_k(t) = \int_{0}^\pi \frac{\partial^2 x}{\partial z^2}\,\psi_k(z)dz = -u(t)\,\frac{d\psi_k(z)}{dz}\Big|_{z=\pi} + \int_0^\pi x(z)\,\frac{d^2\psi_k(z)}{dz^2}\,dz$$

$$= -u(t)\psi_k'(\pi) - k^2 x_k(t)$$

$$= -k^2 x_k(t) + (-1)^k k u(t) \qquad (16)$$

This equation describes the evolution of the k^{th} subsystem.

Let $x_o(z)$ and $x_1(z)$ be equally decomposed in

$$x_o(z) = \sum_{k=0}^{\infty} x_{ok}\phi_k(z) \qquad \phi_k(z) = \psi_k(z)$$

$$x_1(z) = \sum_{k=0}^{\infty} x_{1k}\phi_k(z)$$

The initial condition for system (16) reads then

$$\begin{cases} x_k(o) = x_{ok} \\ \dot{x}_k(o) = x_{1k} \end{cases} \qquad (17)$$

Taking the Fourier transform of (16) and (17)

$$-\omega^2 x_k(\omega) - j\omega x_{ok} - x_{1k} = -k^2 x_k(\omega) + (-1)^k k u(\omega)$$

or

$$x_k(\omega) = \frac{1}{k^2-\omega^2}\ [j\omega x_{ok} + x_{1k} + (-1)^k k u(\omega)]$$

Entirety requires

$$j\omega x_{ok} + x_{1k} + (-1)^k k u(\omega)\Big|_{\omega=\pm k} = 0$$

or

$$\begin{cases} u(k) = (-1)^{k+1}[j x_{ok} + \frac{x_{1k}}{k}\ ] \\ u(-k) = (-1)^{k+1}[-j x_{ok} + \frac{x_{1k}}{k}] \end{cases}$$

The interpolation formula, and the gap-width is given by an inverse application of the sampling theorem well known in system theory, as follows: There is given in the frequency domain a discrete countable sample set, equally spaced, with anti-symmetric imaginary and symmetric real parts (This is necessary for the corresponding time function to be real). One wants to define the minimum T such that these coefficients specify $u(t)$, $t \in [0,T]$. The sample theorem applies usually to a band limited signal and sampling in time. We have here the roles reversed.

$$U(j\omega) = \frac{1}{2\pi}\int_0^T u(t)e^{-j\omega t}dt$$

$$u(t) = \sum_{-\infty}^{+\infty} c_n e^{jn\omega_s t} \text{ with } c_n = \frac{1}{T}\langle u(t), e^{-jn\omega_s t}\rangle = \frac{2\pi}{T}U(n\omega_s) \;;\; \omega_s = \frac{\pi}{T/2} = 1.$$

or $$u(t) = \sum_{-\infty}^{+\infty} \frac{2\pi}{T} U(n\omega_s)e^{jn\omega_s t}$$

Then $$U(j\omega) = \frac{1}{T}\sum_{-\infty}^{+\infty} U(n\omega_s)\int_0^T e^{jn\omega_s t}e^{-j\omega t}dt$$

$$= \frac{1}{T}\sum_{-\infty}^{+\infty} U(n\omega_s).\ \frac{e^{j(n\omega_s-\omega)T}-1}{j(n\omega_s-\omega)}$$

This sample-interpolation formula allows us to find

$$u(t) = \int_{-\infty}^{+\infty} U(j\omega)e^{j\omega t}d\omega\ .$$

From the sample-theorem it follows that the minimum time for these sample points to define $u(t)$ over $[0,T]$ is for $\underline{T = 2\pi}$.

This is twice the time necessary for the wave to travel with its propagation speed c (here normalized to 1) twice the length of the system. Thus results confirm the findings of Russell [R2].

Remark : It may be observed that the expression for u(t) obtained is in fact the control u(t) satisfying the necessary requirements and in addition has minimum norm, so that the interpolation formula may solve other problems than specifying the width. These applications are under investigation.

Conclusion

In this chapter an attempt has been made to introduce a new notion in connection with control for infinite dimensional systems, namely the notion of a controllability gap. Two conditions have been specified under which such gaps cannot occur: if ϕ is analytic, or if $B^{-1}(R(B) \cap Q)$ is dense in H_u; in this expression Q denotes the set of analytic vectors of ϕ. In search for the simplest infinite dimensional system where such phenomenon may occur we considered infinite dimensional systems isomorphic to a countable direct sum of finite dimensional systems. A method is suggested which would provide an explicit expression for the control involved in reducing a point in the neighborhood of an arbitrary initial state to 0 and allowing some conclusions as to the width of the gap. It is suggested that this method may be of great interest in related applications. The theorems 1-5 moreover state some potentially useful necessary and sufficient conditions for complete controllability.

CHAPTER III

OBSERVABILITY AND RELATED NOTIONS

3.1. Introduction

The general aim pursued here is to reconstruct the state of a system from observations of the output over a certain time interval. For distributed parameter systems observations are (usually) made via a finite number of sensors, having their own dynamics and placed at "strategic" locations not just to make it possible to deduce the "state" from the observations but to make this operation least sensitive to errors in the readings. These readings are usually a weighted average of the state over a limited region of the spatial domain. The sensor or transducer in general will represent a non-linear time varying mapping from state to output. We will not incorporate these complications in our model but only consider linear, bounded, memoryless and, where stated, time-invariant mappings. To justify this neglect one can state that this approach is valid insofar as the transducer dynamics are high-frequency effects as compared to the system dynamics. Nonlinearities, hysteresis, lags, can to a certain degree be taken care of in the modelling phase by linearization around some nominal operating curve.

For finite dimensional systems the study of observability is summarized in the Kalman criterion: a system in R^n, $\dot{x} = Ax + Bu$, $x(0) = x_o$, $y = Cx$ where $y \in R^m$ is observable if rank $[C' \vdots A'C' \dots \vdots A'C'] = n$.

The main usefulness of the property of observability lies in the fact that it guarantees full access to the state through output measurements, i.e., the state to output mapping is "in a sense" invertible. This knowledge of the state via the output can then be used in a feedback control law to "optimize" the behavior of the system or "steer it" in a desired way.

If then state-reconstruction is desired only the state-component in R(C) is "directly" accessible. If the pair (A, C) is observable one can [L3] identify the remaining part of the state by constructing an $n-m^{th}$-order dynamical system. Moreover, this "observer" may be given completely arbitrary spectrum in the class of symmetric sets of (n-m) complex numbers. If measurement noises are taken into account, state-reconstruction is usually identified with Kalman-filtering.

For F.D.S. observability means that a finite set of values can be determined, e.g., for $H_x = R^n$ specification of x_o means specification of n numbers. For D.P.S. x_o is a function over a spatial domain. Specification of x_o means that a (countable or uncountable) infinite set of values have to be obtained. If one restricts a priori the state space to the span of a finite number of eigen-functions of A then the D.P.S. is approximated (well or not well) by a F.D.S. which may or may not be observable depending on the number and placements of the sensors. Propagation of information over the space-time domain may be with finite speed for D.P.S. Lumped systems can be thought of as having infinite speed of propagation. It is in this context that the notion of "observability-gap" can be introduced. This concept expresses the fact that there may be a "waiting period" equal to the gap-width before the approximate system is observable, a fact that is not "visible" from the F.D. model used.

3.2. Mathematical Model - Observability

The mathematical model from Chapter I is used, i.e., the system is represented by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ x(0) &= x_o \varepsilon D_o(A) \\ y &= Cx \end{aligned} \tag{1}$$

where A: $D_o(A) \subset H_x \to H_x$

B: $H_u \to H_x$

C: $H_x \to H_y$

H_x, H_u, H_y are Hilbert spaces. A, B(t), C(t) are linear operators, B and C are bounded. A is time-independent and assumed to be the infinitesimal generator of a strongly continuous semigroup $\{\phi\}$ of bounded operators. If $x_o \varepsilon D_o(A)$ the solution to $\dot{x} = Ax$, $x(0) = x_o$ is $x(t) = \dot{\phi}(t)x_o$ in the sense that $\dot{\phi}(t)x_o = A\,\phi\,(t)x_o = \phi(t)Ax_o$

$||\,x(t) - x_o\,|| \to 0$ for $t \to 0^+$

If system (1) is considered over a time inter al $[0,T]$, minimal assumptions on the nonhomogeneous term are that $< x, (Bu)(t) >$ is a measurable function in t on $[0,T]$ for all $x \in H_x$ and $u \in L^2([0,T]; H_u)$, in which case the solution has the form $x(t) = \phi(t)x_o + \int_0^t \phi(t-\sigma)(Bu)(\sigma)d\sigma$ satisfying (1) in the sense that

$$\frac{d}{dt} < x(t), y >_{H_x} = < x(t), A^*y>_{H_x} + < Bu(t), y >_{H_x} \qquad (2)$$

a.e. in $[0,T]$, all $y \in D_o(A^*)$
and that $||x(t) - x_o|| \to 0$ for $t \to 0^+$.

<u>Definition 1.</u>

System (1) is called observable if $C(t)\phi(t)\, x_o = 0$, $t > 0$ implies $x_o = 0$. It is observable on $[0, T]$ if $C(t)\phi(t)\, x_o = 0$, $t \in [0, T] \Rightarrow x_o = 0$. ▮

The problem of observability is that of finding x_o given the output $y(t)$, $t > 0$ and the input $u(t)$, $t > 0$. Then $z(t) = y(t) - C(t) \int_0^t \phi(t-\sigma)\, Bu(\sigma)d\sigma = C(t)\phi(t)\, x_o$; $z(t)$ is the known term. Thus x_o will be given uniquely if the system is observable.

Define $L_t : H_x \to L^2(0, t; H_y)$ by

$$L_t \triangleq C(t)\phi(t). \qquad (3)$$

Then $L^*_t : L^2(0, t; H_y) \to H_x$ is given by

$$L^*_t \cdot = \int_0^t \phi^*(\sigma)\; C^*(\sigma).\, d\sigma \qquad (4)$$

and $$(L^*_t L_t) = \int_0^t \phi^*(\sigma)C^*(\sigma)C(\sigma)\phi(\sigma)d\sigma \tag{5}$$

Theorem 1.

System (1) is observable on $[0, T]$ if one of the following equivalent conditions is satisfied:

1. $N(L_T) = \{0\}$ (*)

2. $N(L^*_T L_T) = \{0\}$

3. $\overline{R(L^*_T)} = \overline{R(L^*_T L_T)} = H_x$

4. The operator $L^*_T L_T$ is positive. ∎

Corollary

If the system is observable on $[0, T_1]$, it is observable on $[0, T_1]$, $T_1 > T_1$.

Proof: This follows trivially from the fact that

$$N(L_{T_1}) \subset N(L_{T_1}),\ T_1 < T_1$$ ∎

Scholium 1

The operator $F^*_T \triangleq \int_0^T \phi^*(s)C^*(s) \cdot ds$ is compact in $L^2(0, T_1; H_x)$, $T \in [0, T_1]$

Proof: Let $s = T-\sigma$ in the expression of F^*_T. Then for some $M, \omega_o > 0$, $\int_0^{T_1}\int_0^{T_1} ||\phi^*(T-\sigma)C^*(T-\sigma)||^2 \ d\sigma dT \leq$

(*) N( ) in this section stands for Nulspace of the operator in parenthesis.

$$\max_{\sigma\varepsilon[0,\, T_1]} \quad ||C^*(\sigma)||^2 \int_o^{T_1}\int_o^{T_1} M^2 e^{2\omega_o(T-\sigma)} d\sigma dT \quad \leq$$

$$\max_{\sigma\varepsilon[0,\, T_1]} \quad ||C^*(\sigma)||^2 \frac{M^2}{(2\omega_o)^2}(e^{2\omega_o T} - 1) \quad < \infty$$

Hence F^*_T is a Hilbert-Schmidt operator and is therefore compact. ▮

Remark: The scholium does not say that the operator $L^*_T : L^2(0, T; H_x) \to H_x$ is compact. This would imply that $(L^*_T)^* = C(t)\phi(t)$. were compact which in general is not true. It is true if case A has a normal spectrum. This fact will be elaborated upon in the next paragraph.

3.3 On the Recuperation of the Initial Function; the Existence of $(L^*_T L_T)^{-1}$.

Since $L^*_T L_T$ is symmetric, bounded and at least positive semi-definite, $\sigma(L^*_T L_T) \subset [0, ||L^*_T L_T||]$. The recuperability of the initial function is related to the invertibility of $L^*_T L_T$ as expressed in the following scholium.

Scholium 2.

If the system is observable on $[0, T]$ then

$$x_o = (L^*_T L_T)^{-1} L^*_T y \tag{6}$$

Proof: Since $L^*_T L_T$ is bounded, symmetric and at least positive semi-definite, its spectrum is a subset of a compact set of R^+, i.e., $\sigma(L^*_T L_T) \subset [0, ||L^*_T L_T||]$.

If the system is observable on $[0, T]$, theorem 1 says that $R(L^*_T L_T)$ is dense in H_x. Hense $(L^*_T L_T)^{-1}$ exists (at least as an unbounded operator), and $(\varepsilon I + L^*_T L_T)^{-1}$ is a bounded operator whose limit for $\varepsilon \to 0$ is $(L^*_T L_T)^{-1}$. Hence, $L_T x_o = y$ of $L^*_T L_T x_o = L^*_T y$. Clearly $L^*_T y \in D_o((L^*_T L_T)^{-1})$ so that (6) holds, even if the inverse operator is unbounded. ∎

It is interesting to elaborate on the invertibility. It will be easily shown that if A has a normal spectrum and with the usual assumptions of completeness of rootvectors that $L^*_T L_T$ is compact and an expression for the inverse can be given easily.

Let A have a normal spectrum and a complete set of rootvectors in the (infinite dimensional) separable Hilbert space H_x. Let the rootspace of $\lambda_i \varepsilon \sigma(A)$ be $M_i(A)$ with basis $\Phi_i = [\phi_{i,1} \cdots \phi_{i,m_i}]$, where m_i is the dimension of $M_i(A)$. Similarly, let $M_j(A^*)$ be the rootspace of $\lambda_j \varepsilon \sigma(A^*)$ with basis $\Psi_j = [\psi_{j,1} \cdots \psi_{j,m_j}]$. Then clearly Ψ_j is orthogonal to Φ_i for $i \neq j$ and for $i = j$, $(\Psi_i, \Phi_i) \neq 0$ (1). We assume the basis elements normalized so that

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(1)

$$(\Psi_i, \Phi_i) = \begin{bmatrix} \langle \psi_{i,1}, \phi_{i,1} \rangle_{H_x} & \cdots & \langle \psi_{i,1}, \phi_{i,m_i} \rangle_{H_x} \\ \vdots & & \vdots \\ \langle \psi_{i,m_i}, \phi_{i,1} \rangle_{H_x} & \cdots & \langle \psi_{i,m_i}, \phi_{i,m_i} \rangle_{H_x} \end{bmatrix}$$

$(\Psi_i, \Phi_i) = I$, I being the identity on R^{m_i} spanned by basis Φ_i.

Let $A_i \triangleq A\restriction_{M_i(A)}$ and $(\Psi_i, y)_{H_x} \triangleq \begin{bmatrix} <\psi_{i,1}, y>_{H_x} \\ \cdot \\ \cdot \\ <\psi_{i,m_i}, y>_{H_x} \end{bmatrix}$

Then the solution $x(t)$ of $\dot{x} = Ax$, $x(0) = x_o$ can be written as

$$x(t) = \sum_{i=0}^{\infty} \Phi_i \, (\Psi_i, x(t)) = \sum_{i=0}^{\infty} \Phi_i \, (\Psi_i, e^{At} x_o) \; ^{(1)}$$

$$= \sum_{i=0}^{\infty} \Phi_i \, e^{A_i t} \, (\Psi_i, x_o).$$

Let $T_N(t) x_o \triangleq \sum_{i=0}^{N} \Phi_i e^{A_i t} \, (\Psi_i, x_o)$

Clearly $T_N : H_x \rightarrow \underset{\oplus\, i=1}{\overset{N}{\Sigma}} M_i$ is compact since its rangespace is finite dimensional. Since $T_N(t) \rightarrow e^{At}$ uniformly on compact subsets of $(0, \infty)$, e^{At} is compact. $C(t)$ being bounded $C(t)e^{At} \triangleq L_T$, $t\varepsilon[0, T]$, is compact. But then so are also L^*_T and $L^*_T L_T$. If $L^*_T L_T$ is compact, $0 \ \varepsilon \ \rho(L^*_T L_T)$ [T_1, p. 286, exercise 9]. Since $L^*_T L_T$ is selfadjoint and bounded, its spectrum is a subset of the real line, in fact $\sigma(L^*_T L_T) \subset [0, ||L^*_T L_T||]$.

---

(1) We use the symbolic expression e^{At} for the semigroup $\phi(t)$ of A. in order to avoid confusion with the basis vectors ϕ_i. If A_i is a restriction of A, then the restriction of $\phi(t)$ to the same subspace is $(\phi(t))_i = \phi_i(t)$ where $\phi_i(t)$ is the semigroup generated by A_i or, written symbolically, $\phi_i(t) = e^{A_i t}$.

The whole question of observability revolves around 0 being an eigenvalue or not. If 0 is an eigenvalue the corresponding rootspace represents unobservable states. If 0 is not an eigenvalue it is an accumulation point of the spectrum. It constitutes, in fact, the continuous spectrum. In that case $(L^*_T L_T)^{-1}$ exists as in unbounded operator on a dense domain. In fact, if $\Phi_i = [\phi_{i,1}, \dots \phi_{i,m_i}]$ is a basis for the rootspace of $\lambda_i \varepsilon\ \sigma(L^*_T L_T)$ then

$$(L^*_T L_T)^{-1} = \sum_{i=0}^{\infty} \frac{1}{\lambda_i} \Phi_i (\Phi_i, .) \tag{7}$$

where $\lambda_o = ||L^*_T L_T||$ and $\lambda_o > \lambda_1 > \lambda_2 \dots \to 0$.

This expression clearly reflects the unboundedness of the inverse. But since $-\varepsilon \notin \sigma(L^*_T L_T)$, $\varepsilon > 0$, $(\varepsilon I + L^*_T L_T)^{-1}$ is bounded and has (7) as its limit for $\varepsilon \to 0$.

3.4. Differential Observability and the Observability-Gap

From definition 1 and theorem 1 it followed that observability is equivalent to

$$\bigcap_{t \geq 0} N(L_t) = \{0\} \tag{8}$$

The question here is if the intersection over $t \varepsilon [0,\infty)$ can be replaced by intersection over $[0, t_1]$ for some $t_1 > 0$. If t_1 cannot be chosen arbitrarily small but (9) holds, we will say that <u>there is an</u>

observability-gap. If t_1 may be chosen arbitrarily small, ie. $t_1 = \varepsilon$

$$\bigcap_{t > 0} N(L_t) = \bigcap_{\varepsilon > t > 0} N(L_t) = N(L_\varepsilon) = \{0\}$$

we will say that the system is <u>differentially observable</u>. The following definition states this in a slightly different way.

<u>Definition 2</u>

The system is said to be differentially observable at t_o if observation of the output over an interval $[t_o, t_o + \varepsilon]$ for arbitrary small $\varepsilon > 0$ suffices to determine the state at t_o.

We henceforth consider only linear time-invariant systems.

<u>Scholium 3</u>

Let the semigroup ϕ be analytic. Then, if the system is observable, it is differentially observable.

<u>Proof</u>:

$$\bigcap_{t > 0} N(L_t) = \{0\} \quad \text{is equivalent to}$$

$$C\phi(t)x = 0 \quad \text{all } t > 0 \Rightarrow x = 0 \tag{9}$$

If ϕ is analytic $C\phi(t)x = 0$ for $t \in [0,\varepsilon]$ implies (9). Hence,

$$N(L_\varepsilon) = \bigcap_{t>0} N(L_t). \text{ Then if } \bigcap_{t>0} N(L_t) = \{0\}, \; N(L_\varepsilon) = \{0\}. \blacksquare$$

<u>Remark</u>

Since for a F.D.S., $H_x = R^n$, ϕ is always analytic scholium 3 is applicable. Moreover, the condition $N(L_\varepsilon) = \{0\}$ can be checked at

$t = 0$ by looking at the Taylor expansion-coefficients of $C\phi(t)x$ around $t = 0$, i.e., if $C\phi(t)x = 0$, $t\varepsilon[0,\varepsilon]$ then

$$\left[(C\phi(t))_{t=0} \quad (C\dot{\phi}(t))_{t=0} \quad \ldots \quad (C\phi(t)^{(k)})_{t=0} \quad \ldots \right] x = 0 \quad (10)$$

By the Cayley-Hamilton theorem only the coefficients $(C\phi(t))^{(k)}$, $k < n-1$ have to be considered. Therefore, $N(L_\varepsilon) = \{0\}$ implies and is implied by (10) or

$$\text{rank}\left[C' \,\vdots\, A'C' \,\vdots\, \ldots \vdots\, A'^{n-1}C' \right] = n \qquad (11)$$

For infinite dimensional systems the Taylor-expansion for $C\phi(t)x_o$ still works if x_o is an analytic vector of ϕ, in which case we can extend the mentioned criterion to infinite dimensions. This will be done in a series of steps.

<u>Theorem 2</u>

Let $P \triangleq \bigcap_{n=0}^{\infty} N(CA^n)$ and $S \triangleq \bigcap_{n=1}^{\infty} D_o(A^n)$

Then

1). $P \supset S \cap N(L_T)$ for any $T > 0$. (12)

2). $P = \{0\}$ is a <u>sufficient</u> condition for differential observability.

<u>**Proof**</u>: To prove (1) let $x \,\varepsilon\, S \cap N(L_T)$. Since $x\varepsilon S$ the expression $C\phi(t)x$ is infinitely differentiable. Evaluating the derivatives at $t=o$ leads to the implication

$$C\phi(t)x = 0 \text{ for all } t\varepsilon(0, T) \Rightarrow CA^n x = 0 \text{ for all } n\varepsilon I^+$$

Hence $x\varepsilon P \cap S$ or $x \,\varepsilon\, P$, since $P \subset S$.

For 2), observe that $P = \{0\}$ implies $S \cap N(L_T) = \{0\}$. Since for the systems under consideration S is dense, and since $N(L_T)$ is closed this implies $N(L_T) = \{0\}$. ∎

Let Q denote the set of analytic vectors of the semigroup .
The following theorem expresses then that if Q is dense, $P = \{0\}$ is also a necessary condition for differential observability.

Theorem 3

Let Q be dense in H_x.

Then $N(L_T) = \{0\} \Rightarrow P = \{0\}$.

Proof:

Let $x \in Q \cap P$ then $x \in N(L_T) \cap Q$ by the properties of analytic vectors (see Chapter II). Using (12) then

$$P \supset S \cap N(L_T) \supset Q \cap N(L_T) \supset Q \cap P \tag{13}$$

If Q is dense then $N(L_T) = \{0\} \Rightarrow P = \{0\}$. ∎

Corollary 1

The condition

$$\overline{\sum_{n=1}^{\infty} R(A^{*n}C^*)} = H_x \tag{14}$$

is sufficient for differential observability.

Proof: Since $N(CA^n) \subset R((CA^n)^*)^{\perp}$

and $R((CA^n)^*) \supset R(A^{n*}C^*) \supset R(A^{*n}C^*)$

clearly $\left[\sum_{i=n}^{\infty} R(A^{*n}C^*)\right]^{\perp} \supset \bigcap_{n=0}^{\infty} N(CA^n) \supset \bigcap_{t>0} N(L_t)$

Therefore $\left[\sum_{i=n}^{\infty} R(A^{*n}C^{*})\right]^{\perp} = \{0\}$ or (14) implies differential observability. ∎

The density of Q in H_x, used in theorem 3 may be relaxed. In the following corollary C^{*-1} stands for "the inverse image of C*".

Corollary 2

If $M \triangleq C^{*-1}(R(C^{*}) \cap Q)$ is dense in H_y condition (14) is a necessary and sufficient condition for differential observability.

Proof: We need only to concentrate on the necessity condition.

