CONTROL OF DELAY SYSTEMS FOR MINIMAX SENSITIVITY

by

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ABSTRACT

The problem considered is the minimization of the maximum weighted sensitivity in the frequency domain for single input/single output linear time invariant systems with a linear time-invariant feedback compensator. The plant is considered to be a finite dimensional (rational transfer function) linear time-invariant system in cascade with a pure delay. The minimization is done in the $\mathcal{H}^{\infty}$ norm, which is the supremum norm for stable transfer functions evaluated on the imaginary axis. The sensitivity weighting function is taken to be a proper stable rational function bounded away from 0 at ∞.

We apply the theory of [Sarason 1967] to find solutions. Our method consists principally of solving an eigenvalue/eigenfunction problem by transforming it into a two-point boundary value problem.

In the case of a 1 pole/1 zero weighting function, when the plant has no right half plane poles or zeros, we have a complete solution to this problem. We find explicit expressions for the ideal closed loop sensitivity and feedback compensator. The optimal compensator is unstable and improper. When the plant has right half plane poles and zeros, we show how to solve the critical two-point boundary value problem, but we have not computed the explicit sensitivity and compensator as of yet.

In the case of more general rational weighting functions, the problem is partially solved. When the weighting function magnitude approaches 1 from above at high frequency, we show how to find the optimal sensitivity and compensator by solving a set of simultaneous transcendental equations. We show the general form of the sensitivity and compensator. The optimal compensator is unstable and improper.

When the weighting function magnitude approaches 1 from below at high frequency, the problem can have non-unique solutions, and we do not have a general solution. We solve a special case, and indicate the possibility of finding other all-pass solutions by means of a limit of quotients of functions related to the eigenfunctions. We show how to make the problem solvable by slightly modifying the weighting functions.

We show how to approximate the optimal weighted sensitivity using proper finite dimensional compensators.

We show that the optimal compensator for problems with a rational plant having no delay can result in an ill-posed feedback system. This is not the case when the plant model contains a delay. We also show that the $\mathcal{H}^{\infty}$ minimal sensitivity feedback system can generally have zero "delay margin."

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CHAPTER 1.

INTRODUCTION

A. Purpose of the investigation.

The goal of this work is to obtain and analyze explicit compensators for delay systems which achieve or approximate a closed loop sensitivity function minimal in the supremum ($\mathcal{H}^\infty$) norm. An underlying premise of this investigation is that all real systems contain delays, so that it is only by examining the optimum for such systems that an understanding can be obtained of how delays limit achievable performance.

We also regard the delay problems considered here as a first step towards considering similar design issues for other infinite dimensional plants.

B. Problem Considered.

We consider the single-input/single output $\mathcal{H}^\infty$ optimal sensitivity control problem formulated in [Zames 1981], but with plants of the form

$$P(s) = e^{-s\Delta} \cdot A(s) \cdot B^{-1}(s) \qquad (1.1)$$

where A and B are stable proper rational functions, and $\Delta > 0$.[1] (A brief summary of the theory of $\mathcal{H}^\infty$ appears in Chapter 2.) The block diagram in

[1] We assume throughout this paper that A(s) and B(s) have no zeros on the imaginary axis. The inclusion of such zeros in our work can be done exactly as in the case where the plant has no delay. See, for example, [Francis and Zames 1984, pp. 13-15].

Figure 1 shows the feedback system models we are considering.

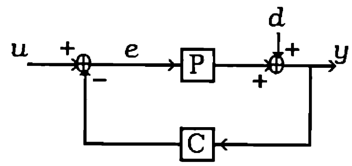

Figure 1. Feedback system considered.

The closed loop sensitivity S(s) is the transfer function from d to y. The weighted sensitivity X(s) for the weighting function W(s) is given by

$$X(s) = W(s)S(s) = W(s)[1+P(s)C(s)]^{-1} \tag{1.2}$$

The problem is to minimize the $\mathcal{H}^{\infty}$ norm of X(s) over all stabilizing proper feedbacks C(s).

C. <u>Discussion of Optimality Criterion</u>

The criterion of minimizing the norm of (1.2) is introduced and motivated in [Zames 1981, pp. 585-586]. In summary, the sensitivity criterion is intended to resemble the classical use of sensitivity specifications as a design requirement. Furthermore the use of the $\mathcal{H}^{\infty}$ norm allows the criterion to be handled in an analytical way, since there is mathematical theory which allows explicitly computation of the optimal compensators. In that paper it is recognized that there are other important design criteria besides sensitivity. It is pointed out in particular that the $\mathcal{H}^\infty$ norm may be useful in analyzing the effects of plant uncertainty on interconnections of systems. This is because the function space $\mathcal{H}^\infty$ is a Banach algebra, and hence the cascade of systems in the algebra belongs to the algebra. Therefore the norm of a cascade of transfer function models satisfies a norm inequality involving the individual norms of the components.

The motivation here in considering the sensitivity (1.2) is an attempt to extend existing results to systems with delays. Since the current theory allowing various $\mathcal{H}^\infty$ norm criteria to be optimized originated with this basic problem, it seems appropriate to also begin the extension with this problem.

The presence of the weighting function W(s) in (1.2) serves an essential purpose. A realistic requirement on closed loop sensitivity for a feedback system will impose more demanding tolerance on some frequencies than others. The weighting function serves to incorporate such specifications.

D. Reason for Delay in Model

We have three reasons for inserting a delay into the nominal plant model for the $\mathcal{H}^\infty$ problem: 1) as the beginning of the development of solutions to the $\mathcal{H}^\infty$ problem for more general infinite dimensional plants, 2) out of concern for the well-posedness of the optimal systems, and 3) as a potential design aid in accommodating uncertain excess phase in the plant, which is a characteristic of various systems of interest. We discuss each of these reasons in turn.

1. A start on $\mathcal{H}^\infty$ design for infinite dimensional plants. Our initial motivation for considering plants with delays was that we wanted to extend solutions to the optimal sensitivity problem existing for finite dimensional models to infinite dimensional ones. The delay seemed to be the simplest and most ubiquitous of such systems.

At the same time we were also concerned that we might run into difficulties because of the fact that our plant models are not continuous in the uniform operator topology[2] as the delay present in the plant varies. [Willems 1971, pp. 93-94] For stable systems this is the same topology induced by the $\mathcal{H}^\infty$ norm in which we have defined the sensitivity minimization problem.

2. Well-posedness of feedback systems. There are two reasons for the requirement for continuity with respect to delay variations of the solutions to our mathematical models. First, all real systems contain what are perceived as time delays, although it may be difficult to isolate a physical delay element. Second, when one wishes to interconnect models into feedback configurations, one needs conditions on the models to guarantee the existence of solutions to the resulting equations. Continuity (with respect to variations of parameters and the insertion of a small delay into the loop) of solutions is part of the accepted definition of well-posedness of feedback systems. [Willems 1971, p. 90-91] See also [Zames 1964]. Well-posedness guarantees

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[2]In this thesis we are considering plant models whose input and output spaces are the "extended" L^2 space $L^2_e(0,\infty)$. Since this space is not a normed space, what is meant by the uniform and strong operator topologies is not obvious. See Chapter 9 Appendix.

existence and uniqueness of solutions, making only physically justifiable assumptions.

Because of the lack of continuity in the uniform operator topology for most systems we would want to consider, this topology is not used in the definition of well-posedness. [Willems 1971, p. 94] The strong operator topology is used instead.

When the component systems of a single loop feedback system all have rational transfer functions, strict properness of the loop gain is sufficient for the system to be well-posed. For this reason, and also, not coincidentally, because strict properness is physically realistic, this assumption is usually made in the finite dimensional plant theory.

In the $\mathcal{H}^{\infty}$ theory for finite dimensional plants, one obtains loop transfer functions which are not strictly proper as the limiting case which attains the optimal sensitivity. The solution to the $\mathcal{H}^{\infty}$-minimal sensitivity problem for finite dimensional plants will then be ill-posed if the memoryless part of the loop transfer function has magnitude greater than 1. We give an example where this occurs in Chapter 9.

The presence of a delay in the plant model, on the other hand, ensures that our solutions give well-posed feedback systems, even in the limit, as we also discuss in Chapter 9.

E. Perspective and Contributions.

The $\mathcal{H}^{\infty}$ sensitivity minimization problem has been extensively studied by others for the case of purely rational plant with multiple inputs and multiple outputs. For a tutorial and bibliography see [Francis and Doyle 1985].

We extend the conventional $\mathcal{H}^{\infty}$ minimal sensitivity solution for single-input/single-output finite dimensional plants to such systems with a single delay added in cascade with the input to the plant. We believe that the work on which we report here is the first application of the $\mathcal{H}^{\infty}$ minimal sensitivity theory to a delay problem. The basic technique used here first appeared in [Flamm 1985].

Similar work has been done independently by Foias, Tannenbaum and Zames. In [Foias, Tannenbaum and Zames 1985a], these authors consider the cases covered by our Chapters 3 and 5. In [Foias, Tannenbaum and Zames 1985b] they consider the cases of our Chapters 6 and 7. These authors do not derive the explicit optimal sensitivities and compensators, nor do they consider the issues of compensator stability and approximation of the optimal sensitivity with proper finite dimensional compensators.

We view the main contributions of this thesis as the following:

1. This work (reported initially in incomplete form in [Flamm 1985]) is the first treatment of the $\mathcal{H}^{\infty}$ norm minimal weighted sensitivity problem with a delay in the input to the plant. The theory of the solution is due to [Sarason 1967], but the formulation and solution of our problem as a two point boundary value problem is new.

2. In the case of a 1 pole/1 zero weighting function when the plant has no right half plane poles or zeros, we have a complete solution to this $\mathcal{H}^{\infty}$ norm minimal weighted sensitivity problem. We provide explicit expressions for the ideal closed loop sensitivity and feedback

compensator, and we show that the optimal compensators are unstable and improper. When the plant has right half plane poles or zeros, we solve the crucial eigenvalue/eigenfunction problem. (We have not yet calculated the explicit sensitivity and compensator.)

3. In the case of more general rational weighting functions, the problem is not completely solved.

When the weighting function magnitude approaches 1 from above at high frequency, we can provide a complete solution. We do so explicitly for plants without right half plane poles or zeros. For plants with right half plane poles or zeros, we solve the part of the problem that differs from the no right half plane poles or zeros case, but we do not explicitly compute the sensitivity and compensator.

When the weighting function magnitude approaches 1 from below at high frequency, the problem can generally have a non-unique solution. For the case of no right half plane poles or zeros, when the $\mathcal{H}^\infty$ norm of the weighting function is equal to 1, the problem has a trivial solution. We also have a conjecture that would give another solution which is all-pass.

4. We show how to approximate the optimal weighted sensitivity using proper finite dimensional compensators.

5. We show that the optimal compensator for problems with a rational plant having no delay can result in an ill-posed feedback system. We show this does not happen when the plant model contains a delay. We

also show that the optimal system with or without a delay can generally have zero "delay margin."

F. Outline of the Dissertation

The development starts in Chapter 2 with a review of the mathematical theory underlying our solution. We transform the problem we are considering in the now standard way to the appropriate form for the application of the results of [Sarason 1967]. We lay out the structure of our solution technique to be applied in subsequent chapters.

Chapter 3 begins the detailed exposition of our solutions, with the simplest delay example, that of a stable minimum phase plant with a delay added at the input. The complexity of the sensitivity weighting function shapes the computations we must perform, and in this chapter we consider the simplest weighting function which gives an acceptable design, a 1 pole/1 zero rational function. After normalization of the frequency variable, the weighting function has a single free parameter. Depending upon the parameter, there are two cases: one leads to a unique optimal sensitivity, the other to a non-unique one. In either case there is an optimal sensitivity which is a constant times a function of constant magnitude on the imaginary axis. We find that the optimal compensator is generally improper and unstable.

Chapter 4 covers the finite dimensional approximation of the optimal compensator for the plant and weighting functions of Chapter 3. As in the case of finite dimensional plants, for implementation we want to approximate the ideal compensator with a proper one. In the process, we show that the approximation technique for the finite dimensional case does not work here. We describe a new one that does, and since the optimal compensator contains a delay, we also show one way to approximate it with a finite dimensional one.

Following the analysis of the prototype solution, we extend the class of plants and weighting functions allowed. In Chapter 5 we extend the 1 pole/1 zero weighting function case to more general rational weighting functions. In Chapter 6 we extend the solution for the 1 pole/1 zero weighting function case of Chapter 3 by adding right half plane zeros to the plant. In Chapter 7 we cover the case where the plant has right half plane poles.

In Chapter 8 we discuss conditions which allow us to compute solutions. In Chapter 9 we cover in more detail the issue of well-posedness for these feedback systems. Chapter 10 summarizes salient points about this work, and describes unsolved problems and work in progress.

CHAPTER 2

FORMULATION OF THE SOLUTION

A. $\mathcal{H}^\infty$ Theory

The work to be presented requires a basic knowledge of the theory of $\mathcal{H}^p$ spaces, the Hardy spaces of analytic functions. In particular, we work with the spaces $\mathcal{H}^\infty$ and $\mathcal{H}^2$. Our main reference is [Hoffman 1964]. [Helson 1983] is also useful at the level of this work. Here we shall summarize without proof some essential facts and definitions.

$\mathcal{H}^p(\Omega)$ spaces $(1 \leq p \leq \infty)$ are Banach spaces of functions analytic on the region $\Omega \subset \mathbb{C}$. There are two types of $\mathcal{H}^p$ spaces used, those of the unit disk, $\mathcal{H}^p(\mathbb{D})$, and those of the half plane, $\mathcal{H}^p(\Pi^+)$. In some of the mathematical literature the half plane Π^+ is the upper half plane $\{z \in \mathbb{C} \mid \mathcal{Im}(z) > 0\}$. In our work we shall use the half plane $\mathbb{C}^+$, the right half plane $\{z \in \mathbb{C} \mid \mathcal{Re}(z) > 0\}$.

For $f(z)$ defined on $\mathbb{C}^+$, let $f_x(y) = f(x+jy)$. $f \in \mathcal{H}^p(\mathbb{C}^+)$ if f is analytic in $\mathbb{C}^+$ and $f_x(y)$ is bounded in L^p norm as $x \downarrow 0$. $\mathcal{H}^p(\mathbb{C}^+)$ is a Banach space under the norm $\|f\| = \lim_{x \downarrow 0} \|f_x\|_p$. $\mathcal{H}^p(\mathbb{C}^+)$ can be identified with a subspace $\mathcal{H}^p(j\mathbb{R})$ of the corresponding L^p space on the line, $L^p(j\mathbb{R})$. The description of this subspace is most easily given for the case of $p = 2$. A theorem of Paley and Wiener says [Hoffman 1962, p. 131]:

> *A complex-valued function f in the right half-plane belongs to $\mathcal{H}^2$ if and only if f is the Laplace transform of a function in $L^2(0,\infty)$.*

The interpretation of this is that $\mathcal{H}^2$ is the space of Laplace transforms of test functions used to evaluate the L^2 stability of a system. $\mathcal{H}^2$ is a Hilbert space.

The space $\mathcal{H}^\infty(j\mathbb{R})$ is the subspace of $L^\infty(j\mathbb{R})$ whose harmonic extensions are analytic in $\mathbb{C}^+$.

We shall refer to $\mathcal{H}^p(\mathbb{C}^+)$ as simply $\mathcal{H}^p$ from now on.

The basic reason that $\mathcal{H}^\infty$ is important to us is that it is the space of transfer functions of linear systems which in the time domain are L^2-stable, causal, and time-invariant, and the $\mathcal{H}^\infty$ norm is the induced L^2 norm (the "gain" of the system). This fact seems to be widely known, although the elements of the proof do not seem to be collected in the same place. For this reason we list them: (1) A function $\varphi(s) \in \mathcal{H}^\infty$ induces a bounded map on $\mathcal{H}^2$ via multiplication. (2) A bounded multiplication operator on L^2 is the Laplace transform of a time-invariant operator on L^2. [Bochner and Chandrasekharan 1947, p. 143] (3) A convolution operator on L^2 corresponds to a multiplication operator under Laplace transform. [Schwartz 1966, p. 308] (4) Causality of a convolution operator on L^2 corresponds to analyticity of its Laplace transform in the right half plane. [Fourès and Segal 1955, p. 389]

A function f in $\mathcal{H}^\infty$ is called *inner* if $|f(j\omega)| = 1$ a.e. for $\omega \in \mathbb{R}$. Inner functions can be factored into the product of a Blaschke product and a singular part. A *Blaschke product* is a function of the form

$$B(s) = \left[\frac{s-1}{s+1}\right]^k \prod_{n=1}^{\infty} \frac{|1-\beta_n^2|}{|1-\beta_n^2|} \cdot \frac{s-\beta_n}{s+\overline{\beta}_n} ,$$

with $1 \notin \{\beta_i\}$, $\{\beta_i\}$ having $\mathcal{Re}(\beta_i) > 0$, and $\sum_{n=1}^{\infty} \frac{\mathcal{Re}(\beta_n)}{1 + |\beta_n^2|} < \infty$. A singular inner function is of the form

$$S(s) = e^{-\rho s} \exp\left[-\int_{-j\infty}^{j\infty} \frac{ts + j}{t + js}\, d\mu(t)\right],$$

where μ is a finite singular positive measure on the imaginary axis and ρ is a non-negative real number.

A function f in $\mathcal{H}^p$ (for $1 \leq p \leq \infty$) is called outer if $\{e^{-s\Delta}f(s),\ \Delta \geq 0\}$ spans a dense subspace of $\mathcal{H}^p$. (For $p = \infty$, the subspace should be dense in the weak* topology.) Also, $f\mathcal{H}^p$ is dense in $\mathcal{H}^p$ is f is outer and bounded.

In the case of rational functions in $\mathcal{H}^\infty$, inner functions are those with unit magnitude on the imaginary axis and left half plane poles which are reflections of right half plane zeros across the imaginary axis (finite Blaschke products). For rational functions in $\mathcal{H}^p$, outer functions are those with no right half plane zeros. Scalar multiples of inner functions are known as all-pass functions, and outer functions are known as minimum phase functions. Any function in $\mathcal{H}^p$ can be factored uniquely (up to scalar multiples) into an inner and an outer part.

B. Transformation of the Original Problem

We now briefly summarize the transformation of the problem of minimizing the weighted sensitivity (1.2) to a problem of the form

$$\inf_{H\in\mathscr{H}^\infty} \|W(s)V(s) - e^{-s\Delta}A(s)H(s)\|_\infty \tag{2.1}$$

where W(s), V(s) and A(s) are fixed $\mathscr{H}^\infty$ functions. This argument appears, for example, in [Francis and Zames 1984, p. 10].

The first step is to use the "Q-parametrization" of all stabilizing compensators introduced in this context by [Zames 1981, pp. 305-306]. Then using a coprime factorization of the plant over stable transfer functions, we can express the problem of minimizing the closed loop sensitivity as a minimization problem affine with respect to a parameter-function which can vary freely in $\mathscr{H}^\infty$. As this free parameter varies, the attainable sensitivity functions range over a subspace of $\mathscr{H}^\infty$. The form of this subspace is $\varphi+\psi\mathscr{H}^\infty$, where φ and ψ are fixed $\mathscr{H}^\infty$ functions. After modifying ψ to make the subspace closed, we can apply the theory of [Sarason 1967] on the dilation (generalized extension — see [Halmos 1967, p. 118]) of operators on certain subspaces of $\mathscr{H}^2$. In the special cases of interest to us, we obtain an explicit expression for the infimal sensitivity in our modified problem, from which we calculate the (improper and infinite dimensional) compensator which attains this sensitivity for the original problem. We can then examine how to find a realizable (proper and finite dimensional) compensator that approximates the infimal sensitivity norm.

At this point, before continuing with theoretical background we first state our assumptions on the weighting function W(s).

C. The Choice of Weighting Function.

We assume that the weighting function is a rational minimum phase function in $\mathcal{H}^\infty$ with real coefficients, and bounded away from zero at ∞. We normalize the weighting function so that $\lim_{|s|\to\infty} W(s) = 1$.

The minimum phase assumption is without loss of generality, since multiplication of (1.2) by an inner function does not change the $\mathcal{H}^\infty$ norm.

The reason for the rationality assumption is simple convenience, and the fact that nothing seems to be lost from this restriction. If, however, strictly proper weighting functions are considered,[3] so that $W(\infty) = 0$, since the resulting optimal weighted sensitivities will be all-pass, the corresponding unweighted sensitivities will be unbounded at ∞. This would be an unacceptable design, were it really to be used.

It could be argued that real implementations will use proper approximations to the optimal compensators. Then the unweighted sensitivity will not only be bounded at infinity, it will have value 1 there.

However, since the approximation techniques presented in the literature, [Zames and Francis 1983] and [Vidyasagar 1985], achieve sensitivity with norm arbitrarily close to the optimal by approximating the optimal weighted sensitivity to higher and higher frequencies, these techniques cause the unweighted sensitivity to grow as the approximation improves. A trade-off would then be required. Since bounded sensitivity is a real design constraint, which proper choice of *non-strictly* proper weighting function accommodates, the use of such a

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[3]It would simplify our problem to take $W(s)$ strictly proper, since then it would be a compact operator on $\mathcal{H}^2$. (See [Rudin 1973, p. 197] for a definition of compact operator. See section F below for the advantage of this.)

weighting function seems to be the natural way of providing the trade-off.

D. The Q-parameterization of Stabilizing Compensators

If C is a feedback compensator which stabilizes the closed loop system (in the sense of both input-output and internal stability) and results in closed loop sensitivity $S = (1+PC)^{-1}$, there is a transfer function Q such that

$$S = (1-PQ). \tag{2.2}$$

Q and C are related by

$$Q = \frac{C}{1+PC} \quad \text{and} \quad C = \frac{Q}{1-PQ}. \tag{2.3}$$

When P is stable, any stable Q results in a stabilizing C, and conversely. When P is unstable, the stability of the closed loop imposes additional constraints on Q in order to ensure that the sensitivity has zeros at the unstable poles of P. Several versions of these constraints appear in [Zames and Francis 1983, pp. 589-590].

We shall not use the constraints explicitly here because the introduction of a coprime factorization in the next section eliminates the need to deal with constraints explicitly.

Remark: The use of this "Q-parametrization" when the plant (or compensator) is stable dates back at least as far as [Newton, Gould and Kaiser 1956, p. 35]. Perhaps the easiest way to see how it arises naturally is via the block diagram Figure 2. Here $\frac{PC}{1+PC}$ is the transfer

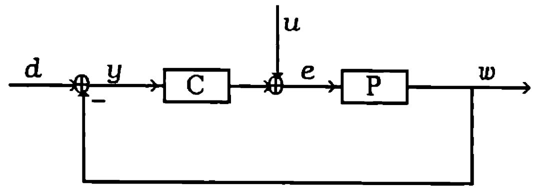

Figure 2. Transformation of Feedback System

function from d to w. In picking C, we might know what we would like the closed loop transfer function to be, and so be able to directly specify a cascade compensator Q, as in Figure 3. Equating the cascade

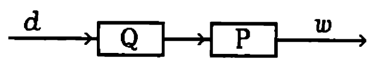

Figure 3. Equivalent Cascade Control System

and feedback compensated system transfer functions, we find $Q = \frac{C}{1+PC}$.

E. Use of Coprime Factorization

By next introducing a coprime factorization [Callier and Desoer 1978] of the (unstable) plant transfer function it is possible to obtain an expression (2.4) for the closed loop sensitivity function which is

$$X(s) = W(s)V(s) - e^{-s\Delta}A(s)H(s) \qquad (2.4)$$

unconstrained as a function of the stable transfer function-parameter H(s). [Francis and Zames 1984, p. 10].

We describe this transformation using the explicit assumed form of our plant given in (1.1), $P(s) = e^{-s\Delta}A(s)\cdot B^{-1}(s)$.

Here A(s) is the stable rational factor of the plant, and V(s) in (2.4) comes from a coprime factorization of the plant over the ring of stable transfer functions via

$$e^{-s\Delta}A(s)U(s) + B(s)V(s) = 1. \tag{2.5}$$

This factorization is called coprime because as a result of (2.5) the functions $e^{-s\Delta}A(s)$ and B(s) have no common factors which are stable transfer functions, except for scalars. We can and do assume that B(s) is a finite Blaschke product. The idea of parameterizing stabilizing compensators with such factorizations is due to [Youla, Bongiorno and Jabr 1976, p. 7]. In our case the computational method to find this factorization appears in [Callier and Desoer 1978, p. 655].

The result we need appears in [Desoer, Liu, Murray and Saeks 1980, p. 404], that (using our notation above) all stabilizing compensators are given by $C(s) = \dfrac{U(s)-H(s)B(s)}{V(s)+H(s)e^{-s\Delta}A(s)}$, as H(s) varies over all functions in $\mathcal{H}^{\infty}$. Using the formula (2.3) for Q(s) with this expression for C(s) we find that

$$Q(s) = BU + B^2H. \tag{2.6}$$

Finally, substituting (2.6) into the expression (2.2) for the sensitivity, and using (1.1) with (2.5), we find that

$$S(s) = 1 - P(BU + B^2H)$$

$$= B(V - e^{-s\Delta}AH)$$

ranges over all sensitivity functions attainable with a stable closed loop as H varies over $\mathcal{H}^{\infty}$.

To obtain the expression (2.1), we merely note that

$$\inf_{H\in\mathcal{H}^{\infty}} \|W(s)S(s)\|_{\infty} = \inf_{H\in\mathcal{H}^{\infty}} \|W(BV - Be^{-s\Delta}AH)\|_{\infty}$$

$$= \inf_{H\in\mathcal{H}^{\infty}} \|WV-We^{-s\Delta}AH)\|_{\infty}$$

(since B is a Blaschke product)

$$= \inf_{H\in\mathcal{H}^{\infty}} \|WV-e^{-s\Delta}AH)\|_{\infty}$$

(since we have assumed W is invertible in $\mathcal{H}^{\infty}$). This is (2.1).

The infimal value in (2.1) is not generally attained for any $H \in \mathcal{H}^{\infty}$ when P is strictly proper, since then $A\mathcal{H}^{\infty}$ is not closed. (This is also the case when the plant has a pole or zero on the imaginary axis. This does not differ form the finite dimensional plant case, and we only handle the strictly proper plant case here.)

If we allowed improper analytic functions H, the infimum would be attained. From (2.6) an improper H means an improper Q parameter, and from (2.3) this means an improper compensator.

We will proceed to find a limiting sensitivity which corresponds to such an improper compensator. Afterwards we will consider the issue of approximation of this improper compensator by proper ones.

Suppose $A = \psi P_o$, where ψ is a finite Blaschke product and P_o is a rational outer function. Let k be the degree difference between the denominator and numerator polynomials of $P_o(s)$. Our solution then proceeds by looking for the solution to

$$\inf_{H \in (s+1)^k \mathcal{H}^\infty} \|W(s)V(s) - e^{-s\Delta}A(s)H(s)\|_\infty \tag{2.7}$$

which is equivalent to

$$\inf_{H \in \mathcal{H}^\infty} \|Y(s) - e^{-s\Delta}\psi(s)H(s)\|_\infty \tag{2.8}$$

with $Y(s) = W(s)V(s)$. This follows since $P_o(s+1)^k\mathcal{H}^\infty = \mathcal{H}^\infty$.

Since $f\mathcal{H}^\infty$ is dense in $\mathcal{H}^\infty$ for any outer function $f \in \mathcal{H}^\infty$ the infimal sensitivities for the two problems are in fact equal.

We shall solve (2.8), and find an $\mathcal{H}^\infty$ function which attains the infimum. We shall think of this function as a limit for the desired sensitivity function, and look for attainable sensitivity functions whose norm approximates the infimum.

F. Results from [Sarason 1967]

Our solution to (2.8) is based on [Sarason 1967], to which the reader is referred for the detailed mathematical justification of our work. Here we briefly summarize the essential features of the theory.

Most of the material in [Sarason 1967] is developed in terms of the space $\mathcal{H}^\infty(T)$, the space of boundary functions on the unit circle for functions bounded and analytic in the unit disk. However in the current work we are concerned with continuous time systems and their Laplace transforms, so we deal with the space $\mathcal{H}^\infty(j\mathbb{R})$, the space of boundary functions on the imaginary axis for functions bounded and analytic in the right half plane.

In [Sarason 1965], summarized in [Sarason 1967, p. 192], the author translates his work for the space $\mathcal{H}^\infty(T)$ to $\mathcal{H}^\infty(\mathbb{R})$ for the special case equivalent to the case that arises for the plants we shall consider in Chapter 3. (Note that Sarason uses $\mathcal{H}^\infty(\mathbb{R})$, $\mathcal{H}^\infty$ of the upper half plane.) We shall cite relevant results in [Sarason 1967] in a form translated to $\mathcal{H}^\infty(\mathbb{C}^+)$, restricted to the special cases of interest to us.

Suppose K is a closed subspace of a Hilbert space H, and W is an operator on H. Let Π_K be the projection operator from H to K. $T = \Pi_K \circ W|_K$ is an operator on K, called the *compression* of W, and W is a *dilation* of T.

Viewed as operators which act by multiplication, the space $\mathcal{H}^\infty$ is a space of bounded linear transformations of $\mathcal{H}^2$ which commute with multiplication by $e^{-s\Delta}$, $\Delta \geq 0$. Let K be a closed subspace of $\mathcal{H}^2$. Take S_Δ to be the operator on K defined by $S_\Delta f = \Pi_K(e^{-s\Delta} \cdot f)$. Let T be an operator on K which commutes with S_Δ, $\Delta \geq 0$. [Sarason 1967] is concerned with dilations (which commute with multiplication by $e^{-s\Delta}$) of such operators on K to all of $\mathcal{H}^2$.

If $\varphi \in \mathcal{H}^\infty$ and $f \in \mathcal{H}^2$, $\varphi \cdot f \in \mathcal{H}^2$, and we denote this multiplication operator on $\mathcal{H}^2$ by M_φ.