$\langle x, A^{*n}C^{*}y\rangle_{H_x} = 0$ for $y \in H_y \cap M$ implies $\langle x, \phi^{*}(s)C^{*}y\rangle_{H_x} = 0$, $s>0$. If now y is arbitrary in H_y, let $\{y_n\}$ be a sequence in M, $y_n \to y$. Then $\lim_{n\to\infty} \langle x, \phi^{*}(s)C^{*}y_n\rangle = \langle x, \phi^{*}(s)C^{*}y\rangle$. Hence $\langle C\phi(s)x, y\rangle_{H_y} = 0$ $s \geq 0$ for all $y \in H_y$ or $x \in \bigcap_{s>0} N(L_s)$. Therefore $N(L_T) = \{0\} \Rightarrow (14)$. ∎

3.5. Reducing a System. "Minimal" Representation.

A "reduced" realization is a realization restricted to its controllable and observable subspaces and "realizing" thus the same input-output relation as the unreduced system. In analogy with F.D.S. such systems could be called "minimal" (although this term in the case of I.D.S. does not reflect necessarily a lower dimensionality, since the reduced system will in general still be a I.D.S.). Let $H_o \triangleq \left[\bigcap_{t\geq 0} N(L_t)\right]^{\perp}$ be a nontrivial subspace of H_x. H_o represents the observable states. Let P be the projection operator $PH_x = H_o$. If $x \in H_o$ then $Ax \in H_o$ so that H_o is A-invariant. Therefore A can be represented on $H_x = H_o \oplus H_o^{\perp}$ as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad , \quad \begin{cases} A_{11} = (I-P)A \restriction_{H_o^{\perp}} \\ A_{12} = (I-P)A \restriction_{H_o} \\ A_{22} = PA \restriction_{H_o} \end{cases}$$

Denote by $x_2 \stackrel{\Delta}{=} Px$, $x_1 = (I-P)x$.

Then

$$\begin{cases} \dot{x}_2 = A_{22}x_2 + PBu(t) \\ x_{20} = Px_o \\ y = CP^{-1}x_2 \ , \ P^{-1}x_2 = \{x \mid Px = x_2\} \end{cases}$$

represents the observable part of the system. Similarly let $H_1 \stackrel{\Delta}{=} \bigcup_{t \geq 0} R(F_t)$, where $F_t \stackrel{\Delta}{=} \int_0^t \phi(t-s)B.ds$. Then clearly H_1 is ϕ-invariant because, if $x = \int_0^{t_1} \phi(t_1-s)Bu_1(s)ds$. Then

$$\phi(t)x = \int_0^{t_1} \phi(t_1+t-s)Bu_1(s)ds = \int_0^{t_1+t} \phi(t_1+t-s)Bu_2(s)ds$$

where $u_2(s) = u_1(s)$, $s \in (0,t_1)$

$= 0$ otherwise.

Hence ϕ can be triangulated on $H_1 \oplus H_1^{\perp} = H_x$ as

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ 0 & \phi_{22} \end{bmatrix}$$

Let A_{11} be the infinitesimal generator of ϕ_{11}. Since clearly $R(B) \subset H_1$ the input term on which ϕ_{12}, ϕ_{22} operate is zero and although

$\phi_{12} \neq 0$ and $\phi_{22} \neq 0$ in general, they have no effect. Hence,

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + Bu(t), \; x_{10} = Px_o \\ y = CP^{-1}x_1 \end{cases}$$

where $x_1 = Px$

$PH_x = H_1$ is a controllable realization.

If this restriction operation was done starting from an observable system, the result would be a controllable and observable system. We called such representation "minimal".

3.6. Observability and Spectral Decompositions

The results stated here are more or less dual to controllability-results stated in Chapter II. Therefore only the results will be stated and instead of the proofs some applications will be considered in the form of examples.

Assume A has normal spectrum and the rootvectors of A form a complete set in H_x. Let M_i be the rootspace corresponding to λ_i and $\mathfrak{Z}_{\lambda_i}$ the eigenspace. Let $A_i \overset{\Delta}{=} A|_{M_i}$ and P_i a projector such that $P_iH_x = M_i$.

The following assertions are then clear. The proofs are almost identical to the proofs of the corresponding statements for controllability in Chapter 2.

Statement 1

The total system is observable (not necessarily differentially) if each spectral subsystem

$$\begin{cases} \dot{x}_i = A_i x_i & y = C_{\Gamma_{m_i}} x_i \\ x_{io} = P_i x_o , & x_i \triangleq P_i x \end{cases}$$

is observable

Statement 2

The i^{th} spectral subsystem is observable if

$$N(C) \cap \mathfrak{Z}_i = \{0\} \tag{15}$$

Hence the total system is observable if (15) holds for all i.

Statement 3

If the output is obtained via of finite number of output sensors in a necessary condition for observability is that $m > \sup_i \nu_i$, ν_i the dimensions of $\mathfrak{Z}_i$.

Example 1. (Diffusion equation on $[0,1]$)

Consider $\dfrac{\delta x(t,z)}{\delta t} = \dfrac{\delta^2 x(t,z)}{\delta z^2} + u(t,x)$

$$x(t,o) = x(t,1) = o$$

$$y(t) = \int_o^1 \delta(z-z_1)\; x\;(t,z)dz = x(t,z_1)$$

i.e., the output is obtained via a single transducer and is a pointwise reading. The output operator is $\mathcal{C} \triangleq \int_o^1 \delta(z-z_1).dz$ and is clearly bounded. The spectrum is normal (in fact simple) and the eigenfunctions $\{\sqrt{2}\ \sin n\pi z\}_{n=1}^{\infty}$ span $H_x = L^2(0,1)$. Statement 1. and 2. says that, since $\mathfrak{Z}_i$ is the one dimensional space spanned by $\sin(i\pi z)$, that $C(\sin i\pi z) = \sin i\pi z_1, \neq 0$ for all i. Since the spectrum is simple, the condition in statement 3 is satisfied with one sensor.

Example 2

Diffusion-equation on spatial domain [0,1] x [0,1].

$$\frac{\delta x(t, z_1, z_2)}{\delta t} = \left(\frac{\delta^2}{\delta z_1^2} + \frac{\delta^2}{\delta z_2^2}\right) x + u(t, z_1, z_2)$$

$$x(t, 0, z_2) = x(t, 1, z_2) = x(t, z_1, 0) = x(t, z_1, 1) = 0$$

$$y(t) = \int_0^1 \int_0^1 \delta(z_1 - \hat{z}_1)\, \delta(z_2 - \hat{z}_2)\, x(t, z_1, z_2) dz_1 dz_2 = x(t, \hat{z}_1, \hat{z}_2)$$

The spectrum is $\{(p^2 + q^2)\pi^2\}_{p,q=1}^{\infty}$ and $\{2 \sin \pi p z_1 \cdot \sin \pi q z_2\}_{p,q=1}^{\infty}$ are the eigenfunctions. Clearly if $p_1^2 + q_1^2 = m^2$ then $p_2^2 + q_2^2 = m^2$ for $p_2 = q_1$, $q_2 = p_1$. Hence the multiplicity of each eigenvalue $m^2\pi^2$ is at least 2. From statement 3 it follows that this system cannot be observable with one sensor. In fact, we need an unbounded number of sensors (see Appendix B, section B.3, example 1).

Example 3

Consider the higher order system

$$\begin{cases} \sum_{i=0}^{n} p_i \frac{\delta^i}{\delta t^i} x + Ax = 0 \quad , \qquad p_n = 1 & \\ & \text{(a)} \\ x^{(i)}(0) = f_i \quad \text{for } i = 0, \ldots, n-1 & \\ y = \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix} x & \text{(b)} \\ A \text{ is selfadjoint; } C_i,\ i = 1, \ldots m \text{ are bounded operators.} & \end{cases}$$

Since we did not consider such higher-order systems in Chapter I, a word on well-posedness is in order.

A higher order system

$$x^{(n)}(t) = Ax(t) \;, \quad t \in R^{+}$$
$$x^{(k)}(o) = f_u \;, \; k = 0, \ldots n-1 \qquad \text{(c)}$$

will be said to be wellposed if there exists a dense subspace D of H_x such that one <u>has</u> a <u>unique</u> solution for $f_u, \ldots, f_{n-1} \in D$ and, whenever $\left\{ \{f_k^{\alpha}\}_{k=0,\ldots,n-1} \right\}_{\alpha\varepsilon J}$ is a net of initial data in D tending to zero then $x^{\alpha}(t) \to 0$ for each $t\varepsilon R^{+}$. System (c) then is "wellposed" in the above sense for $n > 3$ iff A is bounded. (Fattorini, [F2]). We henceforth assume A is bounded. By standard manipulation (a) can be brought into the form

$$\dot{x} = \mathcal{A}x$$
$$y = Cx \qquad \text{(d)}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & & 1 \\ -(p_o + A) & -p_1 & \cdots & & -p_{n-1} \end{bmatrix}_{n\times n}$$

$$C = \begin{bmatrix} c_1 & 0 & 0 \\ & & \\ c_m & 0 & 0 \end{bmatrix}_{m\times n}$$

Let A have a discrete spectrum $\{\lambda_i\}_{i=0}^{\infty}$. Then $\mathcal{A}$ is isomorphic to a matrix representation $\tilde{\mathcal{A}}$ = diag $[\mathcal{A}_1, \mathcal{A}_2, \ldots]$ where

$$\mathcal{A}_j = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & & \ddots & \vdots \\ & & & 1 \\ -(p_o+\lambda_j) & \cdots & & -p_{n-1} \end{bmatrix}$$

It is now easy to see that the spectral multiplicity of A determines the spectral multiplicity of $\mathcal{A}$ and therefore by statement 3 determines the minimal number of sensors needed for observability.

3.7. Partial Controllability and Observability. On Design Procedures for I.D.S.

For many applications, like stabilizing the system by pole-relocation through use of a state-feedback or building an observer to observe a subspace of a state-space, the requirements that the system be controllable and observable in the sense used hitherto are not minimal. One can introduce the notion of partial controllability and observability meaning that only certain subspaces of the state-spave need to be controllable for certain operations to be possible. If N poles need to be relocated, it suffices that the corresponding rootspace S is controllable. e.g., if one wants to stabilize a system having poles $\{\lambda_i\}_{i=1}^{N}$, $\text{Re}\lambda_i > 0$ ... this partial controllability suffices to relocate these poles to any preassigned symmetric set of N complex values, $\{\mu_i\}_{i=1}^{N}$, for example with $\text{Re}\,\mu_i < 0$. The resulting system is stabilized. If one is only interested in a stable output this condition can be relaxed to $S \cap W^{\perp}$ being controllable where $W \subset N(C)$ and W is the maximum (A,B)-invariant subspace in N(C). This result is discussed in detail in connection with a type of I.D.S. in Chapter IV.

A dual problem to that of pole-relocation is that of dynamic observers (Wonham, [W7]). Suppose one is interested only in the behavior of the statecomponent x_N in the rootspace corresponding to N eigenvalues. If the system reduced to this subspace is a F.D.S. say of dimension $\tilde{N}$ one can build a Luenburger observer to "reconstruct" the state. Let dim. R(C) = m then the observer has a minimal dimension $\tilde{N} - m$. Since x_N is an approximation for the state x, the designer might want to make a trade-off between cost and accuracy, i.e., between more sensors ($m\uparrow$), and the cost associated with the number of dynamics of the observer ($\tilde{N} - m\uparrow$) and accuracy ($N\uparrow$) by taking a larger part of the spectrum into account.

Most design procedures available for D.P.S. e.g., for state-estimation, reconstruction, optimal control, start off by forming a finite dimentional plant model. This model may be obtained via spectral theory in case the root-functions are readily computable. This leads to the easiest implementation, an important reason for this being the invariance of the rootspace under the system operator A. Other methods include the Bubnov-Galerkin method [M_1, P_2]. On this model all finite dimensional control techniques can then be applied without any difficulty, e.g., to solve a quadratic regulator problem, build a deterministic observer to estimate the model state, then design the lumped controller. The observer uses output-data from the actual plant and the controller operates on the inputs of the actual plant. The design of the interface between the D.P.S. and the F.D.S. that is designed to operate with the D.P.S. is a worthwhile aspect of the problem: the actual location of input zones and output transducers and their number will largely determine the overall quality of the design. For D.P.S. over a multidimensional spatial domain this may not be an easy problem and very little is available in the literature. The location problem may be considered as a sensitivity problem assuming the basic requirements like partial controllability or observability on certain rootspaces are fulfilled (and which excludes already certain locations).

Consider first the case of a reconstructor for $x_1 \in H_1$ = the direct sum of rootspaces of N poles (dimension $\tilde{N}$). Let $A_{11} \triangleq A\restriction_{H_1}$, $A_{22} \triangleq A\restriction_{H_1^\perp}$.

Then $x = x_1 + x_2$, $x_1 \in H_1$, $x_2 \in H_2 \triangleq H_1^\perp$.

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \qquad y = [C_1 \; \vdots \; C_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The reconstructor for x_1 has the form $\dot{z} = A_{11}z + B_1u + Hy - HC_1z$. Let $z = \hat{x}_1$, the estimate for x_1. The error $e \triangleq x_1 - \hat{x}_1$ satisfies then $\dot{e} = (A_{11} - HC_1)\, e + HC_2x_2$ and is therefore not limited to a transient effect if $HC_2 \neq 0$. If $u = K\,\hat{x}_1$ the overall feedback compensator is described by

$$\begin{bmatrix} \dot{x}_1 \\ e \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11}-B_1K & B_1K & 0 \\ 0 & A_{11}-HC_1 & HC_2 \\ B_2K & B_2K & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ e \\ x_2 \end{bmatrix}$$

Unless HC_2 or B_2K is zero the spectrum of the overall system is <u>not</u> $\sigma(A_{11}-B_1K) \cup \sigma(A_{11}-HC_1) \cup \sigma(A_{22})$. Therefore the spectrum of the "neglected" part of the system is affected and the "intended" pole-locations of the system in H_1 and the reconstructor are not realized exactly. However, the operators H and K are at our choice and although their primary function[(1)] is to specify the spectrum of the part of the I.D.S. modelled by the F.D. plant and the spectrum of the reconstructor, they may be used to cancel entries in B_2K respectively HC_2. <u>Most effective to this purpose is to chose the locations of input-zones and output-transducers since they affect directly the entries of B_2 respectively C_2</u>.

---

(1) If the reconstructor serves to provide the state for a feedback law in a quadratic regulator problem K has to satisfy a Riccati-type equation instead.

In case one wants to use a minimal order observer to achieve the same purpose of reconstructing x_1 let $C = [C_1 \vdots C_3]$, $C_1 \triangleq C\restriction_{H_1}$ $C_3 \triangleq C\restriction_{H_1}$ Then

$$H_x = N(C_1) \oplus N(C_1)^{\perp} \cap H_1 \oplus H_3 = H_1 \oplus H_3 \;, \quad H_3 \triangleq H_1^{\perp}$$

H_1 is an observable subspace. Assume $d(C)$[1] = m (all measurements are independent) and for simplicity let $d(C_1) = m$ (otherwise the measurements would not be independent with respect to the approximate plant model and some transducers where not necessary). Let $\tilde{N} > m$, otherwise there would be no need for a dynamic reconstructor. In H_1 one may change the coordinate system to obtain a new C_1 denoted
$\tilde{C}_1 = [I_{m\times m} \vdots 0]$.

Since clearly $N(C_1)$ and $N(C_1)^{\perp} \cap H_1$ are not A-invariant $\tilde{A}$ on $N(\tilde{C}_1) \oplus N(\tilde{C}_1)^{\perp} \cap H_1$ is then no longer block-diagonal but

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \quad . \quad \text{Similarly} \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}$$

$B_3 \triangleq P_3B$ where P_3 projects $H_x \to H_3$.

The observer is then easily derived to be

$\dot{z} = F\,z + Gu + R_1y_1 + R_3y_3$ with

---

(1) $d(C) \triangleq$ dimension of range of C.

$$\begin{cases} z = \hat{x}_2 + Px_1 \\ F = \tilde{A}_{22} + P\tilde{A}_{12} \\ G = \tilde{B}_2 + P\tilde{B}_1 \\ R_1 = P\tilde{A}_{11} - \tilde{A}_{22}P + \tilde{A}_{21} - P\tilde{A}_{12}P. \\ R_3 = P \end{cases} \qquad \begin{cases} Y_1 = \tilde{x}_1 \\ Y_3 = C_3x_3, \quad x_3 \text{ being the} \\ \text{component of } x \text{ in } H_3. \end{cases}$$

where the gain matrix P is an $\tilde{N}$-m x m-matrix to be chosen freely to regulate the error dynamics.

Again, the estimation error satisfies

$$(\dot{x_2 - \hat{x}_2}) \triangleq \dot{e}_2 = (\tilde{A}_{22} - P\tilde{A}_{12})e_2 + PC_3x_3$$

so that the overall feedback compensator with $u = \begin{bmatrix} K_1, & K_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$ and $\hat{x}_1 = \tilde{x}_1$ is given by

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \tilde{x}_2 \\ e \\ x_3 \end{bmatrix} \begin{bmatrix} \tilde{A}_{11}+\tilde{B}_1K_1 & \tilde{A}_{12}+\tilde{B}_1K_2 & -\tilde{B}_1K_2 & 0 \\ \tilde{A}_{21}+\tilde{B}_2K_1 & \tilde{A}_{22}+\tilde{B}_2K_2 & -\tilde{B}_2K_2 & 0 \\ 0 & 0 & A_{22}-PA_{12} & PC_3 \\ B_3K_1 & B_3K_2 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ e \\ x_3 \end{bmatrix}$$

As for the case of the nonminimal observer the same observations on the role of sensor locations hold.

Remark

If the observer is used in line with a quadratic regulator it would be an easy matter to calculate the costincrement due to the error of the estimate. For F.D.S. similar results were obtained by Bongiorno and Youla [B3].

We conclude this paragraph with a remark on sensitivity. It is advantageous to give the observer fast dynamics, but not too fast (spectrum $\to -\infty$), in which case it starts behaving as differentiators and will be very sensitive to disturbances. Overall sensitivity of observers to sensor location has not been studied and rests on concepts as yet not developed. The system observed has to be observable. Sensitivity depends on "how observable" it is. We comment on this in section 8.

3.8. Quality of Observability

In this paragraph we attempt to give some meaning tothe notion of "quality" of observability (and controllability). Let A_i, $C_i \triangleq C\Gamma_{\mathcal{M}_i}$ be as in section 6. Let $Q_i \triangleq [C'_i \vdots A'_i C'_i \vdots A'^{n-1}_i C'_i]$ and rank $Q_i = n_i = \dim \mathcal{M}_i$ for all $i \in I^+$ corresponding to $\lambda_i \in \sigma(A)$. Let C'_i be an $n_i x m$ matrix, where m is the number of output sensors. Define a vector $\underline{a}^i \triangleq x'_i Q_i$ for arbitrary $x_i \in \mathcal{M}_i$.

Definition

The "worst observable" direction in $\mathcal{M}_i$ is the direction x^*_i minimizing $||a^i|| = (\sum_{j=1}^{mn_i} (a^i_j)^2)^{1/2}$ i.e., $||a^i(x^*_i)|| =$

$$\inf_{\|x_i\|=1} ||a^i(x_i)||$$

Let α^i indicate, for a choice of C (or sensor locations $\hat{z}_1, \ldots, \hat{z}_m$) this infimum. Clearly, $\alpha^i > 0$ if $\mathcal{M}_i$ is an observable subspace. However, α^i small is an indication of "almost" linear dependence of any set of n_i columns of Q_i spanning $\mathcal{M}_i$.

Since the entries of C (in the "spectral" representation of the system) are determined by the sensor location, we can by changing the location manipulate the "quality" of observability. Depending on the problem, possible sensor locations $\hat{\underline{z}}$ may be restricted (e.g., for a problem involving measurements on 3-dimensional bodies, it may be imposed that the sensors are located on the outer surface). If this admissible area is denoted by $\hat{\Omega}$, one has to chose in $\hat{\Omega}$ m points to maximize a function α of α^i $(\hat{\underline{z}}_1, \ldots \hat{\underline{z}}_m)$ for $i \in I^+$. If the plant model is based on N poles then $i \in [0, \ldots N]$. Then α could for example be chosen to be $\alpha \triangleq \min_{i\in[1,\ldots N]} \alpha^i$, pointwise over $[\hat{z}_1 \ \ldots \ \hat{z}_m]' \in (\hat{\Omega})^m$.

Example.

Consider again example 1 in section 6. Then $\alpha^i = \sin i\pi z$.

If N poles are considered for the plant model $\alpha(z) = \bigwedge_{i=1}^{N} |\sin i\pi z|$ and "maximal" observability corresponds to chosing the sensor location z_1 so that $\alpha(z_1) = \max_{z\varepsilon[0,1]} \alpha(z)$. This coincides here with chosing the sensor location as to maximize collectively the output readings for all N modes.

Conclusion

In this chapter the general theory of observability of I.D.S. has been considered. Considerable attention was given to conditions for differential observability. The notions of "minimal" representations and "quality" of observability were presented. The spectral theory was mostly restricted to the case the system operator has normal spectrum. Explicit formulae for the recuperation of the initial function were then obtained. Section 7 gave some remarks on design procedures for D.P.S. and the usefullness of partial observability.

CHAPTER IV

ON QUALITATIVE PROPERTIES OF DELAY-SYSTEMS

In this chapter some general controllability criteria are developed for a general type of systems with hereditary dependence. Next these results are specialized for a type of system for which spectral decomposition methods can be used. On the basis of this some stabilizability-results are derived. An important contribution of this chapter is the result on output-stabilizability. At the end of the chapter a comprehensive review is given of existing criteria interpreted in the light of our present treatment.

4.1 System Description

Introduction. Systems of the form

$$\dot{x} = \sum_{i=1}^{N} A_i \, x\,(t-h_i) + Bu(t) \qquad (1)$$

$$0 \leq h_1 < h_2 \; . \; . \; . < h_N < \infty$$

$A_i(t)$ n x n matrices, i = 1, . . . N.

$\underline{x}(t)$ an n-vectorvalued function

$B(t)$ n x m matrix

are called delay-systems or systems with hereditary dependence. They represent the simplest form of delay-systems we have in mind: the instantaneous rate of change $\dot{x}\,(t)$ depends on a discrete set of past values. Systems of this type were intensively studied as for existence and uniqueness of solutions and their spectral properties [ref.B2,H2,H3,O1]. Many physical systems can be modelled as delay-systems. Such models are used often as "refinements" or "exacter" models for systems considered before as ordinary lumped systems. For some ecological and biomedical applications see ref. [W1, W2, W5].

More complicated models consider nonlinear systems for addition, for example a retarded argument for the control function u(t), or with the delay itself a time function as, for example, in

$$\dot{x}(t) = f(t, x(\alpha_1(t)), x(\alpha_2(t)), \ldots x(\alpha_N(t)). \quad (2)$$

where $\alpha \leq \alpha_i(t) \leq t$ for $t \geq \alpha$ and $\alpha_i(t)$ continuous functions.

Krasovski and later Hale [H3] introduced a new technique for solving such and more general types of delay-systems by considering them as functional differential equations. They introduce the notion of the "state" as a piece of a trajectory. The instantaneous rate of change at t ($\dot{x}(t)$) is then determined by the "state". We develop some notation.

Let $[t_1,t_2]$ be a finite interval on the real line and τ a positive real number. Let $C([t_1 - \tau, t_2]; R^n)$ denote the space of continuous linear mappings from $[t_1 - \tau, t_2] \to R^n$. Consider on this C the norm of uniform convergence to make it a Banach space, i.e. $\| \phi \| = \max_{t \in [t_1-\tau, t_2]} | \phi(t) |$ where $|\cdot|$ is any norm R^n.

Define $x_t(\theta) \in C([-\tau,0]; R^n)^*$ for $t \in [t_1,t_2]$ as

$$x_t(\theta) \triangleq x(t+\theta) \qquad -\tau \leq \theta \leq 0. \tag{3}$$

mostly denoted by x_t. It is called the "state" of the system in the following model.

Mathematical model considered. Consider the system model

$$\dot{x}(t) = L(t,x_t) + Bu(t) \tag{4}$$

where L is a bounded mapping from $R \times C \to R^n$, continuous in t. B is an n x m - matrix, continuous in t. Let the initial condition be given as a function in C.