In the cases of interest to us, T will be the compression of a multiplication operator on $\mathcal{H}^2$ to K, and K will have the special form $K = \mathcal{H}^2 \ominus \psi\mathcal{H}^2$, where ψ is an inner function. Alternatively we write $K = (\psi\mathcal{H}^2)^\perp$, where the orthogonal complement is taken in $\mathcal{H}^2$. (T will commute with S_Δ because $\psi\mathcal{H}^2$ is invariant under multiplication by $e^{-s\Delta}$.)

Let $\|T\|$ refer to the induced operator norm of T. We can state a version for $\mathcal{H}^\infty(\mathbb{C}^+)$ of the main result in [Sarason 1967, Theorem 1] as,

> *If* T *is an operator on K which commutes with* S_Δ*, for all* $\Delta \geq 0$*, then there is a* $\varphi \in \mathcal{H}^\infty$ *such that* $\|\varphi\|_\infty = \|T\|$ *and* $T = \Pi_K \circ M_\varphi|_K$.

If f is a function such that $\|Tf\| = \|T\| \cdot \|f\|$, we call f a *maximal vector* for T. Compact bounded operators have maximal vectors. ([Rudin 1973, p. 313] using also $\|T^* T\| = \|T\|^2$ from [Rudin 1973, p. 297].)

The operators which we consider are not compact, and in the general non-compact case the existence of such a norm-preserving dilation is the best we can do. In the general case we do not know how to construct the dilation.

We subsequently establish that the particular operators in which we are most interested have maximal vectors, in spite of not being compact. In this case we do know how to construct the dilation, and it is really Proposition 5.1 in [Sarason 1967, p. 188] that we require. Our version of that proposition is

> *Let* T *be the compression of a multiplication operator on* $\mathcal{H}^2$ *to K with* $\|T\| = 1$*. If* T *has a maximal vector* f*, then there is a unique* $\varphi \in \mathcal{H}^\infty$ *with* $\|\varphi\|_\infty = 1$ *such that the compression of* M_φ *to K is equal to* T*.* φ *is inner, and it is given by*

$$\varphi = \frac{Tf}{f}. \tag{2.9}$$

G. Application of [Sarason 1967] to Problem

In order to follow [Sarason 1967], we view $\left[Y(s)-e^{-s\Delta}\psi(s)H(s)\right]$ in (2.8) as an operator on $\mathcal{H}^2$. The compression of this operator to

$K = \mathcal{H}^2 \ominus e^{-s\Delta}\psi\mathcal{H}^2$ is equal to the compression of Y(s) on the same subspace. Call this latter operator $T = \Pi_K Y|_K$. (We note that $\|T\| \leq \|Y\|$.) The infimum in (2.8) cannot be less than the operator norm of T. Theorem 1 in [Sarason 1967] says that the desired infimum is in fact equal to $\|T\|$.

Following [Sarason 1967, §7] a way to find this supremum is to use the facts that $\|T\|^2 = \|T^*T\|$ (via the definition of adjoint) [Rudin 1973, p. 297] and that $\rho(T^*T) = \|T^*T\|$ since T^*T is normal [Rudin 1973, p. 282]. Therefore $\|T\| = \rho(T^*T)^{1/2}$.

If T is compact we need only find the largest eigenvalue of T^*T, but this will not be the general case.

In our case we normalize W(s) with $W(\infty) = 1$. Since W(s) is rational it is then equal to 1 plus a strictly proper stable rational function. Therefore T is the identity plus a compact operator, and we have only slightly more complication: Since we shall have T^*T-I is compact, we know from Weyl's theorem that the spectrum of T^*T and I differ only by eigenvalues. [Halmos 1967, pp. 92 & 295]. Therefore, $\sigma(T^*T) \subseteq \{\eta+1: \eta \in \sigma(T^*T-I)\} \cup \{1\}$. Also, 1 is the only cluster point of $\rho(T^*T)$.

Thus the idea will be to examine the eigenvalues of T^*T for a maximum. If none exists, we will have $\|T\| = 1$.

The operator T is equivalent to an operator $V: \mathcal{L}^{-1}(K) \rightarrow \mathcal{L}^{-1}(K)$ via the inverse Laplace transformation. $\mathcal{L}^{-1}(K)$ is a subspace of $L^2(0,\infty)$. One can think of V acting on time functions via convolution and T on transfer functions via multiplication, each followed by the corresponding projection. In particular, there is a one-to-one correspondence between eigenvectors, and the eigenvalues of T^*T and V^*V are the same. Furthermore, compactness of V is equivalent to compactness of T.

If T^*T has a largest eigenvalue, say λ^2, then $\|T\|^2 = \lambda^2$, and the corresponding eigenfunction will be a maximal vector for T. According to Proposition 5.1 in [Sarason 1967], in this case $\frac{T}{\|T\|}$ will be dilated by an inner function given by $\frac{T\hat{f}}{\|T\|\hat{f}}$. This is the unique minimal dilation. In our case, $\frac{T\hat{f}}{\hat{f}}$ would be the optimal sensitivity.

Thus when T^*T has a largest eigenvalue, we can find the minimal dilation of T. For computational reasons we prefer to use V^*V. We then have four steps to find the optimal sensitivity:

- Find K, and the corresponding subspace of $L^2(0,\infty)$,
- Compute V^*V,
- Solve the eigenvalue/eigenfunction problem for this operator,
- Find the maximal eigenvalue and a corresponding eigenfunction, and compute the minimal dilation of T. Since 1 is the only cluster point of $\rho(T^*T)$, if $\|T\| > 1$, T^*T has a largest eigenvalue. Then T has a unique minimal dilation.

When T does not have a maximal vector, we still know from Theorem 1 in [Sarason 1967] that a minimal dilation of T exists, but we don't know that it is unique, or how to compute it in general.

In this latter case the problem is not completely solved. In Chapter 8 we have a practical observation on how to pick W so as to ensure the existence of a maximal vector and a conjecture about the form of a solution for general W.

When $|W(j\omega)| > 1$ for ω large enough T^*T has infinitely many eigenvalues greater than 1. Since the only cluster point of the eigenvalues is 1, this means that T^*T has a largest eigenvalue, and thus a maximal vector. We observe that (since $W(\infty) = 1$) we can always pick $W(s)$ so that $|W(j\omega)|$ eventually approaches 1 from above, while affecting $|W(j\omega)|$ only arbitrarily little at any frequencies of interest, by introducing one additional pole/zero pair at high frequency. In other words, we can always pick a W close to one having the desired magnitude, which results in a solvable problem.

When $|W(j\omega)|$ does not approach 1 from above at infinity, it must do so from below. Then there we get the other two cases. If $|W(j\omega)| < 1$ for all ω, the magnitude of the infimal sensitivity is 1, a maximal vector does not exist, and an optimal sensitivity is obtained with the open loop system (for a stable plant). $|W(j\omega)| > 1$ over some frequency band. In this second case a maximal vector may or may not exist.

H. Computation of the Optimal Compensator

The result of the application of the above theory is a multiple of an inner function, which is either the optimal sensitivity or the optimal sensitivity divided by the Blaschke product resulting from the plant poles. This sensitivity is that for a modified problem ((2.8) instead of (2.1)), where we have extended $\mathcal{H}^\infty$ just enough to allow inversion of the outer part of the plant.

We use this to obtain a sequence of approximations to the optimal solution of our original problem in two steps. First we compute an improper compensator which would give us the computed optimal sensitivity as described above. Then we show how to find a sequence of proper compensators for which the closed loop sensitivity approaches the optimal.

The computation of the improper compensator involves finding the value of the free parameter which gives rise to the optimal sensitivity, using the formula $X(s) = Y(s) + e^{-s\Delta}\psi H$ (when $B = 1$), using the notation of (2.4) and (2.8). Since our computation of X did not involve finding an H which realizes the infimum, the computation $H = \frac{X-Y}{e^{-s\Delta}\psi}$ requires some justification in itself. Suppose we have computed the optimal sensitivity $X(s)$ as above. We know that $\Pi_K(X|_K) = \Pi_K(Y|_K)$, so $\Pi_K((X-Y)|_K) = 0$. We can also see that $\Pi_K(X-Y) = 0$ since $K^\perp$ is invariant under multiplication by $\mathcal{H}^\infty$ functions. Therefore $(X-Y)\mathcal{H}^2 \subseteq K^\perp = e^{-s\Delta}\psi\mathcal{H}^2$. But then $e^{-s\Delta}\psi$ divides $(X-Y)$ in $\mathcal{H}^\infty$ (this follows from the uniqueness of the inner-outer factorization), and the computation works. From this point computation of a compensator is a matter of algebra.

CHAPTER 3

THE FIRST CASE

In this chapter we consider the problem where the plant is given by

$$P(s) = e^{-s\Delta}P_o(s) \tag{3.1}$$

where $P_o(s)$ is a minimum phase and stable function, and the weighting function is given by

$$W(s) = \frac{s+1}{s+\beta} \tag{3.2}$$

with $0 < \beta$.

The content of this chapter follows the general scheme outlined in Chapter 2:

1. Identify the subspace of $\mathscr{H}^2$ given by $K = (e^{-s\Delta}\mathscr{H}^2)^{\perp}$, and the corresponding subspace of $L^2(0,\infty)$ given by $\mathscr{L}^{-1}(K)$.

2. Explicitly compute the compression T of the weighting function W(s), viewed as an operator on $\mathscr{H}^2$, to the subspace K, and the adjoint operator T^*. We also compute the corresponding operators V and V^* on $\mathscr{L}^{-1}(K)$.

3. Using these computations, solve the eigenvalue/eigenfunction problem for V^*V. We do this by converting the problem originally posed for an integral operator to a two-point boundary value problem.

4. Pick the largest eigenvalue if it exists, and compute the

optimal sensitivity and the corresponding optimal compensator. If there is not a largest eigenvalue, we can still find an optimal sensitivity and corresponding compensator, but these will not be unique. In this case we find that W(s) itself in an optimal sensitivity, and we also find a sequence of suboptimal sensitivities which converge to an all-pass optimal sensitivity.

5. Approximate the optimal compensator with a proper and finite dimensional compensator.

Remarks:

1. We have normalized the frequency scale to put the zero of the weighting function at the point -1, so (3.2) is a completely general 1 pole/1 zero weighting function, subject to the stability and minimum phase conditions.

2. In the case of a stable plant P, in the formula for the coprime factorization equation (2.5), we can set $A = P_o$, $B = 1$, $U = 1$, $V = 1-e^{-s\Delta}P_o$. Then our minimization problem becomes

$$\inf_{H \in \mathcal{H}^\infty} \|W(1-P) - PH\| = \inf_{H' \in \mathcal{H}^\infty} \|W - PH'\| \tag{3.3}$$

since for every $H \in \mathcal{H}^\infty$, there is a $H' \in \mathcal{H}^\infty$ such that $P(W+H) = PH'$, and conversely. Thus we proceed with the function Y in (2.7) set equal to W and $\psi = 1$.

3. We shall see that in step (4.), for the 1 pole/1 zero weighting function, there are two cases, depending upon whether the pole lies

closer to the origin than the zero or farther from the origin. The former case, that of $\beta < 1$, is the only one of interest since the latter case has the trivial solution of no feedback giving the optimal sensitivity. (We also demonstrate a non-trivial solution for the $\beta > 1$ case.) Furthermore, we shall see that in the case $\beta < 1$ there always exists a largest eigenvalue.

A. Calculation of K.

The result of this section is that

$$K = \mathcal{L}(L^2(0,\Delta)) \tag{3.4}$$

where $\mathcal{L}(\cdot)$ denotes Laplace transform.

We prove this as follows: Suppose $f,\ h \in L^2(0,\infty)$. K is defined by

$$\langle \hat{f}(s), e^{-s\Delta}\hat{h}(s)\rangle_{\mathcal{H}^2} = 0, \text{ for all } \hat{f}(s)\in K \text{ and } \hat{h}\in\mathcal{H}^2$$

Then

$$0 = \langle \hat{f}(i\omega), e^{-i\omega\Delta}\hat{h}(i\omega)\rangle_{L^2}$$

$$= \langle f(t), h(t-\Delta)\rangle_{L^2}$$

by Parseval's theorem. Therefore $f(t) = 0$ a.e. $[\Delta,\infty)$. Conversely, if $f(t)\in L^2$ and $supp(f)\subseteq[0,\Delta]$, then $\hat{f}\in(e^{-s\Delta}\mathcal{H}^2)^{\perp}$. Thus

$$K = (e^{-s\Delta}\mathcal{H}^2)^{\perp}$$

$$= \{\hat{f}: f \in L^2[0,\infty) \text{ and } supp(f) \subseteq [0,\Delta]\}$$

$$= \left[\frac{1-e^{-s\Delta}}{s}\right] * \mathcal{H}^2$$

where "$*$" denotes convolution.

B. Computation of Operators.

The computation of the operators T, T^*, V and V^* proceeds in a straightforward manner. The results of this section are contained in the expressions for V and V^*, equations (3.5) and (3.6).

Accordingly, for $\hat{f} \in K$,

$$T\hat{f} = \mathcal{L}\left[[u(t)-u(t-\Delta)]\int_0^t w(t-\tau)f(\tau)d\tau\right],$$

where $w = \mathcal{L}^{-1}(W)$, and $u(t)$ is the unit step function,

$$u(t) = \begin{cases} 0 \text{ for } t < 1 \\ 1 \text{ for } t \geq 1. \end{cases}$$

This can be seen simply by projecting $W\hat{f}$ onto K. It is easy to see that T is given by $T = \Pi_K(W|_K) = \frac{1-e^{-s\Delta}}{s} * W(s)|_K$,

and V is the operator on $L^2(0,\Delta)$ defined by

$$(Vf)(t) = \int_0^t \dot{w}(t-\tau)f(\tau)d\tau. \qquad (3.5)$$

We show in the appendix to this chapter that

$$V^* y = \int_t^\Delta y(\tau) w(\tau - t) d\tau. \tag{3.6}$$

C. Solution of eigenvalue problem.

We want to solve

$$\lambda^2 f = V^* V f. \tag{3.7}$$

We do so by using a realization of the operators V and V^* as a pair of differential equations on the interval $(0,\Delta)$, with appropriate boundary conditions. After formulating these differential equations, we find the solution by taking Laplace transforms, after justifying the extension of the domain of the equations. We then explicitly apply the boundary conditions to show that all eigenfunctions are sinusoids and to show that the frequencies and phases of these functions are the solutions of equations (3.17) and (3.19) below, respectively.

Now let

$$y = Vf = \int_0^t w(t-\tau) f(\tau) d\tau \text{ for } t \in [0,\Delta],$$

and let

$$z = V^* y = \int_t^\Delta w(\tau - t) y(\tau) d\tau \text{ for } t \in [0,\Delta].$$

Since $w(t) = \delta_t + (1-\beta) \cdot e^{-\beta t}$, we can take

$$\frac{d}{dt} x_1 = -\beta \cdot x_1 + (1-\beta) f$$

$$y = x_1 + f$$

$$\frac{d}{dt} x_2 = \beta \cdot x_2 - (1-\beta) \cdot y$$

$$z = x_2 + y$$

as a state space model for V^*V valid on $[0,\Delta]$.

Boundary conditions are given by $y(0) = f(0)$ and $z(\Delta) = y(\Delta)$. This is equivalent to $x_1(0) = 0$ and $x_2(\Delta) = 0$ (if $\beta \neq 1$).

Remark: We exclude the case $\beta = \pm 1$ because otherwise the frequency domain weighting function could be $W(s) = \frac{s-1}{s+1}$ or $W(s) = 1$. These are of no interest since we only allow weighting functions which are non-constant and outer.

More concisely,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -\beta & 0 \\ \beta-1 & \beta \end{bmatrix} \cdot \boldsymbol{x} + \begin{bmatrix} 1-\beta \\ \beta-1 \end{bmatrix} \cdot f \tag{3.8a}$$

$$z = [1 \quad 1] \cdot \boldsymbol{x} + f \tag{3.8b}$$

with

$$0 = \begin{bmatrix} x_1(0) \\ x_2(\Delta) \end{bmatrix}. \tag{3.8c}$$

Equations (3.8) simply constitute a realization of the operator V^*V, so that for any $f(t)$ defined on $(0,\Delta)$, the solution to (3.8) is $z(t) = (V^*V)f(t)$.

Now we set $z = \lambda^2 f$ in order to find the eigenfunctions and eigenvalues of V^*V. Then

$$(\lambda^2-1)f = [1 \quad 1] \cdot \boldsymbol{x} \tag{3.9}$$

replaces (3.8b).

If the equations (3.8)-(3.9) were defined on $(0,\infty)$, we could take Laplace transforms and solve for $\hat{f}(s)$ in terms of $x(0)$. This not being the case, we shall still do so, but we require a little more justification. The justification is as follows:

Every solution to (3.8)-(3.9) has some initial value $x(0)$. Using (3.9) we can write (3.8) as

$$\dot{x} = \left[\begin{bmatrix}-\beta & 0\\ \beta-1 & \beta\end{bmatrix} + \frac{1-\beta}{\lambda^2-1}\cdot\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\right]\cdot x \qquad (3.10)$$

and the solution to (3.10) with initial condition $x(0)$ gives us the same $x(t)$ on $(0,\Delta)$. We can extend this solution to $(0,\infty)$ just by integrating (3.10).

This shows that all solutions to (3.8)-(3.9) have such an extension. Conversely, taking any solution to (3.10) with initial condition $x(0)$, we obtain a solution to (3.8)-(3.9) since the value $x_2(\Delta)$ will match (3.8c). Thus we conclude that we can obtain all solutions to (3.8)-(3.9) by taking Laplace transforms as if the problem were defined on $(0,\infty)$, if we can find the correct value for $x(0)$.

Therefore, taking Laplace transforms in (3.8a)

$$s\hat{x} - x(0) = \begin{bmatrix}-\beta & 0\\ \beta-1 & \beta\end{bmatrix}\cdot\hat{x} + \begin{bmatrix}1-\beta\\ \beta-1\end{bmatrix}\cdot\hat{f}$$

and so

$$\begin{bmatrix}s+\beta & 0\\ 1-\beta & s-\beta\end{bmatrix}\cdot\hat{x} = \left[\begin{bmatrix}1-\beta\\ \beta-1\end{bmatrix}\cdot\hat{f} + x(0)\right].$$

Using (3.9)

$$(\lambda^2-1)\hat{f} = [1\ \ 1]\cdot\begin{bmatrix} s+\beta & 0 \\ 1-\beta & s-\beta \end{bmatrix}^{-1} \left(\begin{bmatrix} 1-\beta \\ \beta-1 \end{bmatrix}\cdot\hat{f} + x(0)\right)$$

$$= [1\ \ 1]\cdot\begin{bmatrix} \dfrac{1}{s+\beta} & 0 \\ \dfrac{\beta-1}{s^2-\beta^2} & \dfrac{1}{s-\beta} \end{bmatrix}\cdot\left(\begin{bmatrix} 1-\beta \\ \beta-1 \end{bmatrix}\cdot\hat{f} + x(0)\right)$$

$$= \left(\frac{1-\beta}{s+\beta} - \frac{1-\beta}{s-\beta} - \frac{(1-\beta)^2}{s^2-\beta^2}\right)\hat{f} + \begin{bmatrix} \dfrac{s-1}{s^2-\beta^2} & \dfrac{1}{s-\beta} \end{bmatrix} x(0)$$

$$= -\frac{1-\beta^2}{s^2-\beta^2}\cdot\hat{f} + \begin{bmatrix} \dfrac{s-1}{s^2-\beta^2} & \dfrac{1}{s-\beta} \end{bmatrix} x(0)$$

and finally (assuming $\lambda^2 \neq 1$ — see remark below)

$$\hat{f} = \left(\lambda^2 - \frac{s^2-1}{s^2-\beta^2}\right)^{-1} \begin{bmatrix} \dfrac{s-1}{s^2-\beta^2} & \dfrac{1}{s-\beta} \end{bmatrix} x(0)$$

$$= \left(\lambda^2 - \frac{s^2-1}{s^2-\beta^2}\right)^{-1}\cdot\frac{1}{s^2-\beta^2}\cdot[s-1\quad s+\beta]\cdot x(0).$$

Remark: If $\lambda^2 = 1$ then $(\beta^2-1)f = 0$ implies $f = 0$ or $\beta^2 = 1$. From the remark on page 17, $\beta^2 = 1$ does not correspond to a weighting function which we consider. Thus we can assume that $\lambda^2 \neq 1$.

So $\hat{f}$ has for its poles the zeros of $(\lambda^2-1)s^2 + 1-\lambda^2\beta^2$, that is

$$s^2 = \frac{1-\lambda^2\beta^2}{1-\lambda^2}. \tag{3.11}$$

Taking $\pm\gamma$ to be the two roots of this equation, we either have $\gamma^2 = 0$, or

$$f(t) = e^{\gamma t} + \alpha \cdot e^{-\gamma t}.$$

to within a constant multiplier.

The possibility of these cases and, in the second case, the coefficient α are determined by the boundary conditions. To determine the boundary conditions, we start with (3.9) and differentiate, then substitute (3.8a). We get

$$(\lambda^2-1)f = [1 \quad 1]\cdot \boldsymbol{x}$$

and

$$\begin{aligned}(\lambda^2-1)\dot{f} &= [1 \quad 1]\cdot \dot{\boldsymbol{x}} \\ &= [1 \quad 1]\cdot \left[\begin{bmatrix}-\beta & 0 \\ \beta-1 & \beta\end{bmatrix}\cdot \boldsymbol{x} + \begin{bmatrix}1-\beta \\ \beta-1\end{bmatrix}\cdot f\right] \\ &= [-1 \quad \beta]\cdot \boldsymbol{x}\end{aligned}$$

We write these as

$$\begin{bmatrix}\lambda^2-1 & 0 \\ 0 & \lambda^2-1\end{bmatrix}\cdot \begin{bmatrix}f \\ \dot{f}\end{bmatrix} = \begin{bmatrix}1 & 1 \\ -1 & \beta\end{bmatrix}\boldsymbol{x}.$$

Now we apply (3.8c) to obtain

$$(\lambda^2-1)\cdot \begin{bmatrix}f(0) \\ \dot{f}(0)\end{bmatrix} = \begin{bmatrix}1 \\ \beta\end{bmatrix}\boldsymbol{x}_2(0)$$

and

$$(\lambda^2-1)\cdot \begin{bmatrix}f(\Delta) \\ \dot{f}(\Delta)\end{bmatrix} = \begin{bmatrix}1 \\ -1\end{bmatrix}\boldsymbol{x}_1(\Delta).$$

We conclude that

$$\dot{f}(0) = \beta \cdot f(0) \qquad (3.12)$$

and

$$\dot{f}(\Delta) = -f(\Delta). \tag{3.13}$$

Now it appears that we could have.

$$\text{(i) } \frac{\beta^2\lambda^2-1}{\lambda^2-1} = 0, \text{ (ii) } \frac{\beta^2\lambda^2-1}{\lambda^2-1} > 0 \text{ or (iii) } \frac{\beta^2\lambda^2-1}{\lambda^2-1} < 0.$$

We consider each of these cases in turn.

<u>Case (i).</u> This implies $\ddot{f}=0$. Then $f(t)=k\cdot t + m$, but $\dot{f}(0)=\beta f(0)$ implies $f(t)=m(\beta t+1)$, and we normalize by assuming $m=1$. Then (i) gives

$$\lambda^2\beta^2 = 1,$$

or

$$\lambda^2 = \frac{1}{\beta^2}.$$

The boundary condition at $t = \Delta$ gives us

$$\beta\cdot\Delta + 1 = -\beta.$$

Since $\beta > 0$ by assumption, case (i) is excluded.

<u>Case (ii).</u> In this case

$$f = e^{\gamma t} + \alpha \cdot e^{-\gamma t}$$

with $\gamma^2 = \left[\frac{\beta^2\lambda^2 - 1}{\lambda^2 - 1}\right] > 0$. We assume without loss of generality that $\gamma > 0$. Then the boundary condition (3.12) gives us

$$\gamma - \alpha \cdot \gamma = \beta \cdot (1 + \alpha), \text{ or } \alpha = \frac{\gamma - \beta}{\gamma + \beta}. \tag{3.14}$$

Boundary condition (3.13) gives

$$e^{\Delta\gamma} + \alpha \cdot e^{-\Delta\gamma} = -\gamma \cdot e^{\Delta\gamma} + \alpha \cdot \gamma \cdot e^{-\Delta\gamma}.$$

Substitution of (3.14) and a little algebra gives us

$$\gamma(1+\beta)(e^{\gamma\Delta} + e^{-\gamma\Delta}) = -(\beta+\gamma^2)(e^{\gamma\Delta} - e^{-\gamma\Delta}).$$

So

$$\coth(\gamma\Delta) = -\frac{\gamma^2+\beta}{\gamma(1+\beta)}. \tag{3.15}$$

Now $\gamma > 0$, and $\beta > 0$ by assumption, so $\frac{\gamma^2+\beta}{\gamma(1+\beta)} > 0$. Also $\Delta > 0$, so $\coth(\gamma\Delta) > 1$. This is inconsistent with (3.15), and thus case (ii) does not occur.

<u>Case (iii).</u> In this case we follow the steps in case (ii) except with γ purely imaginary, say $\gamma = j\omega$ with ω real. That is,

$$\omega^2 = \frac{1-\beta^2\lambda^2}{\lambda^2-1} \tag{3.16}$$

Then (3.15) gives us

$$\coth(j\omega\Delta) = \frac{\omega^2-\beta}{j\omega(1+\beta)}$$

or

$$\frac{-j\sin(2\omega\Delta)}{1 - \cos(2\omega\Delta)} = \frac{\omega^2-\beta}{j\omega(1+\beta)}$$

and finally

$$\frac{2\sin(\omega\Delta)\cos(\omega\Delta)}{2\sin^2(\omega\Delta)} = \frac{\omega^2-\beta}{\omega(1+\beta)}.$$

This last equation gives

$$\boxed{\cot(\omega\Delta) = \frac{\omega^2-\beta}{\omega(1+\beta)}}. \tag{3.17}$$

For given β we can numerically solve this equation for ω, finding multiple solutions. From the definition of ω^2 in (3.16), these solutions give us the eigenvalues of V^*V via

$$\lambda^2 = \frac{\omega^2+1}{\omega^2+\beta^2}. \tag{3.18}$$

The corresponding eigenfunctions are given by $f(t) = \cos(\omega t+\varphi)$, where φ is given by

$$\tan(\varphi) = -\frac{\beta}{\omega} \tag{3.19}$$

from (3.12).

The solutions of the equation (3.17) can be characterized graphically as indicated in Figure 4.

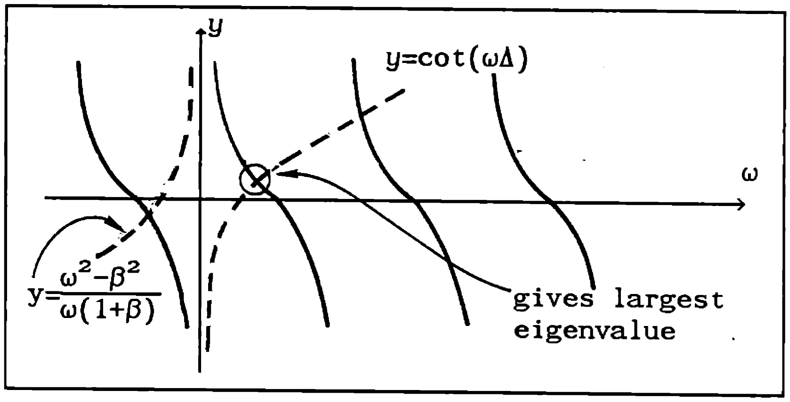

Figure 4. Graphical solution of (3.17).

D. Existence of a largest eigenvalue.

We wish to pick the one, if it exists, among solutions to (3.17) that gives the largest eigenvalue. We see that there are two cases:

- If $\beta<1$ then we should pick the solution for ω of smallest magnitude ($\omega=0$ has been excluded by our consideration of case (iii)). In general this must be done numerically.
- If $\beta>1$, we should pick the largest solution for ω, but one can see from Figure 2 that there is no upper bound on solutions to (3.17). Thus there is an infinite sequence of eigenvalues approaching $\lambda^2=1$ from below. (This means that V^*V is not compact, as we already knew.)

As pointed out in Chapter 2, the spectrum of V^*V is the set of eigenvalues augmented possibly by $\{1\}$. Thus for $\beta>1$, the spectral radius of V^*V is 1, and therefore $\|V\|=1$. Since the open loop system has sensitivity of norm 1 for the case $\beta>1$, the optimal sensitivity is attained by H=0 in (2.2), that is, zero feedback. In other words, $\|W\|_\infty = 1$ when $\beta > 1$, and this is the infimal sensitivity according to the above argument. (We note that W is not inner. For non-compact operators, minimal dilations are not necessarily unique or inner. In section H below we find an inner sensitivity as well for this case, with non-trivial feedback.)

For $\beta<1$, we also have $1 \in \sigma(V^*V)$, but this does not affect the spectral radius since the largest eigenvalue is greater than 1. It is easy to see that the eigenvalues of V^*V are contained in $[\lambda^2_{min}, 1)$ for $\beta>1$, and in $(1, \lambda^2_{max}]$ for $\beta<1$.

To summarize the situation, we have found two cases of interest: either $\beta < 1$, in which case there is an eigenvalue of V^*V equal to the norm of T, or $\beta \geq 1$, in which case $\|T\| = 1$. This will be used below to compute the minimal dilation of T to $\mathcal{H}^2$ in the first case. In the second case W itself is a minimal dilation of T to $\mathcal{H}^2$.

Remark: When $\beta \leq 1$, $|W(j\omega)|$ approaches 1 from above as $\omega \to \infty$. When $\beta > 1$, $|W(j\omega)|$ approaches 1 from below as $\omega \to \infty$. We discuss the generalization of these characteristics for more general W in Chapter 5.

E. Calculation of optimal sensitivity for $\beta<1$.

When $\beta<1$ we can find the eigenfunction f for the largest eigenvalue of V^*V by solving (3.17) and (3.19). This f is a maximal vector for V, and so we can follow the method in Sarason to compute the minimal dilation of T to $\mathcal{H}^2$. This minimal dilation will be the infimal sensitivity which we seek. It is given below in equation (3.21).

Let ω_0 be a solution to (3.17). When ω_0 corresponds to a maximal vector, according to the proof of Proposition 5.1 in [Sarason 1967], $T/\|T\|_\infty$ will be interpolated by an inner function given by $\frac{\widehat{Tf}}{\|T\|_\infty \hat{f}}$. The minimal dilation of T will be given by $\frac{\widehat{Tf}}{\hat{f}}$. We now calculate $\widehat{Tf}$ as $\mathcal{L}(Vf)$ and compute this quotient.

$$Vf = \int_0^t [\delta_{(t-\tau)} + (1-\beta)e^{-\beta(t-\tau)}]\cos(\omega_0\tau+\varphi)d\tau$$

$$= \cos(\omega_0 t+\varphi) + (1-\beta)\frac{\sin(\omega_0 t)}{(\omega_0{}^2+\beta^2)^{1/2}}$$

Then

$$\frac{\widehat{Tf}}{\hat{f}} = \frac{\mathcal{L}(Vf)}{\mathcal{L}(f)}$$

$$= 1 + \frac{1-\beta}{(\omega_0{}^2+\beta^2)^{1/2}} \cdot \frac{\mathcal{L}[\sin(\omega_0 t)\cdot(u(t)-u(t-\Delta))]}{\mathcal{L}[\cos(\omega_0 t+\varphi)\cdot(u(t)-u(t-\Delta))]}$$

$$= 1 + \frac{(1-\beta)\mathcal{L}[\sin(\omega_0 t)\cdot(u(t)-u(t-\Delta))]}{\mathcal{L}[(\omega_0\cos(\omega_0 t)+\beta\sin(\omega_0 t))\cdot(u(t)-u(t-\Delta))]} \tag{3.20}$$

After some detailed calculation (presented in the appendix to this chapter) we find that

$$\frac{\widehat{Tf}}{\hat{f}} = \lambda\cdot\frac{s+1 - e^{-s\Delta}\lambda(s-\beta)}{\lambda(s+\beta) - e^{-s\Delta}\cdot(s-1)} \tag{3.21}$$

When f is a maximal vector for V, (3.21) is the optimal weighted sensitivity, and $\left|\frac{T\hat{f}}{\hat{f}}\right|^2 = \lambda^2_{max}$. We can then compute the optimal feedback compensator for the case $\beta < 1$ as in the next section.

Remark: As $\Delta \to 0$, $\lambda^2_{max} \to 1$, and as $\Delta \to \infty$, $\lambda^2_{max} \to \frac{1}{\beta}$. This can be seen as follows: From Figure 4, as $\Delta \to 0$, ω_0^2 increases. Since $\lambda^2 = \frac{\omega^2+1}{\omega^2+\beta^2}$, as $\omega_0^2 \to \infty$, $\lambda^2 \to 1$. When $\Delta \to \infty$, we see from Figure 4 that $\omega_0^2 \to \beta$. This gives $\lambda^2 \approx \frac{1+\beta}{\beta+\beta^2} = \frac{1}{\beta}$.

F. Calculation of Optimal Feedback Compensator for $\beta<1$.

We want to find the feedback C which results in this weighted sensitivity. We have $X = W(1-PQ)$ and $C = \frac{Q}{1-PQ}$, so

$$C = \frac{W-X}{PX}. \tag{3.22}$$

Recall that we are assuming $P(s) = e^{-s\Delta}\cdot A(s)$, with A(s) stable and minimum phase.

Using (3.21) and (3.2) in (3.22) and simplifying, we get for the optimal compensator

$$\overline{C} = \frac{1}{A\lambda}\cdot\frac{s^2-1-\lambda^2(s^2-\beta^2)}{-(s+1)(s+\beta)+e^{-s\Delta}\lambda(s^2-\beta^2)}$$

Substituting for the λ^2 using (3.18) we get

$$\overline{C} = \frac{\lambda}{A}\cdot\frac{\beta^2-1}{\omega_0^2+1}\cdot\frac{s^2+\omega_0^2}{-(s+1)(s+\beta)+e^{-s\Delta}\lambda(s^2-\beta^2)}$$

Taking

$$\zeta = \frac{\lambda(1-\beta^2)}{\omega_o^2+1} = \frac{\lambda^2-1}{\lambda}$$

this is

$$\overline{C} = \zeta \cdot A^{-1} \cdot \frac{s^2+\omega_o^2}{s^2+(\beta+1)s+\beta} \cdot \frac{1}{1 + e^{-s\Delta} \cdot \lambda \cdot \frac{\beta-s}{s+1}},$$

which can be realized as shown in Figure 5.

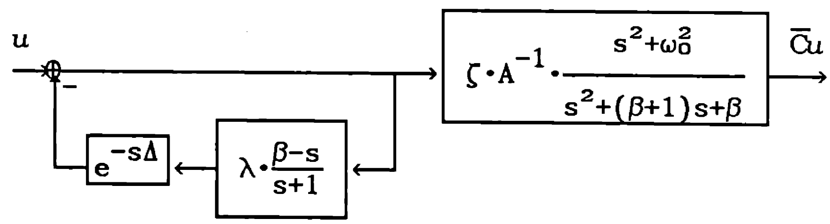

Figure 5. Realization of Optimal Compensator

Since A (the outer part of P) will generally be strictly proper, C and Q will in general be improper. It is necessary to find a proper Q for the compensator to be physically implementable. This issue is covered in Chapter 4.

G. Stability of Optimal Feedback Compensator.

We present three ways to see that the optimal compensator is unstable.

The first is essentially just an extension of the situation for purely rational plants. The idea is that the optimal sensitivity is a constant times a (infinite) Blaschke product, and the numerator of the sensitivity appears in the denominator of the compensator. Therefore the compensator has right half plane poles. The details are as follows. We know from above that $C = \frac{W-X}{PX}$, so

$$C = \frac{1}{P}\left[\frac{W}{X} - 1\right].$$

Since W and P have only left half plane zeros and poles by assumption, if we can show that X is a constant times a Blaschke product, we can conclude that C is unstable.

Let $\varphi = \lambda^{-1}X$. Then φ is inner. We argue (following [Sarason 1967, p. 194]) that $\varphi(s)$ is a Blaschke product: Since $\varphi(s)$ is continuous on the imaginary axis, the only singular inner functions that can divide it are of the form $e^{-s\alpha}$ with $\alpha>0$. But $e^{s\alpha}\varphi(s)$ is unbounded on the positive real axis, so φ is purely a Blaschke product.

We can further show that φ is an infinite Blaschke product by applying Picard's theorem to its numerator. Thus we prove that $1-e^{-s\Delta}\lambda\frac{s-\beta}{s+1}$ has finitely many zeros in the closed left half plane, and then conclude by appealing to Picard's theorem that $1-e^{-s\Delta}\lambda\frac{s-\beta}{s+1}$ has infinitely many zeros in the right half plane.

First we note that $|e^{-s\Delta}| > 1$ for s in the left half plane, and $|e^{-s\Delta}| < 1$ in the right half plane. Now all zeros must satisfy $e^{-\Re e(s\Delta)}\cdot\left|\frac{s-\beta}{s+1}\right| = \frac{1}{|\lambda|}$, and therefore all closed left half plane zeros satisfy $\left|\frac{s-\beta}{s+1}\right| \leq \frac{1}{|\lambda|}$. Since $|\lambda| > 1$ the locus $\left|\frac{s-\beta}{s+1}\right| = \frac{1}{|\lambda|}$ is an ellipse, and so all closed left half plane zeros lie on or inside the intersection of the ellipse $\left|\frac{s-\beta}{s+1}\right| = \frac{1}{|\lambda|}$ with the closed left half plane. See Figure 6.

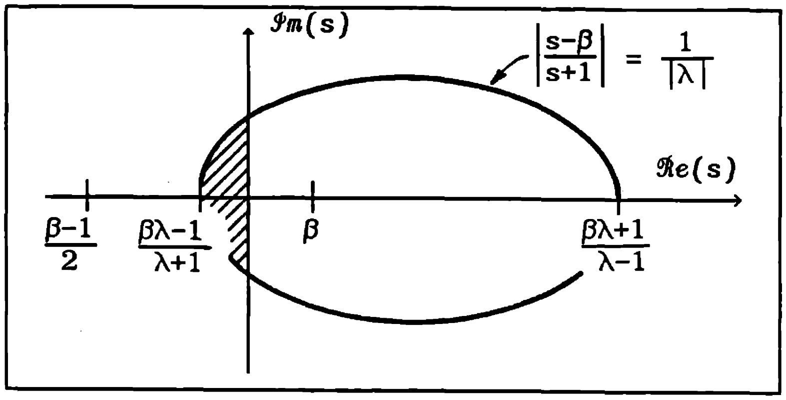

Figure 6. Region for possible left half plane zeros.

Thus all closed left half plane zeros lie in a compact region, and we conclude from analyticity that there are only finitely many in that region. Picard's theorem tells us that there are infinitely many zeros, so that we must conclude that there are infinitely many in the open right half plane.

(The same conclusion might be reached by looking at the Nyquist plot for the feedback loop in Figure 5, although we know of no version of the Nyquist stability criterion which applies to the case of infinitely many right half plane poles.)

A second instability proof gives us more detailed information about the distribution of the right half plane zeros without much more trouble. As stated above, the right half plane zeros must satisfy $\left|\frac{s-\beta}{s+1}\right| > \frac{1}{|\lambda|}$. As $|s| \to \infty$, $\left|\frac{s-\beta}{s+1}\right| \to 1$. Therefore as $|s| \to \infty$, the zero set $\{z_i\}$ approaches the line $|e^{-s\Delta}| = \frac{1}{|\lambda|}$, which is the same as $\mathcal{R}e(s) = \frac{\ln|\lambda|}{\Delta}$, and $\mathcal{I}m(z_i) \to (2n+1)\pi$. Also, since right half plane zeros must satisfy $\left|\frac{s-\beta}{s+1}\right| < 1$ (by comparing distances from β and from the point -1), we have $|\lambda|\cdot\left|\frac{s-\beta}{s+1}\right| < |\lambda|$. Then since $|e^{-s\Delta}|\cdot\left|\frac{s-\beta}{s+1}\right| = \frac{1}{|\lambda|}$,

$|e^{s\Delta}| < |\lambda|$, and we conclude $\mathcal{Re}(s) < \frac{\ln|\lambda|}{\Delta}$ for these zeros.

The third proof uses the fact that $H(s) = (s+1)-e^{-s\Delta}\lambda(s-\beta)$ has right half plane zeros can also be obtained from results on the distribution of zeros of entire functions. (See [Levin 1980] Chapter 7, §4, p. 323, Example 1.) For our purposes it is more convenient to refer to the earlier work [Pontryagin 1955].

According to Theorem 7 in this latter paper it is sufficient to show that the function $G(y) = y\cdot\cos(y\Delta) + \sin(y\Delta) + \lambda\cdot y$ has zeros which are not purely real.[4] To see this all we need is to recognize that G has infinitely many zeros. (To show this we can use Theorem 3 in the same paper.[5]) Setting $G(y) = 0$ we get

$$-y\cdot\cos(y\Delta) = \sin(y\Delta) + \lambda\cdot y.$$

For real values of y, the left hand side of this equation has the lines $z = \pm y$ for an envelope, whereas the right hand side oscillates with deviation 1 about the line $z = \lambda\cdot y$. Since $|\lambda| > 1$, for some value y_0 there are no more real zeros for $|y| > y_0$.

---

[4] Theorem 6 says that if we evaluate H(s) on the imaginary axis, and split the resulting $H(iy)$ $(y\in\mathbb{R})$ into real and imaginary parts, $H(iy) = F(y) + iG(y)$ with $F(y)$ and $G(y)$ taking only real values, then necessary and sufficient conditions for H to have all its zeros in the left half plane are that (i) F and G have <u>only</u> real zeros, that (ii) these zeros alternate, and (iii) for at least one value of y, $G'(y)F(y) - F'(y)G(y) > 0$. Thus we need only show that G has a complex zero to establish that H has a right half plane zero.

[5] Theorem 3 applied to $G(y)$ says that for sufficiently large values of the integer k, $G(y)$ has exactly $4k+1$ zeros on the strip in $\mathbb{C}$ $-2k\pi + \epsilon \leq \mathcal{Re}(y) \leq 2k\pi + \epsilon$. Therefore for all the zeros of $G(y)$ to be real it is necessary and sufficient for $G(y)$ to have exactly $4k+1$ real zeros on the interval in $\mathbb{R}$ $-2k\pi + \epsilon \leq y \leq 2k\pi + \epsilon$.

H. Another Solution for $\beta>1$

We show now that when $\beta>1$ the choice of $\lambda = 1$ in equation (3.24) gives an infimal sensitivity and a corresponding improper compensator. There are essentially two things to show: First, that for $\lambda = 1$ $X(s)$ is in $\mathcal{H}^\infty$ (in fact we show it is inner). Second, that this $X(s)$ is a dilation of T — equivalently, that $e^{-s\Delta}$ divides $(W(s)-X(s))$ in $\mathcal{H}^\infty$.

1. X(s) is inner. For $\lambda = 1$, (3.21) becomes

$$X(s) = \frac{s+1 - e^{-s\Delta}(s-\beta)}{s+\beta - e^{-s\Delta}(s-1)}.$$

Let $e(s)$ be the numerator of $X(s)$ and $g(s)$ be the denominator. Then we see that $e(s) = -e^{-s\Delta}g(-s)$, so it is clear that $|X(j\omega)| = 1$. We need only check that $g(s)$ has no zeros in the right half plane. The zeros of $g(s)$ are given by the solutions to

$$e^{-s\Delta} = \frac{s+\beta}{s-1}.$$

Since the $|e^{-s\Delta}| < 1$ in the right half plane, and $|\frac{s+\beta}{s-1}| > 1$ in the right half plane, there are no right half plane zeros.

2. X(s) is a dilation of T. We show that $e^{-s\Delta}$ divides $(W(s)-X(s))$ in $\mathcal{H}^\infty$.

$$W(s) - X(s) = \frac{s+1}{s+\beta} - \frac{s+1 - e^{-s\Delta}(s-\beta)}{s+\beta - e^{-s\Delta}(s-1)}$$

$$= \frac{[(s^2-\beta^2) - (s^2-1)]e^{-s\Delta}}{(s+\beta)[s+\beta - e^{-s\Delta}(s-1)]}$$

$$= \frac{(1-\beta^2)e^{-s\Delta}}{(s+\beta)[s+\beta - e^{-s\Delta}(s-1)]}.$$

This is divisible by $e^{-s\Delta}$ in $\mathcal{H}^\infty$ since $\dfrac{(1-\beta^2)}{(s+\beta)[s+\beta - e^{-s\Delta}(s-1)]}$ is in $\mathcal{H}^\infty$.

Thus we have a second minimal dilation of T for the case $\beta>1$, the first being given by W(s) itself.

The compensator corresponding to this inner sensitivity is

$$C(s) = \frac{W-X}{PX} = \frac{(1-\beta^2)e^{-s\Delta}}{(s+\beta)[s+\beta - e^{-s\Delta}(s-1)]}\cdot\left[e^{-s\Delta}A\cdot\frac{s+1 - e^{-s\Delta}(s-\beta)}{s+\beta - e^{-s\Delta}(s-1)}\right]^{-1}$$

$$= A^{-1}\frac{(1-\beta^2)}{(s+\beta)[s+1 - e^{-s\Delta}(s-\beta)]}.$$

Remark: If we take λ in (3.21) (call this function $X_\lambda(s)$) for which λ^2 is not equal to the spectral radius of T^*T, the equation which determines the zeros of the denominator of X_λ is

$$e^{-s\Delta} = \frac{\lambda(s+\beta)}{s-1}.$$

In the right half plane $\mathcal{Re}(s) > 0$, $|e^{-s\Delta}| < 1$, so in order for X_λ to have right half plane poles, we must have $\left|\frac{\lambda(s+\beta)}{s-1}\right| < 1$. But $\left|\frac{s+\beta}{s-1}\right| > 1$ in the right half plane, since $\beta > 0$. If $\beta > 1$, we know all eigenvalues are less than 1, but cluster at 1. Therefore for all but finitely many eigenvalues λ the function X_λ is stable, and we actually can find a sensitivity.

CHAPTER 3 Appendix

Details of Calculations

A. Calculation of Adjoint to V.

V^* is defined by $\langle x, V^* y\rangle = \langle Vx, y\rangle$, so we just compute:

$$\langle Vx, y\rangle = \int_0^\Delta y(t)\cdot(w(t)*x(t))dt$$ [We have used the fact that both w and y have their support on $[0,\infty)$.]

Using the fact that

$$w(t-\tau) = \delta_{(t-\tau)} + w_0(t-\tau) = \delta_{(t-\tau)} + (1-\beta)\cdot e^{-\beta(t-\tau)}$$

we then have

$$\int_0^\Delta y(t)\cdot(w(t)*x(t))dt = \int_0^\Delta y(t)\cdot[\int_0^t w_0(t-\tau)x(\tau)d\tau + \delta_t * x(t)]dt$$

$$= \int_0^\Delta y(t)\cdot[(1-\beta)\cdot e^{-\beta t}\cdot\int_0^t e^{\beta\tau}\cdot x(\tau)d\tau + x(t)]\ dt$$

$$= [(\int_0^t y(\tau)e^{-\beta\tau}d\tau)(\int_0^t e^{\beta\tau}\cdot x(\tau)d\tau)]_0^\Delta -$$

$$\int_0^\Delta[\int_0^t y(\tau)e^{-\beta\tau}d\tau]\cdot[e^{\beta t}\cdot x(t)]\cdot dt + \int_0^\Delta y(t)x(t)dt$$

$$= \int_0^\Delta x(t)\int_0^\Delta y(\tau)w_0(\tau-t)d\tau dt - \int_0^\Delta x(t)\int_0^t y(\tau)w_0(\tau-t)d\tau dt + \int_0^\Delta x(t)[\delta(t)*y(t)]dt$$

$$= \int_0^\Delta x(t)[\int_t^\Delta y(\tau)w_0(\tau-t)d\tau + (\delta*y)(t)]dt$$

$$= \langle \int_t^\Delta y(\tau)w(\tau-t)d\tau, x(t)\rangle$$

and we conclude

$$V^* y = \int_t^\Delta y(\tau)w(\tau-t)d\tau.$$

B. Computation of Optimal Sensitivity

We now explicitly compute and simplify the expressions in equation (3.20).

$$\mathcal{L}[\sin(\omega_0 t)\cdot(u(t)-u(t-\Delta))] = \frac{1-e^{-s\Delta}}{s} * \frac{\delta_{(s-j\omega)} - \delta_{(s+j\omega)}}{2j}$$

$$= \frac{1}{2j}\cdot[\frac{1-e^{-(s-j\omega_0)\Delta}}{s-j\omega_0} - \frac{1-e^{-(s+j\omega_0)\Delta}}{s+j\omega_0}]$$

$$= \frac{2j\omega_0 - e^{-s\Delta}\cdot[(s+j\omega_0)e^{j\omega_0\Delta}-(s-j\omega_0)e^{-j\omega_0\Delta}]}{2j(s^2+\omega_0^2)}$$

$$= \frac{\omega_o - e^{-s\Delta}[s\cdot\sin(\omega_o\Delta)+\omega_o\cdot\cos(\omega_o\Delta)]}{s^2+\omega_o^2} \quad (3.22)$$

$$\mathcal{L}[\cos(\omega_o t)\cdot(u(t)-u(t-\Delta))] = \frac{1-e^{-s\Delta}}{s} * \frac{\delta_{(s-j\omega_o)} + \delta_{(\omega+j\omega_o)}}{2}$$

$$= \frac{1}{2}\cdot\left[\frac{1-e^{-(s-j\omega_o)\Delta}}{s-j\omega_o} + \frac{1-e^{-(s+j\omega_o)\Delta}}{s+j\omega_o}\right]$$

$$= \frac{s - e^{-s\Delta}[s\cdot\cos(\omega_o\Delta) - \omega_o\cdot\sin(\omega_o\Delta)]}{s^2+\omega_o^2} \quad (3.23)$$

Using (3.22) and (3.23) in (3.21), we get

$$\frac{T\hat{f}}{\hat{f}} = 1 + \frac{(1-\beta)[-\omega_o+ e^{-s\Delta}\cdot(s\cdot\sin\omega_o\Delta+\omega_o\cos\omega_o\Delta)]}{\omega_o[-s+e^{-s\Delta}(s\cdot\cos\omega_o\Delta-\omega_o\sin\omega_o\Delta)] + \beta[-\omega_o+e^{-s\Delta}(s\cdot\sin\omega_o\Delta+\omega_o\cos\omega_o\Delta)]}$$

$$= 1 + \frac{(1-\beta)[-\omega_o+ e^{-s\Delta}(s\cdot\sin\omega_o\Delta+\omega_o\cos\omega_o\Delta)]}{-\omega_o(\beta+s)+e^{-s\Delta}[s(\omega_o\cos\omega_o\Delta+\beta\sin\omega_o\Delta)+\omega_o(\beta\cos\omega_o\Delta-\omega_o\sin\omega_o\Delta)]}. \quad (3.24)$$

The optimal weighted sensitivity in (3.24) can be written

$$\overline{X} = \frac{\omega_o[-s+e^{-s\Delta}(s\cdot\cos\omega_o\Delta-\omega_o\sin\omega_o\Delta)] + e^{-s\Delta}(s\cdot\sin\omega_o\Delta+\omega_o\cos\omega_o\Delta) - \omega_o}{\omega_o[-s+e^{-s\Delta}(s\cdot\cos\omega_o\Delta-\omega_o\sin\omega_o\Delta)] + \beta[-\omega_o+e^{-s\Delta}(s\cdot\sin\omega_o\Delta+\omega_o\cos\omega_o\Delta)]}$$

$$= \frac{s\cdot(-\omega_o+e^{-s\Delta}\omega_o\cos\omega_o\Delta+e^{-s\Delta}\sin\omega_o\Delta) + \omega_o e^{-s\Delta}(\cos\omega_o\Delta-\omega_o\sin\omega_o\Delta) - \omega_o}{s\cdot(-\omega_o+e^{-s\Delta}\omega_o\cos\omega_o\Delta+e^{-s\Delta}\beta\sin\omega_o\Delta) + \omega_o e^{-s\Delta}(\beta\cos\omega_o\Delta-\omega_o\sin\omega_o\Delta) - \beta\omega_o}$$

where ω_o is the smallest positive solution of (3.13).

We can further simplify this expression, but we first need to establish the fact that

$$\frac{1}{\lambda} = \cos(\omega_0\Delta) + \frac{\beta}{\omega_0}\sin(\omega_0\Delta)$$

This just requires the calculation

$$\cos(\omega_0\Delta) + \frac{\beta}{\omega_0}\sin(\omega_0\Delta) = \cos(\omega_0\Delta)\left[1 + \frac{\beta}{\omega_0}\cdot\frac{\omega_0(1+\beta)}{\omega_0^2-\beta}\right]$$

$$= \cos(\omega_0\Delta)\left[\frac{\omega_0^2+\beta^2}{\omega_0^2-\beta}\right]$$

$$= \left[\frac{1}{1 + \tan^2(\omega_0\Delta)}\right]^{1/2}\cdot\left[\frac{\omega_0^2+\beta^2}{\omega_0^2-\beta}\right]$$

$$= \left[\frac{(\omega_0^2-\beta)^2}{(\omega_0^2+1)(\omega_0^2+\beta^2)}\right]^{1/2}\cdot\left[\frac{\omega_0^2+\beta^2}{\omega_0^2-\beta}\right]$$

$$= \left[\frac{\omega_0^2+\beta^2}{\omega_0^2+1}\right]^{1/2} = \lambda^{-1}$$

Let $\zeta_1 = \frac{\sin(\omega_0\Delta) + \omega_0\cdot\cos(\omega_0\Delta)}{\omega_0}$. We use equation (3.19) and substitute for λ^2 using (3.16) to obtain $\sin(\omega_0\Delta) = \frac{(1+\beta)\omega_0}{\omega_0^2-\beta}\cdot\cos(\omega_0\Delta)$, from which we have:

$$\frac{\omega_0\cos(\omega_0\Delta)+\sin(\omega_0\Delta)}{\omega_0} = \left[1 + \frac{1+\beta}{\omega_0^2-\beta}\right]\cos(\omega_0\Delta)$$

$$= \frac{1+\omega_0^2}{\omega_0^2-\beta}\cos(\omega_0\Delta)$$

$$= \cos(\omega_0\Delta)\cdot\frac{\omega_0^2+\beta^2}{\omega_0^2-\beta}\cdot\frac{1+\omega_0^2}{\omega_0^2+\beta^2}$$

$$= \lambda^{-1}\cdot\lambda^2 = \lambda$$

$$\cos(\omega_0\Delta)-\omega_0\sin(\omega_0\Delta) = \left[1-\frac{(1+\beta)\omega_0^2}{\omega_0^2-\beta}\right]\cos(\omega_0\Delta)$$

$$= -\frac{\beta(1+\omega_0^2)}{\omega_0^2-\beta}\cos(\omega_0\Delta)$$

$$= -\beta\lambda$$

$$\omega_0\cos(\omega_0\Delta)+\beta\sin(\omega_0\Delta) = \left[\omega_0 + \frac{\beta(1+\beta)\omega_0}{\omega_0^2-\beta}\right]\cos(\omega_0\Delta)$$

$$= \omega_0\frac{\beta^2+\omega_0^2}{\omega_0^2-\beta}\cos(\omega_0\Delta)$$

$$= \omega_0\frac{\beta^2+\omega_0^2}{\omega_0^2+1}\zeta_1$$

$$= \frac{\omega_0}{\lambda}$$

and

$$\beta\cos(\omega_0\Delta)-\omega_0\sin(\omega_0\Delta) = \left[\beta-\frac{(1+\beta)\omega_0^2}{\omega_0^2-\beta}\right]\cos(\omega_0\Delta)$$

$$= -\frac{\beta^2+\omega_0^2}{\omega_0^2-\beta}\cos(\omega_0\Delta)$$

$$= -\frac{1}{\lambda^2}.$$

This gives us

$$X = \frac{\omega_0 s(e^{-s\Delta}\lambda-1) - \beta\omega_0\lambda\cdot e^{-s\Delta} - \omega_0}{\omega_0 s(e^{-s\Delta}\cdot\frac{1}{\lambda} - 1) - \omega_0 e^{-s\Delta}\cdot\frac{1}{\lambda} - \beta\omega_0}$$

$$= \lambda\cdot\frac{s+1 - e^{-s\Delta}\lambda(s-\beta)}{\lambda(s+\beta) - e^{-s\Delta}\cdot(s-1)} \tag{3.25}$$

(3.25) is the same as (3.21).

CHAPTER 4

REALIZATION OF THE COMPENSATOR

A. Motivation

Now we consider the problem of approximating the optimal compensator with a real system. Our discussion here will be for the 1 pole/1 zero weighting function case for a stable plant with minimum phase rational factor covered in Chapter 3. However, the analysis extends to more general cases.

There are two problems with the optimal compensator: First, it is generally improper. The physical interpretation of this is that it would have to contain differentiators, which can only be approximated with real systems[6]. The second problem is that the optimal compensator contains a ideal delay, and again this cannot be constructed exactly.

The best we can hope for is that we can approximate the ideal compensator over a finite bandwidth, and design the system so that the behavior outside this band does not significantly affect performance. We would like to describe an approximation procedure such that we can pick whatever finite bandwidth we want, and the performance will approach the optimum as the bandwidth grows.

Of course, we must explain in what sense we are approximating the

[6] The problem with realization of the delayed differentiators we would need is not a lack of causality, since a differentiator cascaded with a delay is even strongly causal. The problems that make differentiators impossible to realize are infinite bandwidth and unboundedness.

optimal compensator. What we really want is to describe a sequence of compensators for which the weighted $\mathcal{H}^\infty$ norm of the closed loop sensitivity approaches the infimal value.

Since the infimal weighted sensitivity is unique (when it has norm greater than 1), and the corresponding compensator is improper when the plant is strictly proper, there can be no proper compensator which achieves the infimum.

From the viewpoint of the minimization problem originally posed, it must be true that when P is strictly proper the set $P\mathcal{H}^\infty$ is not closed in the norm topology on $\mathcal{H}^\infty$. (It turns out that the condition that P is not strictly proper and does not have any zeros or poles on the imaginary axis is sufficient to guarantee that set is closed.)

The rest of the chapter is as follows: We motivate our construction of approximating proper compensators with a discussion of why a version of the usual approximation procedure, which works for the case of rational plants, does not work for the delay case. We then show one way to construct a sequence of compensators for which performance approaches the optimum. We do this in two steps. First we produce a sequence of strictly proper compensators which contain pure delays, which sequence approaches the optimum. Then we show how to approximate the delays with strictly proper systems so as to preserve the good performance.

B. The conventional proper approximation technique.

1. Summary

The only procedures in the literature for this purpose seem to be those given in [Zames and Francis 1983, p. 591] and [Vidyasagar 1985, p. 178].

The procedure in [Zames and Francis 1983] requires the evaluation of the term $B_z(\infty)$, where $B_z(s)$ is the Blaschke product formed from plant zeros. In our case there is no Blaschke product involved, but rather a singular inner function. If we interpret $B_z(s)$ to be this inner function, $B_z(\infty)$ is not defined. There is no apparent way to fix this problem for our case.