If L is linear in x_t the system is called linear.

If L is linear (and bounded) one may apply Riesz representation theorem and write (4) as

$$\dot{x}(t) = \int_{-\tau}^{0} dy\,(t,\theta)\; x_t(\theta) + Bu(t) \tag{5}$$

where $y(t,\theta)$ is an nxn - matrix of functions of bounded variation and d denotes the Stieltjes integral with respect to variable θ.

Remark: Discrete time lag systems as (1) are obviously a special case of the continuous time lag systems (4)(5). To see this, define

---

*Henceforth C stands for $C([-\tau,0];R^n)$

$$y(t,\theta) = \sum_{i=0}^{N} A_i(t) \qquad \text{for } \theta \geq 0$$

$$= \sum^{N} A_i(t) \qquad \text{for } -h_1 \leq \theta < 0$$

$$\vdots$$

$$= \sum_{i=k}^{N} A_i(t) \qquad \text{for } -h_k \leq \theta < -h_{k-1}$$

$$\vdots$$

$$= A_N(t) \qquad \text{for } -h_N \leq \theta < -h_{N-1}$$

$$= 0 \qquad \text{for } \theta < -h_N$$

Then $\sum_{i=0}^{N} A_i(t)\, x\,(t-h_i) = \int_{-h_N}^{0} dy(t,\theta)x_t(\theta)$

where $y(t,\theta)$ is piecewise constant in θ. The important special case of (5) is the case of time-invariant systems, i.e.

$$\begin{aligned} \dot{x} &= L(x_t) + Bu(t) \\ &= \int_{-\tau}^{0} dy(\theta)x_t(\theta) + Bu(t) \end{aligned} \tag{6}$$

This model will allow us to apply spectral theory for unbounded operators and apply some results obtained for systems with normal spectra in Chapter II.

4.2 Existence, uniqueness and continuity with respect to initial data of the solutions.

Consider again model (4)*

Definition 1. Solution.

x is said to be a solution of (4) if there are real numbers

$\sigma \in R$, $A > 0$ such that $x \in C([\sigma - \tau, \sigma + A]); R^n)$ and $x(t)$ for $t \in [\sigma, \sigma + A]$ satisfies (4). x is then said to be a solution of (4) on $[\sigma - \tau, \sigma + A]$.

Definition 2. Solution with initial function given.

Given the function $\psi \in C([-\tau, 0]; R^n)$, we say $x(\sigma, \psi)$ is a solution of (4) with initial condition ψ at σ if there exists an $A > 0$ such that $x(\sigma, \psi)$ is a solution of (4) as $[\sigma - \tau, \sigma + A]$ with $x_\sigma(\sigma, \psi) = \psi$.

We state three existence and uniqueness theorems, respectively for models (4)(5)(6), each time obtaining more explicit results. In the next paragraphs we will only be able to use the results for (5)(6). The results for (4) are only stated for completeness. For proofs we refer to [H3].

*The term $Bu(t)$ can be considered part of $L(t,x_1)$ in this model.

Definition 3. Carotheodory condition.

Let D be an open set in R x C.

$L(t,x_t) : D \to R^n$ is said to satisfy the Carotheodory condition on D if

- $L(t,x_t)$ is measurable in t for fixed x_t and continuous in x_t for fixed t.
- for fixed $(t,x_t) \in D$ there exists a neighborhood $N(t,x_t;\varepsilon)$ and an integrable function $\ell(s)$ such that

$$|L(s,\psi)| \leq \ell(s) \text{ for all } (s,\psi) \in N(t,x_t;\varepsilon).$$

Theorem 1. (Thm. 5.1, 5.2 and §7 in Hale [43])

(Local existence).

Consider the system

$$\begin{aligned} x(t) &= \phi(0) + \int_\sigma^t L(s,x_s)ds \quad , \; t \geq \sigma \\ x_\sigma &= \phi \qquad\qquad\qquad\qquad\quad \phi \in C \end{aligned} \tag{7a}$$

If L satisfies the Carotheodory condition on an open set $D \subset R \times C$ and $L(t,\phi)$ in Lipschitz in ϕ on each compact set in D there exists a unique solution of (7) through each point $(\sigma,\phi) \in D$. ∎

If model (4) is specialized to (5) the local solutions on $[\sigma, \sigma + \varepsilon]$ can be extended to $[\sigma,\infty)$. This is stated in Theorem 2.

Definition 4. Let $L_1^{loc}([\sigma,\infty); R^n)$ denote the space of functions $g: [\sigma,\infty) \to R^n$ which are integrable over arbitrary compact subsets of $[\sigma,\infty)$.

Theorem 2. (Version of Theorem 16.1, Hale [H3]).

Consider the homogeneous form of model (5), denoted by (5'). If $|L(t,\phi| \leq \ell(t) \; |\phi|$ for all $t \in (-\infty, +\infty)$, $\phi \in C$ and $\ell \in L_1^{loc}((-\infty, +\infty); R)$

Then for any $(\sigma,\phi) \in R \times C$ there exists a unique solution $x(\sigma,\phi)$ defined and continuous on $[\sigma - \tau, \infty)$ satisfying (5') on $[\sigma,\infty]$. ∎

Remark 2. In the context of controllability a special version of (5') will be considered, namely the case where $L(t,x_t)$ has the form

$$L(t,x_t) = \sum_{k=1}^{N} A_k(t) \; x \; (t - w_k) + \int_{-\tau}^{0} A(t,\theta) \; x \; (t + \theta)d\theta$$

where $0 \leq w_k \leq \tau$ and $A(t,\theta)$ is integrable in θ for each t, and

$$\left| \int_{-\tau}^{0} A(t,\theta)\phi(\theta)d\theta \right| \leq a(t)\,|\phi| \text{ for all } t \in R, \; \phi \in R$$

and some $a(t) \in L_1^{loc}((-\infty, +\infty); R)$.

Hale states: "This is the most general type of linear system with finite lag known to be useful."

The next corollary states the variation of constants formula i.e. gives the expression for the solution of (5), if $u(t)$ is considered as the control function.

Corollary 1. (Thm. 16.1, 16.3, Hale [H3]).

Let L be as in Theorem 2. Let $Bu(t) \overset{\Delta}{=} h$ in (5) be an element of $L_1^{loc}([\sigma,\infty]; R^n)$. Then there exists a unique solution $x(\sigma,\phi,h)$ to (5) defined and continuous on $[\sigma - \tau, \infty)$ satisfying (5) on $[\sigma,\infty)$. Moreover,

$$x(\sigma,\phi,h) = x(\sigma,\phi,0) + x(\sigma,0,h) \text{ where}$$

$$x(\sigma,\phi,0)(t) : C \to R^n \text{ and}$$

$$x(\sigma,0,h)(t) : L_1^{loc}([\sigma,t]; R^n) \to R^n$$

are continuous linear maps

$$x(\sigma,0,h) = \int_\sigma^t K(t,s)h(s)ds \qquad t \geq \sigma \tag{7b}$$

with

$$\frac{\delta K}{\delta t}(t,s) = L(t,K_t(.,s))^{(1)} \qquad t \geq s \quad \text{a.e. in } t \text{ and } s,$$

$$K(t,s) = 0 \text{ for } t < s \qquad \text{and } K(t,s) = I \quad \text{for } t = s. \blacksquare$$

The expression for the solution to model (6) will play a very important role and is stated in the next paragraph.

---

(1) $K_t(.,s)(\theta) \overset{\Delta}{=} K(t+\theta,s), \ -\tau \leq \theta \leq 0.$

4.3 Semigroup representation - Infinitesimal generator

Consider

$$\dot{x} = L(x_t) + Bu(t) \qquad (6)$$

and let

L be linear and continuous in x_t

B be a n x m matrix

$u(t) \in L_1^{loc}((0,\infty);\ R^m)$

Then

Theorem 3. (§19 in Hale [H3]).

The solution $x(\sigma,\phi,h)$ has the form

$$x(\sigma,\phi,u) = x(\sigma,\phi,0) + \int_\sigma^t T(t-\tau)\ Bu(\tau)d\tau \qquad (7)$$

and T is a one-parameter semigroup of bounded linear operators satisfying

$$\begin{aligned} T(0) &= 1 \\ T(s) &= 0 \quad s < 0 \\ \frac{dT(t)\phi}{dt} &= L(T(t)\phi) \text{ for } \theta = 0 \\ &= \frac{d}{d\theta}(T(t)\phi)(\theta^+),\ -\tau \le \theta < 0. \end{aligned} \qquad (9)$$

Also T(t) is compact for $t \geq \tau$. ∎

Infinitesimal generator of T. The infinitesimal generator of T is defined by

$$A\phi = \lim_{h \to 0^+} \frac{1}{h} [T(h)\phi - \phi] \tag{10}$$

and $D_0(A)$ as the set of functions ϕ for which the limit exists. Hale [13] shows that

$$D_0(A) = \{\phi \in C \mid \phi \text{ has continuous derivative on } [-\tau,0] \text{ with } \dot{\phi}(0) = L(\phi).\}$$

and

$$A\phi(\theta) \begin{cases} = \dfrac{d\phi(\theta)}{d\theta}, & -\tau \leq \theta < 0 \\ = L(\phi) & \theta = 0 \end{cases} \tag{11}$$

One can write the solution $x(\sigma,\phi,u)$ as given in Theorem 3 in a different manner, namely

$$\begin{cases} x(\sigma,\phi,u)(t+\theta) = x(\sigma,\phi,0)(t+\theta) + \displaystyle\int_{\sigma}^{t+\theta} T(t+\theta-s)\, Bu(s)ds \\ \qquad \text{for } t+\theta > \sigma. \\ x(\sigma,\phi,u)(t+\theta) = \phi(t+\theta), \quad \sigma - \tau \leq t+\theta \leq \sigma, \quad -\tau \leq \theta \leq 0 \end{cases} \tag{12}$$

Actually, $x(\sigma,\phi,0)(t+\theta) \begin{cases} = (T(t-\sigma)\phi)(\theta), & t+\theta > \sigma \\ = \phi(t+\theta), & \sigma - \tau \leq t+\theta < \sigma. \end{cases}$

One can denote $T(t + \theta - s) \stackrel{\Delta}{=} T(t - s)X_o(\theta)$. The integral-term of $x(\sigma,\phi,u)$ can then be written as

$$\int_{\sigma}^{t} T(t - s)\, X_o\, Bu(s)\,ds$$

and denotes an element in C if $X_o(\theta)$ is defined as

$$X_o(\theta)\begin{cases} = I & \text{for } \theta = 0 \\ = 0 & \text{for } -\tau \leq \theta < 0 \end{cases}$$

Hence (12) can be rewritten as

$$x_t(\sigma,\phi,u) = T(t - \sigma)\phi + \int_{\sigma}^{t} T(t - s)\, X_o\, Bu(s)\,ds \qquad (13)$$

which is a solution of a functional differential equation of the type

$$\begin{cases} \dot{x}_t = A\,x_t + X_o\,Bu \\ x_{t=\sigma} = \phi \end{cases} \qquad (14a)$$

Expression (14) is an equivalent form to (6) and is more suitable in a control-theoretic context. Indeed x_t is the "state" so that (14) is the usual "state-equation". The problem of existence and uniqueness of solutions has been solved, the solution being (13).

Remark: The solution (7b) for timevarying systems can be written in a functional form similar to (13).

$$x_t(\theta) = (T(t,\sigma)\phi)(\theta) + \int_{\sigma}^{t} T(t+\theta,s)\ Bu(s)ds \qquad (14b)$$

or

$$x_t = T(t,\sigma)\phi + \int_{\sigma}^{t} T(t,s)\ X_o\ Bu(s)ds \qquad (14c)$$

with the same definition for X_o .

4.4 Simplifying assumptions on $L(x_t)$ - Adjoints

It is well known that if a space C is decomposed in a direct sum of A-invariant subspaces called spectral subspaces the spectral components of an element in the space are obtained with the help of eigenvectors of the adjoint of A in C*.
In case $C \triangleq C([-\tau,0];\ R^n)$, C* is the Banach space of functions $\psi : [-\tau,0] \to R^{n*}$ of bounded variation on $[-\tau,0]$, A* is the topological adjoint of A. This setting results in very complicated analysis. However Hale has shown that by restricting $L(x_t)$ further to be of the form

$$L(\phi) = \sum_{k=1}^{\infty} A_k\ \phi(-\tau_k) + \int_{-\tau}^{0} A(\xi)\phi(\xi)d\xi \quad , \text{where} \qquad (15)$$

$$\begin{cases} 0 \leq \tau_k < \tau \text{ for all } k \\ A_k \text{ are n x n matrices} \\ A(\xi) \text{ is a continuous matrix function in R} \end{cases}$$

the adjoint can be defined with respect to a new bilinear form. The dual of C is then $C^* \triangleq C((0,\tau);\ R^{n*})$ and the adjoint

A^* is defined via

$$L^*(\psi) = -\sum_{k=1}^{\infty} \psi(+\tau_k)A_k - \int_{-\tau}^{0} \psi(-\xi)A(\xi)d\xi \tag{16}$$

The bilinear form in question is

$$(\psi,\phi) = \psi(0)\phi(0) - \sum_{k=1}^{\infty} \int_{0}^{\tau_k} \psi(\xi)A_k\phi(\xi - \tau_k)d\xi$$

$$- \int_{-\tau}^{0} \int_{-\theta}^{0} \psi(\xi)A(\theta)\phi(\xi + \theta)d\xi d\theta \tag{17}$$

for $\psi \in C^*$, $\phi \in C$.

One can put (15)(16)(17) in a more standard form.

With
$$L(x_t) = \int_{-\tau}^{0} d_\theta\, y \cdot x(t + \theta) \tag{18}$$

$$\dot{x}\,(t) = \int_{-\tau}^{0} d_\theta\, y \cdot x(t + \theta) \tag{16'}$$

The adjoint equation is then

$$\dot{y}\,(t) = -\int_{-\tau}^{0} y(t - \theta)d_\theta y \tag{17'}$$

with respect to inner product

$$(\psi,\phi) = \psi(0)\phi(0) - \int_{-\tau}^{0}\int_{0}^{\theta} \psi(\xi - \theta)[d_\theta y]\phi(\xi)d\xi \qquad (18)'$$

In state-space formulation (10) becomes

$$\dot{x}_t = A\, x_t \qquad (16)''$$

If we define

$$y^t \in C^* \text{ as}$$

$$y^t(\xi) \triangleq y(t + \xi) \quad , \quad \xi \in [0,\tau]$$

then $\dot{y}^t = A^* y^t$ (17)''

where A^* (and $D_o(A^*)$) satisfies

$$(\psi, A\phi) = (A^*\psi, \phi) \text{ for } \phi \in D_o(A),\ \psi \in D_o(A^*)$$

using inner product (18)'.

Hale [H3] shows that

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds} & 0 < s \leq \tau \\ \displaystyle\int_{-\tau}^{0} \psi(-s)\,dy(s) & s = 0 \end{cases}$$

with $D_o(A^*) = \{\psi \in C([0,\tau];\ R^{n*})/\psi$ has continuous derivative and $\dfrac{d\psi(0)}{ds} = -\displaystyle\int_{-\tau}^{0} \psi(-\theta)\,dy(\theta).\}$.

The solution to (16)'', for initial function $\phi \in C$, on $[\sigma - \tau, \infty]$ was shown to be of the form

$$x_t = T(t - \sigma)\phi \tag{16'''}$$

To (17)'' for $\psi \in C^*$, on $(-\infty, \sigma + \tau)$, the solution is

$$y^t = T^*(t - \sigma)\psi \tag{17'''}$$

This $T^*(t)$ is a semigroup satisfying

$$\frac{dT^*(t)}{dt}\psi = -A^*T^*(t)\psi = -T^*(t)A^*\psi \tag{19}$$

This T^* is not the topological adjoint of T, however.

The notions of A^*, T^* on C^* will be the important tools for the spectral decomposition.

T and T^* are closely related in the sense that, if

x_t is a solution to (16) on $\omega - \tau \leq t < \infty$
y^t is a solution to (17) on $-\infty < t \leq \sigma + \tau$

then

$$(y^t, x_t) = \text{constant on } [\omega, \sigma].$$

4.5 Salient properties of the spectrum of A.

We continue with the simplified form for L introduced in 4.4.

The spectrum follows from the study of $R(\lambda, A) = (\lambda I - A)^{-1}$. If $(\lambda I - A)^{-1}$ exists but $R(\lambda I - A)$ is dense there can be no residual spectrum. If on this dense range $(\lambda I - A)^{-1}$ is continuous, its domain of definition can be extended to the whole space, there is no continuous spectrum. The values of λ where $(\lambda I - A)^{-1}$ does not exist constitute then the discrete spectrum.

Properties of $\sigma(A)$, A as in 4.4.

1. $\sigma(A)$ is a countable point spectrum of finite algebraic multiplicity determined from the entire function $\phi(\lambda) = 0$ where

$$\phi(\lambda) = \det\,[\lambda I - \int_{-\tau}^{0} e^{\lambda\theta}\, dy(\theta)]. \tag{20}$$

2. $\sigma(A) \subset \{\lambda \mid \text{Re}\,\lambda < a \text{ for some real } a\}$.

3. $\sigma(A) \cap \{\lambda \mid \ell < \text{Re}\,\lambda < c \text{ for some real } \ell, c\}$ is finite.

4. If $\phi(0) \in \mathcal{N}(\lambda I - \int_{-\tau}^{0} e^{\lambda\theta}\, dy(\theta))$ and $\lambda \in \sigma(A)$ then

   $e^{\lambda\theta}\,\phi(0)$ is an eigenvector, $-\tau \leq \theta \leq 0$. (21)

5. The generalized eigenvectors $\phi(\theta)$ have the general form

$$\sum_{j=1}^{K} \gamma_j \frac{\theta^{j-1}}{j-1!} e^{\lambda\theta} \qquad -\tau \leq \theta \leq 0. \tag{22}$$

Where $\underline{\gamma} \in \mathcal{N}(A_k)$

$$A_k \triangleq \begin{bmatrix} P_1 & P_2 & \cdots & P_k \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & P_2 \\ 0 & \cdots & & P_1 \end{bmatrix}, \qquad P_k \triangleq \frac{1}{k!}\frac{d^k}{d\lambda^k}(\lambda I - \int_{\tau}^{0} e^{\lambda\theta}\, dy(\theta)).$$

Many of these properties could be derived from the fact that $T(t)$, the semigroup of A, is compact for $t \geqslant \tau$.

First, observe [H5,p467] the following relation between the spectrum of $T(t)$ for $t \geqslant 0$ and $\sigma(A)$:

$$\sigma(T(t)) = e^{t(\sigma(A)-\{0\})} \underset{*}{\cup} \{0\}$$

where $\underset{*}{\cup}$ means "plus eventually the point zero." Next, observe that for $s > \tau$, $T(s)$ is compact. Let $\tilde{r} = \lim_{n\to\infty} ||T^n(s)||^{1/n}$ denote its spectral radius. Then [T 1] $\sigma(T(s))$ is a point spectrum and included in the set $C_1 \triangleq \{\lambda / |\lambda| < \tilde{r}\}$. Also $0 \in \sigma(T(s))$ and if $\sigma(T(s))$ is infinite, 0 is its only cluster point.

If $\lambda \neq 0$, $\lambda \in \sigma(T(s))$ then λ is an eigenvalue ; then there exist two closed subspaces $M(\lambda,s)$, $N(\lambda,s)$ with $M(\lambda,s)$ finite dimensional, reducing $T(s)$. The restriction of $T(s)$ to $N(\lambda,s)$ is again compact and has spectrum $\sigma(T(s)) - \{\lambda\}$.

We concentrate now on statements 2 and 3 on the spectral properties of A. Statement 2 follows from $\sigma(T(s)) \subset C_1$ since this implies $\sigma(A) \subset \{\lambda / \operatorname{Re} \lambda < \frac{\ln \tilde{r}}{\tau}\}$. For statement 3 observe that the number of eigenvalues μ of $T(s)$ satisfying $e^{\ell s} < |\mu| < e^{cs}$ for some real ℓ, c, $\infty > c > \ell > 0$, is finite and for each eigenvalue μ satisfying that condition the root space $M(\mu,s)$ is finite dimensional. Let now $\{\lambda_n\}_{n\in Q}$ be distinct points in $\sigma(A)$ such that $e^{\lambda_n s} = \mu$, $\mu \in \sigma(T(s))$ and such that $e^{\ell s} < |\mu| < e^{cs}$. Then $M(\mu,s) = \sum_{n\in Q_\mu} \mathfrak{M}_{\lambda_n}(A)$ where $\mathfrak{M}_{\lambda_n}(A)$ is the root space for $\lambda_n \in \sigma(A)$. The index set Q_μ therefore is finite. Since the number of μ satisfying the above conditions is finite statement 3 is seen to hold.

4.6 Direct decomposition of C and C* - Completeness of root-vectors.

If $\lambda_o \neq 0$ is a pole of $R(\lambda,A)$ of order m then the Laurent expansion for $|\lambda - \lambda_0| < \varepsilon$, ε sufficiently small, is

$$R(\lambda,A) = \sum_{i=1}^{m} (\lambda - \lambda_0)^{-n} P_n + \sum_{0}^{\infty} (\lambda - \lambda_0)^n A_n \tag{24}$$

(Taylor #5.8, [T1]) and $P_1 C = \mathcal{N}(A - \lambda_0 I)^m$ (25)

The latter is the root-space of A for λ_0 for which we used the notation $\mathcal{M}_{\lambda_0}(A)$. Then (Taylor, [T1])

$$C = \mathcal{M}_{\lambda_0}(A) \oplus \mathcal{R}(\lambda_0 I - A)^m. \tag{26}$$

Similarly

$$C^* = \mathcal{M}_{\lambda_0}(A^*) \oplus \mathcal{R}(\lambda_0 I - A^*)^m, \tag{27}$$

since $\sigma(A) = \sigma(A^*)$ and (Hale [H3], lemma 21.2)

$$\mathcal{M}_{\lambda_0}(A)^{\perp} = \mathcal{R}(\lambda_0 I - A^*)^m. \tag{28}$$

Here orthogonality is with respect to the inner product defined in 4.4.

Let ψ_λ = rows $(\psi_1, \ldots, \psi_p)$ be a basis for $\mathcal{M}_\lambda(A^*)$ and Φ_λ = columns $(\psi_1, \ldots, \psi_p)$ be a basis for $\mathcal{M}_\lambda(A)$ and $(\psi_\lambda, \Phi_\lambda) = [(\psi_j, \phi_i)]$, then this matrix is nonsingular and may be taken I, the identity matrix.

It follows that for $\phi \in C$, $\phi = \phi_1 + \phi_2$ with $\phi_1 \in \mathcal{M}_\lambda(A)$, $\phi_2 \in \mathcal{M}_\lambda(A^*)^\perp$ and ϕ_1 is given by

$$\phi_1 = \Phi_\lambda(\psi_\lambda, \phi) = P_{1,\lambda}\phi. \tag{29}$$

$$\phi_2 = \phi - \phi_1.$$

Scholium 1. Let $\Phi_\lambda, \psi_\lambda$ be as before.

Define $P_\lambda = \Phi_\lambda(\psi_\lambda, \cdot)$ for $\lambda \in \sigma(A)$.

Then $P_{\lambda i} P_{\lambda j} = \delta_{ij} P_{\lambda i}$, $\lambda_i, \lambda_j \in \sigma(A)$.

Motivation: This scholium assures that if we use the family of projections $\{P_\lambda\}_{\lambda \in \sigma(A)}$, the decomposition thus obtained is unique. i.e. for $x \in \overline{\sum_\oplus \mathcal{M}_\lambda(A)}$ then $x = \sum_\oplus (P_\lambda x)$. We obviously would like

$\overline{\sum_{\oplus\, \lambda \in \sigma(A)} \mathcal{M}_\lambda(A)} = C$ which is the statement for completeness of root vectors. This will be treated later. This scholium corresponds to Hale [H3], lemma 21.6 the proof of which was left as an exercise.