The procedure in [Vidyasagar 1985] does not work for our case either. The essence of the difficulty is the same as in the Zames-Francis procedure — the inner factor of the plant is not continuous at infinity. We examine this difficulty in detail, because it motivates our own approach.[7]

In the case of a stable plant, the Vidyasagar procedure consists of multiplying the optimal "Q-parameter" by a rational function, the magnitude of which decreases with increasing frequency at a sufficiently high rate. As the breakpoint of this "roll-off" function increases, the $\mathcal{H}^{\infty}$-norm of the rolled-off sensitivity function approaches the minimum.

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[7] Another way to view the difficulty is the following: Strict properness of the rational part of the plant amounts to zero(s) at ∞. This can be handled in the same way as zeros on the imaginary axis. The easiest way to see this is to transform the problem to $\mathcal{H}^{\infty}$ of the unit disk, for then a strictly proper plant with one more pole than zero has a simple zero at the point 1. Zeros on the imaginary axis transform to other points on the unit circle. The delay makes the zero at ∞ different because the delay has an essential singularity at ∞. This transforms to an essential singularity at 1 on the unit circle. It is because of the collocation of the essential singularity with the zero that the procedure that works in the purely rational plant case for zeros on the axis does not work for the zero at ∞, but does work for other zeros on the axis. See footnote 1 in Chapter 1.

The idea is then to compute the compensator which yields this suboptimal Q-parameter, and in the rational plant case one will have a satisfactory sequence of approximating compensators.

The following paragraphs use some approximations to illustrate how Vidyasagar's technique fails to approach the optimal sensitivity in the limit for the 1 pole/zero weighting function case we have explicitly solved.

2. Application to the delay case.

This section refers to [Vidyasagar 1985, pp. 176-178].

The sensitivity minimization problem (3.3) amounts to finding $\inf_{H\in\mathcal{H}^\infty} \|W - P_i H\|_\infty$. We are looking at the case when the plant P is stable, with $P = P_o e^{-s\Delta}$, $P_i = e^{-s\Delta}$ the inner factor of the plant, and W is the sensitivity weighting function. Suppose $\overline{R}$ yields the infimal value. Since $W - e^{-s\Delta}\overline{R}$ is all-pass (in the cases of interest), $\overline{R}$ is bounded away from 0 at infinity. If P_o is strictly proper, then $P_o^{-1}\overline{R}$ is improper. This will generally be the case with realistic plant models. The optimal Q parameter would be given by $\overline{Q} = P_o^{-1}\overline{R}$, except for the lack of properness. We relax our terminology and refer to this function as a Q parameter.

We address the case of P_o strictly proper. $h_\epsilon(s)$ (defined for an example below) denotes Vidyasagar's roll-off function which multiplies the parameter, in the case of a stable plant.

In the case when P_o has a single zero at ∞

$$h_\epsilon(s) = \frac{\frac{s-1}{s+1} - 1}{\frac{s-1}{s+1} - (1+\epsilon)} = \frac{2}{\epsilon s + 2+\epsilon} = \frac{2/\epsilon}{s + 1 + 2/\epsilon}.$$

Multiple zeros at ∞, whose number n equals the difference of degrees between denominator and numerator of $P_o(s)$, result in the above function raised to the power n.

Let $\overline{R}$ be the optimizing function in the minimization. We first note that the infimal weighted sensitivity magnitude $|W(1-e^{-s\Delta}\overline{R})|$ is a constant. Recall that we are always assuming $W(\infty) = 1$. In Vidyasagar's method, to approximate the optimal sensitivity one lets the ϵ in h_ϵ go to 0. This means that the roll-off break frequency goes to ∞. Therefore, if the method were to work, we could assume that W is approximately 1 before any frequency at which there is significant deviation in h_ϵ from 1.

Thus[8] we consider the situation at frequencies $\omega \geq \omega_r$ where the roll-off starts to take effect, and assume $W(j\omega) = 1$. At the frequencies under consideration we have that

$$|1-e^{-s\Delta}\overline{R}| \approx |W(1-e^{-s\Delta}\overline{R})| = \text{constant}.$$

Therefore for $\omega \geq \omega_r$, $(1-e^{-s\Delta}\overline{R})(\omega)$ lies on a circle about the origin in the complex plane, with radius $\|W(1-e^{-s\Delta}\overline{R})\|_\infty$, and $(-e^{-s\Delta}\overline{R})(\omega)$ lies on a circle with the same radius about the point -1.

The magnitude of the rolled-off sensitivity, $|W(1-e^{-s\Delta}\overline{R}h_\epsilon)|$, can be

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[8]The following argument applies to the case where the inner factor of the plant has a rational part, simply by taking ω_r large enough.

approximated in this frequency range as $|1-e^{-s\Delta}\overline{R}h_\epsilon|$, which in turn can be understood by considering how $e^{-s\Delta}\overline{R}h_\epsilon$ deviates from the circle $e^{-s\Delta}\overline{R}$ about the point -1.

To continue this argument we specialize to the case of $e^{-s\Delta}\overline{R}$ at high frequency for the 1 pole/1 zero weighting function case we have solved. An approximation gives the function

$$(e^{-s\Delta}\overline{R})_\infty(\omega) = \frac{(1-\beta)(\sin\omega_0)/\omega_0}{\cos\omega_0 + (\beta\sin\omega_0)/\omega_0 - e^{j\omega}}$$

for the high frequency behaviour of $e^{-s\Delta}\overline{R}$. We see that the frequency dependence of the phase of this function is in the term $e^{j\omega}$, so that it is periodic with period 2π. For the range $0<\beta<1$, the image of $\omega\in\mathbb{R}$ loops around the entire circle. The point is that $(e^{-s\Delta}\overline{R})_\infty$ changes in phase by 2π radians as ω increases from ω_1 to $\omega_1+2\pi$, whereas h_ϵ is approximately constant over this range if ω_1 is sufficiently large. We shall examine the behaviour of $e^{-s\Delta}\overline{R}h_\epsilon$ for an interval of width 2π at high frequency, and thus assume that h_ϵ is constant over that interval.

We now assume that h_ϵ rolls off steeply enough so that there is almost no gain reduction at the frequency ω_{180} at which h_ϵ contributes 180° of phase shift. (This is not quite realistic — a 47 pole roll-off is required for h_ϵ to contribute 180° of phase shift with only 10% gain reduction.) When $\omega\in[\omega_{180}, \omega_{180}+2\pi]$ the circle on which $((e^{-s\Delta}\overline{R})_\infty h_\epsilon)(\omega)$ lies is the circle of $(e^{-s\Delta}\overline{R})_\infty$ (which is about the point -1) rotated by 180° about the origin. This is a circle of the same radius about the point 1. To form the corresponding sensitivities for points on this circle we add 1, which amounts to translating the center of the circle to the point 2. The points on the circle are the values of the approximate sensitivity function evaluated on $\omega \in [\omega_{180}, \omega_{180}+2\pi]$, and it is clear that the $\mathcal{H}^{\infty}$ norm of this function is the radius of the circle plus 2.

More realistically, h_{ϵ} will not have 47 poles, and we must take into account the contribution of h_{ϵ} to gain reduction while it contributes phase shift. In Figure 7 we show the function $(1-e^{-s\Delta}\overline{R}h_{\epsilon})(j\omega)$ over the range $[\omega_0-\pi, \omega_0+\pi]$ for the indicated values of ω_0, with h_{ϵ} having 4 poles. (β = 0.1 and ϵ = 0.001 π.) As we

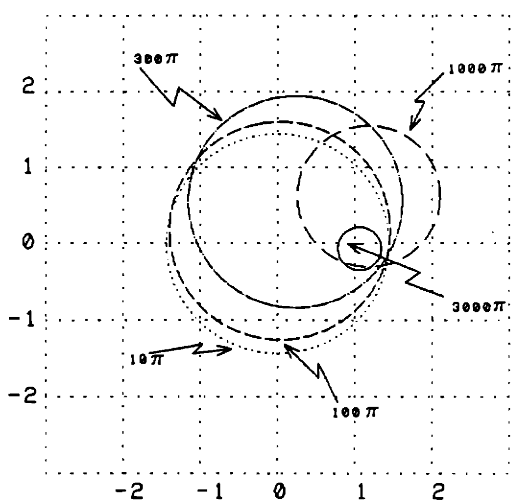

Figure 7. Approximate sensitivity over various ranges.

can see from Figure 7, when h_{ϵ} results in

$$|(e^{-s\Delta}\overline{R}h_{\epsilon})(\omega)| \leq \|1-e^{-s\Delta}\overline{R}\|_{\infty} - 1,$$

whatever the phase shift at ω the sensitivity at this frequency is not greater than $\|1-e^{-s\Delta}\overline{R}\|_{\infty}$. Furthermore, there is some ω_m for which $e^{-s\Delta}\overline{R}h_{\epsilon}$ will satisfy this inequality for all $\omega \geq \omega_m$. Corresponding to ω_m is some angle φ_m for which additional phase shift is not accompanied by sensitivity increase. At high frequency, these constants are a function of β and the number of poles in h_{ϵ}.

When ϵ is small, the maximum sensitivity increase over the optimal value depends on the order of h_{ϵ} and is independent of ϵ, so letting $\epsilon \to 0$ does not yield sensitivities with $\mathcal{H}^{\infty}$ norm approaching the optimum.

This argument has been limited to the 1 pole/1 zero weighting function case because we have used our explicit solution in this case to show how the image of br_0 revolves entirely around the circle. A detailed analysis will show in the general rational weighting function case that $e^{-s\Delta}\overline{R}$ has the same "wrapping" behaviour at high frequency.

C. A proper approximation procedure that works.

1. Summary

We now describe how to construct an approximating sequence of compensators that *does* produce sensitivities which approach the optimum.

The essential idea is to roll-off H in the ideal Q-parameter by multiplying it by a stable transfer function for which the Bode magnitude plot has slope less than 1 (such as (4.2) below), so as to limit the phase deviation due to the roll-off, until sufficient attenuation has been obtained. This this can be accomplished, for example, with a lead-lag network which approximates such compensation by having average slope magnitude less than 1.

4. Realization of the Compensator

We note that any stable Q-parameter results in a stable closed loop system, so that this roll-off technique preserves stability just as the procedures for the isolated right half plane zero case do.

2. Details of the approximation

Now we assume $\overline{Q}$ is the optimal (improper) Q parameter resulting in the optimal weighted sensitivity $\overline{X} = W(1-e^{-s\Delta}\overline{Q})$, and that $|\overline{X}| = k \geq 1$. We can write $\overline{X}(j\omega) = ke^{j\alpha(\omega)}$, where $\alpha(\omega)$ is real.

In the following we will make certain approximations in order to make the argument easier to follow. One can dispense with the approximations at the expense of additional details.

We shall assume

$$\omega > \omega_\epsilon, \text{ where } W(j\omega) \approx 1 \text{ for } \omega > \omega_\epsilon. \tag{4.1}$$

Using this approximation

$$1-e^{-s\Delta}\overline{Q} \approx ke^{j\alpha(\omega)}.$$

We now examine the effect of multiplying the parameter $\overline{Q}$ by the roll-off function

$$h_n(s) = [\gamma/(s+\gamma)]^{1/n}. \tag{4.2}$$

The critical feature of h_n is that we can make the magnitude of $\arg(h_n)$ as small as we want by taking n sufficiently large. We examine the magnitude squared of the approximate sensitivity,

$$|1 + hn(j\omega)(ke^{j\alpha(\omega)}-1)|^2.$$

Suppose ω_m is the frequency at which the worst sensitivity occurs, and define

$$h = |hn(j\omega_m)|,$$

$$\delta = \arg[hn(j\omega_m)]$$

and
$$\alpha = \alpha(\omega_m).$$

Note that h, δ and α are functions of n, as is ω_m. However ω_m is finite since the sensitivity function is 1 at ∞, and δ satisfies $0 \leq \delta \leq \frac{2\pi}{n}$. We henceforth assume we have taken n large enough so that

$$\cos(\delta) \approx 1 \text{ and } \sin(\delta) \approx \delta.$$

Now we show that this approximate sensitivity approaches the infimal sensitivity as n increases.

$$|1 + hn(j\omega)(ke^{j\alpha(\omega)}-1)|^2 \leq |1 + h\cdot e^{j\delta}(ke^{j\alpha}-1)|^2$$

$$= |1 - h\cdot e^{j\delta} + h\cdot e^{j\delta}ke^{j\alpha}|^2$$

$$= |1 - h\cdot\cos(\delta) - jh\cdot\sin(\delta) + hk\cdot\cos(\delta+\alpha) + jhk\cdot\sin(\delta+\alpha)|^2$$

$$= [1 - h\cdot\cos(\delta) + hk\cdot\cos(\delta+\alpha)]^2 + [h\cdot\sin(\delta)hk\cdot\sin(\delta+\alpha) - h\cdot\sin(\delta)]^2$$

$$= 1 - 2h\cos\delta + h^2 + 2hk\cos(\delta+\alpha) + h^2k^2 - 2h^2k[\cos(\delta)\cos(\delta+\alpha) + \sin(\delta)\sin(\delta+\alpha)]$$

$$= 1 - 2h\cos\delta + h^2 + 2hk[\cos(\delta)\cos(\alpha)-\sin(\delta)\sin(\alpha)] + h^2k^2 - 2h^2k\cdot\cos(\alpha)$$

$$= 1 -2h + 2h(1-\cos\delta) + h^2 + h^2k^2 + 2hk\Big[\cos(\alpha) + [\cos(\delta)-1]\cdot\cos(\alpha) - \delta\cdot\sin(\alpha) + [\delta-\sin(\delta)]\cdot\sin(\alpha) - h\cdot\cos(\alpha)\Big]$$

$$\approx 1 -2h + h^2 + h^2k^2 + 2hk[(1-h)\cdot\cos(\alpha) - \delta\cdot\sin(\alpha)]$$

$$\lesssim 1 -2h + h^2 + h^2k^2 + 2hk[(1-h)\cdot|\cos(\alpha)| + \delta\cdot|\sin(\alpha)|]$$

$$\lesssim (1-h+hk)^2 + 2hk\cdot\delta$$

$$= [1 + h(k-1)]^2 + 2hk\cdot\delta$$

$$\lesssim k^2 + 2k\delta$$

$$\lesssim k^2 + 4k\pi/n$$

Thus the squared magnitude of the sensitivity of the rolled-off compensator differs from the squared magnitude of the infimal sensitivity by a term of the order of $1/n$. (The "$\approx$" symbol can be removed by placing conditions on how small δ must be, and adding terms to the right hand expression of the order of δ^2.)

Also the roll-off can be fast after the loop gain has decreased sufficiently: Since $|h(j\omega)| \to 0$ as $\omega \to \infty$ and $|e^{-s\Delta}\overline{Q}|$ is bounded, as ω increases and after $|h(j\omega)|$ is sufficiently small, say

$$|h(j\omega)| < \frac{k-1}{2} \text{ for } \omega > \omega_r,$$

we can allow $|h(j\omega)|$ to decrease arbitrarily fast without increasing the $\mathcal{H}^\infty$-norm of the resulting sensitivity.

We note that in our case of a stable plant stable roll-off of the compensator results in a stable closed loop system since the modified compensator corresponds to a Q-parameter which is in $\mathcal{H}^\infty$. Furthermore, the argument works with a stable finite dimensional approximation to h_n. From now on we assume h_n is such an approximation. We also note that roll-off of the parameter $\overline{Q}$ ensures a proper compensator: Let $Q_n = \overline{Q}h_n$, the rolled-off Q parameter. Then if C_n is the resulting compensator, $C_n = \frac{Q_n}{1-PQ_n}$, so if Q_n is proper, so is C_n since P is proper. It is important to note that since PQ_n is strictly proper and Q_n is stable, C_n can have only finitely many poles. This is in contrast to $\overline{C}$, and is important in part 2 of section D below.

3. Computation of the proper compensator.

Our proper, but infinite dimensional compensator is given by

$$C_n = \frac{W-X}{PX} = \frac{W-W(1-PQ_n)}{PW(1-PQ_n)}$$

$$= \frac{Q_n}{1-PQ_n} = \frac{h_n\overline{Q}}{1-Ph_n\overline{Q}}.$$

Recall that we are assuming $W(s) = \frac{s+1}{s+\beta}$. Take $n(s) = s+1$ and $d(s) = s+\beta$. Let $\check{f}(s)$ denote $f(-s)$. Then from (3.21)

$$\overline{X} = \lambda \cdot \frac{n + \lambda e^{-s\Delta}\check{d}}{\lambda d + e^{-s\Delta}\check{n}}.$$

Using this,

$$\overline{Q} = \frac{W-\overline{X}}{PW}$$

$$= \frac{n - d\overline{X}}{e^{-s\Delta}n} \cdot P_o^{-1}$$

$$= \frac{n(\lambda d - e^{-s\Delta}\check{n}) - \lambda d(n + \lambda e^{-s\Delta}\check{d})}{e^{-s\Delta}n(\lambda d + e^{-s\Delta}\check{n})} \cdot P_o^{-1}$$

$$= \frac{(n\check{n} - \lambda^2 d\check{d})}{n(\lambda d + e^{-s\Delta}\check{n})} \cdot P_o^{-1}$$

Thus

$$C_n = \frac{h_n\overline{Q}}{1-Ph_n\overline{Q}}$$

$$= \frac{h_n(\lambda^2 d\check{d} - n\check{n})P_o^{-1}}{n(\lambda d - e^{-s\Delta}\check{n}) - e^{-s\Delta}h_n(\lambda^2 d\check{d} - n\check{n})}$$

$$= \frac{h_n(\lambda^2 \check{d}d - \check{n}n)P_o^{-1}}{\lambda nd + e^{-s\Delta}[(1-h_n)\check{n}n - \lambda^2 \check{d}d]}. \tag{4.3}$$

D. A Finite Dimensional Compensator

We next discuss the requirement that maintenance of closed loop stability and approximation of optimal sensitivity impose on rational approximation of the delay in the feedback compensator (4.3). We see that the restriction is that the delay must be approximated closely enough until h_n is sufficiently small (see equation (4.9)), and then the delay approximation must not exceed 1 in magnitude.

1. Need for further approximation.

From these formulas, C_n will in general contain a pure delay. Even though this rolled-off compensator is physically more realistic than the ideal compensator, it would be useful to show how to further approximate the ideal compensator with one which is finite dimensional. Since C_n rolls off, we could approximate it by a finite dimensional compensator directly. We would then have to separately check that the sensitivity is close to the optimal and that stability is preserved. An obvious question is, can we obtain a sequence of stabilizing compensators for which the closed loop sensitivity approaches the optimal, just by using the above formulas with a finite dimensional approximation to the delay?

We repeat that approximation of the delay in our sense means only that our closed loop sensitivity approximates the infimal norm of the sensitivity.

2. Approximation of the delay.

In the 1 pole/zero case we have solved explicitly, we have a formula for the optimal compensator, and so we know the exact dependence on the delay term. We now examine the effect of approximations to this delay on the closed loop sensitivity.

We show one way to approximate the delay in the denominator of (4.3). Note that our strictly proper but non-rational Q parameter is

$$Q_n = \bar{Q}h_n = h_n \cdot P_o^{-1} \cdot \frac{(\lambda^2 d\check{d} - n\check{n})}{n(\lambda d + e^{-s\Delta}\check{n})}$$

In approximating $e^{-s\Delta}$ with a rational function, we must be concerned with two things: First, we must preserve the stability of $\bar{Q}h_n$, and second, we must preserve the approximation of the closed loop sensitivity to the optimal sensitivity. The first concern amounts to maintaining the closed loop stability.

Let us approximate the delay by replacing $e^{-s\Delta}$ with the rational function $\rho(s)$. Let $\tilde{Q}_n$ be Q_n with $\rho(s)$ substituted for $e^{-s\Delta}$, and let $\tilde{C}_n$ be the resulting compensator. Let $\tilde{Q}$ represent $\bar{Q}$ with $\rho(s)$ substituted for $e^{-s\Delta}$. The following discusses one procedure for selecting $\rho(s)$.

Let $\omega_c > \omega_\varepsilon$ (see (4.1)) be the frequency at which the roll-off due to h_n results in both

$$|P(\omega)C_n(\omega)| < 1$$

and

$$|P(\omega)Q_n(\omega)| = |P(\omega)\overline{Q}(\omega)h_n(\omega)| = |P(\omega)\overline{X}(\omega)\overline{C}(\omega)h_n(\omega)| < \|X\| - 1$$

for

$$\omega > \omega_c.$$

(At the end we shall also require that for $\omega > \omega_0$ h_n satisfy (4.9) below. For a given h_n, this can be taken as part of the definition of ω_c.)

The first condition will guarantee that $(1 + P\tilde{C}_n)$ has no right half plane zeros if $\tilde{C}_n$ is "close enough" to C_n, thus ensuring stability. The second condition will guarantee that $|1 - P(\omega)\tilde{Q}_n(\omega)| < \|\overline{X}\|$, if $\tilde{Q}_n$ is "close enough" to Q_n, ensuring approximation of sensitivity.

We have assumed $W(\omega) \approx 1$ for $\omega > \omega_c$, since $\omega_c > \omega_\epsilon$, and we assume now that

$$\rho(j\omega) \approx e^{-j\omega} \text{ for } \omega < \omega_c,$$

so that $\tilde{C}_n(\omega) \approx C_n(\omega)$ for $\omega < \omega_c$. We shall ensure closed loop stability with $\tilde{C}_n$ by requiring

$$|P(\omega)\tilde{C}_n(\omega)| < 1 \quad \text{for } \omega > \omega_c. \tag{4.4}$$

From the Nyquist locus, we see that this guarantees stability: Since $\rho(j\omega) \approx e^{-j\omega\Delta}$ until $|P\tilde{C}_n| < 1$ and $|PC_n| < 1$, PC_n and $P\tilde{C}_n$ are approximately equal and have the same number of encirclements of the point -1.[9] We shall ensure that the sensitivity is not increased by requiring

$$|P\tilde{Q}_n| < \|X\| - 1 \text{ for } \omega > \omega_c. \tag{4.5}$$

Equation (4.4) is the same as

$$\left|\frac{P\tilde{Q}_n}{1-P\tilde{Q}_n}\right| < 1.$$

or

$$|P\tilde{Q}h_n| < |1 - P\tilde{Q}h_n|$$

Now for $\omega > \omega_c$

$$\tilde{Q} \approx \frac{\lambda^2-1}{\lambda+\rho(s)} \cdot P_o^{-1}$$

so substituting for P and $\tilde{Q}$ we get

$$|(\lambda^2-1)e^{-j\omega\Delta}h_n| < |\lambda+\rho-(\lambda^2-1)e^{-j\omega\Delta}h_n|$$

and finally

$$\left|\frac{\lambda^2-1}{\lambda} \cdot h_n e^{-j\omega\Delta}\right| < \left|1 + \frac{\rho}{\lambda} - \frac{\lambda^2-1}{\lambda} \cdot h_n e^{-j\omega\Delta}\right|.$$

Thus we can ensure stability by requiring

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[9] A Nyquist argument is valid here because C_n and $\tilde{C}_n$ have only finitely many poles, in contrast to $\overline{C}$.

$$2 \cdot \frac{\lambda^2-1}{\lambda} \cdot |h_n e^{-j\omega\Delta}| < |1 + \frac{\rho}{\lambda}|. \tag{4.6}$$

Upon the same substitutions equation (4.5) becomes

$$\frac{\lambda^2-1}{\lambda} \cdot \frac{1}{\|X\|-1} \cdot |h_n e^{-j\omega\Delta}| < |1 + \frac{\rho}{\lambda}|. \tag{4.7}$$

Let $\mu = \max(\frac{1}{\|X\|-1}, 2)$. Then we satisfy both (4.6) and (4.7) by requiring that $\rho(j\omega)$ satisfy

$$\mu \cdot \frac{\lambda^2-1}{\lambda} \cdot |h_n e^{-j\omega\Delta}| < |1 - \frac{\rho}{\lambda}|. \tag{4.8}$$

Since $\lambda > 1$ and $|\rho(j\omega)| \leq 1$, the curve $1+\rho(j\omega)/\lambda$ lies entirely in the right half plane. Thus to satisfy both (4.4) and (4.5) it is sufficient to take

$$|h_n(j\omega)| < \frac{1}{(\lambda-1)\mu} \text{ for } \omega > \omega_c. \tag{4.9}$$

To summarize we have three ranges of frequency over which our approximation of the optimal compensator takes effect. For $|\omega| < \omega_\epsilon$ the approximating compensator is very close in magnitude and phase to the optimal compensator. Over $\omega_\epsilon \leq |\omega| \leq \omega_c$ the compensator starts to roll off until by ω_c (4.8) is satisfied. From then on, $|\omega| \geq \omega_c$, so long as $|\rho(j\omega)| \leq 1$, $\rho(j\omega)$ need not be close to $e^{-j\omega\Delta}$.

CHAPTER 5

GENERAL RATIONAL WEIGHTING FUNCTIONS

In this chapter we expand the solution for the case in Chapter 3 of a 1 pole/1 zero weighting function by allowing the weighting function to have finitely many pole/zero pairs, so long as it is a proper stable minimum phase rational function which is not strictly proper. We start from equations (3.1) - (3.3), with a $w(t)$ appropriate for this more general case.

In this chapter we assume that the sensitivity weighting function is given by

$$W(s) = \prod_{i=1}^{n} \frac{s-\zeta_i}{s-\beta_i}, \tag{5.1}$$

where $\{\zeta_i\}$ and $\{\beta_i\}$ are in the left half plane, and occur in complex conjugate pairs.

Our approach will be to realize the operators V and V^* with a system of differential equations, find appropriate boundary conditions, and pose the eigenvalue/function problem in this framework.

If a maximal eigenvalue for T^*T exists, we can find the (unique) maximal weighted sensitivity by applying (2.9) as in Chapter 3. Since the only cluster point for eigenvalues of T^*T is the point 1 (see Chapter 2 section F) and T^*T is bounded, there will be maximum eigenvalue of T^*T if $\|T\| > 1$ or 1 itself is an eigenvalue of T^*T. Otherwise there will be no maximum eigenvalue.

In Chapter 8 we give a sufficient (and weak, practically speaking) condition on W(s) which will guarantee $\|T\| > 1$, and thus the existence of a maximal eigenvalue.

The case of no maximal eigenvalue occurred in Chapter 3 when $\beta > 1$. In that case we found two functions which were minimal weighted sensitivities, W itself and also an all-pass function obtained by letting $\lambda \to 1$ in the quotient $\frac{TF_\lambda}{F_\lambda}$, where F_λ is an eigenfunction for T corresponding to the eigenvalue λ.

When we have no maximal eigenvalue in the case of this chapter, it is easy to see that W will be a minimal weighted sensitivity when $|W(j\omega)| \leq 1$ for $\omega \in \mathbb{R}$. It is similarly easy to see that $|W(j\omega)| < 1$ guarantees that there is no maximal eigenvalue. We present below in section H a conjecture which allows us to find another minimal weighted sensitivity in this latter case.

Our goal now is to solve the problem given by equation (2.8), with $Y(s) = W(s)$ as in (5.1) and $\psi(s) = 1$. Following the development in Chapter 3, the space K is the same, and the first difference is in expressing the operator T. We proceed directly to computing V^*V, the time domain version of T^*T.

We shall see that the eigenfunctions for V^*V are linear combinations of exponentials with frequencies being the roots of the characteristic equation (5.13). The eigenvalues and eigenfunctions are determined by solving (5.13) with the boundary conditions (5.19). These are simultaneous transcendental equations. We examine the example of the 2 pole/zero weighting function case, and we see that it appears difficult to solve algebraically. We also compute the explicit optimal sensitivity and compensator when T^*T has a maximal eigenvalue. The structure of these functions is the same as the case in Chapter 3. We finally state a conjecture which simplifies our formulas and allows us to find an all-pass minimal weighted sensitivity when $|W(j\omega)| < 1$.

A. Realization of V^*V

1. Formulation of differential equations. Taking $y = Vf$ and $z = V^*y$, we have $z = V^*Vf$. Explicitly,

$$\left.\begin{aligned} y &= Vf = \int_0^t w(t-\tau)f(\tau)d\tau \text{ for } t\in[0,\Delta], \\ z &= V^*y = \int_t^\Delta w(\tau-t)y(\tau)d\tau. \end{aligned}\right\} \tag{5.2}$$

We assume $\beta_i \neq \beta_j$ for $i \neq j$. If $w(t)$ is the inverse Laplace transform of $W(s)$, we can write

$$w(t) = \delta_t + \sum_{i=1}^{n} \alpha_i \cdot e^{-\beta_i t}, \tag{5.3}$$

or $w(t) = \delta_t + \mathbf{c}^T e^{-\mathbf{D}t}\mathbf{b}$, where $\mathbf{c} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ and

$$\mathbf{D} = \begin{bmatrix} \beta_1 & 0 & \cdots & & 0 \\ 0 & \beta_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \beta_{n-1} & 0 \\ 0 & \cdots & & 0 & \beta_n \end{bmatrix}.$$

Thus we can take

$$\frac{d}{dt}x_1 = -\mathbf{D}\cdot x_1 + \mathbf{b}\cdot f$$

$$y = \mathbf{c}^{\mathsf{T}} \cdot x_1 + f \tag{5.4}$$

$$\frac{d}{dt} x_2 = \mathbf{D} \cdot x_2 - \mathbf{b} \cdot y$$

$$z = \mathbf{c}^{\mathsf{T}} \cdot x_2 + y$$

as a state space model valid on $(0, \Delta)$. (If $\beta_i = \beta_j$ for some i and j, $\mathbf{D}$ will be a more general Jordan form, and $\mathbf{b}$ and $\mathbf{c}$ will change appropriately. The following also applies with modifications in such a case.)

2. <u>Formulation of boundary conditions</u> Boundary conditions are obtained by equating derivatives in (5.2) and (5.4):

$$y(0) = f(0) + \int_0^0 w(t-\tau) f(\tau) d\tau$$

and

$$y(0) = \mathbf{c}^{\mathsf{T}} \cdot x_1(0) + f(0)$$

imply

$$\mathbf{c}^{\mathsf{T}} \cdot x_1(0) = 0.$$

$\dot{y}(0) = \dot{f}(0) + w(0) f(0)$ and $\dot{y}(0) = \mathbf{c}^{\mathsf{T}} \cdot \left[-\mathbf{D} x_1(0) + \mathbf{b} f(0) \right] + \dot{f}(0)$ imply

$$\mathbf{c}^{\mathsf{T}} \mathbf{D} x_1(0) = 0$$

since $\mathbf{c}^{\mathsf{T}} \mathbf{b} = w(0)$. In general we obtain

$$y^{(i)}(t) = f^{(i)}(t) + \int_0^t w_0^{(i)}(t-\tau)\cdot f(\tau)d\tau + \sum_{j=0}^{i-1} w_0^{(j)}(0)f^{(i-1-j)}(t)$$

and

$$y^{(i)}(t) = \mathbf{c}^{\mathsf{T}}(-\mathbf{D})^{i}\boldsymbol{x}_1(t) + \sum_{j=0}^{i-1} w^{(j)}(0)f^{(i-1-j)}(t) + f^{(i)}(t),$$

and so

$$y^{(i)}(0) = f^{(i)}(0) + \sum_{j=0}^{i-1} w_0^{(j)}(0)f^{(i-1-j)}(0)$$

and

$$y^{(i)}(0) = \mathbf{c}^{\mathsf{T}}(-\mathbf{D})^{i}\boldsymbol{x}_1(0) + \sum_{j=0}^{i-1} w^{(j)}(0)f^{(i-1-j)}(0) + f^{(i)}(0).$$

Thus $\mathbf{c}^{\mathsf{T}}(-\mathbf{D})^{i}\boldsymbol{x}_1(0) = 0$, and since $(\mathbf{c}^{\mathsf{T}},\mathbf{D})$ is observable, we conclude that

$$\boldsymbol{x}_1(0) = 0.$$

Similar computations allow us to conclude

$$\boldsymbol{x}_2(\Delta) = 0.$$

More concisely,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -\mathbf{D} & \mathbf{0} \\ -\mathbf{b}\cdot\mathbf{c}^{\mathsf{T}} & \mathbf{D} \end{bmatrix}\cdot\boldsymbol{x} + \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}\cdot f \tag{5.5}$$

$$z = (\mathbf{c}^{\mathsf{T}} \ \mathbf{c}^{\mathsf{T}})\cdot\boldsymbol{x} + f \tag{5.6}$$

with $$\begin{bmatrix} \mathbf{x}_1(0) \\ \mathbf{x}_2(\Lambda) \end{bmatrix} = 0. \tag{5.7}$$

In this form we have a standard two-point boundary value problem, and for any f we obtain a unique solution z to (5.5)-(5.7). When we impose the condition that f be an eigenfunction of $V^{*}V$, we expect to see only countably many fs which provide solutions.

B. Incorporation of the eigenvalue problem

We set $z = \lambda^2 f$. Then

$$(\lambda^2-1)f = (\mathbf{c}^T\ \mathbf{c}^T)\cdot x \tag{5.8}$$

replaces (5.6).

Differentiating (5.8) we get

$$(\lambda^2-1)\dot{f} = (\mathbf{c}^T\ \mathbf{c}^T)\cdot\dot{x}$$

$$= (\mathbf{c}^T\ \mathbf{c}^T)\cdot\left(\begin{bmatrix} -\mathbf{D} & 0 \\ -\mathbf{b}\cdot\mathbf{c}^T & \mathbf{D} \end{bmatrix}\cdot x + \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}\cdot f\right)$$

$$= (\mathbf{c}^T\ \mathbf{c}^T)\cdot\begin{bmatrix} -\mathbf{D} & 0 \\ -\mathbf{b}\cdot\mathbf{c}^T & \mathbf{D} \end{bmatrix}\cdot x \tag{5.9}$$

$$(\lambda^2-1)\ddot{f} = (\mathbf{c}^T\ \mathbf{c}^T)\cdot\begin{bmatrix} -\mathbf{D} & 0 \\ -\mathbf{b}\cdot\mathbf{c}^T & \mathbf{D} \end{bmatrix}\cdot\left(\begin{bmatrix} -\mathbf{D} & 0 \\ -\mathbf{b}\cdot\mathbf{c}^T & \mathbf{D} \end{bmatrix}\cdot x + \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}\cdot f\right)$$

$$= (\mathbf{c}^T\ \mathbf{c}^T)\cdot\begin{bmatrix} -\mathbf{D} & 0 \\ -\mathbf{b}\cdot\mathbf{c}^T & \mathbf{D} \end{bmatrix}^2\cdot x + (\mathbf{c}^T\ \mathbf{c}^T)\cdot\begin{bmatrix} -\mathbf{D} & 0 \\ -\mathbf{b}\cdot\mathbf{c}^T & \mathbf{D} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}\cdot f$$

$$= (\mathbf{c}^T \ \mathbf{c}^T)\cdot\begin{bmatrix} \mathbf{D}^2 & 0 \\ \mathbf{b}\mathbf{c}^T\mathbf{D}-\mathbf{D}\mathbf{b}\mathbf{c}^T & \mathbf{D}^2 \end{bmatrix}\cdot x + (\mathbf{c}^T \ \mathbf{c}^T)\cdot\begin{bmatrix} -\mathbf{D} & 0 \\ -\mathbf{b}\cdot\mathbf{c}^T & \mathbf{D} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}\cdot f \quad (5.10)$$

Remark: In the 1 pole/zero weighting function case this agrees with (3.8)-(3.9) with

$$\mathbf{D} = \beta, \qquad \mathbf{c} = 1 \quad \text{and} \quad \mathbf{b} = 1-\beta.$$

C. Solution of the two point boundary value problem.

In the following, we shall want to take Laplace transforms in (5.5)-(5.8). However, since these equations are defined only on the interval $(0,\Delta)$, we cannot without extending the equations to $(0,\infty)$.

We can take the equations (5.5)-(5.8) to be defined on $[0,\infty)$, and we can solve the resulting eigenvalue/function problem. We would like to use this solution to obtain the solution to the problem on $[0,\Delta]$. It is obvious that every solution on $[0,\infty)$ induces a solution on $[0,\Delta]$, but we do not know *a priori* that every solution on $[0,\Delta]$ can be obtained in this way. We will show this to be the case, and then we can take Laplace transforms.

Suppose f is solution to the problem on $[0,\Delta]$, with eigenvalue $\lambda^2 \neq 1$. Then (5.8) gives $f = \frac{1}{\lambda^2-1}\cdot(\mathbf{c}^T \ \mathbf{c}^T)\cdot x$. Substituting this into (5.5) we get

$$\dot{x} = \begin{bmatrix} -\mathbf{D} & 0 \\ -\mathbf{b}\cdot\mathbf{c}^T & \mathbf{D} \end{bmatrix}\cdot x + \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}\cdot\frac{1}{\lambda^2-1}\cdot(\mathbf{c}^T \ \mathbf{c}^T)\cdot x. \quad (5.11)$$

Since f is a solution on $[0,\Delta]$, there is some $x(0)$ corresponding to this solution. Let $x(t)$ on $[0,\infty)$ be the solution to (5.11) with initial condition $x(0)$. Then we simply take $f(t)$ on $[0,\infty)$ to be given by $f = \frac{1}{\lambda^2-1}\cdot(\mathbf{c}^T\ \mathbf{c}^T)\cdot x$. This f is the desired extension, and Laplace transforms of f and x on $[0,\infty)$ exist because the components of x are finite sums of polynomials times exponentials.

Now we consider (5.5) on $[0,\infty)$ and take Laplace transforms. We have

$$s\cdot\hat{x} - \hat{x}(0) = \begin{bmatrix} -\mathbf{D} & \mathbf{0} \\ -\mathbf{b}\cdot\mathbf{c}^T & \mathbf{D} \end{bmatrix}\cdot\hat{x} + \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}\cdot\hat{f},$$

and so

$$\hat{x} = (s\mathbf{I} - \begin{bmatrix} -\mathbf{D} & \mathbf{0} \\ -\mathbf{b}\cdot\mathbf{c}^T & \mathbf{D} \end{bmatrix})^{-1}\cdot\left[\begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}\cdot\hat{f} + \hat{x}(0)\right].$$

Taking Laplace transforms in (5.8) and using this we get

$$(\lambda^2-1)\cdot\hat{f} = (\mathbf{c}^T\ \mathbf{c}^T)\cdot(s\mathbf{I} - \begin{bmatrix} -\mathbf{D} & \mathbf{0} \\ -\mathbf{b}\cdot\mathbf{c}^T & \mathbf{D} \end{bmatrix})^{-1}\cdot\left[\begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}\cdot\hat{f} + \hat{x}(0)\right].$$

Some calculations give us

$$(s\mathbf{I} - \begin{bmatrix} -\mathbf{D} & \mathbf{0} \\ -\mathbf{b}\cdot\mathbf{c}^T & \mathbf{D} \end{bmatrix})^{-1} = \begin{bmatrix} s\mathbf{I}+\mathbf{D} & \mathbf{0} \\ \mathbf{b}\mathbf{c}^T & s\mathbf{I}-\mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} (s\mathbf{I}+\mathbf{D})^{-1} & \mathbf{0} \\ -(s\mathbf{I}-\mathbf{D})^{-1}\mathbf{b}\mathbf{c}^T(s\mathbf{I}+\mathbf{D})^{-1} & (s\mathbf{I}-\mathbf{D})^{-1} \end{bmatrix}$$

and

$$(\mathbf{c}^T\ \mathbf{c}^T)\cdot\begin{bmatrix} (s\mathbf{I}+\mathbf{D})^{-1} & \mathbf{0} \\ -(s\mathbf{I}-\mathbf{D})^{-1}\mathbf{b}\mathbf{c}^T(s\mathbf{I}+\mathbf{D})^{-1} & (s\mathbf{I}-\mathbf{D})^{-1} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}$$

$$= \mathbf{c}^T(s\mathbf{I}+\mathbf{D})^{-1}\mathbf{b} - \mathbf{c}^T(s\mathbf{I}-\mathbf{D})^{-1}\mathbf{b}\mathbf{c}^T(s\mathbf{I}+\mathbf{D})^{-1}\mathbf{b} - \mathbf{c}^T(s\mathbf{I}-\mathbf{D})^{-1}\mathbf{b}$$

$$= W_0(s) + W_0(-s)W_0(s) + W_0(-s)$$

$$= W(s)W(-s) - 1$$

where we note $W(-s) = \mathcal{L}(w(-t))$. Then

$$\hat{f} = \left[\lambda^2 - W(-s)W(s)\right]^{-1}(\mathbf{c}^T\ \mathbf{c}^T)(sI - \begin{bmatrix} -\mathbf{D} & \mathbf{0} \\ -\mathbf{b}\cdot\mathbf{c}^T & \mathbf{D} \end{bmatrix})^{-1}\mathbf{x}(0) \tag{5.12}$$

The right hand side of this equation has for its poles the solutions of

$$\lambda^2 - W(-s)W(s) = 0 \tag{5.13}$$

since the denominators of the entries of $(sI - \begin{bmatrix} -\mathbf{D} & \mathbf{0} \\ -\mathbf{b}\cdot\mathbf{c}^T & \mathbf{D} \end{bmatrix})^{-1}$ are factors of the denominator of the left hand side of (5.13).

Rather than find $\mathbf{x}(0)$ in order to find $\hat{f}$, in general it will be more direct to write $\hat{f}$ as a partial fraction expansion with undetermined coefficients and poles being the solutions of (5.13), and solve for the coefficients in the time domain using the boundary conditions. Note that (5.13) is really an n^{th} order polynomial equation in s^2.

Remark: In the 1 pole/zero case (5.13) gives us (3.11).

Example. In the 2 pole/zero case, if we let the zeros be $-\zeta_1$ and $-\zeta_2$ and the poles be $-\beta_1$ and $-\beta_2$, and we assume they are real, we get

$$0 = \lambda^2 - \frac{(s+\zeta_1)(s+\zeta_2)}{(s+\beta_1)(s+\beta_2)} \cdot \frac{(s-\zeta_1)(s-\zeta_2)}{(s-\beta_1)(s-\beta_2)}$$

Therefore

$$0 = \lambda^2(s^2-\beta_1^2)(s^2-\beta_2^2) - (s^2-\zeta_1^2)(s^2-\zeta_2^2).$$

$$= (\lambda^2-1)s^4 + [\zeta_1^2+\zeta_2^2-\lambda^2(\beta_1^2+\beta_2^2)]s^2 + \lambda^2\beta_2^2\beta_1^2 - \zeta_1^2\zeta_2^2 \qquad (5.14)$$

(If $\beta_1 = \overline{\beta}_2$ and $\zeta_1 = \overline{\zeta}_2$, the same formula holds.)

The solutions of (5.14) are given by

$$s^2 = \frac{1}{2(\lambda^2-1)} \cdot \{-[\zeta_1^2+\zeta_2^2-\lambda^2(\beta_1^2+\beta_2^2)] \pm$$
$$[[\zeta_1^2+\zeta_2^2-\lambda^2(\beta_1^2+\beta_2^2)]^2 - 4\cdot(\lambda^2-1)\cdot(\lambda^2\beta_2^2\beta_1^2 - \zeta_1^2\zeta_2^2)]^{1/2}\}$$

$$= \frac{\lambda^2(\beta_1^2+\beta_2^2)-(\zeta_1^2+\zeta_2^2) \pm \left[[\lambda^2(\beta_1^2-\beta_2^2)-(\zeta_1^2-\zeta_2^2)]^2 + 4\lambda^2(\beta_1^2-\zeta_1^2)(\beta_2^2-\zeta_2^2)\right]^{1/2}}{2(\lambda^2-1)}$$

Let $\{s_{\lambda,i}\}$ be the set of solutions to (5.11). The possible eigenfunctions f_j must satisfy the boundary conditions (5.7). This requirement implicitly determines what values for λ are possible.

D. A computational form of the boundary conditions.

We now consider the transformation of the boundary conditions (5.7) to constraints on f. From (5.8)

$$(\lambda^2-1)f = \begin{bmatrix} \mathbf{c} \\ \mathbf{c} \end{bmatrix}^T \boldsymbol{x}.$$

Define

$$\mathbf{A} = \begin{bmatrix} -\mathbf{D} & 0 \\ -\mathbf{b}\mathbf{c}^{\mathsf{T}} & \mathbf{D} \end{bmatrix}.$$

Then differentiating, we have the following:

$$(\lambda^2-1)\dot{f} = \begin{bmatrix} \mathbf{c} \\ \mathbf{c} \end{bmatrix}^{\mathsf{T}}(\mathbf{A}\boldsymbol{x} + \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix} f) = \begin{bmatrix} \mathbf{c} \\ \mathbf{c} \end{bmatrix}^{\mathsf{T}}\mathbf{A}\boldsymbol{x}.$$

$$(\lambda^2-1)\ddot{f} = \begin{bmatrix} \mathbf{c} \\ \mathbf{c} \end{bmatrix}^{\mathsf{T}}\mathbf{A}(\mathbf{A}\boldsymbol{x} + \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix} f) = \begin{bmatrix} \mathbf{c} \\ \mathbf{c} \end{bmatrix}^{\mathsf{T}}\mathbf{A}^2\boldsymbol{x} + (2\dot{w}_0(0)-w_0(0)^2)\cdot f$$

$$(\lambda^2-1)f^{(3)} = \begin{bmatrix} \mathbf{c} \\ \mathbf{c} \end{bmatrix}^{\mathsf{T}}\mathbf{A}^2(\mathbf{A}\boldsymbol{x} + \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix} f) + (2\dot{w}_0(0)-w_0(0)^2)\cdot\dot{f}$$

$$= \begin{bmatrix} \mathbf{c} \\ \mathbf{c} \end{bmatrix}^{\mathsf{T}}\mathbf{A}^3\boldsymbol{x} + \cdot(2\dot{w}_0(0)-w_0(0)^2)\dot{f}$$

In general we have

$$(\lambda^2-1)f^{(i)} = \begin{bmatrix} \mathbf{c} \\ \mathbf{c} \end{bmatrix}^{\mathsf{T}}\mathbf{A}^i\boldsymbol{x} + \sum_{j=1}^{i-1} \begin{bmatrix} \mathbf{c} \\ \mathbf{c} \end{bmatrix}^{\mathsf{T}}\mathbf{A}^j \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix} f^{(i-j-1)},$$

although $\begin{bmatrix} \mathbf{c} \\ \mathbf{c} \end{bmatrix}^{\mathsf{T}}\mathbf{A}^j \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix} = 0$ for j even.

Using this formulation we can write

$$\mathbf{M}f = \mathbf{N}\boldsymbol{x},$$

where

$$\boldsymbol{f} = \begin{bmatrix} f \\ \dot{f} \\ \ddot{f} \\ \vdots \\ f^{(2n-1)} \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} \lambda^2-1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda^2-1 & 0 & 0 & \cdots & 0 \\ -\begin{bmatrix}\mathbf{c}\\\mathbf{c}\end{bmatrix}^T \mathbf{A}\begin{bmatrix}\mathbf{b}\\-\mathbf{b}\end{bmatrix} & 0 & \lambda^2-1 & 0 & \cdots & 0 \\ 0 & -\begin{bmatrix}\mathbf{c}\\\mathbf{c}\end{bmatrix}^T \mathbf{A}\begin{bmatrix}\mathbf{b}\\-\mathbf{b}\end{bmatrix} & 0 & \lambda^2-1 & \ddots & 0 \\ -\begin{bmatrix}\mathbf{c}\\\mathbf{c}\end{bmatrix}^T \mathbf{A}^3\begin{bmatrix}\mathbf{b}\\-\mathbf{b}\end{bmatrix} & 0 & -\begin{bmatrix}\mathbf{c}\\\mathbf{c}\end{bmatrix}^T \mathbf{A}\begin{bmatrix}\mathbf{b}\\-\mathbf{b}\end{bmatrix} & 0 & \ddots & 0 \\ 0 & -\begin{bmatrix}\mathbf{c}\\\mathbf{c}\end{bmatrix}^T \mathbf{A}^3\begin{bmatrix}\mathbf{b}\\-\mathbf{b}\end{bmatrix} & 0 & -\begin{bmatrix}\mathbf{c}\\\mathbf{c}\end{bmatrix}^T \mathbf{A}\begin{bmatrix}\mathbf{b}\\-\mathbf{b}\end{bmatrix} & \ddots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ & 0 & -\begin{bmatrix}\mathbf{c}\\\mathbf{c}\end{bmatrix}^T \mathbf{A}^3\begin{bmatrix}\mathbf{b}\\-\mathbf{b}\end{bmatrix} & 0 & -\begin{bmatrix}\mathbf{c}\\\mathbf{c}\end{bmatrix}^T \mathbf{A}\begin{bmatrix}\mathbf{b}\\-\mathbf{b}\end{bmatrix} & \lambda^2-1 \end{bmatrix}$$

and

$$\mathbf{N} = \begin{bmatrix} (\mathbf{c}^T \ \mathbf{c}^T) \\ (\mathbf{c}^T \ \mathbf{c}^T)\mathbf{A} \\ \vdots \\ (\mathbf{c}^T \ \mathbf{c}^T)\mathbf{A}^{2n-1} \end{bmatrix}$$

We note that by minimality of our state space realizations for the convolution operator and its adjoint, $\mathbf{N}$ is always invertible. Thus we have $\mathbf{N}^{-1}\mathbf{M}\boldsymbol{f}(t) = \boldsymbol{x}(t)$. Using the boundary conditions and partitioning N^{-1} as $\mathrm{N}^{-1} = \begin{bmatrix}(\mathrm{N}^{-1})_1 \\ (\mathrm{N}^{-1})_2\end{bmatrix}$, we have

$$(\mathbf{N}^{-1})_1\mathbf{M}f(0) = 0 \text{ and } (\mathbf{N}^{-1})_2\mathbf{M}f(\Delta) = 0.$$

If the submatrices are themselves invertible, a somewhat simpler computation is to partition $\mathbf{M}$ and $\mathbf{N}$ as $\mathbf{M} = \begin{bmatrix}\mathbf{M}_1\\ \mathbf{M}_2\end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix}\mathbf{N}_{11} & \mathbf{N}_{12}\\ \mathbf{N}_{21} & \mathbf{N}_{22}\end{bmatrix}$. Then we get $\mathbf{M}f(0) = \begin{bmatrix}\mathbf{N}_{12}\\ \mathbf{N}_{22}\end{bmatrix}x_2(0)$ and $\mathbf{M}f(\Delta) = \begin{bmatrix}\mathbf{N}_{11}\\ \mathbf{N}_{21}\end{bmatrix}x_1(\Delta)$. These give

$$[\mathbf{M}_2-\mathbf{N}_{22}\mathbf{N}_{12}^{-1}\mathbf{M}_1]f(0) = 0$$

and

$$[\mathbf{M}_2-\mathbf{N}_{21}\mathbf{N}_{11}^{-1}\mathbf{M}_1]f(\Delta) = 0.$$

<u>Remark.</u> In the 1 pole/zero case this becomes

$$\begin{bmatrix}\lambda^2-1 & 0\\ 0 & \lambda^2-1\end{bmatrix}\begin{bmatrix}f\\ \dot{f}\end{bmatrix} = \begin{bmatrix}1 & 1\\ -1 & \beta\end{bmatrix}x,$$

$$\frac{1}{\beta+1}\begin{bmatrix}\beta & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\lambda^2-1 & 0\\ 0 & \lambda^2-1\end{bmatrix}\begin{bmatrix}f\\ \dot{f}\end{bmatrix} = x$$

$$\frac{\lambda^2-1}{\beta+1}\cdot\begin{bmatrix}\beta & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}f\\ \dot{f}\end{bmatrix} = x.$$

Since the boundary conditions are $x_1(0) = 0$ and $x_2(\Delta) = 0$, this implies

$$[\beta \quad -1]\begin{bmatrix}f(0)\\ \dot{f}(0)\end{bmatrix} = 0$$

and

$$[1 \quad 1] \begin{bmatrix} f(\Delta) \\ \dot{f}(\Delta) \end{bmatrix} = 0.$$

Assuming $k \neq 0$ and, $f(t) = a_1 e^{kt} + a_2 e^{-kt}$, this gives

$$\begin{bmatrix} f(t) \\ \dot{f}(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ k & -k \end{bmatrix} \begin{bmatrix} e^{kt} & 0 \\ 0 & e^{-kt} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

Applying this, we find

$$\begin{bmatrix} \beta-k & \beta+k \\ -(k+1)e^{k\Delta} & (k-1)e^{-k\Delta} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0.$$

Therefore

$$\det\left(\begin{bmatrix} \beta-k & \beta+k \\ -(k+1)e^{k\Delta} & (k-1)e^{-k\Delta} \end{bmatrix}\right) = 0,$$

and

$$(k+\beta)(k+1)e^{k\Delta} = (k-\beta)(k-1)e^{-k\Delta}.$$

We write this in a more tractable form as

$$(k^2+k\beta+k+\beta)e^{k\Delta} = (k^2-k\beta-k+\beta)e^{-k\Delta},$$

or

$$(k^2+\beta)(e^{k\Delta}-e^{-k\Delta}) = -k(\beta+1)(e^{k\Delta}+e^{-k\Delta}).$$

We know k is either real or imaginary. If k is real this becomes

$$(k^2+\beta)\sinh(k\Delta) = -k(\beta+1)\cosh(k\Delta),$$

If k is imaginary we have

$$(k^2+\beta)\sin(k\Delta) = -k(\beta+1)\cos(k\Delta).$$

The only real solution is $k = 0$, but this is ruled out. For k imaginary this agrees with (3.17), and there are infinitely many imaginary solutions.

<u>Example.</u> We continue the solution of the 2 pole/zero (distinct poles) case. Now we use the coefficients in the partial fraction expansion of W, $\{\alpha_i\}$ (see (5.4)), instead of the zeros of the weighting function $\{\zeta_i\}$ which appear in (5.14).

$$\mathbf{M} = \begin{bmatrix} \lambda^2-1 & 0 & 0 & 0 \\ 0 & \lambda^2-1 & 0 & 0 \\ w_0(0)^2-2\dot{w}_0(0) & 0 & \lambda^2-1 & 0 \\ 0 & w_0(0)^2-2\dot{w}_0(0) & 0 & \lambda^2-1 \end{bmatrix}$$

and

$$\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\beta_1-\alpha_2-\alpha_1 & -\beta_2-\alpha_2-\alpha_1 & \beta_1 & \beta_2 \\ \beta_1^2+\alpha_2\beta_1-\alpha_2\beta_2 & \beta_2^2+\alpha_1\beta_2-\alpha_1\beta_1 & \beta_1^2 & \beta_2^2 \\ -\beta_1^3-\alpha_2\beta_1^2-\alpha_1\beta_1^2+\alpha_2\beta_2\beta_1-\alpha_2\beta_2^2 & -\beta_2^3-\alpha_1\beta_2^2-\alpha_2\beta_2^2+\alpha_1\beta_1\beta_2-\alpha_1\beta_1^2 & \beta_1^3 & \beta_2^3 \end{bmatrix}.$$

using the notation above.

Let $f(t) = \sum_{i=1}^{2} (a_i e^{s_i t} + b_i e^{-s_i t})$, where $\{\pm s_i\}$ are the solutions of $\lambda^2 = W(s)W(-s)$. Then

$$\begin{bmatrix} f(t) \\ \dot{f}(t) \\ \ddot{f}(t) \\ f^{(3)}(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ s_1 & s_2 & -s_1 & -s_2 \\ s_1^2 & s_2^2 & s_1^2 & s_2^2 \\ s_1^3 & s_2^3 & -s_1^3 & -s_2^3 \end{bmatrix} \begin{bmatrix} e^{s_1 t} & 0 & 0 & 0 \\ 0 & e^{s_2 t} & 0 & 0 \\ 0 & 0 & e^{-s_1 t} & 0 \\ 0 & 0 & 0 & e^{-s_2 t} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix}.$$

Detailed calculations (which appear in the appendix to this chapter) show that f satisfies the boundary conditions $x_1(0) = 0$ and $x_2(\Delta) = 0$ if and only if

$$\det\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = 0. \tag{5.15}$$

where

$$L_1 = \left[(w_0(0)^2 - 2\dot{w}_0(0))I - (\lambda^2-1)K_1 \quad (\lambda^2-1)I \right] \begin{bmatrix} 1 & 1 & 1 & 1 \\ s_1 & s_2 & -s_1 & -s_2 \\ s_1^2 & s_2^2 & s_1^2 & s_2^2 \\ s_1^3 & s_2^3 & -s_1^3 & -s_2^3 \end{bmatrix}$$

and

$$L_2 = \left[(w_0(0)^2 - 2\dot{w}_0(0))I - (\lambda^2-1)K_2 \quad (\lambda^2-1)I \right] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ s_1 & s_2 & -s_1 & -s_2 \\ s_1^2 & s_2^2 & s_1^2 & s_2^2 \\ s_1^3 & s_2^3 & -s_1^3 & -s_2^3 \end{bmatrix} \begin{bmatrix} e^{s_1 \Delta} & 0 & 0 & 0 \\ 0 & e^{s_2 \Delta} & 0 & 0 \\ 0 & 0 & e^{-s_1 \Delta} & 0 \\ 0 & 0 & 0 & e^{-s_2 \Delta} \end{bmatrix}$$

The simultaneous solution of (5.15) and (5.14) determine the eigenfunctions for this problem. This procedure can be extended to the general n pole/zero case, and may be useful for numerical computations.

E. General form of the eigenfunctions of T^*T.

The previous analysis shows one way of calculating the eigenvalues and eigenvectors of T^*T. We are next interested in finding the explicit infimal sensitivity and corresponding compensator, so we examine the general form of the eigenfunctions of T^*T.

We next show that the numerator polynomials of eigenfunctions will be a multiples of the denominator of W(s), as in the case of 1 pole/1 zero weighting function. The interpretation of this is that then the image under T will have no "transient" part. In other words, we think of T as a linear system acting on the input, the eigenfunction $f \in K$. Since the system is causal, we can ignore the fact that the output terminates after time Δ, and just think of inputs and outputs as being exponentials. As such the output will consist of the "steady-state" response at the exponential frequency of the input, and the "transient" part at the exponential frequency of the convolution kernel $\mathcal{L}^{-1}(W)$. The motivation for thinking of it this way is that the eigenfunction condition for T^*T requires that any transient response be annihilated by T^*, thus wasting energy from the input that could have been used to result in a bigger output magnitude. The result applies to all eigenfunctions, not just the maximal one, since T^* actually cannot annihilate a transient from T.