Proof. Clearly P_λ is a projection since it is idempotent and bounded. The first property follows from

$$\Phi_\lambda(\psi_\lambda, [\Phi_\lambda(\psi_\lambda, x)]) = \Phi_\lambda(\psi_\lambda, x).$$

If $P_{\lambda i} P_{\lambda j} \neq 0$ for $\lambda_i \neq \lambda_j$ let $\mathcal{M} \triangleq \mathcal{M}_{\lambda i} \cap \mathcal{M}_{\lambda j}$ be a d-dimensional subspace. Let $\mathcal{M}_{\lambda i} = \mathcal{N}(A - \lambda_i I)^s$ for minimum index s, and $\mathcal{M}_{\lambda j} = \mathcal{N}(A - \lambda_j I)^r$, for minimum r. Since A commutes with

$(A - \lambda_i I)^s$ and $(A - \lambda_j I)^r$, $A\mathcal{M} \subset \mathcal{M}$. Let Φ be a basis for $\mathcal{M}$. Then there exists a d x d matrix B such that

$$A\phi = \phi B$$

This B moreover satisfies

$$(B - \lambda_i I)^d = (B - \lambda_j I)^d = 0$$

which is a contradiction. ∎

On completeness of the system of root vectors we can make the following observations. If $\phi \in C$, $\sum_{\lambda \in \sigma(A)} P_\lambda \phi$ may or may not converge. If it converges, let $r_\phi = \phi - \sum_{\substack{Re\lambda > -a \\ \text{at some big } a>0}} P_\lambda \phi$. Consider the solution $x_t(r_\phi\ \sigma)$ with initial condition r_ϕ at σ. Then

$$|| x_t(r_\phi, \sigma || \leq ke^{-a(t-\sigma)} || \phi ||$$

for some $k > 0$.

Consider $\sum_{\lambda \in \sigma(A)} P_\lambda \phi$ and the corresponding r_ϕ. Then $e^{Mt}||x_t(r_\phi, \sigma)|| \to 0$ as $t \to \infty$ for arbitrary M, i.e. the solution, based on the "rest" of ϕ decays faster than any exponential. This means that the significant part of the solution $T(t,\sigma)\phi$ lies in the root spaces of T and these solutions grow and decay as exponentials. In case $\Sigma P_\lambda \phi$ diverges, partial sums still provide a good estimate to the solution as far as the behavior of $x_t(\phi, \delta)$ is concerned for $t \to \infty$.

The case $\Sigma P_\lambda x$ convergent always occurs if A is a spectral operator (Dunford and Schwartz, [D2] Vol. III, and [S3]). In particular, if T(t) has spectral radius $\lim_{n\to\infty} || T^n ||^{1/n} < 1$ then T is a spectral operator and its quasi-nilpotent part is zero. This implies $r_\phi = 0$ (Hahn [H1]). Let P_1 be the projection corresponding to the eigenvalues of T outside the unitcko. Consider $C_2 = (I - P_1)C$. Let T_2 denote the reduction of T to C_2. Then $\sigma(T_2) = \sigma(T) \cap \{\lambda : |\lambda| < 1\}$ and T_2 has spectral radius less than 1. Since T is a spectral operator, T_2 is and its quasi-nilpotent part is 0. Hence $\sum_{\substack{\lambda \\ \lambda \in \sigma(T_2)}} (P_\lambda \phi)$ converges to $\phi - P_1\phi$ for any $\phi \in C$.

4.7 Reduction of the operators A, T

For $\lambda \in \sigma(A)$ the pair of closed subspaces $(\mathcal{M}_\lambda(A), \mathcal{M}_\lambda(A^*)^\perp)$ reduces A as was demonstrated. Given a basis Φ_λ in $\mathcal{M}_\lambda(A)$ with basis elements $(\phi_1, \ldots, \phi_d)$ the reduction of A to $\mathcal{M}_\lambda(A)$ can be written as a d x d matrix B_λ such that

$$A\Phi_\lambda = \Phi_\lambda B_\lambda \tag{30}$$

and

$$\sigma(B_\lambda) = \{\lambda\}. \tag{31}$$

The Φ_λ-vectors are root vectors and thus have the general form discussed in 4.5. However, using (30) and (11) it is easily derived that

$$\Phi_\lambda(\theta) = \Phi_\lambda(0)e^{B_\lambda \theta} \text{ where } \Phi_\lambda(0) \text{ satisfies} \tag{32}$$

$$\Phi_\lambda(0)B_\lambda = L(\Phi_\lambda(0)e^{B_\lambda \theta}) . \tag{33}$$

Formulae (30) and (32) completely give the reduction of A. The same subspaces also reduce T, the semigroup of A. The following formulae are then easily derived:

$$\frac{d}{dt} T(t)\ \Phi_\lambda = AT(t)\ \Phi_\lambda = T(t)\ \Phi_\lambda B_\lambda \tag{34}$$

or

$$T(t)\ \Phi_\lambda = (T(t)\ \Phi_\lambda)_{t=0}\ e^{B_\lambda t} = \Phi_\lambda\ e^{B_\lambda t} \tag{35}$$

or

$$(T(t)\ \Phi_\lambda)(\theta) = \Phi_\lambda\ (0)\ e^{B_\lambda (t+\theta)} \qquad -\tau \leq \theta \leq 0,\ t \geq 0 \tag{36}$$

This leads to the expression of the solution to $\dot{x}_t = Ax_t$, $x_o = \phi$ in subspace $\mathcal{M}_\lambda$.

Let $\phi \in C$, then its projection on $\mathcal{M}_\lambda$ is $\Phi_\lambda\ (\psi_\lambda, \phi)$. The solution in $\mathcal{M}_\lambda$ is then

$$x_t(\phi,0)\restriction_{\mathcal{M}_\lambda} = \Phi_\lambda\ (0)\ e^{B_\lambda (t+\theta)}\ (\psi_\lambda, \phi).$$

The solution to the forced system $\dot{x}_t = Ax_t + X_o Bu(t)$ (see (14)), with i.c. $x_{t=0} = \phi$ has the form expressed in formula (13).

Then

$$\begin{aligned} x_t(\phi,u)\restriction_{\mathcal{M}_\lambda} &= \Phi_\lambda\ (\psi_\lambda, x_t(\phi,u)) \\ &= \Phi_\lambda(\psi_\lambda, T(t)\phi) + \int_0^t \Phi_\lambda\ (\psi_\lambda, T(t-\sigma)\ X_o\ Bu(\sigma))d\sigma \\ &= \Phi_\lambda \cdot e^{B_\lambda t}\ (\psi_\lambda, \phi) + \int_0^t \Phi_\lambda \cdot e^{B_\lambda (t-\sigma)}\ \psi_\lambda(0) Bu(\sigma)\, d\sigma \end{aligned} \tag{37}$$

Let $(\psi_\lambda, x_t) \overset{\Delta}{=} y(t)$ then (38)

$$y(t) = e^{B_\lambda t} y(0) + \int_0^t e^{B_\lambda (t-\sigma)}\ \psi_\lambda(0) Bu(\sigma)\, d\sigma\ . \tag{39}$$

This is the solution to the finite dimensional system

$$\dot{y}(t) = B_\lambda y(t) + \psi_\lambda(0)Bu(t) \tag{40}$$

$$y(0) = (\psi_\lambda,\phi) \ .$$

The dimension equals the dimension of $\mathcal{M}_\lambda$, i.e. the algebraic multiplicity of eigenvalue λ and has nothing to do with the dimension n of $x(t)$.

The solution x_t^2 in the complementary space $\mathcal{M}_\lambda(A^*)^\perp$ can be seen to satisfy

$$x_t^2 = x_t - \Phi_\lambda(\psi_\lambda,x_t) \tag{41}$$

and in view of (38)(39)(13)

$$x_t^2 = T(t-\sigma)(\phi - \Phi_\lambda(\psi_\lambda,\phi)) + \int_0^t T(t-\sigma)(X_0 - \phi_\lambda\psi_\lambda(0))\ Bu(\sigma)d\sigma \tag{42}$$

4.8 Controllability.

The detailed analysis of the preceding paragraphs will allow a succinct treatment of controllability and related notions. This paragraph is essentially made up by two parts. In the first controllability for the most general type of linear systems for which we have global existence of solutions is considered. These systems correspond to the linear time varying type (5) whose solutions were discussed in Theorem 2. We will call them type A. One could have presented a theory for local controllability as well, corresponding to (4) and Theorem 2. Controllability for nonlinear systems however will not be included. In the second half of this paragraph we restrict attention to systems for which spectral theory is available, i.e. linear, time-invariant system of the type (6) with the additional restriction (15). Systems of this form will be said to be of type B.

4.8.1 Definitions.

The "state-space" is $C \overset{\Delta}{=} C([-\tau,0];R^n)$. A "trajectory-set" is defined as a set in R^n. The space of admissible controls is $U = L^1_{loc}([0,\infty];R^m)$. The solution to (5) was discussed in corollarys and shown (in 14b, 14c) to have the form

$$(x_t(\phi,u))(\theta) = T(t,0)\ \phi(\theta) + \int_0^t T(t+\theta,s)\ Bu(s)$$

on $[o,\infty]$. (43)

If this expression is evaluated at $\theta = 0$, the "trajectory" is obtained. $(x_t)(\theta)$, $-\tau \leq \theta \leq 0$ is the "state".

Definition 5.

The system is said to be (functionally) controllable from $0 \varepsilon C$, if for any given $x_d \in C$ there exists a time $t_1 > 0$ and an admissible control segment $u_{[0,t_1]}$ which will drive the system from state 0 to a state x_{t_1} arbitrarily close to x_d i.e.

$$\|\ x_d - x_{t_1}(0,u_{[0,t_1]})\ \|_C \leq \varepsilon. \tag{44}$$

Definition 6.

The system will be said to be R^n-controllable from $0 \in C$ if for any $x_1 \in R^n$ there exists a $t_1 > 0$ and an admissible $u_{[0,t_1]}$ such that the trajectory $x(0,u_{[0,t]})$ "hits" x_1 at t_1.

Define subspace U_t of U as L^1_{loc} ([0,t]; R^m) and $F_t : U_t \to C$ by

$$(F_t(u))(\theta) \triangleq \int_0^t T(t + \theta, s)\, Bu(s)ds. \qquad (45)$$

4.8.2 General controllability result for type A-systems.

Scholium 2. A system of type A (a fortiori for type B) is functionally controllable from 0 iff

$$\overline{\bigcup_{t \geq 0} R(F_t)} = C$$

In this expression R(.) stands for "range". ∎ Since clearly $R(F_t)$ is a monotonous increasing function of t one has also :

Scholium 3. A sufficient condition for functional controllability from 0 for the system in scholium 2 is that

$$\overline{R(F_T)} = C \text{ for some } T > 0 \qquad ∎$$

If we define

$$F^*_t : C^* \to U^*_t$$

where * denotes dual or adjoint, then, since $\mathcal{N}(F^*_t) = R(F_t)^\perp \subset C^*$; we can formulate equivalent expressions.

Scholium 4.

A necessary and sufficient condition for functional controllability from $0 \in C$ is

$$\bigcap_{t \geq 0} \mathcal{N}(F^*_t) = \{0\}$$

A sufficient condition is $\mathcal{N}(F^*_T) = \{0\}$ for some $T > 0$. ∎

The main difficulty obviously lies in obtaining explicit expressions for the adjoints. We rely on expressions developed by Hale [H3, § 32, 33]. Let B_o denote the Banach space of functions $\psi : [-\tau,0] \to R^{n*}$ of bounded variation on $[-\tau,0]$, left continuous on $(-\tau,0)$ and with norm $|\psi| = \mathrm{var}_{[-\tau,0]}\psi$. We identify C* with B_o. Clearly $U_t^* = L^\infty([0,t]; R^{n*})$.

The topological adjoint F_t^* of F_t is defined using the pairing

$$\langle p,x\rangle_1 = \int_{-\tau}^{0} dp(\theta)\, x(\theta) \qquad \text{for } p \in B_o, \quad x \in C$$

Then
$$\langle p, F_t(\sigma)u\rangle_1 = \langle F_t^*(\sigma)p, u\rangle_2$$

$\langle , \rangle_2$ is the pairing $\langle v,u\rangle_2 = \int_{\sigma}^{t} v(s)\, u(s)\, ds$, $u \in U_t$, $v \in U_t^*$, or

$$\langle F_t^*(\sigma)p, u\rangle_2 = \int_{\sigma}^{t} (F_t^*(\sigma)p)(s)\, u(s)\, ds.$$

Hale shows that

$$\begin{cases} (F_t^*(\sigma)p)(s) = -B'(s)\,[T^*(s,t)p]\,(0^-) \text{ for almost} \\ \qquad \text{all } s \in [\sigma,t] \text{ and any } p \in B_o \end{cases} \tag{46}$$

In this expression $T^*(s,t) : B_o \to B_o$ can be written as

$$T^*(s,t) = (I + \Omega(s))\, \tilde{T}(s,t)\, (I + \Omega(t))^{-1} \tag{47}$$

where $\Omega(s)$ is a quasi-nilpotent operator on B_o defined from

$$(\Omega(s)\psi)(\theta) = \int_{\theta}^{0} \psi(\alpha)\, y(s+\alpha, \theta-\alpha)\, d\alpha \qquad -\tau \le \theta \le 0,\ \psi \in B_o$$

and $\tilde{T}(s,t) : B_o \to B_o$ is a bounded linear operator such that $\tilde{T}(\sigma,t)\psi$ represents the solution on $[-\infty, t-\tau]$ to the "adjoint equation"

$$\begin{cases} z(s) + \int_s^{\infty} z(\alpha)\, y(\alpha, s-\alpha)\, d\alpha = \text{constant} \\ z \in R^{n*} \\ z_t = \psi \end{cases}$$

The y in this expression was defined from

$$L(x_t, t) \triangleq \int_{-\tau}^{0} d_\theta y(t,\theta)\, \mathcal{X}(t+\theta).$$

In view of the characterization of F* in (46) scholium 4 can be formulated to read:

Scholium 5.

A necessary and sufficient condition for functional controllability from 0 for the system considered in scholium 2-4 is that

$$B'[T^*(0,t)](0^-) = 0 \text{ for all } t \in [0,\infty]$$

implies that $p = 0$ in C^*.

Proof. This statement is equivalent to

$$\bigcap_{t \geq 0} \mathcal{N}(F_t^*(\sigma)) = \{0\} \qquad \blacksquare$$

4.8.3 Spectral controllability.

The systems considered in this subparagraph are assumed to be of type B. Let $\lambda \in \sigma(A)$. Then $\mathcal{M}_\lambda(A)$ is finite dimensional and

$\overline{\sum_{\lambda \in \sigma(A)} \mathcal{M}_\lambda(A)} = C$ as was shown in 4.6.

Scholium 6. The system under consideration is functionally controllable from 0 iff each spectral subsystem is controllable.

Proof: The proof is based on formulae (37) to (40) and scholium 2. If $\lambda \in \sigma(A)$ let P_λ be the projector such that $P_\lambda C = \mathcal{M}_\lambda$. Then

$$\overline{\bigcup_{t\geq 0} R(F_t)} = C \text{ implies} \tag{48}$$

$$P\,\overline{(\bigcup_{t\geq 0} R(F_t))} = \overline{\bigcup_{t\geq 0} P\, R(F_t)} + \overline{\bigcup_{t\geq 0} R(F_t^\lambda)}^{(1)} = \mathcal{M}_\lambda$$

where

$$F_t^\lambda \,.\, \triangleq \Phi_\lambda \int_0^t e^{B_\lambda(t-\sigma)} \psi(0)B \,.\, d\sigma.$$

Since $\mathcal{M}_\lambda$ (of dimension, let's say d_λ) is isomorphic to R^{d_λ} and Φ_λ nonsingular, this implies

$$\text{rank}\,[\int_0^t e^{B_\lambda \sigma} \psi(0)\psi'(0)e^{B_\lambda' \sigma} d\sigma] = d_\lambda \tag{49}$$

for all $\lambda \in \sigma(A)$ and some $t > 0$.

Conversely (49) implies, together with $\Sigma_\lambda P_\lambda = I$ expression (48). ∎

Scholium 7.

The system under consideration is functionally controllable from 0 iff for all $\lambda \in \sigma(A)$

$$\mathcal{N}((\psi_\lambda(0)B)^*) \cap \mathcal{N}(\lambda I - B_\lambda^*) = \{0\}. \tag{50}$$

Proof: Let $\mathcal{M}_\lambda$ have dimension d_λ. The corresponding spectral subsystem in R^{d_λ} is of the form (40) for which in Chapter III and expression analogous to (50) was derived which is a necessary and sufficient condition for

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[1] The closure may be omitted in the last two expressions since $R(F_t^\lambda)$ is a finite dimensional subspace and hence closed. From analycity then

$$\overline{\bigcup_{t\geq 0} R(F_t^\lambda)} = R(F_t^\lambda) \text{ for any } T > 0.$$

controllability of the subsystem. If the condition is then satisfied for all $\lambda \in \sigma(A)$ the conclusion follows from scholium 6. ∎

Remark. One would have liked to write $\mathcal{N}(\lambda I - A^*)$ instead of $\mathcal{N}(\lambda I - B_\lambda^*)$. However $\mathcal{N}(\lambda I - A^*) \subset C^*$ and although the system is of dimension d_λ in $\mathcal{M}_\lambda$ it is only isomorphic to a system in R^{d_λ}. One could write

$$\mathcal{N}([\Phi_\lambda \psi_\lambda(0)B]^*) \cap \mathcal{N}(\lambda I - A^*) = \{0\}. \tag{51}$$

Scholium 8. Let B be an n x p matrix. Let γ_λ be the geometric multiplicity of $\lambda \in \sigma(A)$ and

$$\gamma = \sup_{\lambda \in \sigma(A)} \gamma_\lambda$$

Then a necessary condition for functional controllability from 0 is

$$\gamma \leq \min (n,p). \tag{52}$$

Proof. Since $\psi_\lambda(0)$ is an d_λ x n, B an n x p matrix

$$\dim \mathcal{R}[\psi(0)B] \leq \min (n,p)$$

If for some $\lambda \in \sigma(A)$, $\gamma_\lambda > \min (n,p)$ the nulspaces as expressed in (50) do intersect (with nontrivial intersection). Therefore the corresponding subsystem is not controllable in R^{d_λ}. The total system can then not be functionally controllable (from scholium 6). ∎

4.9 Stabilizability using state-feedback.

Now that functional controllability has been reduced to controllability of spectral subsystems, each of which is isomorphic to a finite dimensional system, the stage is set for extending Wonham's result [W6] on stabilizability to delay-systems. The only difficulty lies in the identification of the feedback operator for the infinite dimensional space. The systems considered in 4.9 are type B-systems of 4.8.

Definition 7.

The system $\dot{x}_t = Ax_t + X_o Bu(t)$ will be said to be stabilizable if there exists a bounded operator K such that $u(t) = -Kx_t$ makes the system asymptotically stable or $\lim_{t\to\infty} x_t = 0$.

Theorem 4.

Let $\Lambda \stackrel{\Delta}{=} \sigma(A) \cap \{\lambda : \mathrm{Re}\lambda \geq 0\} = \{\lambda_i\}_{i=1}^{P}$ and let P_Λ be the corresponding project Let

$$C = P_\Lambda C + (I - P_\Lambda)C \stackrel{\Delta}{=} C_\Lambda + C_\pi \tag{53}$$

Let Φ_Λ be a basis for C_Λ and ψ_Λ a basis for C^*_Λ. Let B_Λ be defined from

$$A\Phi_\Lambda = \Phi_\Lambda B_\Lambda$$

Then the system is stabilizable iff $(B_\Lambda, \psi_\Lambda(0)B)$ is a controllable pair.

Specification of what is involved. The objective is to show that the "bad" eigenvalues in $\sigma(A)$ i.e. the ones in Λ can be relocated into the left half plane with a bounded linear feedback and without disturbing the other eigenvalues in $\sigma(A)$. Moreover Λ can be relocated to an arbitrary configuration $\tilde{\Lambda}$, provided $\tilde{\Lambda}$ is a selfadjoint set.

In the notation of the theorem $\Phi_\Lambda = [\phi_{\lambda_1}, \phi_{\lambda_2}, \ldots, \phi_{\lambda_p}]$ where Φ_{λ_j} is a basis in $\mathcal{M}_{\lambda_j}(A)$. Similarly $\psi_\lambda = \mathrm{rows}[\psi_{\lambda_1}, \ldots, \psi_{\lambda_p}]$ where rows $[\psi_{\lambda_j}]$ is a basis in $\mathcal{M}_{\lambda_j}(A^*)$ and $B_\Lambda = \mathrm{diag}[B_{\lambda_i}]$, B_{λ_i} being defined from $A\Phi_{\lambda_i} = \Phi_{\lambda_i} B_{\lambda_i}$. The feedback operator K is required to be a bounded linear map from $C([-\tau,0]; R^n) \to R^m$ such that

$$\sigma(A + X_o BK) = \tilde{\Lambda} \cup \sigma(A \restriction_{C_\pi}). \tag{54}$$

Proof. The projection of $\dot{x}_t = Ax_t + X_o Bu$ on C_Λ is (55)

$$\Phi_\Lambda \dot{y}(t) = \Phi_\Lambda (B_\Lambda y(t) + \psi_\Lambda (0) Bu (t)) \tag{56}$$

Let C_Λ have dimension d_Λ, then system (54) is isomorphic to a system in R^{d_Λ}

$$\dot{y}(t) = B_\Lambda y(t) + \psi_\Lambda (0) Bu (t) \tag{57}$$

Wonham [W6] has shown that if this system is controllable there exists a matrix C such that

$$\sigma(B_\Lambda + \psi_\Lambda (0) BC) = \tilde{\Lambda} .$$

In this expression, $\psi_\Lambda(0) : R^n \to R^{d_\Lambda}$; $B : R^m \to R^n$; $C : R^{d_\Lambda} \to R^m$.

Define $K. \triangleq C e^{B_\Lambda \theta} (\psi_\Lambda, .)$. Then K is bounded and linear. Consider now the decomposition $x = x_1 + x_2$, $x_1 \in C_\Lambda$, $x_2 \in C_\pi$ and

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{with}$$

$$\tilde{A}_{11} = P_\Lambda(A + X_o BK)\restriction_{C_\Lambda} \qquad \tilde{A}_{12} = P_\Lambda(A + X_o BK)\restriction_{C_\pi}$$

$$\tilde{A}_{21} = (1-P_\Lambda)(A + X_o BK)\restriction_{C_\Lambda} \qquad \tilde{A}_{22} = (1-P_\Lambda)(A + X_o BK)\restriction_{C_\pi}$$

Then $\tilde{A}_{22} = (I-P_\Lambda)A\restriction_{C_\pi} = A\restriction_{C_\pi}$, $\tilde{A}_{12} = 0$.

The spectrum of $(A + X_o BK)$ is then the union of the spectrum of $\tilde{A}_{11}$

which is $\tilde{\Lambda}$ since

$$\tilde{A}_{11} = \Phi_\Lambda[B_\Lambda + \psi_\Lambda(0)BC] \tag{58}$$

and of $\tilde{A}_{22}$ which is $\sigma(A\restriction_{C_\pi})$.

Conversely if the system is stabilizable, the subsystem in C_Λ must be controllable. For if it were not, let $C_\Lambda = C_\Lambda^1 \oplus C_\Lambda^2$ where C_Λ^1 is the subspace spanned by

$$\Phi_\Lambda[\psi_\Lambda(0)B \,\vdots\, B_\Lambda\psi_\Lambda(0)B \,\vdots\, \ldots \,\vdots\, B_\Lambda^{d_\Lambda - 1}\psi_\Lambda(0)B]$$

Let the corresponding spaces in R^{d_Λ} be E_1 and E_2 i.e. $R^{d_\Lambda} = E_1 \oplus E_2$. Let P_1 be the projection on R^{d_Λ} such that $P_1 R^{d_\Lambda} = E_1$. Then with $y_1 = P_1 y$, $y_2 = (I-P_1)y$, $y \in R^{d_\Lambda}$ one has, since E_1 is B_Λ-invariant (as is R^{d_Λ})

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} P_1 B_\Lambda\restriction_{E_1} & P_1 B_\Lambda\restriction_{E_2} \\ 0 & (1-P_1)B_\Lambda\restriction_{E_2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} P_1\psi_\Lambda(0)B \\ (1-P_1)\psi_\Lambda(0)B \end{bmatrix}$$

Since $R(\psi_\Lambda(0)B) \subset E_1$, $(I-P_1)\psi_\Lambda(0)B = 0$.

Hence the system for y_2 in E_2 is uncontrollable and because $\sigma((1-P_1)B_\Lambda) \subset \sigma(A|_{C_\Lambda})$ and $\sigma((1-P_1)B_\Lambda)$ is nonempty this system is unstable, and since it is homogeneous it is not stabilizable. ∎

Specification of K

The feedback operator K was defined as

$$K \triangleq Ce^{B_\Lambda \theta}(\psi_\Lambda, \cdot) \tag{59}$$

The justification of this choice is obviously the necessity of obtaining the right finite dimensional expression (58), i.e.

$$P_\Lambda(A+X_o BK)\upharpoonright_{C_\Lambda} = \Phi_\Lambda[B_\Lambda + \psi_\Lambda(0)BC]\upharpoonright_{C_\Lambda} \tag{60}$$

$$P_\Lambda(A+X_o BK)\upharpoonright_{C_\Lambda} = 0 \tag{61}$$

(61) assures that the feedback will leave the spectrum of A on $(I-P_\Lambda)C$ unaltered.

We verify (60) (61) for the choice of K in (59), using

$$P_\Lambda = \Phi_\Lambda(\psi_\Lambda, \cdot) \tag{62}$$

For $x_t \in C_\Lambda$

$$\Phi_\Lambda(\psi_\Lambda, Ax_t + x_o BCe^{B_\Lambda \theta}(\psi_\Lambda, x_t)) = \tag{63}$$

$$\Phi_\Lambda(\psi_\Lambda, Ax_t) + \Phi_\Lambda(\psi_\Lambda, x_o BCe^{B_\Lambda \theta} y) \tag{64}$$

where $y = (\psi_\Lambda, x_t)$ is an element in R^{d_Λ}. (64) = $\phi_\Lambda B_\Lambda y + \phi_\Lambda \psi_\Lambda(0)BCy$ verifying (60) and (61).

To specify K completely one has still to identify C. This was done by Heymann [14]. For the sake of completeness we describe the construction of C.

Let $\tilde{B} \overset{\Delta}{=} \psi_\Lambda(0)B$ with columns $b_1 \ldots, b_m$. Let $S(A,B)$ denote the controllability space of the pair (A,B). Then [14] for every nonzero vector b_i there exists a matrix C_i such that

$$S(B_\Lambda, \tilde{B}) = S(B_\Lambda + \tilde{B}C_i, b_i)$$

If $\tilde{\Lambda}$ is the desired pole configuration, a vector K can be defined such that the scalar input system

$$\begin{cases} \dot{y}(t) = (B_\Lambda + \tilde{B}C_i)y(t) + b_i\tilde{u}(t) \\ \tilde{u}(t) = k^T y(t) \end{cases} \tag{65}$$

has as its spectrum $\tilde{\Lambda}$. Let $b_i = Br$ for some r in the domain of $\tilde{B}$. Then $C = C_i + rk^T$ is the desired expression which completes (59).

Remark :

The same technique can be applied to the problem of pole-reassignment of the spectrum of A in the half plane Re $\lambda > a$, $a > -\infty$, the justification being that $\sigma(A) \cap \{\lambda / \mathrm{Re}\lambda > a\}$ is a finite set with finite multiplicities and the space spanned by the corresponding root vectors is finite dimensional.

4.10 Stabilizability of the Output State-Feedback.

In 4.9 it was established that the system could be stabilized using a bounded feedback operator iff the controllable space covered the subspace spanned by the root vectors of the eigenvalues in the right half plane. This controllability requirement can be relaxed in case output-feedback is used.

Completion of the model

In addition to the state-evolution equation

$$\begin{cases} \dot{x}_t = Ax_t + X_0 Bu \\ x_{t=0} = \phi \in C([-\tau,0];R^n) \end{cases}$$

with
$$\begin{cases} X_0(\theta) = 0 \quad -\tau \leq \theta < 0 \\ \quad\quad = 1 \quad\quad \theta = 0 \\ B \text{ an } n \times m \text{ matrix} \\ A : D_0(A) \subset C \to C \text{ as defined in (11) and (15).} \end{cases} \tag{66}$$

We consider the output $z(t)$ being obtained as a simple linear transformation on the trajectory $x(t)$.

$$z(t) = Mx(t), \quad M \text{ a pxn matrix} \tag{67}$$

Expression (67) is not suitable for the state-space formulation (66). However, let the operator P be defined from

$$(Px_t)(\theta) \begin{cases} = x(t) & \text{for } \theta = 0 \\ = 0 & -\tau \leq \theta < 0 \end{cases} \tag{68}$$

then $P : C \to R^n$ is a projection.

$$\text{Define} \qquad \begin{array}{l} \tilde{M} : C \to R^n \text{ by} \\ \tilde{M} = MP \end{array} \tag{69}$$

<u>Definition 8.</u> System (66,69) is output stabilizable using state feedback if there exists a bounded operator K such that $\lim_{t\to\infty} z(t) = 0$ where

$$z(t) = \tilde{M}x_t \tag{69}$$

and x_t is the solution of

$$\begin{cases} \dot{x}_t = (A+X_o BK)x_t \\ x_{t_o} = \phi \in C \end{cases} \tag{70}$$

whole

<u>The purpose is to specify minimal qualitative properties needed to make output-stabilizability possible.</u>

<u>Remark</u> :

Let η denote the nulspace of $\tilde{M}$ and η^C its complementary space in C, i.e. $C = \eta \oplus \eta^C$. If there exists a bounded operator $K : C \to R^m$ such that

$$(A+X_oBK)\mathcal{M} \subset \mathcal{M}$$

then (69) and (70) become, with $x = x_1+x_2$, $x_1 \in \mathcal{M}$, $x_2 \in \mathcal{M}^c$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} (A+X_oBK)_{11} & (A+X_oBK)_{12} \\ 0 & (A+X_oBK)_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$z = \tilde{M}x_2$$

and the output is stable iff $\sigma((A+X_oBK)_{22})$ is in the left half plane.

This approach fails to make use of the finite dimensionality of the space spanned by the unstable modes.

Let $C = C_\Lambda \oplus C_\pi$ where C_Λ corresponds to the span (70b) of the unstable modes. In C_Λ the system is

$$\Phi_\Lambda\left[\dot{y}(t) = B_\Lambda y(t)+\psi_\Lambda(0)Bu(t)\right] \tag{71}$$

and in C_Λ as in (42). Let $x = x_1 + x_2$, $x_1 \in C_\Lambda$, $x_2 \in C_\Lambda$ and

$$z = \tilde{M}_1x_1 + \tilde{M}_2x_2 \ , \quad \tilde{M}_1 = \tilde{M}_{C_\Lambda} \ , \quad \tilde{M}_2 = \tilde{M}_{C_\pi}$$

Since $x_2(t)\to 0, t\to\infty$ so does $M_2x_2(t)$. We therefore restrict attention to

$$z_1 = \tilde{M}_1x_1 = \tilde{M}_1\Phi_\Lambda y \tag{72}$$

In view of (68), (72) becomes

$$z_1 = M\Phi_\Lambda(0)y \tag{73}$$

The problem of output stabilizability for delay systems is reduced to the same problem for the finite-dimensional system in $R^{\hat{d}}$ as expressed by model (71,73).

Since to the author's best knowledge an adequate treatment of this problem in finite dimensions is not available it will be presented here. The conclusions for delay systems will be drawn later.

Output stabilizability for finite dimensional systems

Consider

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{74}$$

where A,B,C are time-invariant respectively dxd, dxm, dxp matrices.

Let $\mathcal{X}=R^d$ denote the state space.

Let $\sigma(A) \subset \{\lambda : \text{Re } \lambda \geq 0\}$. This specification is no restriction since otherwise the system can be decomposed into an inherent stable and unstable system and a feedback in our context has only to be considered for the latter part.

Let $\mathcal{N}$ denote the nulspace of C. Then

$$R^d = \mathcal{N} \oplus \mathcal{N}^{\perp} \tag{75}$$

Let W be a subspace in $\mathcal{N}$ such that

$$AW \subset W + \mathcal{B} \tag{76}$$

where $\mathcal{B} \triangleq$ Range (B). (77)

Before proceeding, we give an equivalent condition to (76).

Lemma (Wonham, [W6])

$AW \subset W + \mathcal{B}$ iff there exists an mxd-matrix F such that

$$(A+BF)W \subset W \tag{78}$$

Proof : The necessity is clear.

For the sufficiency part, let $(\omega_1,\ldots,\omega_s)$ be a basis in W. Then $Aw_i = Bu_i + v_i$ for some $u_i \in R^m$, $v_i \in W$. Choose F so that $Fw_i = -v_i$, $i=1,\ldots,s$. This defines F on W. Extend its domain to R^d by defining F outside W arbitrary. The lemma is then proven. ▮

Remark :

If $\mathcal{N}(C) = \{0\}$ or if there exists no nontrivial subspace W of $\mathcal{N}$ satisfying (78) then the minimum requirement for stabilizability is that the pair (A,B) be controllable.

If W_1 is a nontrivial subspace of $\mathcal{N}$ satisfying (78) then (A+BF) has an upperdiagonal form with respect to the decomposition

$$X = W_1 \oplus W_1^{\perp}, \quad x = x_1 + x_2, \quad x_1 \in W_1, \quad x_2 \in W_2 \triangleq W_1^{\perp} \tag{80}$$

i.e.

$$(A+BF)x = \begin{bmatrix} (A+BF)_{11} & (A+BF)_{12} \\ 0 & (A+BF)_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{81}$$

Let P_1 be the projection such that $P_1 X = W_1$. If we decompose A, B, F accordingly, then, since $W_1 \subset \mathcal{N}$, output stabilizability means that $\sigma((A+BF)_{22}) \left(= \sigma(A_{22}+B_2F_2),\ \ B_2 \triangleq (I-P_1)B,\ \ F_2 = F\restriction_{W_1^{\perp}}\right)$ lies in the left half plane for some chosen F_2. Such choice is possible for arbitrary A_{22}, B_2 if they form a controllable pair. If $AW_1 \subset W_1$ and $A/\mathcal{B} \triangleq$ controllability space of (A,B) then clearly $(I-P_1)A/\mathcal{B} = A_{22}/\mathcal{B}_2$ so that the system is output-stabilizable if $A/\mathcal{B}$ covers W_1. This is proven in some detail in the next theorem.

Definition 9

A subspace W satisfying $AW \subset W + \mathcal{B}$ is called an (A,B)-invariant subspace.

If W_1 is now chosen to be the maximum (A,B)-invariant subspace in $\mathcal{N}$ the space $W_1^{\perp}$ shrinks and the requirement that (A,B) is controllable on W_1 becomes less stringent.

Given $\mathcal{N}$, a maximum (A,B)-invariant subspace exists, which is easily seen as follows. Let $AV_1 \subset V_1 + \mathcal{B}$ and $AV_2 \subset V_2 + \mathcal{B}$ Then $A(V_1+V_2) \subset (V_1+V_2) + \mathcal{B}$.

Let $\{V_\alpha\}_{\alpha\varepsilon\tilde{I}}$ where $\tilde{I}$ is an index set, be the family of (A,B)-invariant subspaces, then this family is a semilattice with respect to + and inclusion. Then any subfamily closed under addition has a unique largest element with respect to inclusion, namely the sum of all members of the subfamily. Choose as subfamily the (A,B)-invariant subspaces V_i satisfying $V_i \subset \mathcal{N}$ then there exists a unique largest element namely the sum.

Wonham [W6] has given an algorithm for finding the maximum (AB)-invariant subspace W_1 in a space $\mathcal{N}$ namely:

$$\text{Let} \begin{cases} V^{(0)} = \mathcal{N} \\ V^{(i)} = V^{(i-1)} \cap A^{-1}(\mathcal{B} + V^{(i-1)}) \quad i = 1,2,\ldots,\nu \end{cases} \tag{83}$$

where $\nu = \dim \mathcal{N}$. Then $W_1 = V^{(\nu)}$.

Let P_2 denote the projection so that $P_2 X = W_1^{\perp} \triangleq W_2$, where W_1 is the maximum AB-invariant subspace in $\mathcal{N}$.

Define
$$B_2 \triangleq P_2 B \tag{84}$$
$$A_2 \triangleq P_2 A \restriction_{W_2} \tag{85}$$

Theorem 5

Let W_1, P_2, B_2, A_2 as just defined. Assume $AW_1 \subset W_1$. Then the system is stabilizable with a feedback operator F such that $(A+BF)W_1 \subset W_1$ iff (A_2, B_2) is a controllable pair.

Proof : Let (A_2, B_2) be controllable, i.e. $A_2/\mathcal{B}_2 = W_1^{\perp}$. Then $\exists\ \tilde{F} : P_2 X \to R^m$ such that $A_2 + B_2\tilde{F}$ is stable. Let $F \triangleq [0 \mid \tilde{F}]$, $F : X \to R^m$. Then clearly $(A+BF)\restriction_{W_1} = A\restriction_{W_1}$. Therefore since $AW_1 \subset W_1$, $(A+BF)W_1 \subset W_1$.

Conversely if there exists an F, $(A+BF)W_1 \subset W_1$ and $P_2(A+BF)\restriction_{W_2}$ ($= A_{22} + B_2 F\restriction_{W_2}$) is stable for arbitrary A, B then (A_{22}, B_2) is controllable on W_2. ∎

Remark :

The assumption $AW_1 \subset W_1$ was essential to assure that the system in $W_1^{\perp} \supset \mathcal{N}^{\perp}$ is decoupled from the system in $W_1 \subset \mathcal{N}$. If some information on A, B is available like e.g. $R(B_2) \supset R(P_2 A\restriction_{W_1})$ this requirement may be dropped since the system may then be decoupled by the feedback i.e. choose

$$F = [F_1 \mid \tilde{F}]$$

where $\tilde{F}$ is as in the theorem and F_1 so that

$$(P_2 A\restriction_{W_1}) + F_1 B_2 = 0$$

making $(A+BF)W_1 \subset W_1$ still true.

Clearly $\tilde{F}$ may be used to give the system in $W_1^{\perp}$ arbitrary pole-configuration if desired.

Application to delay-systems

From the former section we immediately draw the conclusion.

The system (69) (70) is output stabilizable if its controllability space covers a certain subspace determined by the output operator. The complications arise from the fact that this subspace is not A-invariant. Let $C = C_\Lambda \oplus C_\pi$ as in (70b) and

$$\mathcal{N} = \mathcal{N}(\tilde{M}) \cap C_\Lambda, \quad \tilde{M} \text{ as in (69)} \qquad (86)$$

Then in $\mathcal{N}$ find the maximum $(A, X_0 B)$-invariant subspace W with a construction similar to (83). The minimum qualitative property on the system in order for it to be output stabilizable if $AW \subset W$ is then that $C_\Lambda \cap W^{\perp}$ belongs to the controllable space of the pair $(A, X_0 B)$.

4.11 Review of existing controllability criteria - Our results in perspective.

Although a large amount of literature on systems with hereditary dependence has been generated the results are typically on existence, uniqueness of solutions and well-posedness. There is a dearth of research focussed on controllability of such systems. For systems of the type considered in section 4.8.2 and in a Hilbert space setting results similar to ours and concurrently were obtained by M. Delfour and S. Mitter [Dl]. Functional expressions as in section 4.8.2 are not very satisfactory.

One would like to obtain criteria for controllability not involving the kernels or semigroups but solely based on data provided by the model. Our treatment in section 4.8.3 may be considered as a contribution in that direction. The calculation of the spectrum involves finding the zero's of an entire function which can be obtained often simply via Laplace transforms. The test for functional controllability is essentially reduced to a test in a countable number of finite dimensional spaces. The fact that functional controllability is equivalent to what was called "spectral" controllability rests on the basic fact that the systems under consideration have a complete set of root vectors. To our knowledge no necessary and sufficient condition of this type is available for such general systems. One also has to watch the definition of controllability being used. We review results by F.M. Kirillova and S.V. Churakova [K2] and L. Weiss [W3] and extend somewhat both results.

Kirillova considers R^n-controllability, for systems of the type

$$\begin{cases} \dot{x} = Ax + Bx(t-h) + Cu(t), \quad t > 0 \\ x(t) \in R^n \\ u(t) \in R^p \quad \text{and } u(.) \text{ piecewise continuous} \end{cases} \tag{87}$$

If one forms the sequence of matrices P_j^r, $j=1,..2^{r-1}$, $r=1\dots n$ with $P_1^1 = C$, $P_{2j-1}^{r+1} = AP_j^r$, $P_{2j}^{r+1} = BP_j^r$ and define P as

$$P = \left[P_1^1 \,\vdots\, \cdots \,\vdots\, (P_{2j-1}^{r+1} \,\vdots\, P_{2j}^{r+1})_{\substack{r=1\dots n-1 \\ j=1\dots 2^{n-1}}} \,\vdots\, \cdots \right] \tag{88}$$

and similarly Q_j^k, j=1...k, k=1...n with

$$Q_1^1 = C \; ; \; Q_j^{k+1} = AQ_o^k + BQ_{j-1}^k, \left.\begin{matrix} k=1\ldots n \\ j=1\ldots k \end{matrix}\right\} \; ; \; Q_j^k = 0, \left\{\begin{matrix} j > k \\ y = 0 \end{matrix}\right. \tag{89}$$

and

$$Q = [Q_1^1 \,\vdots\, Q_1^2 \,\vdots\, \ldots \,\vdots\, Q_n^n] \quad \text{then} \tag{90}$$

Theorem [K2]

A necessary condition for R^n-controllability is rank $P = n$.

A sufficient condition for R^n-controllability is rank $\phi = n$, and for $n \leq 2$ rank P = rank O. ∎

Weiss [W3] extends Kirillova's result to the case A, B, C are timevarying continuous matrices. The admissible controls are relaxed to $u(.) \in \mathcal{L}_{loc}^{\infty}(0,\infty; R^{p})$. Weiss developed a sufficient condition for R^n-controllability patterned after Kirillova.

Let $Q(t) = [Q_1^1(t) \,\vdots\, Q_1^2(t) \,\vdots\, \ldots \,\vdots\, Q_n^n(t-(n-1)h)]$ where $Q_j^k = 0$ for $j = 0$, $j > k$.

$$Q_1^1(t) = C(t), \; Q_j^{k+1}(t) = \frac{d}{dt} Q_j^k(t) - A(t+(j-1)h)Q_j^k(t) - B(t+(j-1)h)Q_{j-1}^k(t) \quad j=1\ k, k=1\ldots n \tag{91}$$

then rank $[Q(t_1)] = n$ implies R^n-controllability. If some additional restrictions are imposed on $B(t), C(t)$ this even implies functional controllability. Such restrictions are of the type $R(C(t)) \supset R(B(t))$ for $t \in [t_1, T_1+h]$ if the solution $x(t_1, t_o; \phi, v_{[t_o,t_1]}) = 0$ for any ϕ and some $t_1(\phi)$.

Weiss' result can be easily extended to a countable number of finite delays using the same method. The only real difficulty is in the derivation of the recurrence formula (91).

Consider the system

$$\dot{x}(t) = \sum_{j=0}^{L} A_j(t)x(t-h_j) + C(t)u(t) \qquad (92)$$

with $\quad 0 = h_o < h_1 < \ldots < h_L < \infty \qquad (93)$

$A_j(.)$, $C(.)$ continuous matrices.

We introduce the multi-index $r = [r_1 \ldots r_L]'$ (94)

and the delay vector $\quad h = [h_1 \ldots h_L]$ (95)

and define $rh = hr \stackrel{\Delta}{=} \sum_{i=1}^{L} r_i h_i$ (96)

As before define now $Q_r^k = 0$ for $k = 0$, $k > n$ or if for some i, $r_i > k$ or for some i, $r_i = 0$.

If $r_i = j$ for $i = 1,\ldots L$ we will denote r by j. If $r_i = 0$, $i = 1,\ldots,j-1,j+1,\ldots L$ and $r_i = 1$, $i = j$ we will write $r = e_j$.

One can now set up a recurrence formula similar to (91) namely:

$Q_1^1(t) = C(t)$

$$Q_r^k(t) = \dot{Q}_r^k(t) - \sum_{i=0}^{L} A_i(t+(r-1)h)Q_{r-e_i}^k \qquad (97)$$

for $k \geq r_i$ for all i, and $k \leq n$.

Before stating the result in a scholium, the notion of "break-point" is introduced.

<u>Definition of a break point</u>

A point $t_1 \in [0,T]$ is called a breakpoint if $t_1 = \{T-rh\}$ for some value of multi-index r, $r = (r_1,\ldots,r_L)$ where $r_i \in I^+$ $i = 1,\ldots,L$.

One can order the break points. Let t_i correspond to multi-index r^i.

<u>Scholium 9.</u> A sufficient condition for R^n-controllability of system (92) is rank $Q(T) = n$ for some $T > 0$ where

$$Q(T) = [Q_1^1(T) \mid \dots \mid Q_r^1(T-(r-1)h) \ \dots \ Q_r^n(T-r-1)h) \quad]_{r\varepsilon\alpha}$$

where α represents all values of the multi-index r such that $T-(r-1)h$ is a break point. ∎

<u>Proof</u> : Appendix D.

Other controllability criteria for very restricted systems have been developed. We only mention work by Popov [P 3] and Choudhury [C1].

CONCLUSION and Suggestion for Further Research.

This thesis has been concerned with qualitative properties of linear Infinite Dimensional Systems. One main contribution lies in the generality of the approach to system theory. The system considered includes as special cases D.P.S., F.D.S. and a very general class of delay systems described by functional differential or differential difference equations. It became soon obvious that one of the factors of main importance is the type of spectrum of the system operator, and that certain spectral configurations may or may not exclude certain phenomena. The configuration indeed largely determines the "type" of semigroup ϕ generated by the system operator and this ϕ codetermines the properties of the solution x(t) through the variation of constants formula for operator-differential equations. One new phenomenon discussed was the "property-gap": a system has a certain qualitative property, but this property is only fully established after a certain finite time has elapsed. Results in II.2.4, II.2.5 and III.3.4 then state conditions on the system operators A, B, and C under which no such gap occurs (in case of controllability respectively observability). If a gap can occur, it is of interest to calculate its "width". This was done for the property of controllability (II.2.7) where the "width" was identified with the "type" of an entire function. The gap-phenomenon is possible only for I.D.S. We looked then for I.D.S. which can be considered as the simplest extension of F.D.S. and for which a gap is not excluded: systems obtainable as an infinite direct sum of F.D.S. The corresponding spectrum was called "normal". Such class of systems includes a class of d ffusive and oscillatory type D.P.S. and the most general class of linear time invariant delay-systems known to be useful. The search for a method to specify the gapwidth lead to a new technique to obtain explicitly the control driving a state to zero. Applications to time-optimal and minimum norm control are readily obtainable extensions. The interpolation formulae discussed is available for F.D.S. in the form of the Hermite-formulae. As important side results, we only mention the controllability criteria in II.2.5.2, II.2.6, and II.2.7.1.

Worthwhile results on "partial" observers for I.D.S. are obtained in III.3.7. If it is desired that the observer be a F.D.S., the state-estimation-error is not just a "transient effect" but is "driven" by a function of the state-component of the "neglected modes". Minimization of the error may be tied to the output-sensor-location problem as indicated.

Chapter IV considers delay-systems. Several good controllability results are presented in IV.4.8. The results on point-wise controllability in IV.4.11 are of independent interest (derivation is in the appendix). Minimal conditions on controllability in order for the system to be state- or output-stabilizable are stated in IV.4.9, IV.4.10. These results are new and as yet not available even for F.D.S.

As a suggestion for further research, I can only state that the book [D2, part III] on spectral operators (which unfortunately appeared too late to influence this work) provides the background to extend all this theory for "operators with normal spectra" to "spectral operators". This would allow a more compact and general treatment than the one given in Chapter II and Chapter III.