We know that an eigenfunction for the eigenvalue λ^2 is of the form $F(s) = \mathcal{L}\Big[[u(t)-u(t-\Delta)]f(t)\Big]$, with

$$f(t) = \sum_{i=1}^{n} \left[a_i e^{-s_i t} + b_i e^{s_i t}\right],$$

where $\{\pm s_i\}$ is the set of solutions to the characteristic equation $\lambda^2 = W(s)W(-s)$. (We assume for simplicity that the solutions are distinct.) Let

$$\mathcal{L}(f(t)) = \sum_{i=1}^{n} \left[\frac{a_i}{s+s_i} + \frac{b_i}{s-s_i}\right]$$

$$= \frac{a(s)}{\lambda^2 d(s)d(-s)-n(s)n(-s)},$$

(this defines $a(s)$) where $W(s) = \frac{n(s)}{d(s)}$. Now

$$f(t) = \sum_{i=1}^{n} \left[a_i e^{-s_i \Delta} e^{-s_i (t-\Delta)} + b_i e^{s_i \Delta} e^{s_i (t-\Delta)}\right]$$

$$= g(t-\Delta),$$

where

$$g(t) = \sum_{i=1}^{n} \left[a_i e^{-s_i \Delta} e^{-s_i t} + b_i e^{s_i \Delta} e^{s_i t}\right].$$

Note that $\mathcal{L}(g(t))$ has the same denominator, $\lambda^2 d(s)d(-s)-n(s)n(-s)$, as $\mathcal{L}(f(t))$. Thus

$$F(s) = \mathcal{L}(f(t)) - e^{-s\Delta}\mathcal{L}(g(t))$$

$$= \frac{a(s) - e^{-s\Delta}b(s)}{\lambda^2 d(s)d(-s) - n(s)n(-s)}.$$

Where $b(s)$ is the numerator polynomial of $\mathcal{L}(g(t))$. Letting $x(s)$ be the polynomial defined by

$$\mathcal{L}[u(t-\Delta)(w(t)*f(t))] = \frac{e^{-s\Delta}x(s)}{[\lambda^2 d(s)d(-s) - n(s)n(-s)]d(s)}.$$

we find that

$$TF = \frac{a(s)n(s) - e^{-s\Delta}x(s)}{[\lambda^2 d(s)d(-s) - n(s)n(-s)]d(s)},$$

since

$$\mathcal{L}(w(t)*f(t)) = \frac{a(s)n(s)}{[\lambda^2 d(s)d(-s) - n(s)n(-s)]d(s)}$$

Let $f(\check{t}) = f(-t)$, and let S_Δ be the time shift operator $S_\Delta[f(t)] = f(t-\Delta)$. From the expression for V^* in (5.3) we see that $V^*y = S_\Delta(VS_\Delta\check{y})\check{}$. Let $Y = \hat{y}$. We can write $T^*Y = e^{-s\Delta}[Te^{-s\Delta}\check{Y}]\check{}$. Applying this to the equation for TF above we get

$$\lambda^2 F = T^*TF = e^{-s\Delta}[Te^{-s\Delta}(TF)\check{}\,]\check{}$$

$$= e^{-s\Delta}[Te^{-s\Delta}\frac{a(-s)n(-s) - e^{s\Delta}x(-s)}{[\lambda^2 d(-s)d(s) - n(-s)n(s)]d(-s)}]\check{}$$

$$= e^{-s\Delta}[T\frac{a(-s)n(-s)e^{-s\Delta} - x(-s)}{[\lambda^2 d(-s)d(s) - n(-s)n(s)]d(-s)}]\check{}$$

$$= e^{-s\Delta}\left[\frac{1-e^{-s\Delta}}{s} * \left[\frac{n(s)}{d(s)} \cdot \frac{a(-s)n(-s)e^{-s\Delta} - x(-s)}{[\lambda^2 d(-s)d(s) - n(-s)n(s)]d(-s)}\right]\right]^{\vee}$$

$$= e^{-s\Delta}\left[\frac{-x(-s)n(s) + z(s)e^{-s\Delta}}{[\lambda^2 d(-s)d(s) - n(-s)n(s)]d(-s)d(s)}\right]^{\vee}$$

$$= \frac{-e^{-s\Delta}x(s)n(-s) + z(-s)}{[\lambda^2 d(-s)d(s) - n(-s)n(s)]d(-s)d(s)}$$

for some polynomial $z(s)$. Comparing this with F(s) above, we conclude that $\lambda^2 b(s) = \frac{x(s)n(-s)}{d(s)d(-s)}$. Since $b(s)$ and $x(s)$ are polynomials, $d(s)d(-s)$ divides $x(s)$ and $n(-s)$ divides $b(s)$. Since $d(s)$ divides $x(s)$ and the support of $\mathcal{L}^{-1}(TF)$ is restricted to $(0,\Delta)$, we must have that $d(s)$ divides $a(s)$. Putting these relationships together, for some polynomials $c(s)$, $e(s)$ and $r(s)$ we can write

$$a(s) = c(s)d(s),$$

$$x(s) = e(s)d(s)d(-s)$$

and $$b(s) = r(s)n(-s).$$

These conditions apply for all eigenfunctions.

F. Nature of the optimal sensitivity.

The case in which T does not have a maximal eigenvalue is discussed below in section H. In the following we assume that T has a maximal vector. We now use this assumption to apply the formula of Sarason given in (2.9) to determine the optimal sensitivity.

Given the assumption that F is a maximal vector, we know from [Sarason 1967, Prop. 5.1] that $X(s) = \lambda\cdot\varphi(s)$, where $\varphi(s)$ is inner. Using the formula for φ in (2.9) and the above notation we have

$$\varphi(s) = \lambda^{-1}\cdot\frac{a(s)n(s) - e^{-s\Delta}x(s)}{[a(s) - e^{-s\Delta}b(s)]d(s)}$$

$$= \lambda^{-1}\cdot\frac{c(s)d(s)n(s) - e^{-s\Delta}e(s)d(s)d(-s)}{[c(s)d(s) - e^{-s\Delta}r(s)n(-s)]d(s)}$$

$$= \lambda^{-1}\cdot\frac{c(s)n(s) - e^{-s\Delta}e(s)d(-s)}{c(s)d(s) - e^{-s\Delta}r(s)n(-s)}. \tag{5.21}$$

Now we argue (following [Sarason 1967, p. 194]) that $\varphi(s)$ is a Blaschke product: Since $\varphi(s)$ is continuous on the imaginary axis, the only singular inner functions that can divide it are of the form $e^{-s\alpha}$ with $\alpha>0$. But $e^{s\alpha}\varphi(s)$ is unbounded on the positive real axis, so φ is purely a Blaschke product.

G. The optimal compensator

A simple computation using (5.21) then gives us the optimal compensator:

$$C = \frac{W-X}{PX}$$

$$= \frac{n(s)[a(s)d(s) - e^{-s\Delta}b(s)d(s)] - d(s)[a(s)n(s) - e^{-s\Delta}y(s)]}{e^{-s\Delta}P_iP_oB^{-1}d(s)[a(s)n(s) - e^{-s\Delta}y(s)]}$$

$$= \frac{-n(-s)n(s)r(s) + e(s)d(s)d(-s)}{P_iP_oB^{-1}[c(s)d(s)n(s) - e^{-s\Delta}e(s)d(s)d(-s)}$$

$$= P_i^{-1}P_o^{-1}B\cdot\frac{e(s)d(s)d(-s) - r(s)n(s)n(-s)}{c(s)d(s)n(s)}\cdot\frac{1}{1 - e^{-s\Delta}\frac{e(s)d(-s)}{c(s)n(s)}} \quad (5.22)$$

This has the same structure as the 1 pole/1 zero weighting function case.

Remarks: 1) Since X is an infinite Blaschke product, $C = \frac{1}{P}(\frac{W}{X} - 1)$ is necessarily unstable. 2) If P is strictly proper, C will be improper.

H. Conjecture on Weighted Sensitivity Formula.

We would like to conclude from the fact that $\varphi(s)$ in (5.21) is a Blaschke product that the zeros of the numerator are reflections across the imaginary axis of the zeros of the denominator. However, because of the possibility of cancellations, we are unable to do so.

If we now conjecture that there are no common zeros of the numerator and denominator of (5.21), the fact that this is a Blaschke product implies that

$$\lambda^{-1}[c(-\bar{s})n(-\bar{s}) - e^{\bar{s}\Delta}e(-\bar{s})d(\bar{s})]^{-}$$

and

$$c(s)d(s) - e^{-s\Delta}r(s)n(-s)$$

have the same zeros, which are in the left half plane. Since all the various polynomials above have conjugate symmetry of their zeros, $[c(-\bar{s})]^{-} = c(-s)$, etc., we conclude that

$$\lambda^{-1}[c(-s)n(-s) - e(-s)d(s)e^{s\Delta}] \quad \text{and} \quad c(s)d(s) - e^{-s\Delta}r(s)n(-s)$$

have the same zeros and magnitude on the imaginary axis. Therefore so do

$$\lambda^{-1}[c(-s)n(-s)e^{-s\Delta} - e(-s)d(s)] \quad \text{and} \quad c(s)d(s) - e^{-s\Delta}r(s)n(-s)$$

The reciprocals of these last two expressions are in $\mathcal{H}^{\infty}$, have the same magnitude on the imaginary axis, and have the same poles and zeros. Therefore they can only differ by a singular inner factor. But they are continuous on the imaginary axis, and have no common factor of the form $e^{-s\alpha}$ for $\alpha > 0$, so they have no singular inner factor. Thus they are equal, and we conclude that

$$\lambda^{-1}c(s)n(s) = r(-s)n(s)$$

and

$$\lambda^{-1}e(s)d(-s) = c(-s)d(-s).$$

Therefore $c(s) = \lambda r(-s) = \lambda^{-1}e(-s)$, and (5.21) becomes

$$\varphi(s) = \frac{n(s) - \lambda e^{-s\Delta}d(-s)}{\lambda d(s) - e^{-s\Delta}n(-s)}. \tag{5.23}$$

Notice this implies that $TF = \lambda e^{-s\Delta}\check{F}$, or $(Vf)(t) = \lambda\cdot f(\Delta - t)$ on

$(0,\Delta)$. This appears to be the same as a conjecture made by George Zames[10], that the image under V of an eigenfunction of V^*V must be the eigenfunction reversed in time, except for scaling.

Also, this agrees with the 1 pole/1 zero weighting function case of Chapter 3.

Under the conjecture then, in the case of stable plant with no right half plane zeros (the details for more general cases have not yet been completely worked out), we have from (5.22)

$$X(s) = \lambda\cdot\frac{n(s) - \lambda e^{-s\Delta}d(-s)}{\lambda d(s) - e^{-s\Delta}n(-s)}. \tag{5.24}$$

and we get for the optimal compensator:

$$C = \frac{W-X}{PX}$$

$$= \frac{n(\lambda d-e^{-s\Delta}\check{n})-\lambda(n-\lambda e^{-s\Delta}\check{d})d}{\lambda e^{-s\Delta}d(n-\lambda e^{-s\Delta}\check{d})P_o}$$

$$= \frac{\lambda^2 d\check{d} - n\check{n}}{\lambda d(n-\lambda e^{-s\Delta}\check{d})}\cdot P_o^{-1}$$

$$= P_o^{-1}\cdot\frac{\lambda^2 d\check{d} - n\check{n}}{\lambda n d}\cdot\frac{1}{1 - \lambda e^{-s\Delta}\cdot\frac{\check{d}}{n}}.$$

[10] Conversation with Prof. George Zames, November 1985.

This has the same structure as the 1 pole/1 zero weighting function case.

<u>Remark:</u> As in Chapter 3, $\lambda=1$ in (5.24) gives an inner function which interpolates W on K when $\|W\|_{\infty} = 1$. This gives a second solution ($X(s) = W(s)$ is the other) for this case. In the general case of no maximal eigenvalue we do not yet know whether this procedure works.

Conditions for Solution of Boundary Value Problem

The following calculations derive the conditions (5.15) for the solution of the boundary value problem in the example.

Using $x_1(0) = 0$ and $x_2(\Delta) = 0$ we get

$$(\lambda^2-1)\begin{bmatrix} f(0) \\ \dot{f}(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{bmatrix} x_2(0),$$

$$\frac{\lambda^2-1}{\beta_2-\beta_1}\begin{bmatrix} \beta_2 & -1 \\ -\beta_1 & 1 \end{bmatrix}\begin{bmatrix} f(0) \\ \dot{f}(0) \end{bmatrix} = x_2(0)$$

$$\begin{bmatrix} w_0(0)^2-2\dot{w}_0(0) & 0 & \lambda^2-1 & 0 \\ 0 & w_0(0)^2-2\dot{w}_0(0) & 0 & \lambda^2-1 \end{bmatrix}\begin{bmatrix} f(0) \\ \dot{f}(0) \\ \ddot{f}(0) \\ f^{(3)}(0) \end{bmatrix} =$$

$$\begin{bmatrix} \beta_1^2 & \beta_2^2 \\ \beta_1^3 & \beta_2^3 \end{bmatrix}\cdot\frac{\lambda^2-1}{\beta_2-\beta_1}\cdot\begin{bmatrix} \beta_2 & -1 \\ -\beta_1 & 1 \end{bmatrix}\begin{bmatrix} f(0) \\ \dot{f}(0) \end{bmatrix}$$

$$= \frac{\lambda^2-1}{\beta_2-\beta_1}\begin{bmatrix} \beta_1\beta_2(\beta_1-\beta_2) & \beta_2^2-\beta_1^2 \\ \beta_1\beta_2(\beta_1^2-\beta_2^2) & \beta_2^3-\beta_1^3 \end{bmatrix}\begin{bmatrix} f(0) \\ \dot{f}(0) \end{bmatrix}$$

$$= (\lambda^2-1)\begin{bmatrix} -\beta_1\beta_2 & \beta_2+\beta_1 \\ -\beta_1\beta_2(\beta_1+\beta_2) & \beta_2^2+\beta_1\beta_2+\beta_1^2 \end{bmatrix}\begin{bmatrix} f(0) \\ \dot{f}(0) \end{bmatrix}$$

$$(\lambda^2-1)\begin{bmatrix} \ddot{f}(0) \\ f^{(3)}(0) \end{bmatrix} = (\lambda^2-1)\begin{bmatrix} -\beta_1\beta_2 & \beta_1+\beta_2 \\ -\beta_1\beta_2(\beta_1+\beta_2) & \beta_2^2+\beta_1\beta_2+\beta_1^2 \end{bmatrix}\begin{bmatrix} f(0) \\ \dot{f}(0) \end{bmatrix} - (w_0(0)^2-2\dot{w}_0(0))\begin{bmatrix} f(0) \\ \dot{f}(0) \end{bmatrix}$$

Taking

$$K_1 = \begin{bmatrix} -\beta_1\beta_2 & \beta_1+\beta_2 \\ -\beta_1\beta_2(\beta_1+\beta_2) & \beta_2^2+\beta_1\beta_2+\beta_1^2 \end{bmatrix}$$

we have

$$\begin{bmatrix} (w_0(0)^2-2\dot{w}_0(0))I - (\lambda^2-1)K_1 & (\lambda^2-1)I \end{bmatrix} \begin{bmatrix} f(0) \\ \dot{f}(0) \\ \ddot{f}(0) \\ f^{(3)}(0) \end{bmatrix} = 0 \qquad (5.20)$$

$$(\lambda^2-1)\begin{bmatrix} f(\Delta) \\ \dot{f}(\Delta) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\beta_1-\alpha_1-\alpha_2 & -\beta_2-\alpha_2-\alpha_1 \end{bmatrix} \boldsymbol{x}_1(\Delta)$$

$$\frac{(\lambda^2-1)}{\beta_1-\beta_2} \cdot \begin{bmatrix} -\beta_2-\alpha_2-\alpha_2 & -1 \\ \beta_1+\alpha_1+\alpha_2 & 1 \end{bmatrix}\begin{bmatrix} f(\Delta) \\ \dot{f}(\Delta) \end{bmatrix} = \boldsymbol{x}_1(\Delta)$$

$$\begin{bmatrix} w_0(0)^2-2\dot{w}_0(0) & 0 & \lambda^2-1 & 0 \\ 0 & w_0(0)^2-2\dot{w}_0(0) & 0 & \lambda^2-1 \end{bmatrix}\begin{bmatrix} f(\Delta) \\ \dot{f}(\Delta) \\ \ddot{f}(\Delta) \\ f^{(3)}(\Delta) \end{bmatrix} =$$

$$\begin{bmatrix} \beta_1^2+\alpha_2\beta_1-\alpha_2\beta_2 & \beta_2^2+\alpha_1\beta_2-\alpha_1\beta_1 \\ -\beta_1^3-\alpha_2\beta_1^2-\alpha_1\beta_1^2+\alpha_2\beta_2\beta_1-\alpha_2\beta_2^2 & -\beta_2^3-\alpha_1\beta_2^2-\alpha_2\beta_2^2+\alpha_1\beta_1\beta_2-\alpha_1\beta_1^2 \end{bmatrix} \cdot \frac{(\lambda^2-1)}{\beta_1-\beta_2} \cdot \begin{bmatrix} -\beta_2-\alpha_2-\alpha_1 & -1 \\ \beta_1+\alpha_1+\alpha_2 & 1 \end{bmatrix}\begin{bmatrix} f(\Delta) \\ \dot{f}(\Delta) \end{bmatrix}$$

Taking

$$K_2 = \begin{bmatrix} -(\alpha_1+\alpha_2+\beta_1+\beta_2)(\alpha_1+\alpha_2)-(\alpha_1\beta_1+\alpha_2\beta_2-\beta_1\beta_2) & -(\alpha_1+\alpha_2+\beta_1+\beta_2) \\ (\alpha_1+\alpha_2+\beta_1+\beta_2)(\beta_1\beta_2+\beta_1\alpha_2+\beta_2\alpha_1) & (\beta_1+\beta_2)^2+\beta_2(\alpha_1-\beta_1)+\alpha_2\beta_1 \end{bmatrix}.$$

we have

$$\begin{bmatrix}(w_0(0)^2-2\dot{w}_0(0))I - (\lambda^2-1)K_2 & (\lambda^2-1)I\end{bmatrix}\begin{bmatrix} f(\Delta) \\ \dot{f}(\Delta) \\ \ddot{f}(\Delta) \\ f^{(3)}(\Delta)\end{bmatrix} = 0 \qquad (5.21)$$

Now we note that if $f(t) = \sum_{i=1}^{2} (a_i e^{s_i t} + b_i e^{-s_i t})$, where $\{\pm s_i\}$ are the solutions of $\lambda^2 = W(s)W(-s)$,

$$\begin{bmatrix} f(t) \\ \dot{f}(t) \\ \ddot{f}(t) \\ f^{(3)}(t)\end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ s_1 & s_2 & -s_1 & -s_2 \\ s_1^2 & s_2^2 & s_1^2 & s_2^2 \\ s_1^3 & s_2^3 & -s_1^3 & -s_2^3 \end{bmatrix} \begin{bmatrix} e^{s_1 t} & 0 & 0 & 0 \\ 0 & e^{s_2 t} & 0 & 0 \\ 0 & 0 & e^{-s_1 t} & 0 \\ 0 & 0 & 0 & e^{-s_2 t}\end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix}.$$

Then (5.20) and (5.21) become

$$\begin{bmatrix}(w_0(0)^2-2\dot{w}_0(0))I - (\lambda^2-1)K_1 & (\lambda^2-1)I\end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ s_1 & s_2 & -s_1 & -s_2 \\ s_1^2 & s_2^2 & s_1^2 & s_2^2 \\ s_1^3 & s_2^3 & -s_1^3 & -s_2^3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = 0 \quad (5.22)$$

and

$$\left[(w_0(0)^2-2\dot{w}_0(0))I - (\lambda^2-1)K_2 \quad (\lambda^2-1)I\right]\cdot$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ s_1 & s_2 & -s_1 & -s_2 \\ s_1^2 & s_2^2 & s_1^2 & s_2^2 \\ s_1^3 & s_2^3 & -s_1^3 & -s_2^3 \end{bmatrix} \begin{bmatrix} e^{s_1\Delta} & 0 & 0 & 0 \\ 0 & e^{s_2\Delta} & 0 & 0 \\ 0 & 0 & e^{-s_1\Delta} & 0 \\ 0 & 0 & 0 & e^{-s_2\Delta} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = 0$$

(5.23)

Taking

$$L_1 = \left[(w_0(0)^2-2\dot{w}_0(0))I - (\lambda^2-1)K_1 \quad (\lambda^2-1)I\right] \begin{bmatrix} 1 & 1 & 1 & 1 \\ s_1 & s_2 & -s_1 & -s_2 \\ s_1^2 & s_2^2 & s_1^2 & s_2^2 \\ s_1^3 & s_2^3 & -s_1^3 & -s_2^3 \end{bmatrix}$$

and

$$L_2 = \left[(w_0(0)^2-2\dot{w}_0(0))I - (\lambda^2-1)K_2 \quad (\lambda^2-1)I\right]\cdot$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ s_1 & s_2 & -s_1 & -s_2 \\ s_1^2 & s_2^2 & s_1^2 & s_2^2 \\ s_1^3 & s_2^3 & -s_1^3 & -s_2^3 \end{bmatrix} \begin{bmatrix} e^{s_1\Delta} & 0 & 0 & 0 \\ 0 & e^{s_2\Delta} & 0 & 0 \\ 0 & 0 & e^{-s_1\Delta} & 0 \\ 0 & 0 & 0 & e^{-s_2\Delta} \end{bmatrix}$$

There exists an $\boldsymbol{a} = [a_1 \ a_2 \ b_1 \ b_2]^T$ which satisfies (5.22) and (5.23) if and only if

$$\det\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = 0. \qquad (5.25)$$

This is the same as (5.15).

CHAPTER 6

SOLUTION WITH RHP ZEROS

In earlier chapters we presented the solution of the problem of minimizing the weighted sensitivity function by feedback, when the plant is stable and has no right half plane zeros. In the next two chapters we eliminate the restrictions on right half plane zeros and stability, one at a time.

We carry these solutions only to the point of solving the eigenvalue/eigenvector problem. The explicit computation of sensitivity and compensator for these cases is left to future work.

The analysis in Chapter 8 regarding conditions for the existence of maximal eigenvalues of $T^{*}T$ applies equally well in these cases. The same holds for the realization analysis in Chapter 4.

We now consider the problem

$$\inf_{H\in\mathcal{H}^{\infty}} \|W - e^{-s\Delta}\cdot\psi\cdot H\|_{\infty}$$

with $\psi = \prod_{i=1}^{n} \frac{s-b_i}{s+b_i^{*}}$, where $Re(b_i)>0$.

The idea of our extension of the results in Chapter 3 is as follows. The subspace of $\mathcal{H}^2$ given by $K = \mathcal{H}^2 \ominus e^{-s\Delta}\psi\mathcal{H}^2$ can be decomposed into a direct sum $K = K_1 \oplus K_2$, where K_1 is due to the inner function $e^{-s\Delta}$ and K_2 is due to the inner function ψ. As in Chapter 2, K is the space on which the operator T induced by the weighting function W acts. Using this decomposition T has a triangular form when written as an operator matrix, as does the time domain version V.

We then proceed to explicitly compute the form of V^*V, and separate V^*V into a direct sum of operators corresponding to the decomposition of K. Since K_2 is finite dimensional, the part of V^*V that acts on K_2 is finite dimensional, as is the part of V^*V that acts on K_1 but gets factored through K_2.

Expressing the eigenvalue/eigenfunction conditions for V^*V separately on $\mathcal{L}^{-1}(K_1)$ and on $\mathcal{L}^{-1}(K_2)$, we get one equation on each of the subspaces, linked by $2n$ coupling coefficients when there are n right half plane zeros. Choosing a basis for expressing the coupling, the eigenvalue equation on $\mathcal{L}^{-1}(K_2)$ gives n independent linear relations for the coupling terms, which then allow us to eliminate that equation by substitution into the other. We are left with a single equation expressing the eigenvalue condition, which looks like the case in Chapter 3 except for the addition of a linear combination of n functions (which form a basis for V^*K_2) whose coefficients depend explicitly on the eigenvalues and the singular values of the restriction of V to K_2.

We next assume that the weighting function has one pole/zero pair, and we derive a differential equation for the eigenfunctions. We finally explicitly compute the boundary conditions and derive an algebraic equation, the solutions of which give the eigenvalues as in Chapter 3. At this point the determination of the infimal closed loop sensitivity and corresponding ideal compensator proceeds exactly as in Chapter 3, and the analysis in Chapter 4 also applies directly.

A. Calculation of K.

Proceeding in Chapter 3, the first step is to find $K = (e^{-s\Delta}\cdot\psi\cdot\mathcal{H}^2)^{\perp}$. The results in Chapter 3 for $\psi = 1$ show that $K_1 = \mathcal{L}(L^2(0,\Delta)) \subseteq K$, and it is known [Francis-Zames 1984, p. 11] from the case $\psi = \prod_{i=1}^{n} \frac{s-b_i}{s+\bar{b}_i}$ that

$K_2' = \langle \frac{1}{s+\bar{b}_i},\ i=1,\ldots n\rangle \subseteq K$.

We show that $K = K_1+K_2'$. From the above, it is obvious that $K \supseteq K_1+K_2'$. To see that $K \subseteq K_1+K_2'$, suppose $\hat{k} \in K$, and let k be the inverse Laplace transform of $\hat{k}$. Split k into a sum $k = k_1+k_2$, where k_1 has support on $[0,\Delta]$ and k_2 has support on (Δ,∞). Obviously $\hat{k}_1 \in K_1$, and we need only check that $\hat{k}_2 \perp \psi\mathcal{H}^2$ implies that $\hat{k}_2 \in K_1+K_2'$. But since k_2 is zero on $[0,\Delta]$, the orthogonality condition requires that $e^{s\Delta}\cdot\hat{k}_2 \in \left\{\left[\prod_{i=1}^{n} \frac{s-b_i}{s+\bar{b}_i}\right]\mathcal{H}^2\right\}^{\perp}$, and the case in [Francis-Zames 1984] gives us the desired result.

We take

$$K_2 = \langle \frac{e^{-s\Delta}}{s+\bar{b}_i},\ i=1,\ldots n\rangle,$$

for then $K = K_1 \oplus K_2$.

B. Computation of V^*V.

The $\mathcal{H}^{\infty}$ function W(s) is an operator on $\mathcal{H}^2$, and our next step is to find the norm of the compression of W to K. We call this compression T, $T = \Pi_K W|_K$. The idea is to find the norm (in the case in which T^*T has a largest eigenvalue) by finding the eigenvalues of T^*T, and then picking the largest one. To this end, we compute T and T^*. As in Chapter 3, it will we will ultimately find it convenient to compute T^*T in the time domain version V^*V.

We can represent T as an operator matrix acting on the direct sum decomposition of K given in §A above: $T = \begin{bmatrix} T_1 & T_{21} \\ T_{12} & T_2 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ T_{12} & T_2 \end{bmatrix}$, where $T_{21} = 0$ by causality of T. To be more explicit we define the operators on K

$$T_1 : K \longrightarrow K_1 \; : \; F \longmapsto \Pi_{K_1}[W \cdot (\Pi_{K_1} F)]$$

$$T_{12} : K \longrightarrow K_2 \; : \; F \longmapsto \Pi_{K_2}[W \cdot (\Pi_{K_1} F)]$$

$$T_2 : K \longrightarrow K_2 \; : \; F \longmapsto \Pi_{K_2}[W \cdot (\Pi_{K_2} F)].$$

In preparation for computing T^*T, we note that K_2 is finite dimensional, so that we can split T^*T into three parts -- the infinite dimensional operator $T_1^*T_1$ and the finite dimensional operators $(T_{12}^*T_{12} + T_{12}^*T_2)$ and $(T_2^*T_{12} + T_2^*T_2)$.

We compute the singular value decomposition of T_2. Let $\{G_i, i= 1,\ldots,n\}$ and $\{L_i, i= 1,\ldots,n\}$ be the orthonormal bases for the domain and range of T_2 in the singular value decomposition, and let $\{\sigma_1, \ldots, \sigma_n\}$ be the corresponding singular values. We note further that

$$sp(G_i, i=1,\ldots n) = sp(L_i, i= 1,\ldots,n) = sp(\frac{e^{-s\Delta}}{s+\bar{b}_i}, i=1,\ldots n),$$

and that

$$\sigma_i L_i = T_2 G_i = \sum_{j=1}^{n} \langle WG_i, G_j \rangle \cdot G_j .$$

Suppose $F \in K$. We write $F = F_1 + F_2$, with $F_1 \in K_1$ and $F_2 \in K_2$. Define $P(s) = \frac{1-e^{-s\Delta}}{s}$ so that $\Pi_{K_1}(X) = P * X$. We note that

$$\Pi_{K_2}(X) = \sum_{i=1}^{n} \langle X, G_i \rangle \cdot G_i = \sum_{i=1}^{n} \langle X, L_i \rangle \cdot L_i .$$

Thus

$$TF = \Pi_K WF$$

$$= T_1 F_1 + T_{12} F_1 + T_2 F_2$$

$$= P*(W\cdot(P*F)) + \sum_{i=1}^{n} \langle W\cdot(P*F), L_i \rangle \cdot L_i + \sum_{i=1}^{n} \langle W\cdot(\sum_{j=1}^{n} \langle F, G_j \rangle \cdot G_j), L_i \rangle \cdot L_i$$

$$= P*(W\cdot(P*F)) + \sum_{i=1}^{n} \langle W\cdot(P*F), L_i \rangle \cdot L_i + \sum_{i=1}^{n} \sigma_i \langle F, G_i \rangle \cdot L_i .$$

Now we can rewrite the operators on K

$$T_1 : K \longrightarrow K_1 : F \mapsto P*[W\cdot(P*F)]$$

$$T_{12} : K \longrightarrow K_2 : F \mapsto \sum_{i=1}^{n} \langle W\cdot(P*F), L_i \rangle \cdot L_i$$

$$T_2 : K \longrightarrow K_2 : F \mapsto \sum_{i=1}^{n} \sigma_i \langle F, G_i \rangle L_i .$$

Then we want to explicitly compute

$$T^* T = T_1^* T_1 + T_{12}^* T_{12} + T_{12}^* T_2 + T_2^* T_{12} + T_2^* T_2 .$$

In order to do this we must write down the operators T_1^*, T_{12}^* and T_2^*.

We let V_i be the operator acting on $L^2(0,\infty)$ corresponding to the operator T_i above. Then taking $p = \mathcal{L}^{-1}(P) = u(t)-u(t-\Delta)$, $g_i = \mathcal{L}^{-1}(G_i)$, $\ell_i = \mathcal{L}^{-1}(L_i)$ and $w = \mathcal{L}^{-1}(W)$, and defining $\check{w}(t) = w(-t)$.