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APPENDIX A - MATHEMATICAL NOTIONS

In Appendix A differential equations of the type $\frac{dx(t)}{dt} = Ax(t) + h(t)$ are discussed for existence and uniqueness of solutions and more specifically for the properties of the solution $x(t)$. The study of these properties is after all the major topic of this thesis. The system-operator A will be considered time-independent, the only case which is, thus far, easily amenable to a solution using the results of semigroup operator theory. Conditions on A will be discussed under which strongly continuous (C^o-) and analytic semigroups and C^o- unitary groups are generated describing the solution. These characterizations of the semigroup play a major role in chapters II and III. The definition and a generation theorem for C^o- contraction semi-groups by a dissipative operator A will be stated. Stabilizability (pole-relocation) in chapters III and IV will be the system property indicating the possibility for transforming a non-dissipative into a dissipative operator, thus changing the character of the solution.

Mathematical Notions

Throughout this work the (mostly unbounded) operator A with domain $D_o(A)$ will be <u>closed</u> or at least <u>closable</u> and the <u>infinitesimal generator</u> of an, at least, <u>C^o- semigroup</u> which may be in addition <u>contractive</u>, or <u>analytic</u> or in fact be a <u>group</u>, which can then be <u>unitary</u>. Conditions on A to be the infinitesimal generator of any of these special C^o- semigroups are summarized in generation theorems.

A.1 - Definitions

Let X and Y denote Banach Spaces.

Definition 1. (Closed operator). An operator A on X to Y is said to be closed if its graph $G(A) \triangleq \{(f,Af) | f \in D_o(A)\}$ is a closed subspace in $X \times Y$, or, equivalently, if $\{f_n\} \subset D_o(A)$, $f_n \to f$, $Af_n \to g$ implies that $f \in D_o(A)$ and $Af = g$.

Example [S1]

Let $X = Y = C[0,\infty)$ and $A \triangleq z\frac{d}{dz}$ with $D_o(A) \triangleq \{x \in C'[0,\infty) | z\frac{dx}{dz} \in C[0,\infty)\}$. Then A is closed.

Definition 2. (Closure of an operator). An operator A as in definition 1 is called closable if $\overline{G(A)}$ is a graph i.e. $(0,y) \in \overline{G(A)}$ implies $y = 0$. $\overline{G(A)}$ is then the graph of an operator denoted by $\overline{A}$ which is called the closure of A.

Example

Let $X = Y = L^2[0,1]$. Consider the operator A defined from $Ax = \frac{d^2x}{dz^2}$ for $x \in C^2[0,1]$. Then A is not closed. In order to obtain its closure we close its graph and consider the operator $\overline{A}$ corresponding to that graph. Then $\overline{A} = \frac{d^2}{dz^2}$ with $D_o(\overline{A}) = \{x \in L^2[0,1] / x$ and $\frac{dx}{dz}$ are absolutely continuous a.e. on $[0,1]$ and $\frac{d^2x}{dz^2} \in L^2[0,1]\}$.

<u>Definition 3</u>. (C^o-semigroup). A C^o-semigroup ϕ on X is a family of bounded operation $\phi = \{\phi(t)\}_{t\geq 0}$ on X satisfying $\phi(t)\phi(s) = \phi(t+s)$, $\phi(0) = I$, $\phi(\cdot)x \in C(R^+,X)$ for every $x \in X$, (i.e. $\phi(\cdot)x : R^+ \to X$ is continuous).

<u>Definition 4</u>. (C^o-contraction semigroup) A C^o-contraction semigroup ϕ on X is a C^o-semigroup on X such that $||\phi(t)|| \leq 1$ for $t \in R^+$.

<u>Definition 5</u>. (Infinitesimal generator of a C^o-semigroup). A is said to be the infinitesimal generator of the C^o-semigroup ϕ if $Ax = \lim_{t\to 0} \frac{\phi(t)x-x}{t}$, $D_o(A)$ being the linear subspace of X for which this limit exists.

<u>Definition 6</u>. (Unitary operator). A bounded operator U on a Hilbert space is called unitary if its adjoint U^* equals its inverse U^{-1} i.e. $U^* = U^{-1}$ or iff U is isometric and onto.[G3].

<u>Definition 7</u>. (Analytic semigroups of a certain "type"). In order to make a succinct statement of the generation-theorem for analytic C^o-semigroups possible we define first the class of generators $G_A(\theta)$ generating analytic semigroups ϕ of type $(\theta - \frac{\pi}{2})$. Let $\Sigma_\theta \triangleq \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg\lambda| < \theta\}$.

<u>Definition 7a</u>

A on a complex Banach space X is said to belong to class $G_A(\theta,M)$, $M \geq 1$, $\pi/2 < \theta \leq \pi$, if A is closed and densly defined and for all $\lambda \in \Sigma_\theta$, $\lambda \in \rho(A)$ and $||(\lambda-A)^{-1}|| \leq \frac{M}{|\lambda|}$.

<u>Definition 7b</u>

$A \in G_A(\theta)$, $\pi/2 < \theta \leq \pi$ if for each ε, $0 < \varepsilon < (\theta-\pi/2)$ there exists a $M_\varepsilon > 1$ such that $A \in G_A(\theta-\varepsilon, M_\varepsilon)$.

Definition 7c

Let $0 < \alpha \leq \pi/2$, $M \geq 1$.

$\phi = \{\phi(t) : t \in \Sigma_\alpha \cup \{0\}\}$ satisfying

1) $\phi(t)\phi(s) = \phi(t+s)$ for all $t,s \in \Sigma_\alpha$

$\phi(0) = I$

2) the complex valued function $\langle y, \phi(\cdot)x \rangle$ is analytic on

Σ_α for all $x \in X$, $y \in X^*$.

3) $\lim_{\substack{t\to 0 \\ t\in\Sigma_{\alpha-\varepsilon}}} \phi(t)x = x$ for all $x \in X$, $\varepsilon \in (0,\alpha)$.

4) $\phi(t)X \subset D_o(A)$ and $||\phi(t)|| \leq M$, $||tA\phi(t)|| \leq M$ for all

$t \in \Sigma_\alpha$

is said to be an analytic semigroup of type (α,M).

Definition 7d

$\phi = \{\phi(t) : t \in \Sigma_\alpha \cup \{0\}\}$

is an analytic semigroup of type α where $0 < \alpha \leq \pi/2$ if for each ε, $\alpha > \varepsilon > 0$, there exists an $M_\varepsilon \geq 1$ such that $\{\phi(t) : t \in \Sigma_{\alpha-\varepsilon} \cup \{0\}\}$ is an analytic semigroup of type $(\alpha-\varepsilon, M_\varepsilon)$.

We next specify conditions under which A generates one of the types of semigroups mentioned in definitions 3, 4, 6, 7.

A.2 - Generation Theorems

Theorem 1. [Hille-Yosida]

A is the infinitesimal generator of a contractive C^o-semigroup ϕ iff A is closed, densily defined, and for each $\lambda > 0$, $\lambda \in \rho(A)$ and

$||\lambda(\lambda-A)^{-1}|| \leq 1.$

An alternative form not using the resolvent but instead that A is dissipative is expressed in the following theorem.

Theorem 2. [Lumer-Phillips]

A on Hilbert space H generates a contractive C^o-semigroup if $D_o(A)$ is dense, $Re\langle Ax,x\rangle \leq 0$ for all $x \in D_o(A)$ and $(0,\infty)\cap \rho(A) \neq \phi$.

Theorem 3. [Hille-Phillips; Feller-Mijadera]

A is the infinitesimal generator of a C_o-semigroup iff

1) A is closed and densily defined

2) there exist constants $M \geq 1$ and ω_o such that $\lambda \in \rho(A)$ for $\lambda > \omega_o$ and

3) $||\prod_{i=1}^{n}(\lambda_i-\omega_o)(\lambda_i-A)^{-1}|| \leq M$, $n=1,2,3\ldots$ $\lambda_i > \omega_o$

or 2') there exist constants M, ω_o such that $\lambda \in \rho(A)$ for $\lambda > \omega_o$

3') $||(\lambda-\omega_o)^n(\lambda-A)^{-n}|| \leq M$, $n=1,2,3\ldots$, $\lambda > \omega_o$

In both cases $||\phi(t)|| \leq Me^{\omega_o t}$, $t \in R^+$.

For the generation of groups this theorem can be adapted to read :

Theorem 4. [Hille-Phillips]

A is the infinitesimal generator of a C_o-group ϕ iff

1) A is closed and densily defined

2) there exist constants M, ω_o such that $\lambda \in \rho(A)$ whenever λ is real, $|\lambda| > \omega_o$ and

3) $||(|\lambda|-\omega_o)^n(\lambda-A)^{-n}|| \leq M, \; |\lambda| > \omega_o, \; n = 1,2,\ldots$

In this case $||\phi(t)|| \leq Me^{\omega_o|t|}, \; t \in R.$

Remark: We point out that the generation theorems require the spectrum of A to be restricted to a left half plane for semigroups and to a symmetric vertical strip in the complex plane in the case of generation of groups.

We will, in connection with hyperbolic systems be interested in C_o-unitary groups. We state in this context the well-known theorem by Stone [54].

Theorem 5.

A is the generator of a C_o-unitary group iff A is skewadjoint (i.e. if $A^* = -A$ or iA is selfadjoint). ▮

Finally on the generation of analytic C^o-semigroups:

Theorem 6.

A generates an analytic semigroup of type $(\theta-\pi/2)$ iff $A \in G_A(\theta)$. ▮

Remark: It may be shown [G3] that if ϕ is analytic $\phi(t)(X) \subset \bigcap_{n=1}^{\infty} D_o(A^n)$ and that the latter set is dense in X.

Examples [Y1, p 243]

Let the semigroup ϕ on $C[0,\infty)$ be $(\phi(t)x)(z) = x(t+z)$. Then the generator A is defined by $Ax(z) = \frac{d}{dz}x(z)$ for $x \in D_o(A)$ $D_o(A) = \{x \in C[0,\infty) \mid x \text{ and } \frac{dx}{dz} \in C[0,\infty)\}$. Similarly $\frac{d^2}{dz^2}$ on $C(-\infty,+\infty)$ is the generator of the integral operator associated with the Gaussian kernel

$$(\phi(t)x)(z) = (2\pi t)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-(z-v)^2/2t} x(v)\,dv \quad \text{if } t > 0$$

$$= x(z) \text{ for } t = 0 .$$

A3. Operator Differential Equations

Consider the differential equation

$$\dot{x}(t) = Ax(t) \quad , \quad t \geq 0, \quad x(t) \in H_x \text{ ,a Hilbert space}$$

$$x(t=0) = x_o$$

Let A be the infinitesimal generator of a C^o-semigroup ϕ on X. If $x_o \in D_o(A)$ then $\phi(t)x_o$ is clearly a solution. It is unique in the strong sense namely $||\phi(t)x_o - x_o|| \to 0,\ t \to 0^+$. Suppose one considers next

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (A1)$$

$$x(t=0) = x_o$$

If $Bu(t) \in C(R^+, D_o(A))$, $ABu(t) \in C(R^+, H_x)$ and $x_o \in D_o(A)$ or equivalently $Bu(t) \in C^1(R^+, H_x)$ and $x_o \in D_o(A)$ then a solution exists, is unique, and given by the variation of constants formula (see [C3])

$$x(t) = \phi(t)x_o + \int_0^t \phi(t-\sigma)Bu(\sigma)\,d\sigma \qquad (A2)$$

The solution thus obtained is strongly differentiable (is absolutely continuous). Very often one is satisfied with solutions satisfying (A1) in a weaker sense. For example a reasonable definition of a solution to (A1) on [0,T] is a weaker sense is that x(t) satisfies

$$\frac{d}{dt}\langle x(t),y\rangle = \langle x(t),A^*y\rangle + \langle Bu(t),y\rangle \quad \text{a.e. for } t \in [0,T] \text{ and all } y \in D_o(A^*).$$

Clearly the requirements on Bu(t) can then be relaxed. It was shown in [Bi] that, in order for (A2) to be a solution to (A1) in the above weak sense and $x \in L^2(0,t;H_x)$, minimal assumptions on Bu(t) are that $\langle Bu(t),y\rangle$ is a measurable function in t for all $y \in H_x$ and $\int_0^T ||Bu(t)||^2 dt < \infty$.

APPENDIX B - SPECTRAL THEORY

Spectral theory, especially for non-self-adjoint operators is one of the hardest and most in-accessible branches of functional analysis. We will expose only the notions used in later chapters and give some examples. The notion of spectral projections will be discussed, its relation to the Riesz-integral and the concept of a reduction of an operator. Some theorems are presented for operators with normal spectra. It is shown as illustration how a non-self adjoint operator with normal spectrum may be obtained by perturbing a self-adjoint operator.

B.1. Projections in Banach Spaces and Hilbert Spaces

Definition 1. (Direct Sum)

A Banach space B is a direct sum of two subspaces M and N (written $B = M \oplus N$) if

$$M \cap N = \phi$$

$$f \in B \rightarrow f = g + h, \quad g \in M, \quad h \in N$$

Such representation is clearly unique. One calls M and N a complementary pair.

Definition 2. (Projection)

A bounded linear transformation P on Banach space B satisfying $P^2 = P$ is called a projection.

To a complementary pair (M,N) in B we can associate a projection P by $Pf = g$, $f = g + h$, $g \in M$, $h \in N$. Clearly P is linear, closed and since defined over all of B, it is bounded. P is idempotent, and in M is identity, in N it is 0. Conversely, a bounded linear idempotent transformation P from space B into B defines an associated complementary pair by

$$M = \{f : Pf = f\}$$
$$N = \{f : Pf = 0\}$$

and the projection associated to (M,N) is P. Indeed, $f \in M$ implies $Pg = f$ for some $g \in B$, hence $M = \text{range}(P)$ and $M = \text{range}(I-P)$. Clearly M,N are linear, closed and $M \cap N = \phi$ $f = Pf+(I-P)f$ with $Pf \in M$, $(I-P)f \in N$; hence $M \oplus N = B$.

Theorem 1.

Let P be a projection in B and let (M,N) be its associated pair. Let P* denote its adjoint defined on B*, the dual of B. Then P* is a projection with (M*,N*) as its associated pair and $M^* = N^{\perp}$, $N^* = M^{\perp}$. ∎

Partial orderings in projections.

A binary relation > introduces a partial ordering in a set S provided

$$a > a, \quad \forall\ a \in S$$

$$\left.\begin{array}{l} a > b \quad , \quad b > c \Rightarrow a > c \\ a > b \quad , \quad b > a \Rightarrow a = b \end{array}\right\} a,b,c \in S$$

Consider the set S of all projections acting on B. We say $P_1 > P_2$ if $M_1 \supset M_2$, $N_1 \subset N_2$. It is easy to see that P_1P_2 is a projection if P_1 and P_2 commute. If $P_1 > P_2$ then P_1 and P_2 commute and $P_1P_2=P_2$. Conversely if $P_1P_2 = P_2P_1 = P_2$ then $P_1 > P_2$. These notions are relevant when discussing spectral resolutions.

Reduction of a bounded linear transformation.

Let S be a subspace in B. Let TS be its image under a bounded mapping $T : B \to B$. Clearly TS may be anywhere in B.

Definition 3. (Reduction)

Given the complementary pair (M,N) of closed linear manifolds. Let $T \in \mathcal{L}(B)$. Suppose $\begin{cases} T(N) \subset N \\ T(M) \subset M \end{cases}$ then (M,N) is said to reduce B.

Scholium 1.

Let (M,N) be a complementary pair and P the associated projection. Let $T \in \mathcal{L}(B)$. Then (M,N) reduces T iff PT = TP. ∎

Theorem 2. [Lorch, L2]

Let B be reflexive and $T \in \mathcal{L}(B)$. Let H denote the nulspace of T, R its the closure of its range. Let T* denote the adjoint of T and N*, R* its nulspace and closure of the range, then

(a) $N^{\perp} = R^*$, $R^{\perp} = N^*$

(b) $N^{*\perp} = R$, $R^{*\perp} = N$. ∎

Reduction for unbounded spectral operators is done in a similar way.

Suppose T is unbounded, the complementary pair (M,N) reduces T if $T(M \cap D_o(T)) \subset M$ and $T(N \cap D_o(T)) \subset N$. Clearly, if for example M is finite dimensional and $D_o(T)$ is dense the first inclusion would reduce to $T(M) \subset M$. For a complete account on reduction of unbounded operators we refer to Dunford and Schwartz, Vol. III [D2].

Definition 4. (Projection in a Hilbert space)

A projection on a Hilbert space A is a bounded linear transformation P on M such that $P^2 = P$ and $P^* = P$.

Scholium 2.

In a Hilbert space a projection P is specified by one closed linear manifold and the norm of P is 1. ∎

Definition 5.

A family of projections E_λ on M, $\lambda \varepsilon (-\infty, +\infty)$ is called a resolution of identity if

1) $\lambda < \mu \quad E_\lambda < E_\mu$ in the sense of ordering of projections.
2) $\lambda_n \to -\infty \quad E_{\lambda_n} \to 0$; $\lambda_n \to +\infty \quad E_{\lambda_n} \to I$.
3) If $\{\lambda_n\}$ is an increasing sequence converging to λ, then $E_{\lambda_n} \to E_\lambda$.

All convergences are to be taken in the strong operator topology. Condition 3) indicates that, in fact, we have defined a left continuous resolution. We have $E_\lambda = E_{\lambda-}$ by definition. If $E_{\lambda+} \neq E_\lambda$ then the resolution has a jump at λ. $E_{\lambda-\varepsilon}$ means that the resolution is constant in the ε-neighborhood of λ. In separable Hilbert space the number of jumps is denumerable. In finite dimensions, the number of finite and everywhere else the resolution is constant.

One of the main results of spectral theory in Hilbert spaces related to self adjoint ($T = T^*$) and normal operators ($TT^* = T^*T$) is expressed in the following theorems:

Theorem 3a

Let E be a resolution of identity. The operator $\int \lambda dE_\lambda$ exists as a Lebesque-Stieltjes integral; convergence being in the uniform topology if E is a bounded resolution and in the strong topology otherwise. The result is a self adjoint operator. Conversely any self adjoint operator can be brought into this form. ∎

Theorem 3b

Let E be a bounded resolution of identity and $E_o = 0$, $E_{2\pi+} = I$. Then $U = \int_0^{2\pi} e^{i\theta} dE_\theta$ exists in the uniform topology and represents a unitary transformation. Conversely every unitary transformation can be brought into this form. ∎

Next consider normal operators (both self adjoint and unitary operators are normal).

Theorem 3c

Let E_λ and E_μ be resolutions commuting for each λ and μ. Then $T = \int r dE_r$, $r = \lambda + i\mu$, $dE_r = dE_\lambda \cdot dE_\mu$ is normal and every normal operator can be brought into this form. ∎

These theorems play an important role in the operational calculus [D2, T1].

B.2 Reduction of linear operators

A projection P reducing a linear operator A was characterized by the fact that PA = AP. We next present a class of operators obtained via the Riesz integral on a "spectral set", defined as a set s in the complex plane described by closed rectifiable curve in $\rho(A)$. Only nontrivial spectral sets will be considered, that is, spectral sets s for which $\rho(A) \cap s \neq \phi$.

Theorem 4. (The Riesz integral)

Let C be a curve describing a spectral set s of A. Let $P = \frac{1}{2\pi i}\int_C R(\xi,A)d\xi$, where $R(\xi,A)$ is the resolvent of A at ξ. Then P is a projection commuting with A and any transformation commuting with A. The pair of closed linear manifolds associated with P reduces A.

Proof : We provide a short elegant proof of this important theorem. Clearly P is linear, bounded and commutes as stated. Hence P reduces A if it is still shown that P is idempotent. Consider inside C a curve C' obtained by "allowable" deformation in $\rho(A)$ (i.e. rectifiable, enclosing the same part of $\sigma(A)$). Then

$$P = \frac{1}{2\pi i}\int_C R(\eta,A)d\eta = \frac{1}{2\pi i}\int_{C'} R(\xi,A)d\xi \ . \qquad (B1)$$

$$(2\pi i)^2P^2 = \int_C R(\eta,A)d\eta\int_{C'} R(\xi,A)d\xi$$

$$= \int_C\int_{C'} [R(\eta,A)-R(\xi,A)][\xi-\eta]^{-1}d\eta d\xi \ .$$

using Hilbert's identity

$$R(\lambda,A)-R(\mu,A) = (\lambda-\mu)\ R(\lambda,A).R(\mu,A) \qquad (B2)$$

$$(2\pi i)^2 P^2 = \int_{C'} R(\xi,A) \int_{C} (\eta-\xi)^{-1} d\eta d\xi - \int_{C} R(\eta,A) \int_{C'} (\eta-\xi)^{-1} d\xi d\eta$$

The second term on the rhs being zero this equals

$$2\pi i \int_{C'} R(\xi,A)\,d\xi = (2\pi i)^2 P \quad . \blacksquare$$

Corollary 1.

Let F,F' be two closed rectifiable curves in $\rho(A)$ of spectral sets s,s' with their associated projections P and P'. Then, if F,F' are exterior to each other, P and P' are orthogonal in the sense that $PP' = 0$. If F' is interior to F, the $PP' = P'$.

Proof : Clearly P and P' commute as follows directly from scholium 1. The rest of the proof is a trivial modification of the proof of scholium 1. ∎

Let M,N be the pair associated with the projection P. Assume that $T(N) \subset N$, $T(M) \subset M$. If we write $T' = T_{\restriction N}$, $T'' = T_{\restriction M}$, $P' = P_{\restriction N}$, $P'' = P_{\restriction M}$ where $T_{\restriction M}$ means the restriction of T to the closed subspace M, then clearly $P' = 0'$, $P'' = I''$ where I'' is the identity for M and $0'$ is the zero operator for N. Define similarly the resolvents $R'(\xi,T)$, $R''(\xi,T)$, then $R'(\xi,T)(I'\xi-T') = I'$. The same holds for $T''(\xi,T)$.

Corollary 2.

Let C be the curve defining the projection P.

Let T' and T'' be the corresponding reduction of T.

If ξ lies outside C, $\quad \xi \in \rho(T')$

If ξ lies inside C, $\xi \varepsilon \rho(T'')$.

In fact $\sigma(T')$ equals the portion of $\sigma(T)$ inside C. ∎

Bounded operators may have pointspectra, continuous spectra, and residual spectra. This is even more true for unbounded operators. Insofar as our projections can be identified with stepfunctions on measurable sets on the complex field (the measure being a Lebesque-Stieltjes measure) one can separate the discrete from the continuous measure. We give an application of this idea. First we state that a self adjoint operator cannot have a residual spectrum.

Theorem 5.

A self adjoint densily defined operator cannot have a residual spectrum, i.e. the range $R(\lambda I-A)$ is at least dense.

Proof : If $\lambda \varepsilon \sigma(A)$ it will be shown that it must belong to the pointspectrum $P\sigma(A)$ or the continuous spectrum $C\sigma(A)$ i.o.w. $\lambda \varepsilon \sigma(A)$, $\lambda \notin P\sigma(A)$, $\lambda \notin C\sigma(A)$ is impossible.

Indeed if there exists a $g \neq 0$ belonging to $R(\lambda I-A)^{\perp}$, then we have, for f in a dense set

$$<(\lambda I-A)f,g> = 0 = <f,(\lambda I-A)g>$$

and

$$<f,(\lambda I-A)g> = 0$$

implies

$$(\lambda I-A)g = 0$$

which means that λ belongs to the pointspectrum of A; which leads us to the contradiction. (We have used the fact that $\lambda \epsilon \sigma(A)$ implies that λ is real when A is self-adjoint.) ∎

Theorem 6. [Lorch, L2]

Let A be self adjoint on Hilbert space H with domain $D \subset H$. Then there exist two closed linear manifolds $\{M_1, M_2\}$, $M_1 = M_2^{\perp}$ (and $M_2 = M_1^{\perp}$) and $A_1 \stackrel{\Delta}{=} A_{\upharpoonright (M_1 \cap D)}$ has pure pointspectrum and $A_2 \stackrel{\Delta}{=} A_{\upharpoonright (M_2 \cap D)}$ had pure continuous spectrum. Furthermore A_1 is self-adjoint on M_1, A_2 is self-adjoint on M_2. ∎

B.3 Normal spectra

A large portion of this work considers operators with normal spectra. This notion will be defined shortly.

Consider an operator A on a complex Banach space X. A nonzero function $\phi \in X$ is called an eigenvector for an eigenvalue λ_o if $\lambda_o \in P\sigma(A)$ and $(A-\lambda_o 1)\phi = 0$. The eigenspace $\mathfrak{Z}_{\lambda_o}$ for λ_o is the subspace of X spanned by $\phi = 0$ and the eigenvectors for λ_o. The dimension of $\mathfrak{Z}_{\lambda_o}$ is called the proper or geometric multiplicity of λ_o.

A nonzero vector $\phi \in H$ is called rootvector of A for eigenvalue λ_o if $(A-\lambda_o I)^n \phi = 0$ for some positive integer n. The set of all rootvectors for λ_o, together with $\phi = 0$ form a lineal, that is a linear manifold that is not necessarily closed. The dimension $\nu_{\lambda_o}(A)$ of rootlineal $\mathfrak{M}_{\lambda_o}$ is called the algebraic multiplicity of eigenvalue λ_o. If $\nu_{\lambda_o}(A) < \infty$, $\mathfrak{M}_{\lambda_o}$ is closed, in which case it is called rootsubspace.

Definition 6. (Normal eigenvalue)

An eigenvalue λ_o of a linear operator A on a Banach space X is called normal if it is an isolated point of $\sigma(A)$, its algebraic multiplicity ν_{λ_o} is finite and two complementary subspaces (M,N) exist, $M = \mathcal{M}_{\lambda_o}$ and $N = (A-\lambda_o I)^{\nu_{\lambda_o}} X$.

If λ_o is normal and $\lambda_o \in \sigma(A)$, there exists a neighborhood $N_\varepsilon(\lambda_o)$ such that $N_\varepsilon(\lambda_o) \cap \sigma(A) = \{\lambda_o\}$. One has then that $\mathcal{M}_{\lambda_o} = P_{\lambda_o} X$, where P_{λ_o} is the Riesz integral

$$P_{\lambda_o} = \frac{1}{2\pi i} \int_{|\lambda-\lambda_o|<\varepsilon} R(\lambda,A)\,d\lambda \; . \tag{B3}$$

Using a residue formula this expression may be simplified in the case of normal spectrum to

$$P_{\lambda_o} = \frac{1}{\nu_{\lambda_o}-1!} \frac{d^{\nu_{\lambda_o}-1}}{d\lambda^{\nu_{\lambda_o}-1}} (\lambda-\lambda_o)^{\nu_{\lambda_o}} R(\lambda,A) \Big|_{\lambda=\lambda_o} \tag{B4}$$

Remark : This theory obviously is applicable to finite dimensional systems where it is easier to illustrate the notions that have been presented. This is done in the following example.

Example.

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ on $X = R^3$. Then $\sigma(A) = \{1,3\}$, $\nu_1 = 2$, $\nu_3 = 1$.

$$P_{\lambda=1} = \frac{d}{d\lambda} (\lambda-1)^2 R(\lambda,A) \Big|_{\lambda=1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

The rootspace of eigenvalue $\lambda = 1$ is

$$\begin{bmatrix} x_1 & -x_2 & -x_3 \\ -x_1 & +x_2 & -x_3 \\ & 2x_3 & \end{bmatrix} = \mathcal{N}(A-I)^2 \triangleq \mathcal{M}_1$$

and the eigenspace is $\begin{bmatrix} x_1 \\ -x_1 \\ 0 \end{bmatrix} = \mathcal{N}(A-I) \triangleq \mathfrak{Z}_1$. It is easily checked that P_1 is indeed a projection $(P_1^2=P_1)$ and that $(A-I)^2 P_1 R^3 = \underline{0}$ or

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & -x_2 & -x_3 \\ -x_1 & +x_2 & -x_3 \\ & 2x_3 & \end{bmatrix} = \underline{0} \ .$$

Similarly

$$P_{\lambda=3} = (\lambda-3)R(\lambda,A)\big|_{\lambda=3} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{x} = P_{\lambda=3}\underline{x} + P_{\lambda=1}x = \begin{bmatrix} \frac{1}{2}(x_1+x_2+x_3) \\ \frac{1}{2}(x_1+x_2+x_3) \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(x_1-x_2-x_3) \\ \frac{1}{2}(-x_1+x_2-x_3) \\ x_3 \end{bmatrix}$$

and $P_{\lambda=1}P_{\lambda=3} = \underline{0}$. ∎

In general $\lambda_o \in \sigma(A)$ is called an isolated singularity if there exists a neighborhood $N_\varepsilon(\lambda_o)$ such that

$$\sigma(A) \cap N_\varepsilon = \{\lambda_o\} \quad .$$

Since $R(\lambda,A)$ is analytic in $\rho(A)$ there is a Laurent expansion [T1, Y1]

$$R(\lambda,A) = \sum_{n=0}^{\infty} (\lambda-\lambda_o)^n A_n + \sum_{n=1}^{\infty} (\lambda-\lambda_o)^{-n} B_n \qquad (B5)$$

A_n, B_n are bounded operators defined by

$$\begin{cases} A_n = \frac{1}{2\pi j} \int_{|\lambda-\lambda_o|<\delta} (\lambda-\lambda_o)^{-n-1} R(\lambda,A)\,d\lambda & (B6) \\ \\ B_n = \frac{1}{2\pi j} \int_{|\lambda-\lambda_o|<\delta} (\lambda-\lambda_o)^{n-1} R(\lambda,A)\,d\lambda. & (B7) \end{cases}$$

This expression is valid for $|\lambda-\lambda_o| < \delta$ for any δ such that $\sigma(A) - \{\lambda_o\}$ lies outside the circle $\{\lambda : |\lambda-\lambda_o| < \delta\}$.

A_n, B_n satisfy the recurrence-relations

$$\begin{cases} (A-\lambda_o)A_{n-1} = A_n \\ (A-\lambda_o)B_n = B_{n+1} = (A-\lambda_o)^n B_1 \\ (A-\lambda_o)A_o = B_1 - I \,. \end{cases} \qquad (B8)$$

The <u>case of normal spectrum</u> corresponds to the case $B_m \neq 0$, $B_{m+1} = 0$, in which case all $B_n \neq 0$, $n \leq m$, and all $B_n = 0$ $n > m$, $m = \nu_{\lambda_o}$ (the algebraic multiplicity). It is easily seen from the definitions that B_1 is a Riesz integral-projection with range $\mathcal{N}(\lambda_o I-A)^m$ i.e.

$$(\lambda_o I-A)^m B_1 x = 0 \qquad \text{for all } x \in \mathcal{X} \;.$$

Of particular interest is the case of compact resolvents. If $R(\lambda,A)$ is compact for $\lambda = \lambda_o$ it can be shown by analytic extension that $R(\tilde{\lambda},A)$ is compact for $\tilde{\lambda}$ belonging to the connected part of $\rho(A)$ containing λ_o.

If $R(\lambda,A)$ is a compact operator, the spectum of A consists of isolated singularities which are not essential. In addition the range of B_1 for each singularity is finite dimensional.

The case of normal spectrum allows us to write out immediate the semigroup in function of the singularities of $R(\lambda,A)$ which coincide with the eigenvalues of A (Since $R(\lambda,A)$ in this case has no essential singularities the poles of $R(\lambda,A)$ are the eigenvalues of A).

A basic requirement is that $\sigma(A)$ lies in a left half plane. Let $\sigma(A) \subset \{\lambda : \mathrm{Re}\,\lambda < \omega_0\}$, ω_0 a real number. Then $T(t)$, the semigroup of A, is related to $\tilde{T}(t)$, the semigroup of $\tilde{A} \triangleq A-\omega_0 I$, by $\tilde{T}(t) = e^{-\omega_0 t}T(t)$. $\tilde{T}(t)$ is then a contraction-semigroup.

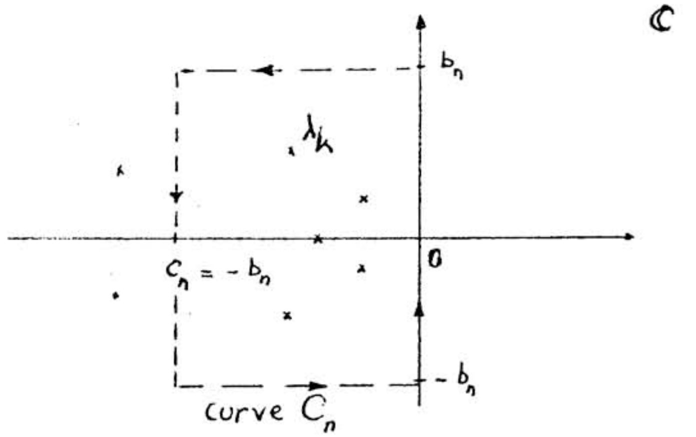
Since $\lambda^2 R(\lambda,\tilde{A})x-\lambda x$ converges to $\tilde{A}x$ for $\lambda \to \infty$ and $x \in D_0(\tilde{A})$

$$e^{\lambda^2 R(\lambda,\tilde{A})x-\lambda x} \xrightarrow[\lambda \to \infty]{} e^{\tilde{A}t}x \triangleq \tilde{T}(t)x$$

This intuitive formula for approximating $\tilde{T}(t)$ is the basis of the generation theorems for semigroups. Clearly for $\lambda \to \infty$ $\lambda \in \rho(\tilde{A})$ and $R(\lambda,\tilde{A})$ is analytic. Next we use values of λ where $R(\lambda,\tilde{A})$ is singular. Since the semigroup $\tilde{T}(t)$ is the inverse Laplace transform of $R(\lambda,\tilde{A})$ one may write

$$\underset{x\in D_0(\tilde{A})}{\tilde{T}(t)x} = \underset{b\to\infty}{\text{s-lim}} \frac{1}{2\pi j} \int_{-jb}^{+jb} e^{\lambda t} R(\lambda,\tilde{A})x\,d\lambda \qquad (B9)$$

Consider the contour integral over curve C_n for kernel $R(\lambda,\tilde{A})$



$$I_{C_n} = \int_{-jb_n}^{+jb_n} = \sum_{C_n} \text{residues}_{\lambda_k} - \int_{cb_n}^{jb_n+c_n} - \int_{jb_n+c_n}^{-jb_n+c_n} - \int_{-jb_n+c_n}^{-jb_n} \qquad (B10)$$

For each singularity in C_n one can use the expansion formula for $R(\lambda,A)$ to calculate the residue. Then

$$I_{\lambda_k} = \frac{1}{2\pi j}\int_{|\lambda-\lambda_k| < \delta} e^{t\lambda_k}R(\lambda,A)x d\lambda = \sum_{n=1}^{m_{\lambda_k}} \frac{t^{n-1}}{n-1!} e^{\lambda_k t} B_{k,n}x$$

$$= \sum_{n=1}^{m_{\lambda_n}} \frac{t^{n-1}}{n-1!} (A-\lambda_k)^{n-1} B_{k,1}x$$

Since the contribution of the three integrals on the right hand side of (10) goes to zero for $b_n \to \infty$ for example if

$$||R(\lambda,A)|| \leq \frac{m}{|\lambda|^r} \quad \text{for} \quad r > 0$$

one has

$$\tilde{T}(t)x = \lim_{b_n \to \infty} \sum_{C_n} I_{\lambda_k} \quad \text{and} \quad T(t)x = \lim_{b_n \to \infty} e^{\omega_o t} \sum_{C_n} I_{\lambda_k}$$

under that condition.

This convergence is not necessarily uniform in t if the left half plane $\{\lambda : \text{Re } \lambda < \omega_o\}$ covers part of $\{\lambda : \text{Re } \lambda \geq 0\}$ or $\omega_o \geq 0$. This does not contradict our intuition since then the system is unstable and T(t) grows without bound with t.

Example 1 Operators with normal spectrum.

The best-known examples of operator A having a normal spectrum are that of a self adjoint differential operator on a compact spatial domain. The best reference is Dunford and Schwartz, II [D2] and Courant and Hilbert [C2]. We only state one specific example.

Let $X = L^2(K)$, K being the rectangle $0 \leq x_j \leq \frac{\pi}{a_j}$, $j = 1,2,\ldots r$, A = Laplacian $\Delta = \sum_{j=1}^{r} \frac{\partial^2}{\partial x_j^2}$ and $D_o(A) = \{x \in L^2(K) \mid \Delta x \in L^2(K),\ x=0$ in the boundary of $K\}$. Then A is clearly self adjoint and has pure point spectrum. [Courant, Hilbert chapter 2, section 5.4 in [C2]]. The eigenvalues are $\lambda = -\sum_{j=1}^{r} a_j^2 n_j^2$ for $n_1, n_2, \ldots, n_r = 1,2,3,\ldots$ corresponding to eigenfunctions $\phi = \prod_{j=1}^{r} \sin a_j n_j x_j$. If $a_1^2, a_2^2, \ldots, a_r^2$ are linearly independent over the integers (e.g. if r = 1) then the multiplicity of all eigenvalues is 1. If $a_j^2 = a^2$ for $j = 1, \ldots r$ then $\lambda = -a^2 \sum_{j=1}^{r} n_j^2$, and the number of ways λ can be decomposed as a sum of r squares is unbounded with λ. Hence $\sup_\lambda \nu_\lambda = \infty$, where ν_λ is the algebraic multiplicity.

Example 2 (Sturm-Liouville operators over a compact interval of R.)

From reading the classical literature of mathematical physics one gets the impression that all relevant differential operators are symmetric and have only real pointwise spectrum away from zero. The reason for this is that a certain class of 2nd order differential operators known as Sturm-Liouville operators have been apt to describe a most important series of physical problems. The class is characterized by the fact that separation of variables applies reducing the p.d.e. to an infinite set of ordinary differential equations. Although the solution often cannot be given in closed form, it can be approximated in terms of elementary functions to arbitrary high degree of accuracy.

In this general class fall also types of 2nd order operators for which the elementary functions are respectively Bessel functions, Jacobi, Legendre, Chebyshev, Laguerre and Hermite polynomials and 2nd order operators known as Mathieu's and Gauss' differential operator. The solutions of the latter are the well-known Gauss-hypergeometric series.

Lanczos [L1] has pointed out that the eigenvalue problem of the most general 2nd order differential operator

$$Dx(z) = A(z)\ddot{x}(z) + B(z)\dot{x}(z) + C(z)x(z)$$

with $A(z) > 0$ on the interior of the domain for z can always be formulated as for a formally self adjoint operator.

This can be done by weighing the inner product defining the adjoint, or do a transformation on the independent variable z or do a transformation on the dependent variable x. In all these operations the eigenvalues and eigenfunction system remain unchanged (with different interpretations).

The most general 2nd order formally self adjoint differential operator can be presented by

$$Dx(z) = \frac{d}{dz}\left(A_1 \frac{d}{dz} x(z)\right) + Cx(z) \qquad \text{(b)}$$

If we transform z to z_1 by $z = \phi(z_1)$ then $dz = \phi'(z_1)dz_1 \triangleq \omega(z_1)dz_1$, and (b) becomes

$$Dx(z_1) = \frac{1}{\omega(z_1)} \frac{d}{dz_1}\left(A_1 \frac{d}{dz_1} x(z_1)\right) + Cx(z_1) \ . \qquad \text{(c)}$$

If $\omega(z_1) = \frac{1}{A(z_1)} e^{\int \frac{B}{A} dx}$ then (a) is transformed into (c) which is self adjoint (using as weighting function $\omega(z_1)$.).

It should be pointed out that we have as yet only formal self adjointness and that the boundary conditions have to be chosen appropriately.

<u>Example 3</u> (Unbounded perturbance of a self adjoint operator)

<u>Definition</u> : A (in general unbounded) linear operator B is A-compact, A being linear and closed if for some regular point λ_o of A the operator

$$B(A-\lambda_o I)^{-1}$$

is compact.

It can then be shown that $B(A-\lambda I)^{-1}$ is compact for every regular λ of A. (i.e. a point $\lambda \in \rho(A)$ or a normal eigenvalue.)

Then, if L is a self adjoint operator, and T is an L-compact operator such that

$p[(L-aI)^{-1}T(L-aI)^{-1}] < \infty$, a being any regular point of L, then the entire spectrum of $A = L + T$ consists of normal eigenvalues, all of them, except a finite number lying in sectors $-\varepsilon < \arg\lambda < +\varepsilon$, $\pi-\varepsilon < \arg\lambda < \pi+\varepsilon$ (where ε is a function of A) and the system of root vectors of A is complete.

In the above expression p(.) stands for the order of the operator within its brackets. If $\{s_n(Q)\}$ is the set of s-numbers of Q (if Q is compact its s-numbers are the eigenvalues of $(Q^*Q)^{1/2}$) then the order of Q equals the infimum of numbers r such that $\sum_{n=1}^{\infty} s_n^r(Q) < \infty$. Stronger statements on spectra of unbounded non-self adjoint operators and completeness of root vectors can be found in [Gl.]

Example 4

Another important example of an unbounded operator having normal spectrum arises in functional differential equations of delay-type. We refer to chapter IV for details.

APPENDIX C-

The Variation of Constants formula for a class of D.P.S.

In appendix A.3 the Cauchy problem in its homogeneous and non-homogeneous form where treated for an abstract differential equation

$$\dot{x} = Ax + Bu, \quad x(0) = x_o$$

The solution to this equation was called the variation of constants formula.

In this section some nontrivial results for parabolic and hyperbolic partial differential equations are cited as an application of the semigroup theory of Appendix A. For more details we refer to [G3].

C.1 Parabolic initial boundary value problems

The main result here is that if A is an at least weakly elliptic operator the semigroup generated is analytic. This is of vital importance for controllability. Control systems that can be classified as parabolic are for example diffusive-type systems. The nicest operators involved are self adjoint with constant coefficients. It can be shown that even if such operator A is perturbed by B and B (unbounded) satisfies certain regularity conditions with respect to A, the resultant semigroup is still analytic. The spectrum may then spread out over the plane however rather than be restricted to the real line.

We restrict ourselves to the following two statements.

Statement 1 [G3]

Let P(D) be a weakly elliptic homogeneous polynomial of degree 2m (m=0,1,2,...) with real coefficients, i.e.,

$$P(\xi) < 0 \quad \text{for all} \quad \xi \in R^n - \{0\}$$

Then $P(D)$ generates an <u>analytic</u> uniformly bounded semigroup $\{T(t)\}_{t>0}$ such that

- for all $f \in H_x$, $g \in H_x^*$, $\langle \phi(.)f, g \rangle$ is analytic on

$$\Sigma_{\pi/2-\varepsilon} = \{t \in \mathbb{C}\ ,\ t \neq 0,\ |\arg t| < \frac{\pi}{2} - \varepsilon\}$$

- for all $t \in \Sigma_{\pi/2-\varepsilon}$, $\quad \phi(t)H_x \subset D_o(A)$

$$||\phi(t)|| \leqslant M_\varepsilon$$

$$||tA\phi(t)|| \leqslant M_\varepsilon$$

- $\lim\limits_{\substack{t \to 0 \\ t \in \Sigma_{\pi/2-\varepsilon}}} \phi(t)f = f$ for all $f \in H_x$ and

$\phi(t)\phi(s) = (t+s)$ for all $t,s \in \Sigma_{\pi/2}$

$\phi(0) = I$

If e.g. $P(\xi) = -\sum\limits_{j=1}^{n} \xi_j^2$

then $\phi(t)f(x) = \sqrt{4\pi t} \int_{R^n} e^{\frac{-|x-y|^2}{4t}} f(y)dy$. Thus, parabolic p.d.e. of the type

$$x_t = P(D)x + Bu$$

$$x(0) = x_o$$

are governed by analytic semigroups which gives the corresponding system all kinds of regular properties. In particular, with respect to controllability at time t they behave very much like finite dimensional systems.

<u>Statement 2</u>. (Perturbed systems).

Let $x \in R^n$; $\alpha \triangleq (\alpha_1, \ldots, \alpha_n)$ where $\alpha_1, \ldots, \alpha_n \in \{0,1,2,\ldots\}$

$D^\alpha \triangleq D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$ and $D_j \triangleq i \frac{\partial}{\partial x_j}$

Let $P(D)$ be a strictly elliptic real homogeneous polynomial of order $2m$ $(m=1,2,\ldots)$.

Let $Q(x,\xi) = \sum_{|\alpha|<2m-1} b_\alpha(x)\xi^\alpha$ where $b_\alpha \in L^\infty(R^n)$, then $P(D)+Q(.,D)$ generates a semigroup analytic in the right half plane. So, we see that the perturbation $Q(.,D)$ with variable coefficients does not alter the basic properties of $P(D)$.

C.2 Second order hyperbolic initial boundary value problems.

We intend, while studying 2nd order hyperbolic systems, to keep in mind as an example the infinite dimensional oscillator, which, in turn, is a generalization of the lumped parameter two dimensional oscillator

$$\ddot{x} = -ax, \qquad a > 0$$
$$x(0) = x_{10} \tag{C1}$$
$$\dot{x}(0) = x_{20}$$

where $\dot{x} \triangleq \frac{dx}{dt}$.

The usual approach for solving system (11) is to reduce the equations to

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

with

$$\begin{bmatrix} x(t=0) \\ \dot{x}(t=0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

and the well-known solution

$$x(t) = x_{10} \cosh \sqrt{-a}\, t + \frac{1}{\sqrt{-a}} x_{20} \sinh \sqrt{-a}\, t .$$

The system considered here has a similar formal expression for the system equations and the solution. Consider

$$\ddot{x} = -Ax, \quad A : D_o(A) \subset X \to X \tag{C2}$$

$$x(0) = x_{10} \in X$$

$$\dot{x}(0) = x_{20} \in X$$

System (12) can be rewritten as

$$\dot{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \tag{C3}$$

or as

$$\dot{\underline{x}} = \mathcal{A}\underline{x}, \quad \underline{x}(0) = \underline{x}_o \tag{C4}$$

with the obvious definitions for $\mathcal{A}$, $\underline{x}$, $\underline{x}_o$. Assuming that $B \triangleq \sqrt{-A}$ is an infinite-simal generator of a C^o-group in $\mathcal{L}(\mathcal{X},\mathcal{X})$ and that $\sqrt{-A}^{-1}$ exists then the C_o-group expression for $\mathcal{A}$ is

$$e^{t\mathcal{A}} = \cosh(t\sqrt{-A}) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \sqrt{-A}^{-1} \sinh(t\sqrt{-A}).\mathcal{A} \tag{C5}$$

where the operator functions $\cosh t\sqrt{-A}$, $\sqrt{-A}^{-1} \sinh t \sqrt{A}$ are defined by, for instance, their power series

$$\cos t\sqrt{-A} = \sum_{k=0}^{\infty} \frac{(-A)^k t^{2k}}{2k!}$$

$$(-A)^{-\frac{1}{2}} \sinh t\sqrt{-A} = \sum_{k=0}^{\infty} \frac{(-A)^k t^{2k+1}}{2k+1!}$$

Extensive results in this direction were recently presented in ref. [F2]. We only state one theorem which indicates the right choice of topology for wellposedness. It is an adapted version of a theorem in [G4] from a Banach to a Hilbert space setting.

Theorem 1 [G4]

Let $\mathcal{A} \triangleq \begin{bmatrix} 0 & I \\ B^2 & 0 \end{bmatrix}$ with $D_o(\mathcal{A}) = D_o(B^2) \times D_o(B)$, and let B be the infinitesimal generator of a C^o-group in $\mathcal{L}(X,X)$ and $0 \in \rho(B)$. Then $\mathcal{A}$ is the infinitesimal generator of a C^o-group in $\mathcal{L}(Z,Z)$ given by (5), where $Z = Y \times X$ with norm $||.||$, $||{y \atop x}|| = (|||y|||^2 + |x|^2)^{1/2}$ and $Y = D_o(B)$ with norm $|||.|||$, $|||y||| = (|y|^2 + |By|^2)^{1/2}$ where $|.|$ is the norm in X.