We see in the appendix to this chapter that we can write

$$V^* V f = V_1^* V_1 f + \sum_{i=1}^{n} \tilde{\ell}_i (\langle f, \tilde{\ell}_i \rangle + \sigma_i \langle f, g_i \rangle) \qquad \text{on } [0,\Delta]$$

and

$$V^* V f = \sum_{i=1}^{n} \sigma_i \cdot g_i (\langle f, \tilde{\ell}_i \rangle + \sigma_i \langle f, g_i \rangle) \qquad \text{on } (\Delta,\infty).$$

where $$\tilde{\ell}_i = p \cdot (\tilde{w} * \ell_i) = V_{12}^* \ell_i .$$

C. Application of Eigenvalue Condition

Taking $f = f_1 \oplus f_2$, where f_1 has support on $[0,\Delta]$ and $f_2 = \sum_{i=1}^{n} \alpha_i g_i$, we now set $V^* V f = \lambda^2 f$. Note that $\langle f, g_i \rangle = \alpha_i$. Thus

$$\lambda^2 f_1 = V_1^* V_1 f_1 + \sum_{i=1}^{n} \tilde{\ell}_i(\langle f_1, \tilde{\ell}_i\rangle + \sigma_i \cdot \alpha_i) \text{ on } [0, \Delta]$$

and

$$\lambda^2 \sum_{i=1}^{n} \alpha_i g_i = \sum_{i=1}^{n} \sigma_i \cdot g_i(\langle f_1, \tilde{\ell}_i\rangle + \sigma_i \cdot \alpha_i) \text{ on } (\Delta, \infty).$$

Since the g_i are linearly independent, we must have $\lambda^2\alpha_i = \sigma_i \cdot (\langle f_1, \tilde{\ell}_i\rangle + \sigma_i \cdot \alpha_i)$ for each i. Our eigenvalue/eigenfunction problem has become

$$\lambda^2 f_1 = V_1^* V_1 f_1 + \sum_{i=1}^{n} \tilde{\ell}_i(\langle f_1, \tilde{\ell}_i\rangle + \sigma_i \cdot \alpha_i)$$

for $t \in [0, \Delta]$ subject to

$$\left\{\alpha_i = \frac{\sigma_i \langle f_1, \tilde{\ell}_i\rangle}{\lambda^2 - \sigma_i^2},\ i=1, \ \ldots, \ n\right\}. \tag{6.1}$$

Thus the problem is

$$\lambda^2 f_1 = V_1^* V_1 f_1 + \sum_{i=1}^{n} \tilde{\ell}_i \frac{\lambda^2 \langle f_1, \tilde{\ell}_i\rangle}{\lambda^2 - \sigma_i^2}.$$

D. Derivation of differential equations.

At this point it is evident that we need to invert the operator $(\lambda^2 - V_1^* V_1)$, which we do by writing the operator equation as a set of differential equations. We work out all the details for the case $W(s) = \frac{s+1}{s+\beta}$, and at the end we obtain equation (6.15) as the eigenvalue-determining equation. This is the extension of equation (3.17) to the case of right half plane zeros in the plant.

Let

$$y = V_1 f_1$$

and

$$z = V_1^* y + \sum_{i=1}^{n} \tilde{\ell}_i \frac{\lambda^2 \langle f_1, \tilde{\ell}_i \rangle}{\lambda^2 - \sigma_i^2},$$

where the support of the functions in these equations is restricted to $[0, \Delta]$. We can write

$$y = w * f_1$$

and

$$z = \check{w} * y + \sum_{i=1}^{n} \tilde{\ell}_i \frac{\lambda^2 \langle f_1, \tilde{\ell}_i \rangle}{\lambda^2 - \sigma_i^2}.$$

In order to illustrate the technique for realization of these equations and solution of the eigenvalue problem we assume

$$W(s) = \frac{s+1}{s+\beta},$$

and therefore

$$w(t) = \delta_t + w_0(t) = \delta_t + (1-\beta)e^{-\beta t}.$$

Remark: The case of general (ration, stable and minimum phase) W(s) is covered by the direct extension of the methods of Chapter 5.

Thus we can obtain $V^* V f$ as the solution z to

$$\frac{d}{dt} x_1 = -\beta \cdot x_1 + (1-\beta) \cdot f_1$$

$$y = x_1 + f_1$$

$$\frac{d}{dt} x_2 = \beta \cdot x_2 - (1-\beta) \cdot y$$

$$z = x_2 + y + \sum_{i=1}^{n} \tilde{\ell}_i \frac{\lambda^2 \langle f_1, \tilde{\ell}_i \rangle}{\lambda^2 - \sigma_i^2}.$$

More concisely,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -\beta & 0 \\ -(1-\beta) & \beta \end{bmatrix} \cdot \boldsymbol{x} + \begin{bmatrix} 1-\beta \\ \beta-1 \end{bmatrix} \cdot f_1 \tag{6.2}$$

$$z = (1 \quad 1) \cdot \boldsymbol{x} + f_1 + \sum_{i=1}^{n} \tilde{\ell}_i \frac{\lambda^2 \langle f_1, \tilde{\ell}_i \rangle}{\lambda^2 - \sigma_i^2}. \tag{6.3}$$

Boundary conditions are given by $y(0) = f(0)$ and

$$z(\Delta) = y(\Delta) + \sum_{i=1}^{n} \tilde{\ell}_i(\Delta) \frac{\lambda^2 \langle f_1, \tilde{\ell}_i \rangle}{\lambda^2 - \sigma_i^2},$$

or

$$x_1(0) = x_2(\Delta) = 0.$$

Now setting $z = \lambda^2 f_1$ in (6.3), we obtain

$$(\lambda^2 - 1) f_1 = (1 \quad 1) \cdot \boldsymbol{x} + \sum_{i=1}^{n} \tilde{\ell}_i \frac{\lambda^2 \langle f_1, \tilde{\ell}_i \rangle}{\lambda^2 - \sigma_i^2}. \tag{6.4}$$

Differentiating and substituting (6.2),

$$(\lambda^2-1)\dot{f}_1 = (1 \quad 1)\left(\begin{bmatrix}-\beta & 0\\ -(1-\beta) & \beta\end{bmatrix}\cdot \boldsymbol{x} + \begin{bmatrix}1-\beta\\ \beta-1\end{bmatrix}\cdot f_1\right) + \sum_{i=1}^{n} \dot{\tilde{\ell}}_i \frac{\lambda^2\langle f_1,\tilde{\ell}_i\rangle}{\lambda^2-\sigma_i^2}$$

$$= (-1 \quad \beta)\cdot \boldsymbol{x} + \sum_{i=1}^{n} \dot{\tilde{\ell}}_i \frac{\lambda^2\langle f_1,\tilde{\ell}_i\rangle}{\lambda^2-\sigma_i^2}.$$

Differentiating and substituting (6.2) again,

$$(\lambda^2-1)\ddot{f}_1 = (-1 \quad \beta)\left(\begin{bmatrix}-\beta & 0\\ -(1-\beta) & \beta\end{bmatrix}\cdot \boldsymbol{x} + \begin{bmatrix}1-\beta\\ \beta-1\end{bmatrix}\cdot f_1\right) + \sum_{i=1}^{n} \ddot{\tilde{\ell}}_i \frac{\lambda^2\langle f_1,\tilde{\ell}_i\rangle}{\lambda^2-\sigma_i^2}$$

$$= \beta^2(1 \quad 1)\cdot \boldsymbol{x} + (\beta^2-1)f_1 + \sum_{i=1}^{n} \ddot{\tilde{\ell}}_i \frac{\lambda^2\langle f_1,\tilde{\ell}_i\rangle}{\lambda^2-\sigma_i^2}$$

Now substituting for $(1 \quad 1)\cdot \boldsymbol{x}$ using (6.4),

$$(\lambda^2-1)\ddot{f}_1 = \beta^2\cdot\left((\lambda^2-1)\cdot f_1 - \sum_{i=1}^{n} \tilde{\ell}_i \frac{\lambda^2\langle f_1,\tilde{\ell}_i\rangle}{\lambda^2-\sigma_i^2}\right) + (\beta^2-1)f + \sum_{i=1}^{n} \ddot{\tilde{\ell}}_i \frac{\lambda^2\langle f_1,\tilde{\ell}_i\rangle}{\lambda^2-\sigma_i^2}$$

$$= (\beta^2\lambda^2-1)\cdot f_1 + \sum_{i=1}^{n} \tilde{\ell}_i \frac{\lambda^2\langle f_1,\tilde{\ell}_i\rangle}{\lambda^2-\sigma_i^2}\cdot(\ddot{\tilde{\ell}}_i - \beta^2\tilde{\ell}_i). \qquad (6.5)$$

To simplify, we calculate $\tilde{\ell}_i$ as follows.

$$\tilde{\ell}_i = 0 \qquad \begin{cases} t < 0 \\ t > \Delta \end{cases}$$

$$\tilde{\ell}_i = p\cdot(\tilde{w}*\ell_i)$$

Since $\ell_i \in K_2$, we can write

$$\ell_i(t) = \sum_{j=1}^{n} \gamma_{ij} e^{-b_i(t-\Delta)} u(t-\Delta),$$

(where this equation defines $\{\gamma_{ij}\}$) and therefore

$$\tilde{\ell}_i(t) =$$

$$[u(t)-u(t-\Delta)] \sum_{j=1}^{n} \gamma_{ij} \int_{-\infty}^{\infty} [\delta_{(\tau-t)} + (1-\beta)e^{-\beta(\tau-t)} u(\tau-t)] e^{-b_j(\tau-\Delta)} u(\tau-\Delta) d\tau$$

$$= [u(t)-u(t-\Delta)] \sum_{j=1}^{n} \gamma_{ij} \int_{\max(t,\Delta)}^{\infty} (1-\beta)e^{-\beta(\tau-t)} u(\tau-t) e^{-b_j(\tau-\Delta)} u(\tau-\Delta) d\tau$$

$$= (1-\beta) \sum_{j=1}^{n} \gamma_{ij} \int_{\Delta}^{\infty} e^{-\beta(\tau-t)} e^{-b_j(\tau-\Delta)} d\tau \qquad \text{for } 0 \le t \le \Delta$$

$$= (1-\beta) \sum_{j=1}^{n} \gamma_{ij} e^{(b_j\Delta+\beta t)} \int_{\Delta}^{\infty} e^{-(b_j+\beta)\tau} d\tau$$

$$= (1-\beta) \sum_{j=1}^{n} \gamma_{ij} e^{(b_j\Delta+\beta t)} \left[\frac{-1}{b_j+\beta} \cdot e^{-(b_j+\beta)\tau} \right]_{\tau=\Delta}^{\infty}$$

$$= (1-\beta)e^{\beta(t-\Delta)} \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta} \tag{6.6}$$

We see at this point that $\ddot{\tilde{\ell}} = \beta^2\tilde{\ell}$, and we conclude from (6.5) that

$$f_1(t) = \cos(\omega t+\varphi),$$

where

$$-\omega^2 = \frac{\beta^2\lambda^2-1}{\lambda^2-1} \quad \text{or} \quad \lambda^2 = \frac{\omega^2+1}{\omega^2+\beta^2}. \tag{6.7}$$

Boundary conditions come from

$$f_1(0) = \frac{1}{\lambda^2-1}x_2(0) + \frac{1}{\lambda^2-1} \sum_{i=1}^{n} \tilde{\ell}_i(0)\frac{\lambda^2\langle f_1,\tilde{\ell}_i\rangle}{\lambda^2-\sigma_i^2},$$

$$\dot{f}_1(0) = \frac{\beta}{\lambda^2-1}x_2(0) + \frac{1}{\lambda^2-1} \sum_{i=1}^{n} \dot{\tilde{\ell}}_i(0)\frac{\lambda^2\langle f_1,\tilde{\ell}_i\rangle}{\lambda^2-\sigma_i^2},$$

$$f_1(\Delta) = \frac{1}{\lambda^2-1}x_1(\Delta) + \frac{1}{\lambda^2-1} \sum_{i=1}^{n} \tilde{\ell}_i(\Delta)\frac{\lambda^2\langle f_1,\tilde{\ell}_i\rangle}{\lambda^2-\sigma_i^2},$$

and

$$\dot{f}_1(\Delta) = \frac{-1}{\lambda^2-1}x_1(\Delta) + \frac{1}{\lambda^2-1} \sum_{i=1}^{n} \dot{\tilde{\ell}}_i(\Delta)\frac{\lambda^2\langle f_1,\tilde{\ell}_i\rangle}{\lambda^2-\sigma_i^2}.$$

These imply

$$\dot{f}_1(0) = \beta \cdot f(0) + \frac{1}{\lambda^2-1} \cdot \sum_{i=1}^{n} \frac{\lambda^2 \langle f_1, \tilde{\ell}_i \rangle}{\lambda^2-\sigma_i^2} (\dot{\tilde{\ell}}_i(0) - \beta \tilde{\ell}_i(0)), \tag{6.8}$$

and

$$\dot{f}_1(\Delta) = -f(\Delta) + \frac{1}{\lambda^2-1} \cdot \sum_{i=1}^{n} \frac{\lambda^2 \langle f_1, \tilde{\ell}_i \rangle}{\lambda^2-\sigma_i^2} (\tilde{\ell}_i(\Delta) + \dot{\tilde{\ell}}_i(\Delta)). \tag{6.9}$$

The solutions to the equation (6.5), subject to the boundary conditions (6.8) and (6.9), determine the eigenvalues and eigenfunctions of V^*V.

E. Simplification of boundary conditions.

Next we see from (6.8) that

$$\tilde{\ell}_i(0) = (1-\beta) e^{-\beta\Delta} \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta},$$

$$\dot{\tilde{\ell}}_i(0) = (1-\beta)\beta e^{-\beta\Delta} \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta},$$

$$\tilde{\ell}_i(\Delta) = (1-\beta) \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta},$$

and

$$\dot{\tilde{\ell}}_i(\Delta) = (1-\beta)\beta \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta}.$$

So the boundary conditions from (6.7)and (6.8) amount to

$$\dot{f}_1(0) = \beta \cdot f_1(0) + \frac{1}{\lambda^2-1} \cdot \sum_{i=1}^{n} \frac{\lambda^2 \langle f_1, \tilde{\ell}_i \rangle}{\lambda^2 - \sigma_i^2} \left[(1-\beta)\beta e^{-\beta\Delta} \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta} - \beta(1-\beta) e^{-\beta\Delta} \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta} \right]$$

$$= \beta \cdot f_1(0) \tag{6.10}$$

and

$$\dot{f}_1(\Delta) = -f_1(\Delta) + \frac{1}{\lambda^2-1} \cdot \sum_{i=1}^{n} \frac{\lambda^2 \langle f_1, \tilde{\ell}_i \rangle}{\lambda^2 - \sigma_i^2} \left[(1-\beta) \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta} + (1-\beta)\beta \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta} \right]$$

$$= -f_1(\Delta) + (\omega^2+\beta^2)\lambda^2 \sum_{i=1}^{n} \frac{\langle f_1, \tilde{\ell}_i \rangle}{\lambda^2 - \sigma_i^2} \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta} \tag{6.11}$$

F. Determination of eigenvalues.

Using (6.10), we see

$$-\omega \cdot \sin(\varphi) = \beta \cdot \cos(\varphi) \tag{6.12}$$

and from (6.11)

$$-\omega \cdot \sin(\omega\Delta+\varphi) = -\cos(\omega\Delta+\varphi) + (\omega^2+\beta^2)\lambda^2 \sum_{i=1}^{n} \frac{\langle f_1, \tilde{\ell}_i \rangle}{\lambda^2 - \sigma_i^2} \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta}. \tag{6.13}$$

Expanding sin and cos and using (6.12), (6.13) becomes

$$\frac{-\omega}{(\beta^2+\omega^2)^{1/2}} \left[\omega \cdot \sin(\omega\Delta) - \beta \cdot \cos(\omega\Delta) \right]$$

$$= \frac{-1}{(\beta^2+\omega^2)^{1/2}} \left[\omega\cos(\omega\Delta) + \beta\sin(\omega\Delta) \right] + (\omega^2+\beta^2)\lambda^2 \sum_{i=1}^{n} \frac{\langle f_1, \tilde{\ell}_i \rangle}{\lambda^2 - \sigma_i^2} \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta}$$

$$= \frac{-1}{(\beta^2+\omega^2)^{1/2}}\Big[\omega\cos(\omega\Delta) + \beta\sin(\omega\Delta)\Big] + (\omega^2+\beta^2)\lambda^2 \sum_{i=1}^{n} \frac{\langle f_1,\tilde{\ell}_i\rangle}{\lambda^2-\sigma_i^2} \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta}.$$

Substituting for λ^2 from (6.7) and collecting terms,

$$\frac{-1}{(\beta^2+\omega^2)^{1/2}}\Big[(\omega^2-\beta)\cdot\sin(\omega\Delta) - \omega(\beta+1)\cdot\cos(\omega\Delta)\Big]$$

$$= (\omega^2+\beta^2)(\omega^2+1) \sum_{i=1}^{n} \frac{\langle f_1,\tilde{\ell}_i\rangle}{\omega^2+1 - \sigma_i^2(\omega^2+\beta^2)} \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta} \qquad (6.14)$$

Using (6.6), we now compute an explicit expression for $\langle f_1,\tilde{\ell}_i\rangle$.

$$\langle f_1,\tilde{\ell}_i\rangle = \int_0^\Delta \cos(\omega t+\gamma)(1-\beta)e^{\beta(t-\Delta)} \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta} dt$$

$$= (1-\beta)e^{-\beta\Delta} \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta} \cdot \int_0^\Delta \cos(\omega t+\gamma)e^{\beta t} dt$$

$$= (1-\beta)e^{-\beta\Delta} \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta} \cdot \left[\frac{\omega\sin(\Delta\omega+\varphi) + \beta\cos(\Delta\omega+\varphi)}{e^{-\beta\Delta}(\omega^2+\beta^2)} - \frac{\omega\sin(\varphi) + \beta\cos(\varphi)}{\omega^2+\beta^2}\right]$$

$$= (1-\beta)e^{-\beta\Delta} \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta} \cdot \frac{1}{(\beta^2+\omega^2)^{1/2}} \cdot \{e^{\beta\Delta}[\cos(\varphi)\sin(\omega\Delta+\varphi)$$
$$- \sin(\varphi)\cos(\omega\Delta+\varphi)] - [\cos(\varphi)\sin(\varphi) - \sin(\varphi)\cos(\varphi)]\}$$

$$= (1-\beta) \sum_{j=1}^{n} \frac{\gamma_{ij}}{b_j+\beta} \cdot \frac{1}{(\beta^2+\omega^2)^{1/2}} \cdot \sin(\omega\Delta)$$

Now substituting this in (6.14),

$$\frac{-1}{(\beta^2+\omega^2)^{1/2}}\left[(\omega^2-\beta)\cdot\sin(\omega\Delta) - \omega(\beta+1)\cdot\cos(\omega\Delta)\right]$$

$$= (\omega^2+\beta^2)(\omega^2+1)\sin(\omega\Delta)(1-\beta)\ \frac{1}{(\beta^2+\omega^2)^{1/2}}\cdot\sum_{i=1}^{n}\frac{1}{\omega^2+1-\sigma_i^2(\omega^2+\beta^2)}\left[\sum_{j=1}^{n}\frac{\gamma_{ij}}{b_j+\beta}\right]^2$$

$$(\omega^2-\beta)\cdot\sin(\omega\Delta) - \omega(\beta+1)\cdot\cos(\omega\Delta)$$

$$= (\omega^2+\beta^2)(\omega^2+1)(1-\beta)\sin(\omega\Delta)\cdot\sum_{i=1}^{n}\frac{-1}{\omega^2+1-\sigma_i^2(\omega^2+\beta^2)}\left[\sum_{j=1}^{n}\frac{\gamma_{ij}}{b_j+\beta}\right]^2$$

$$\omega(\beta+1)\cos(\omega\Delta) = \left[\omega^2-\beta + (\omega^2+\beta^2)(\omega^2+1)(1-\beta)\sum_{i=1}^{n}\frac{1}{\omega^2+1-\sigma_i^2(\omega^2+\beta^2)}\left[\sum_{j=1}^{n}\frac{\gamma_{ij}}{b_j+\beta}\right]^2\right]\sin(\omega\Delta).$$

$$\cot(\omega\Delta) = \frac{\left[\omega^2 - \beta + (\omega^2+\beta^2)(\omega^2+1)(1-\beta)\sum_{i=1}^{n}\frac{1}{\omega^2+1-\sigma_i^2(\omega^2+\beta^2)}\left[\sum_{j=1}^{n}\frac{\gamma_{ij}}{b_j+\beta}\right]^2\right]}{\omega(\beta+1)} \qquad (6.15)$$

This equation determines the eigenvalues of V^*V via

$$\lambda^2 = \frac{\omega^2+1}{\omega^2+\beta^2},$$

and the eigenfunctions via

$$f(t) = [u(t) - u(t-\Delta)]\cos(\omega t+\varphi) + \sum_{i=1}^{n} \alpha_i g_i$$

where φ is given by (6.12) and $\{\alpha_i\}$ is given by (6.1).

G. Other Analysis.

From this point calculation of the optimal sensitivity and compensator would proceed as in Chapter 3. The approximation analysis in Chapter 4 applies directly. Conditions for the existence of maximal eigenvalues are discussed in Chapter 8.

CHAPTER 6 Appendix

Computation of Adjoint Operators

T_2^* is defined by (the inner product being in K)

$$\langle F, T_2^* X\rangle = \langle T_2 F, X\rangle$$

$$= \langle \sum_{i=1}^{n} \sigma_i \langle F, G_i\rangle L_i, X\rangle$$

$$= \sum_{i=1}^{n} \sigma_i \langle F, G_i\rangle \langle L_i, X\rangle$$

$$= \langle F, \sum_{i=1}^{n} \sigma_i \langle X, L_i\rangle G_i\rangle$$

from which we conclude

$$T_2^* X = \sum_{i=1}^{n} \sigma_i \langle X, L_i\rangle G_i .$$

We know T_1^* from Chapter 3 equation (3.6). T_{12}^* is most conveniently derived in the time domain, which we do as follows.

$$\langle V_{12} f, x\rangle = \langle w*(p\cdot f), x\rangle$$

$$= \langle p\cdot f, \check{w}*x\rangle$$

$$= \langle f, p\cdot(\check{w} * x)\rangle$$

Therefore $V_{12}^{*}x = p\cdot(\check{w} * x)$.

Now we compute $V^{*}V = V_1^{*}V_1 + V_{12}^{*}V_{12} + V_{12}^{*}V_2 + V_2^{*}V_{12} + V_2^{*}V_2$.

$$V_{12}^{*}V_{12}f = V_{12}^{*}\Big(\sum_{i=1}^{n} \langle w*(p\cdot f), \ell_i\rangle\cdot\ell_i\Big)$$

$$= p\cdot\check{w} * \Big(\sum_{i=1}^{n} \langle w*(p\cdot f), \ell_i\rangle\cdot\ell_i\Big)$$

$$= \sum_{i=1}^{n} \langle w*(p\cdot f), \ell_i\rangle\cdot p\cdot(\check{w} * \ell_i)$$

$$= \sum_{i=1}^{n} \langle f, p\cdot(\check{w} * \ell_i)\rangle\cdot p\cdot(\check{w} * \ell_i)$$

$$V_{12}^{*}V_2 f = V_{12}^{*}\Big(\sum_{i=1}^{n} \sigma_i\langle f, g_i\rangle\cdot\ell_i\Big)$$

$$= p\cdot\Big(\check{w} * \Big(\sum_{i=1}^{n} \sigma_i\langle f, g_i\rangle\cdot\ell_i\Big)\Big)$$

$$= \sum_{i=1}^{n} \sigma_i\langle f, g_i\rangle\cdot p\cdot(\check{w} * \ell_i)$$

$$V_2^{*}V_{12}f = \sum_{i=1}^{n} \sigma_i\langle V_{12}f, \ell_i\rangle\cdot g_i$$

$$= \sum_{i=1}^{n} \sigma_i \langle \sum_{j=1}^{n} \langle w*(p\cdot f), \ell_j\rangle\cdot \ell_j, \ell_i\rangle\cdot g_i$$

$$= \sum_{i=1}^{n} \sigma_i \sum_{j=1}^{n} \langle w*(p\cdot f), \ell_j\rangle\cdot\langle \ell_j, \ell_i\rangle\cdot g_i$$

$$= \sum_{i=1}^{n} \sigma_i \sum_{j=1}^{n} \langle f, p\cdot(\check{w}*\ell_j)\rangle\cdot\langle \ell_j, \ell_i\rangle\cdot g_i$$

$$= \sum_{i=1}^{n} \sigma_i \langle f, p\cdot(\check{w}*\ell_i)\rangle\cdot g_i$$

$$V_2^* V_2 f = \sum_{i=1}^{n} \sigma_i \langle V_2 f, \ell_i\rangle\cdot g_i$$

$$= \sum_{i=1}^{n} \sigma_i \langle \sum_{j=1}^{n} \sigma_j \langle f, g_j\rangle\cdot \ell_j, \ell_i\rangle\cdot g_i$$

$$= \sum_{i=1}^{n} \sigma_i \sum_{j=1}^{n} \sigma_j \langle f, g_j\rangle\cdot\langle \ell_j, \ell_i\rangle\cdot g_i$$

$$= \sum_{i=1}^{n} \sigma_i^2 \langle f, g_i\rangle\cdot g_i$$

Referring to equations (3.5) and (3.6), we see that we can let $f \in L^2(0,\infty)$ and write $V_1 f = p\cdot(w*(p\cdot f))$ and $V_1^* f = p\cdot((p\cdot x)*\check{w})$. Therefore

$$V_1^* V_1 f = p\cdot((p\cdot(w*(p\cdot f))*\check{w}).$$

The net result is

$$V^*V = p\cdot((p\cdot(w*(p\cdot f)))*\check{w} + \sum_{i=1}^{n} \langle f, p\cdot(\check{w}*\ell_i)\rangle\cdot p\cdot(\check{w}*\ell_i) +$$
$$\sum_{i=1}^{n} \sigma_i \langle f, g_i\rangle\cdot p\cdot(\check{w}*\ell_i) + \sum_{i=1}^{n} \sigma_i \langle f, p\cdot(\check{w}*\ell_i)\rangle\cdot g_i + \sum_{i=1}^{n} \sigma_i^2 \langle f, g_i\rangle\cdot g_i$$

$$= p\cdot((p\cdot(w*(p\cdot f)))*\check{w}) + \sum_{i=1}^{n} p\cdot(\check{w}*\ell_i)\cdot(\langle f, p\cdot(\check{w}*\ell_i)\rangle +$$
$$\sigma_i \langle f, g_i\rangle) + \sum_{i=1}^{n} \sigma_i g_i\cdot(\langle f, p\cdot(\check{w}*\ell_i)\rangle + \sigma_i\langle f, g_i\rangle)$$

$$= p\cdot((p\cdot(w*(p\cdot f)))*\check{w}) +$$
$$\sum_{i=1}^{n} (p\cdot(\check{w}*\ell_i) + \sigma_i g_i)\cdot(\langle f, p\cdot(\check{w}*\ell_i)\rangle + \sigma_i\langle f, g_i\rangle)$$

We write

$$V^*Vf = V_1^*V_1 f + \sum_{i=1}^{n} (\tilde{\ell}_i + \sigma_i\cdot g_i)(\langle f, \tilde{\ell}_i\rangle + \sigma_i\langle f, g_i\rangle),$$

where

$$\tilde{\ell}_i = p\cdot(\check{w}*\ell_i) = V_{12}^*\ell_i$$

and

$$\sigma_i = \langle w*g_i, \ell_i\rangle = |V_2 g_i|.$$

Since the support of $\tilde{\ell}_i$ is on $[0,\Delta]$ and the support of g_i is on $(\Delta.\infty)$, we write

$$V^{*}Vf = V_1^{*}V_1 f + \sum_{i=1}^{n} \tilde{\ell}_i(\langle f, \tilde{\ell}_i \rangle + \sigma_i \langle f, g_i \rangle) \qquad \text{on } [0, \Delta]$$

and

$$V^{*}Vf = \sum_{i=1}^{n} \sigma_i \cdot g_i(\langle f, \tilde{\ell}_i \rangle + \sigma_i \langle f, g_i \rangle) \qquad \text{on } (\Delta, \infty).$$

CHAPTER 7

SOLUTION FOR UNSTABLE PLANTS

When the plant is unstable, we must accommodate a non-trivial coprime factorization of the plant. Following [Francis & Zames 1984, p. 10] we see that if a coprime factorization of the plant is $e^{-s\Delta}AB^{-1}$, with $e^{-s\Delta}AU + BV = 1$ and B inner, the Q-parameterization of stabilizing compensators is $C = Q/(1-PQ)$,

$$Q = BU + B^2H, \tag{6.1}$$

where $H \in \mathcal{H}^\infty$. The weighted sensitivity is given by $X(H) = WB(V-e^{-s\Delta}AH)$, and the infimal magnitude is $\inf_{H\in\mathcal{H}^\infty} \|WV - e^{-s\Delta}AH\|$.

Thus the problem looks like the case of stable plants, except with WV substituted for W. However, the situation is complicated by the fact that V is not rational. In this chapter we carry the solution only so far as to see how to apply the results of earlier chapters.

We first find the explicit form of V. We then see that, when there are no right half plane zeros, the compression of V to K acts like that of a rational function. Then we can directly apply the results of Chapter 5.

When there are right half plane zeros, V is not rational on K_2. Still, in that case V can be split into two parts, one of which is rational. The development then would proceed as in Chapter 6, with modifications.

With an unstable plant, the approximation analysis of Chapter 4 must be modified to take into account the more detailed structure of the Q-parameter. We do the additional analysis needed to justify the application of Chapter 4.