The Cauchy-problem $\dot{\underline{x}} = \mathcal{A}\,\underline{x}$ for $\underline{x}(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$ with $x_{10} \in D_o(B^2)$, $x_{20} \in D_o(B)$ is well-posed and $x(.) : R \to D_o(B^2)$ is continuous. ∎

Remark :

In the proof of this theorem an expression for $||e^{\mathcal{A}t}||$ was given which we state for later reference:

$$||e^{\mathcal{A}t}|| \leq 4(|B^{-1}| + I)|\sinh(tB)| + 4|\cosh(tB)|. \qquad (C6)$$

A very important special case is obtained when A in (C2) equals C^2 and C is self adjoint. The solution is then described by a C^o-unitary group $\phi(t)$

$$\phi(t) = \cos t\, C.\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + C^{-1}\sin t\, C.\begin{bmatrix} 0 & 1 \\ -C^2 & 0 \end{bmatrix} \qquad (C7)$$

Here one does not need $0 \in \rho(C)$. The statement of theorem 1 remains valid if $Z = D_o(C) \times X$ is replaced by $\tilde{Z} = D_o(C)/(\text{kernel } C) \times X$. Before, we related generation of a C^o-unitary group to skew-adjointness of its generator. If iC is skew-adjoint, so that it generates a C^o-unitary group, the norm on Z(or $\check{Z}$) can be chosen so that

$\begin{bmatrix} 0 & 1 \\ -C^2 & 0 \end{bmatrix} \triangleq \mathcal{A}$ is skew adjoint. In fact $\left\|\begin{matrix} y \\ x \end{matrix}\right\| = (|iCy|^2+|x|^2)^{1/2}$ is the right norm [G3].

Example (Wave-equation)

Consider the system

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \quad , \quad x \in [0,1]$$

$$y(x,0) = \frac{\partial y}{\partial t}(x,0) = 0 \qquad (C8)$$

$$y(0,t) = \frac{\partial y}{\partial x}(1,t) = 0$$

To stress the importance of the right choice of topology it will be shown that the straight forward attempt to reduce this system to a first order system by introduction of a two component vector $v \triangleq \begin{bmatrix} y \\ \frac{dy}{dt} \end{bmatrix}$, $v \in X \times X$, $X = L^2(0,1)$ gives rise to difficulties with respect to wellposedness. Let the initial condition be

$$v_o(x) = \begin{bmatrix} \varepsilon\sin(2N-1)\,\frac{\pi}{2}\,x \\ 0 \end{bmatrix} \qquad (C9)$$

(i.e. only the Nth mode-component of y is nonzero). The solution v(x,t) is then

$$v(x,t) = \begin{bmatrix} \varepsilon\sin\omega_N x \,.\, \cos\omega_N t \\ -\varepsilon\omega_N \sin\omega_N x . \sin\omega_N t \end{bmatrix} \;,\; \omega_N \triangleq (2N-1)\,\frac{\pi}{2}\,.$$

Since

$$||v_o(x)||_{X\times X} = \varepsilon^2/2 \quad , \quad ||v(x,t)||_{X\times X} = \frac{\varepsilon^2}{2}\,[\cos^2\omega_N t+\omega_N^2\sin^2\omega_N t]$$

It is seen that the solution $v(x,t)$ does not depend continuously on the initial data, since $||v(x,t)||$ is arbitrarily large because of the term ω_N^2 not present in $||v_o(x)||$.

We next study wellposedness of the Cauchy problem (C7) (C9) in the topology of theorem 1 . Clearly $v_o(x) \in D_o(B^2) \times D_o(B)$ where $B = i\frac{\partial}{\partial x}$. Also

$$||v_o(x)||_Z^2 = |\varepsilon \sin\omega_N x|^2 + \varepsilon^2 |B\sin\omega_N x|^2 = \frac{\varepsilon^2}{2}(1+\omega_N^2).$$

and

$$||v(x,t)||^2 = |\varepsilon \sin\omega_N x.\cos\omega_N t|^2$$

$$+ |\varepsilon\omega_N \sin\omega_N x.\cos\omega_N t|^2$$

$$= \frac{\varepsilon^2}{2}[\cos^2\omega_N t+\omega_N^2\cos^2\omega_N t+\omega_N^2\sin^2\omega_N t]$$

$$\leq \frac{\varepsilon^2}{2}(1+\omega_N^2)$$

The conclusion is that future states have a norm not bigger the norm of the initial state. This result expresses the principle of energy conservation. What has been demonstrated is that the problem is wellposed for the indicated choice of topology and auxiliary variables.

As for parabolic systems, we will present two general statements about the Cauchy problem of a 2nd order hyperbolic system, specifying A and the spaces involved.

First we develop some notation.

Let $C^{\infty}(\Omega)$ be the set of infinitely complex valued functions on Ω, a bounded set in R^n.

Let $C_o^{\infty}(\Omega) \subset C^{\infty}(\Omega)$ have compact support in Ω.

Let $1 \leq \alpha_j \leq n$, $\alpha = (\alpha_1, \ldots, \alpha_j)$, $|\alpha| = \sum_{k=1}^{j} \alpha_k$ and $D^{\alpha} \triangleq \frac{\partial^j}{\partial z_{\alpha_1}, \ldots, \partial z_{\alpha_j}}$. Let $H_k = \{f \in C^{\infty}(\Omega) : D^{\alpha} f \in \mathcal{L}^2(\Omega)$, $|\alpha| \leq k\}$ for some positive integer k. Then $(f,g)_k = \sum_{|\alpha|=k} \int_{\Omega} D^{\alpha} f \overline{D^{\alpha} g} dx$ for f, $g \in C^{\infty}(\Omega)$ defines an inner product on H_k. Let $\mathcal{H}_k$ denote the completion of H_k under the norm induced by $(.,.)_k$.

For $f, g \in C_o^{\infty}$ define H_k^o and $\mathcal{H}_k^o$ similarly. $\mathcal{H}_k$, $\mathcal{H}_k^o$ are Hilbert spaces. (For k = 0 $\mathcal{H}_k^o = \mathcal{H}_k = \mathcal{L}^2(\Omega)$.)

Let

$$Ax(z) \triangleq (-1)^p \sum_{\tau=0}^{p} \sum_{|\alpha|=|\beta|=\tau} D^{\alpha}(a_{\alpha\beta}(z) D^{\beta} x(z))$$

where $\alpha\beta = (\alpha_1 \ldots \alpha_j \beta_1 \ldots \beta_k)$.

Assume the following regularity-assumptions

1) $a_{\alpha\beta}$ is complex valued and $\overline{a_{\alpha\beta}(z)} = a_{\beta\alpha}(z)$, $z \in \Omega$ and sufficiently often differentiable.

2)
$$\text{Re}\{\sum_{|\alpha|=|\beta|=p} a_{\alpha\beta}(z) \xi_{\alpha} \xi_{\beta}\} \geq k_o |\xi|^{2p}, \ k_o > 0$$

$$\xi \in R^n, \ z \in \overline{\Omega}$$

3) The boundary $\delta\Omega$ of Ω is smooth (at least of class C^{3p-2}).

An example of such operator is the n-dimensional $\Delta \triangleq \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i^2}$, $z \in R^n$ in the wave equation $\frac{\partial^2 x}{\partial t^2} = \Delta x$.

For this example

$$A = (-1)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial z_i}(a_{ij}(z)\frac{\partial}{\partial z_j}\ \cdot\ ) \quad \text{where } a_{ij}(z) = -\delta_{ij}$$

(Kronecker δ's)

Statement 1. [G4]

The Cauchy problem

$$\ddot{x}(t,z) = -Ax(t,z) \quad ; \quad \dot{x} \triangleq \frac{\delta}{\delta t}$$

$$x(0,z) = f_1$$

$$\dot{x}(0,z) = f_2$$

has a strong (i.e. $\mathcal{L}^2$) solution

$$x(\cdot,z): R \to \mathcal{H}_{2p} \cap \mathcal{H}^o_p \text{ if}$$

$$f_1 \in \mathcal{H}_{2p} \cap \mathcal{H}^o_p \text{ and}$$

$$f_2 \in \mathcal{H}^o_p$$

The solution is governed by a C^o-group. ∎

Statement 2

One can perturb A by P

$$Px(t,z) = \sum_{|\alpha| \leq p} \ell_\alpha(z) D^\alpha x(t,z)$$

where $\ell_\alpha(z)$ is an essentially bounded measurable function on $\Omega \in R^n$ and obtain the same conclusions. ∎

For applications of these statements to Maxwell's equations, elasticity, acoustics, see [W4].

APPENDIX D.

Point Controllability for Hereditary Systems with Discrete Delays

In this section Weiss' sufficiency condition for R^n-controllability for a system with one discrete delay is extended to systems with multiple discrete delays. Conceptually the method of derivation of the criterion is much like that for finite dimensional lumped systems. The kernel however satisfies a delay-equation which makes the case unusual. The challenge present in extending the result to a multiple-delay case is that of extending a criterion for a scalar valued to a vector valued case, the vector here being the delay vector. The difficulties involved are largely of an organizational nature.

This section is intended to complete and provide some details for Chapter IV, section 4.11.

Consider the delay-system

$$\dot{x}(t) = \sum_{j=o}^{N} A_j(t)\, x\,(t-h_j) + B(t)u(t) \qquad t \geq 0 \qquad \text{(D1)}$$

$0 = h_o < h_1 < \ldots < h_N < \infty$

$A_j(.)$ for $j = 0, \ldots N$ are $n \times n$ matrices

$B(.)$ is $n \times m$ matrix

All matric es are assumed to be sufficiently differentiable. In section 4.1 it was shown how to represent this system in the form section 4.1 (5)

$$\dot{x}(t) = \int_{-h_N}^{0} d\eta(t,\theta)\ x_t(\theta) + B(t)u(t) \qquad \text{(D1)'}$$

The solution was discussed in section 4.2, cor. 1 and remark 2, and was shown to be of the form

$$(x(\sigma = 0,\ \psi = 0,\ u))(t) = \int_{o}^{t} U(t,\ s)B(s)u(s)ds \qquad \text{(D2)}$$

where

$$\frac{\partial U(t,s)}{\partial\ t} = \int_{-h_N}^{o} d\eta(t,\theta)U(t+\theta,s) = L(t,U_t(.,s)) \quad t \geqslant 0$$

$$U(t,s) = 0 \quad \text{for } t < s \qquad \text{(D3)}$$

$$U(t,t) = I$$

In our context

$$L(t,\ U_t(\cdot,\ s)) = \sum_{k=1}^{N} A_k(t)\ U(t-h_k,\ s) \qquad \text{(D4)}$$

Let the space of admissible controls be $U_{ad} = L_2([0,\infty);\ R^m)$ the space of square integrable functions on $[0,\infty)$ satisfying

$\int_{o}^{\infty} |\ x(t)|^2\ dt < \infty$, where $|.|$ represents any norm in R^m.

Define the subspace $U_{ad,t} = L_2([0,\ t];\ R^m) \subset U_{ad}$

Define $F_t : U_{ad,t} \to R^n$ by $F_t\ . = \int_{o}^{t} U(t,s)B(s).ds$

Then the "adjoint" $F^*_t : R^{n^*} \to L_2([0,t];\ R^{m^*})$ is $F^*_t\ . = U(t,\ s)B(s)$.

Scholium 1

System (D1) is R^n-controllable if for some $t_1 > 0$ $R(F_{t_1}) = R^n$. ($R(F_{t_1} F^*_{t_1}) = R^n$) or $N(F^*_{t_1}) = \{0\}$.

Proof:

$$R(F_{t_1}) = R(F_{t_1} F^*_{t_1}) = R^n \text{ implies } (F_{t_1} F^*_{t_1})^{-1} \text{ exists.}$$

Let $x_1 = R^n$ be the "target point" for a trajectory corresponding to initial function $\varphi = 0 \in C$, then $u_{[0,t_1]} = F^*_{t_1} (F_{t_1} F^*_{t_1})^{-1} x_1$

accomplishes that $x(\sigma = 0, \varphi = 0, u_{[0,t_1]})(t_1) = x_1$ $\blacksquare$

The condition $F^*_{t_1}(y) = 0 \Rightarrow y = 0$ (D5)

or

$$y\, U(t,s)B(s) = 0 \quad \text{for all } s \in [0,t] \Rightarrow y = 0 \qquad \text{(D6)}$$

can be expressed as an algebraic condition not involving the kernel $U(t,s)$, using equations (D3) and (D4). The derivation of that condition is the main objective of this paragraph.

If (D.6) holds on $[0,t]$ then

$$\frac{\delta^m}{\delta s^m} \left[yU(t,s)B(s) \right] = 0 \quad \text{on} \quad [0,t]\ ,\ m = 0, 1, 2 \ldots \qquad \text{(D7)}$$

Although we have defined already the "adjoint" of F_t, we have not talked about the "adjoint" equation to (D1). Define the "adjoint" equation to (D1) to be

$$\frac{dy(s)}{ds} = -\sum_{k=1}^{N} y(s + h_k)\, A_k(s + h_k) \qquad (D8)$$

having a solution on $(-\infty, t + h_N]$. This adjoint is defined with respect to the pairing

$$(\psi, \phi, t) = \psi(0)\,\phi(0) - \sum_{k=1}^{N} \int_0^{h_k} \psi(\xi)\, A_k(t+\xi)\, (\xi-h_k)\, d\xi \quad , \textit{where}$$

$\psi \in ([o,\tau]; R^{n*})$, $\phi \in C([-\tau,o]; R^{n})$

If $(x_t)(\theta) = x(t + \theta)$, $t \geq 0$, $-h_N \leq \theta \leq 0$ represents the solution to (1) and $(y^t)(\theta) = y(t + \theta)$, $t \leq 0$, $\theta \varepsilon [0,h_N]$ represents the solution to (D8) then Hale shows that (y^t, x_t, t) = constant. This property may be considered as characterizing the "adjoint" solution. The solution y(t) to (D8) with initial function $\psi = y_{[0,h_N]} = 0$ $\varepsilon\, C([0, h_N]; R^{n*})$ and driving term $C(s)\, v(s)$ is

$$y(s) = \int_0^s V(s,t) C(t) v(t)\, dt \qquad (D8)'$$

The kernel $V(s,t)$ satisfies on $(-\infty, t]$ the equation

$$\frac{\delta}{\delta s} V(s,t) = -\sum_{k=0}^{N} V(s + h_k, t)\, A_k(s + h_k) \qquad (D9)$$

$V(s,t) = 0 \quad s > t.$

Moreover $\qquad V(s,t) = U(t,s) \qquad$ (D10)

Condition (D7) can now be elaborated upon using (D9) and (D10).

Special Case: A system with two delays. (N=2)

Define a set of matrix functions by the following recurrence relations

$$Q_{11}^{1}(t) = B(t) \qquad (D11)$$

$$Q_{r_1 r_2}^{k+1}(t) = \dot{Q}_{r_1 r_2}^{k}(t) - A_o(t) Q_{r_1 r_2}^{k}(t)$$

$$- A_1(t + (r_1-1)h_1 + (r_2-1)h_2) Q_{r_1-1, r_2}^{k}(t)$$

$$-A_2(t + (r_1-1)h_1 + (r_2-1)h_2) Q_{r_1, r_2-1}^{k}(t) \qquad (D11)'$$

and

$$Q_{r_1 r_2}^{k} = 0 \quad \text{for } k = 0,\ k > n,\ r_i = 0,\ r_i > k,\ i = 1, 2. \qquad (D12)$$

Lemma 1

$$\frac{\delta^m}{\delta s^m}\left[y\, V(s,t)B(s)\right] = \sum_{r_1=1\ r_2=2}^{n,n} V(s + (r_1-1)h_1 + (r_2-1)h_2,\ t) \cdot Q_{r_1 r_2}^{m+1}(s) \qquad (D13)$$

Proof

For m = 0 (D13) is satisfied using (D11), (D12).

For m = 1 $\quad y\, \dot{V}(s,t)B(s) + y\, V(s,t)\, \dot{B}(s) =$

$$- y \sum_{k=0}^{N} V(s+h_k,t)\ A_k(s+h_k)\ Q_{11}^{1}(s) + y\, V(s,t)\ \dot{Q}_{11}^{1}(s) \qquad (D14)$$

From (A11)' for k = 1

$$\varphi^2_{r_1 r_2}(s) = \dot{\varphi}^1_{r_1 r_2}(s) - A_o(s)\varphi^1_{r_1 r_2}(s)$$

$$- A_1(s + (r_1-1)h_1 + (r_2-1)h_2)\varphi^1_{r_1-1,r_2}(s)$$

$$- A_2(s + (r_1-1)h_1 + (r_2-1)h_2)\varphi^1_{r_1 r_2-1}(s).$$

or

$$\varphi^2_{11}(s) = \dot{\varphi}^1_{11}(s) - A_o\varphi^1_{11}(s)$$

$$\varphi^2_{11}(s) = \qquad - A_1(s+h_1)\varphi^1_{11}(s)$$

$$\varphi^2_{12}(s) = \qquad - A_2(s+h_2)\varphi^1_{11}(s)$$

So that (D14) equals

$$= y\ V(s,t)\varphi^2_{11}(s) + yV(s+h_1,t)\varphi^2_{21}(s)t + V(s+h_2,t)\varphi^2_{12}(s)$$

Using (D12) this is seen to be equal to the r.h.s. of (D13).

<u>For m + 1:</u>

Assume (D13) holds for m = p. The inductive proof will be complete if then (D13) holds for m = p+1. It is to be shown that

$$\frac{d}{ds}\sum_{r_1=1,r_2=1}^{n,n} V(s + (r_1-1)h_1 + (r_2-1)h_2,\ t)\varphi^{p+1}_{r_1 r_2}(s) =$$

$$\sum_{r_1=1,r_2=1}^{n,n} V(s + (r_1-1)h_1 + (r_2-1)h_2,\ t)\varphi^{p+2}_{r_1 r_2}(s)\ . \qquad (D15)$$

Let in the following equalities q stand for the expression

$s + (r_1-1)h_1 + (r_2-1)h_2$. The l.h.s. of (D15) then equals

$$\sum_{r_1=1,r_2=1}^{n,n} \dot{V}(q,t)\varphi^{p+1}_{r_1 r_2}(s) + V(q,t)\dot{\varphi}^{p+1}_{r_1 r_2}(s) =$$

$$\sum_{r_1=1,r_2=1}^{n,n} V(q,t)\dot{\varphi}^{p+1}_{r_1 r_2}(s) - V(q,t)\, A_o(s) - V(q+h_1,t)\, A_1(q+h_1)$$

$$- V(q+h_2,t)\, A_2(q+h_2) =$$

$$\sum_{r_1=1,r_2=1}^{n,n} V(q,t)\Big[\dot{\varphi}^{p+1}_{r_1 r_2}(s) - A_o(s)\varphi^{p+1}_{r_1 r_2} - A_1(q)\varphi^{p+1}_{r_1-1,r_2}(s) -$$

$$A_2(q)\varphi^{p+1}_{r_1,r_2-1}(s)\Big]$$

$$+ \sum_{r_1=1,r_2=1}^{n,n} V(q,t)\Big[A_1(q)\varphi^{p+1}_{r_1-1,r_2}(s) + A_2(q)\varphi^{p+1}_{r_1,r_2-1}(s)$$

$$- \sum_{r_1=1,r_2=1}^{n,n} V(q+h_1,t)\, A_1(q+h_1)\varphi^{p+1}_{r_1 r_2}(s) + V(q+h_2,t)\, A_2(q+h_2)\varphi^{p+1}_{r_1 r_2}(s)$$

The last equality was obtained by adding and subtracting terms. It can now be easily checked that the last two sums cancel out against each other using (D12). The first sum equals the r.h.s. of (D15) using (D11)'; which proves the lemma. ∎

Scholium 2

$$\frac{\delta^m}{\delta s^m}\; y\, V(s,t)B(s)\,\Big|_{(t-s_1h_1 - s_2h_2)^-} = y\, Q^{m+1}_{(s_1+1)+(s_2+1)}(t-s_1h_1 - s_2h_2) \qquad (D16)$$

Proof:

In (D13) all terms for which $(r_1-1)h_1 + (r_2-1)h_2 > s_1h_1 + s_2h_2$ disappear because $V(a,t) = 0$ for $a > t$. All terms for which $(r_1-1)h_1 + (r_2-1)h_2 < s_1h_1 + s_2h_2$ are zero from the evaluation of the m^{th} derivative at $(t - s_1h_1 - s_2h_2)^+$. The only remaining term is for indices r_1, r_2 such that $s_1h_1 + s_2h_2 = (r_1-1)h_1 + (r_2-1)h_2$. Since $V(t,t) = I$ this term equals

$$y\, Q^{m+1}_{(s_1+1) + (s_2+1)}\,(t-s_1h_1 - s_2h_2). \blacksquare$$

Let $t_i = t - (s_1^{\,i} - 1)h_1 - (s_2^{\,i}-1)h_2 \in [0,t]$ be called a breakpoint and $Q^k_{s^i_1 s^i_2}(t_i)$, $k = 1 \ldots, n$ be matrices generated by the recursive formulae (D11)' and (D12).

The following theorem then formulates, an algebraic condition which is sufficient for pointwise controllability.

Theorem 1

Let $Q(t)$ represent the matrix

$$Q(t) \triangleq \left[\cdots \vdots\; Q^1_{s^i_1 s^i_2}(t_i) \;\vdots\; Q^n_{s^i_1 s^i_2}(t_i) \;\vdots\; \cdots \right]_{t_i \,\in\, B_{[0,t]}}$$

where $B_{[0,t]}$ represents the set of breakpoints on $[0,t]$. Then if for some $t > 0$ rank $Q(t) = n$, the delay system is R^n-controllable at t.

Proof:

If not, let $y \neq 0 \in R^{n*}$ such that $y \in N(F^*_t)$ (scholium 1) or $y\,U(t,s)B(s) = 0 \quad \forall s \in [0,t]$ (formula D5, D6). Then using (D7) and (D10) this implies $y \frac{\delta^m}{\delta s^m} V(s,t)B(s) = 0$ for all $s \in [0,t]$ and $m = 0, 1, 2 \ldots$, Scholium 2 then gives $yQ^{m+1}_{s_1^i s_2^i}(t_i) = 0$ for all $t_i \in B_{[0,t]}$ and $m = 0, 1, 2 \ldots$ ($m < n$ using (D12)). Hence, $Q(t)$ cannot be of full rank. ∎

General Case - N delays

The extension from $N = 2$ to the general case is now only a matter of notation. The proofs carry through in exactly the same way. We preferred to treat first $N = 2$ to keep notation simple and proof proofs more transparent.

Let $r \stackrel{\Delta}{=} [r_1 \;, \ldots r_N]$ be a multi-index.

$h \stackrel{\Delta}{=} [h_1 \;, \ldots h_N]'$ the delay vector and $rh \stackrel{\Delta}{=} \sum_{i=1}^{N} r_i h_i$.

If $r_i = j$ for $i = 1 \ldots N$ we will write $r = j$ and if $r_i = 0$ for $i = 1 \ldots j-1, j+1, \ldots N$ and $r_j = 1$ we write $r = e_j$. In an analogeous way as in (D11), (D11)', (D12) define matrices Q^k_r , $k = 1, \ldots n$ by $Q^1_1(t) = B(t)$,

$$Q^{k+1}_r(t) = \dot{Q}^k_r(t) - \sum_{i=0}^{N} A_i(t + (r-1)h)\, Q^k_{r-e_i}(t) \qquad \text{(D17)}$$

$Q^k_r(t) = 0$ for $k > n$, $k = 0$, $r_i = 0$ or $r_i > h$ $i = 1, \ldots N$.

Then

Lemma: $\frac{\delta^m}{\delta s^m}$ y $V(s,t)B(s) = \sum_{r=1}^{n} V(s+(r-1)h,t)\, Q_r^{m+1}(s)$. ∎ (D18)

Theorem 2:

Let multi-index s^i correspond to breakpoint $t_i = t - (s^i - 1)h$ $\varepsilon [0,t]$ and $Q^k_{s^i}(t_i)$, $k = 1, \ldots n$ be matrices generated by (D17).

Let

$$Q(t) \triangleq \left[\ldots \;\vdots\; Q^1_{s^i}(t_i) \cdots\cdots Q^n_{s^i}(t_i) \;\vdots\; \ldots \right]_{t_i \,\varepsilon\, B_{[0,t]}}$$

Then rank $Q(t) = n$ for some $t > 0$ implies R^n-controllability at t.

Proof: similar to the proof in theorem 1. ∎