A. An Example.

In order to extend our work to this case, we examine the most simple example. For $P = \frac{e^{-s}}{s-1}$. a coprime factorization is given by

$$\left(\frac{e^{-s}}{s+1}\right)\cdot(2e) + \left(\frac{s-1}{s+1}\right)\cdot\left(\frac{1+s-2e^{1-s}}{s-1}\right) = 1.$$

Then the optimization problem becomes

$$\inf_{X\in\mathcal{H}^\infty} \left\|W\left(\frac{1+s-2e^{1-s}}{s-1}\right) - e^{-s}X\right\| = \inf_{X\in\mathcal{H}^\infty} \left\|W\left(\frac{1+s}{s-1} + 2\cdot\frac{e^{1-s}}{s-1}\right) - e^{-s}X\right\|.$$

It can be seen that $W\cdot\frac{e^{1-s}}{s-1}$ is zero on $K = (e^{-s}\mathcal{H}^2)^\perp$, so the operator whose norm we seek it that of convolution with $\mathcal{L}^{-1}\left[W\cdot\frac{1+s}{s-1}\right]$ on [0,1].

When we have more unstable poles it is the general case that W is to be modified by a factor which consists of the sum of an $\mathcal{H}^\infty$ function (which is the Laplace transform of the projection of an unstable function onto $[0,\Delta]$) and another stable function (which is rational). We shall see that our technique for the case of rational weighting function then applies.

B. Computation of WV.

We assume that the plant is of the form $P = e^{-s\Delta}P_oP_iB^{-1}$ where P_o is outer, P_i is a finite Blaschke product, and B is a finite Blaschke product. We assume that all complex poles and zeros of P occur in conjugate pairs. Thus we assume $B^{-1}(s) = \frac{m(-s)}{m(s)}$. In order to apply the formulation of [Francis & Zames 1984] we must find a factorization of P which is coprime over $\mathcal{H}^{\infty}$ to find V explicitly.

The computations involved in finding this factorization are given in [Callier and Desoer 1978, pp. 655 and 660].

Following these references, we first decompose P into the sum of an unstable rational part R and a stable but possibly irrational part G. To perform this decomposition we consider the inverse Laplace transform,

$$\mathcal{L}^{-1}(P(s)) = \mathcal{L}^{-1}(e^{-s\Delta}P_o P_i B^{-1})$$

$$= p(t-\Delta)$$

$$= p_s(t-\Delta) + p_u(t-\Delta),$$

where $p_s(t)$ is stable and $p_u(t)$ is unstable. Since $p_s(t-\Delta)$ is stable, we need only consider the additive decomposition of $p_u(t-\Delta)$ into stable and unstable parts.

The idea of this decomposition is the following. Since $p_u(t-\Delta)$ is zero on $(0,\Delta)$, the purely unstable part will consist of $p_u(t-\Delta)$ plus $p_u(t-\Delta)$ propagated backwards to the origin. The stable part will then consist of just −1 times the restriction of the purely unstable part to $(0,\Delta)$.

We now explicitly compute the decomposition. Let $\{d_i\}$ be the zeros of B. Since $B^{-1}(s)$ is rational,

$$p_u(t) = u(t)\cdot \sum_{i=1}^{m} \sum_{j=1}^{k_i} a_{ij} t^j e^{d_i t},$$

where $\sum_{i=1}^{m} k_i = n$ and n is the degree of the denominator of $P_u(s)$. Then (leaving $\{r_{ij}\}$ undefined for the moment)

$$p_u(t-\Delta) = u(t-\Delta) \sum_{i=1}^{m} \sum_{j=1}^{k_i} a_{ij}(t-\Delta)^j e^{d_i(t-\Delta)} - u(t) \sum_{i=1}^{m} \sum_{j=1}^{k_i} r_{ij} t^j e^{d_i t} + u(t) \sum_{i=1}^{m} \sum_{j=1}^{k_i} r_{ij} t^j e^{d_i t}.$$

$$= -[u(t)-u(t-\Delta)] \sum_{i=1}^{m} \sum_{j=1}^{k_i} r_{ij} t^j e^{d_i t} + u(t-\Delta)\left[\sum_{i=1}^{m} \sum_{j=1}^{k_i} a_{ij}(t-\Delta)^j e^{d_i(t-\Delta)} - \sum_{i=1}^{m} \sum_{j=1}^{k_i} r_{ij} t^j e^{d_i t}\right] + u(t) \sum_{i=1}^{m} \sum_{j=1}^{k_i} r_{ij} t^j e^{d_i t}$$

$$= -[u(t)-u(t-\Delta)] \sum_{i=1}^{m} \sum_{j=1}^{k_i} r_{ij} t^j e^{d_i t} + u(t-\Delta)\left[\sum_{i=1}^{m} \sum_{j=1}^{k_i} [a_{ij}(t-\Delta)^j e^{-d_i \Delta} - r_{ij} t^j] e^{d_i t}\right] + u(t) \sum_{i=1}^{m} \sum_{j=1}^{k_i} r_{ij} t^j e^{d_i t}$$

$$= -[u(t)-u(t-\Delta)] \sum_{i=1}^{m} \sum_{j=1}^{k_i} r_{ij} t^j e^{d_i t} + u(t) \sum_{i=1}^{m} \sum_{j=1}^{k_i} r_{ij} t^j e^{d_i t}$$

if

$$\sum_{j=1}^{k_i} [a_{ij}(t-\Delta)^j e^{-d_i \Delta} - r_{ij} t^j] = 0,$$

which serves to define $\{r_{ij}\}$. Clearly

$$g(t) = p_s(t-\Delta) - [u(t)-u(t-\Delta)] \sum_{i=1}^{m} \sum_{j=1}^{k_i} r_{ij} t^j e^{d_i t}$$

is stable and

$$r(t) = \sum_{i=1}^{m} \sum_{j=1}^{k_i} r_{ij} t^j e^{d_i t}$$

is unstable.

In terms of Laplace transforms, for the case where the d_i are distinct, this decomposition may be seen to be

$$P = R + G$$

with

$$R = q(s)/m(s)$$

and

$$G = \left[e^{-s\Delta} P_o(s) P_i(s) m(-s) - q(s) \right] / m(s),$$

where $q(s)$ is the $(k-1)$-degree monic polynomial that takes the value

$e^{-d_i \Delta} P_o(d_i) P_i(d_i) m(-d_i)$ at each point $s = d_i$ for i=1 to k.

It can be seen that when P(s) has repeated unstable roots, G(s) still has the form $G = \left[e^{-s\Delta} a(s) - b(s)\right] / m(s)$, where a and b are polynomials. For simplicity we just consider the case of distinct roots.

Now, suppose a coprime factorization of $q(s)/m(s)$ is $\overline{A} \cdot \overline{B}^{-1}$, and these coprime factors satisfy the generalized "Bezout" identity $\overline{A} \cdot \overline{U} + \overline{B} \cdot \overline{V} = 1$ for some $\overline{U}$ and $\overline{V}$. (Note that $\overline{A}$, $\overline{B}$, $\overline{U}$ and $\overline{V}$ are all rational and in $\mathcal{H}^{\infty}$.) Then a coprime factorization of P is $A \cdot B^{-1}$ with $A = \overline{A} + G \cdot \overline{B}$ and $B = \overline{B}$ [Callier and Desoer 1978, p. 655]. A and B satisfy $A \cdot U + B \cdot V = 1$ with $U = \overline{U}$ and

$$V = \overline{V} - \overline{U} \cdot G.$$

In the example,

$$\frac{e^{-s}}{s-1} = -\left(\frac{1-e^{-s}}{s}\right) * \left(\frac{e^{-1}}{s-1}\right) + \frac{e^{-1}}{s-1}$$

$$= \frac{e^{-s} - e^{-1}}{s-1} + \frac{e^{-1}}{s-1}$$

so that

$$R(s) = \frac{e^{-1}}{s-1} \quad \text{and} \quad G(s) = \frac{e^{-s} - e^{-1}}{s-1}.$$

A coprime factorization of R(s) is $\left(\frac{e^{-1}}{s+1}\right) \cdot \left(\frac{s-1}{s+1}\right)^{-1}$, which satisfies

$$\left(\frac{e^{-1}}{s+1}\right) \cdot 2e + \left(\frac{s-1}{s+1}\right) \cdot 1 = 1,$$

so $\overline{A} = (\frac{e^{-1}}{s+1})$, $\overline{B} = (\frac{s-1}{s+1})$, $\overline{U} = 2e$ and $\overline{V} = 1$. Therefore $U = 2e$ and $V = 1 - 2e\cdot\frac{e^{-s}-e^{-1}}{s-1}$, which agrees with our previous calculations.

As far as we are concerned, the key facts are that $\overline{U}$ and $\overline{V}$ are rational functions, and G is as given above.

C. The compression of WV to K.

Thus $W\cdot V = W\cdot(\overline{V} - \overline{U}\cdot G)$, and we seek the norm of this function viewed as an operator on K.

For the case where $P_i = 1$, K is just the subspace of Laplace transforms of functions with support on $[0,\Delta]$. In this case, the relevant operator on $L^2(0,\Delta)$ is the sum of the operators of convolution with $\mathcal{L}^{-1}(W\overline{V})$ and convolution with $\mathcal{L}^{-1}(-W\overline{U}G)$. However, because $G = \left[e^{-s\Delta}P_o(s)P_i(s)m(-s)-q(s)\right]/m(s)$, the projection onto $L^2(0,\Delta)$ of the convolution of $L^2(0,\Delta)$ with $\mathcal{L}^{-1}(-W\overline{U}G)$ is equal to the projection onto $L^2(0,\Delta)$ of the convolution with $\mathcal{L}^{-1}(W\overline{U}R)$. So we are concerned with the norm of the operator on $L^2(0,\Delta)$ of convolution with $\mathcal{L}^{-1}[W\cdot(\overline{V} + \overline{U}\cdot R)]$. $W\cdot(\overline{V} + \overline{U}\cdot R)$ is a rational function, so we can handle this case with the state space formulation of our solution for general rational W.

We remark that $(\overline{V} + \overline{U}\cdot R) = B^{-1}$. To see this we recall $A = \overline{A} + G\cdot\overline{B}$ and $\overline{B} = B$. Then $\overline{A}\cdot\overline{U} + \overline{B}\cdot\overline{V} = 1$ implies $(A-GB)\overline{U} + B\overline{V} = 1$. But $R = \overline{A}\cdot\overline{B}^{-1} = (A-GB)B^{-1}$, so $B(R\overline{U} + \overline{V}) = 1$.

Based on this remark, our operator on K is given by

$$T = \Pi_K(WB^{-1}) \quad .$$

Notice that for this case our characteristic equation for the eigenvalues is $\lambda^2 = W(s)B^{-1}(s)W(-s)B^{-1}(-s)$. Because of our symmetry assumption on poles and zeros and the fact that B(s) is a Blaschke product, $B^{-1}(s) = B(-s)$, and $B^{-1}(s)B^{-1}(-s) = 1$. Of course the eigenvalues differ from the case in which there are no unstable roots because the term B^{-1} changes the boundary conditions as expressed in Chapter 6.

D. Right half plane zeros.

Now we consider the case where $P_i \neq 1$. The development proceeds exactly as in Chapter 5, with $W(\overline{V} - \overline{U}G)$ substituted for W, up to the point at which we assumed $W = \frac{s+1}{s+\beta}$. Since G is not rational, calculations are more difficult. In fact we see that when we split the domain of G into K_1 and K_2 as in Chapter 5, the action of $\mathcal{L}^{-1}(G)$ on $\mathcal{L}^{-1}(K_1)$ and $\mathcal{L}^{-1}(K_2)$ is convolution with (different) sums of exponentials (and delayed exponentials, for $\mathcal{L}^{-1}(K_2)$).

Now we proceed to the computations. We take

$$P = e^{-s\Delta}P_o P_i B^{-1} = e^{-s\Delta}(P_s + P_u),$$

where P_s and P_u are stable and unstable rational functions, respectively. As above, G = P - R, and we can decompose G as $G = G_1 + G_2$, where $G_1 = e^{-s\Delta}P_s$ and $G_2 = -\frac{1-e^{-s\Delta}}{s}\cdot R$. Note that $supp(G_1) \subseteq K_1^{\perp}$. The operators T_1, T_2 and T_{12} become

$$T_1: \quad F \mapsto \Pi_{K_1}[WB^{-1} \cdot (\Pi_{K_1} F)]$$

$$T_{12}: \quad F \mapsto \Pi_{K_2}[WB^{-1} \cdot (\Pi_{K_1} F)]$$

$$T_2: \quad F \mapsto \Pi_{K_2}[W(\bar{V}-\bar{U}G) \cdot (\Pi_{K_2} F)]$$

The expressions for T_1 and T_{12} follow from the calculations in section C. The formula for T_2 is direct without simplification. We note that $\mathcal{L}^{-1}(T_2)$ is a sum of exponentials plus a sum of exponentials truncated at $t = \Delta$.

The functions L_i and $\tilde{\ell}_i$ are defined analogously to Chapter 6, and we can specify the boundary conditions just as in the case of general right half plane zeros.

E. Completing the solution.

As in the previous cases, the calculations described above give us simultaneous equations which determine the eigenvalues and eigenfunctions of V^*V. In a particular case we must find the largest eigenvalue, and apply Sarason's Proposition 5.1 to compute the infimal sensitivity. This in turn allows us to compute the optimal compensator.

F. Unstable plant: proper approximation.

In this case the procedure in Chapter 4 by which we obtain a sequence of proper compensators for which the closed loop weighted sensitivity approaches the optimal does not apply.

When the plant is unstable, we cannot roll off Q directly, since not every stable Q results in a stable closed loop. In the unstable case, $Q = BU + B^2H$, where $P = e^{-s\Delta}AB^{-1}$ is a coprime factorization of P over $\mathcal{H}^\infty$ and U is the coefficient function in the generalized Bezout identity $e^{-s\Delta}AU + BV = 1$. Here the free parameter is H. We note that both B and U are rational and proper. Since Q is improper when the plant is strictly proper, it follows that so is H.

(Remark: It is tempting to say that since $H = 0$ yields a stable closed loop system, we can stabilize the loop with the suboptimal but proper feedback resulting from $Q_1 = BU$, and then apply the above proper approximation procedure to the optimal compensator for the stabilized loop. It is not necessarily true that the composition overall optimal compensator can be obtained as the of two such feedbacks: If the Q parameter for the second feedback is Q_2, the effective Q parameter is $Q_3 = BU + B^2U^2Q_2$. Therefore, unless the optimal free parameter ("H") lies in the set $V^2\mathcal{H}^\infty$, it cannot be obtained in this manner.

Since BU is proper, to obtain a proper Q, what we want to do is roll off the (improper) free parameter $H = \frac{Q-BU}{B^2}$. Recall that the weighted sensitivity is $X = WB(V-e^{-s\Delta}AH)$. Obviously, if there is a sequence of proper compensators which allows us to approximate the infimal sensitivity $\hat{X} = WB(V-e^{-s\Delta}\psi\hat{H})$, we must have $\|\hat{X}\|_\infty \geq \liminf_{\omega \to \infty} |(WBV)(\omega)|$. We need to examine the behavior of $\liminf_{\omega \to \infty} |(WBV)(\omega)|$.

Now W and B are rational with $W(\infty) = B(\infty) = 1$, so we look at $V(j\omega)$.

From above, $V = \overline{V} - \overline{U} \cdot G$. $\overline{V}$ and $\overline{U}$ are proper and rational, and G is the sum of a proper rational function and a strictly proper (since we assume P is strictly proper) function. Therefore $V(\infty)$ exists.

We therefore require that $\|X\|_\infty \geq V(\infty)$. We can see from above that $V(\infty) = \sum_{i=1}^{k} e^{-d_i \Delta} \cdot P_0(d_i) \cdot d(-d_i)$, and we note in particular that $V(\infty) \in \mathbb{R}$.

We can now simply repeat the argument for stable plants in Chapter 4, substituting $[V(\infty) - A(j\omega)R(j\omega)]$ for $[1 - P(j\omega)Q(j\omega)]$, if we assume that we take our bandwidth of good approximation wide enough for $V(j\omega)$ to be sufficiently close to $V(\infty)$ when ω is beyond this band.

CHAPTER 8

CONSTRUCTION OF SOLUTIONS

A. Summary

In this chapter we discuss the construction of solutions to the sensitivity minimization problem in its form (2.7). We know from the theory of [Sarason 1967] that a solution exists, but unless the operator T^*T has a maximal eigenvalue, we do not necessarily know how to find a solution.

There are three cases, depending upon the weighting function W(s):

(1) W is such that we can guarantee the existence of a maximal eigenvalue.

When $|W(j\omega)| > 1$ for ω large enough T^*T has infinitely many eigenvalues greater than 1. Since the only cluster point of the eigenvalues is 1, this means that T^*T has a largest eigenvalue, and thus a maximal vector. We observe that (since $W(\infty) = 1$) we can always pick W(s) so that $|W(j\omega)|$ eventually approaches 1 from above, while affecting $|W(j\omega)|$ only arbitrarily little at any frequencies of interest, by introducing one additional pole/zero pair at high frequency. In other words, we can always pick a W close to one having the desired magnitude, which results in a solvable problem.

When $|W(j\omega)|$ does not approach 1 from above at infinity, it must do so from below. Then there we get the other two cases.

(2) W is such that we can guarantee there is no maximal eigenvalue. If

$|W(j\omega)| < 1$ for all ω, the magnitude of the infimal sensitivity is 1, a maximal vector does not exist, and an optimal sensitivity is obtained with the open loop system (for a stable plant).

(3) In the third case we cannot tell from examination of W whether or not a maximal eigenvalue exists. In this case $|W(j\omega)| > 1$ over some frequency band. In this case a maximal vector may of may not exist. Then we see two possibilities for solving the problem:

(a) Modify W slightly, so that case (1) above applies. We show below how to do this without significantly changing W.

(b) Solve the eigenvalue problem, and examine the solutions. If a maximum eigenvalue exists, it will lie between 1 and $\|W\|_\infty$. This fact should allow numerical procedures searching for it to be finite. We do not discuss this alternative further. If a maximum eigenvalue does not exist, and the condition of case (2) above is not met, then the procedure employed in Chapter 3 section H to find an all-pass minimal sensitivity may work.

B. Existence of a Maximal Eigenvalues.

We know from Chapter 2 that the spectrum of T^*T consists of eigenvalues union the point 1. We show here a sufficient condition on W(s) for T^*T to have a maximal eigenvalue, namely, that there exists some $\omega_0 > 0$ such that

$$\text{for } \omega > \omega_0, \ |W(j\omega)| > 1. \tag{8.1}$$

There are two steps to see this.

(1) For every eigenvalue λ^2 of T^*T there are imaginary solutions to

the equation $\lambda^2 = W(s)W(-s)$.

(2) The imaginary solutions to $\lambda^2 = W(s)W(-s)$ are not restricted to any finite interval as λ^2 varies over all eigenvalues.

It then follows from (8.1) that there will be eigenvalues greater than 1, since there will be eigenvalues λ_i^2 for corresponding $\omega_i > \omega_0$ with $\lambda_i^2 = W(j\omega_i)W(-j\omega_i) = |W(j\omega_i)|^2 > 1$. Finally, since $W(\infty) = 1$, the existence of one eigenvalue greater than 1 implies the existence of a maximal eigenvalue.

The condition (8.1) should always be possible to meet in practice. If (8.1) does not hold we can modify W as follows: We have that there exists some $\omega_0 > 0$ such that for $\omega > \omega_0$

$$|W(j\omega)| < 1 \tag{8.2}$$

Since (in Chapter 2) we have normalized W so that $W(\infty) = 1$, for any $\epsilon > 0$ we can pick ω_0 large enough so that $|W(j\omega)| < 1 - \frac{\epsilon}{4}$ for $\omega \geq \omega_0$. If we take $W' = W\cdot\left[\frac{s+(1+\epsilon)\omega_0}{s+\omega_0}\right]$, it is easy to see that $\|W-W'\|_\infty \leq \|W\|_\infty\cdot\epsilon$ and $|W'| - |W| \leq |W|\cdot\epsilon$. Furthermore $|W'(j\omega)| > 1$ for $\omega > \omega_0$ and $W'(\infty) = 1$. Thus W' satisfies (8.2) yet can be made close to W by choice of ϵ.

We now proceed to show the facts (1) and (2).

(We say that there is an eigenvalue on the interval $(\omega_1,\omega_2) \subseteq \mathbb{R}$ if for some $\omega \in (\omega_1,\omega_2)$, there is a $\lambda^2 \in \sigma(T^*T)$ such that $|W(j\omega)|^2 = \lambda^2$.)

(1) Existence of imaginary solutions to $\lambda^2 = W(s)W(-s)$

We first prove that every eigenvalue of T^*T has at least one corresponding imaginary solution of (6.13).

First, suppose $\lambda^2 \neq 1$. If λ^2 is an eigenvalue of T^*T, and $y = \lambda^2$ intersects $y = |W(j\omega)|^2$, then there is a solution ω_0 to $\lambda^2 = |W(j\omega)|^2 = W(j\omega)W(-j\omega)$. But then $\lambda^2 = W(s)W(-s)$ has the imaginary solutions $s = \pm j\omega_0$. If $y = \lambda^2$ does not intersect $y = |W(j\omega)|^2$, then either $\lambda^2 \geq \sup_{\omega\in\mathbb{R}} |W(j\omega)|^2$ or $\lambda^2 \leq \inf_{\omega\in\mathbb{R}} |W(j\omega)|^2$, by continuity of $|W(j\omega)|^2$. But $\|W(s)\|_\infty^2 = \sup_{\omega\in\mathbb{R}} |W(j\omega)|^2$ and $\|W^{-1}(s)\|_\infty^2 = \left[\inf_{\omega\in\mathbb{R}} |W(j\omega)|^2\right]^{-1}$, since, by assumption on our weightings, $W^{-1}(s) \in \mathcal{H}^\infty$. Therefore either $\lambda^2 \geq \|W(s)\|_\infty^2$ or $\|W^{-1}(s)\|_\infty^2 \leq \frac{1}{\lambda^2}$.

Since we must have $\|T\| \leq \|W(s)\|_\infty$, the first case would imply $\lambda^2 = \|W(s)\|_\infty^2$. The second case also implies $\lambda^2 = \|W(s)\|_\infty^2$, by applying the same argument to $\frac{1}{\lambda^2}$, T^{-1} and W^{-1}: (Note that T is invertible, since W is causally invertible.) Let f be an eigenvector for T^*T, and let $g = Tf$. g satisfies $|T^{-1}g|^2 = \frac{1}{\lambda^2}|g|^2$, and so $\|T^{-1}\|^2 \geq \frac{1}{\lambda^2}$. For the second case, $\|T^{-1}\| \leq \|W^{-1}(s)\|_\infty$, so we conclude $\|W^{-1}(s)\|_\infty^2 = \frac{1}{\lambda^2}$, and therefore $\lambda^2 = \inf_{\omega\in\mathbb{R}} |W(j\omega)|^2$.

Since W(s) is continuous on the imaginary axis, we see that if $y = \lambda^2$ does not intersect $y = |W(j\omega)|^2$, we must have $\lambda^2 = |W(\infty)|^2 = 1$.

Now suppose $\lambda^2=1$ is an eigenvalue and f is a corresponding unit eigenvector. By the reasoning above, if $W(-j\omega)W(j\omega) = 1$ has no solution, then either $|W(j\omega)| > 1$ for all $\omega \in \mathbb{R}$ or $|W(j\omega)| < 1$ for all

$\omega \in \mathbb{R}$. Suppose $|W(j\omega)| < 1$. (For $|W(j\omega)| > 1$ just use $(T^*T)^{-1}$ and $(\check{W}W)^{-1}$ instead of T^*T and $\check{W}W$.) Since we must have $\|T^*Tf\|_2 \leq \|\check{W}Wf\|_2$, and $\|T^*Tf\|_2 = 1$, we conclude $\|\check{W}Wf\|_2 = 1$. But this is impossible with $|W(j\omega)| < 1$ and $\|f\|_2 = 1$. We conclude that $W(-j\omega)W(j\omega) = 1$ has an imaginary solution.

(2) The set of eigenvalues is not be restricted to a finite interval.

Let $\{s_n\}$ be the solutions of $\check{W}W = 1$. We show that if the eigenvalues $\{\lambda_i\}$ are restricted to a finite interval, the solutions $\{s_{in}\}$ to $\lambda_i^2 = W(s)W(-s)$ are contained in a compact set in $\mathbb{C}$, which is impossible.

Regard $\check{W}W$ as a map from $j\mathbb{R}$ to $\mathbb{R}$. As such, it is continuous; in particular it is continuous on any closed interval of the imaginary axis. If the eigenvalues are restricted to a finite interval, they lie in the image of the closed interval under the continuous map $\check{W}W$, and therefore are contained in a compact set, clustering at the point 1. Now $\{s_{in}\}$ is the set of solutions to the polynomial equation $n(s)n(-s) - \lambda_i^2 d(s)d(-s) = 0$. The solutions are continuous functions of the parameter λ_i (with appropriate ordering of the solutions). As before, the set $\bigcup_i \{s_{in}\}$ must also be contained in a compact set in $\mathbb{C}$.

Now we argue that the $\bigcup_i \{s_{in}\}$ cannot be in a compact set: Every function in $\mathcal{L}^{-1}(K)$ must have a unique expansion as an infinite sum of orthogonal eigenfunctions of V^*V [Rudin 1973, Theorem 12.29(d)]. From the above $\{e^{s_{in}t}\}_{i,n}$ contain $\mathcal{L}^{-1}(K)$ in their span. In particular, $\mathcal{L}^{-1}(K)$ will contain $L^2(0,\Delta)$, and we would be able to represent functions in $L^2(0,\Delta)$ as $\sum a_i e^{\alpha_i t}$, where $|\alpha_i| < \gamma < \infty$ for some $\gamma > 0$. This is impossible, and we conclude that the eigenvalues of T^*T cannot lie on a finite interval.

C. The Indeterminate Case

If the sufficient condition in (A.) on $|W|$ does not hold, i.e., $|W(j\omega)| \to 1$ from below as $\omega \to \infty$, all we can say is that there will be a maximal vector if there is an eigenvalue greater than 1, using the same reasoning as above. Otherwise there may or may not be one.

If $|W(j\omega)| < 1$ for all $\omega \in \mathbb{R}$, then if the plant is stable, it is easy to see that the open loop system attains the minimal sensitivity. In this case, when the plant has no right half plane zeros, we saw in Chapters 3 (and 5, using an unproven conjecture), we can obtain an infimal sensitivity in the following way. Let λ_i^2 be a sequence of eigenvalues of T^*T approaching 1, and let f_i be the corresponding eigenfunctions. Form the quotient $\varphi_i = \frac{Tf_i}{f_i}$. It is easy to see that this will always have magnitude 1 on the imaginary axis. For the case in Chapter 3 section H we also saw that for all but finitely many eigenvalues it is stable. For this case it is possible to show that by evaluating the expression for φ_i at $\lambda = 1$ we get a stable function, and in fact φ_i is generally stable for λ_i close enough to 1.

When $|W(j\omega)| \geq 1$ and there is no maximal eigenvalue, we do not know whether this limiting procedure results in a stable function.

D. A False Conjecture.

At one time we thought that the condition $|W(j\omega)| > 1$ on some finite interval would be sufficient to guarantee the existence of an eigenvalue greater than 1, but this is not the case. We present the following argument to clarify this matter.

Suppose $|W(j\omega)| > 1$ for $\omega \in (a,b)$ and $|W(j\omega)| \leq 1$ otherwise. If this implied that there were an eigenvalue greater than 1, there would be an eigenvalue on the interval (a,b). Then there would be a function $f \in K$ with $\|f\|_2 = 1$ such that $\|Tf\|_2 > 1$. But

$$\|Tf\|_2 \leq \|Wf\|^2 = \int |W(j\omega)f(j\omega)|^2 d\omega.$$

Thus to have an eigenvalue greater than 1, it is necessary for $f(j\omega)$ to have its support sufficiently concentrated on (a,b). But $\mathcal{L}^{-1}(f) \in L^2(0,\Delta)$ (for example), and using any version of the "uncertainty relation" for transforms causes us to conclude that this support requirement will not be generally met.

CHAPTER 9

WELL-POSEDNESS

We stated in Chapter 1 that our investigations were partially motivated by the fact that although the input-output behaviour of linear systems is not generally continuous in the uniform operator topology, this is the topology resulting from the $\mathcal{H}^\infty$ norm on transfer functions.

The importance of understanding $\mathcal{H}^\infty$-minimal sensitivity design for delay systems is emphasized by the fact that designs which achieve $\mathcal{H}^\infty$-minimal sensitivity for finite dimensional systems are not generally continuous even in the strong operator topology when a small delay is added. We illustrate this fact with a simple example.

A. Example of Lack of Continuity

In [Zames and Francis 1983, p. 593] the solution to the $\mathcal{H}^\infty$-minimal sensitivity problem is computed for the case of a stable plant with two right half plane zeros. We consider this example here. Let

$$P_0 = \frac{(b_1-s)(b_2-s)}{(b_1+s)(b_2+s)} \cdot P_1 \tag{9.1}$$

with $\mathcal{R}e(b_i)>0$ and $b_1+b_2 \in \mathbb{R}$, be the plant, where P_1 is stable and minimum phase. Let

$$W(s) = \frac{s+1}{s+\beta}$$

with $\beta > 0$, be the sensitivity weighting function. The optimal "Q" parameter is

$$\overline{Q}(s) = \left[1 - \frac{D(c-s)}{W(s)(c+s)}\right] \cdot \frac{(b_1+s)(b_2+s)}{(b_1-s)(b_2-s)} \cdot P_1^{-1}$$

where D and c are constants determined by β and the b_is. D may be positive or negative, depending on whether $\beta < 1$ or $\beta > 1$. (We will not show it here, but $\beta < 1$ is the only interesting case.) We assume D is positive, for $\beta < 1$. Then the optimal feedback compensator is

$$\overline{C} = \overline{Q}(1-P_0\overline{Q})^{-1}.$$

Now suppose that the true plant is really

$$P_\epsilon(s) = e^{-\epsilon s}P_0(s) \qquad (9.2)$$

with $\epsilon > 0$.

1. Stability with Delay Added to Closed Loop.

We check stability for the closed loop system with true plant (9.2) and compensator designed for the plant (9.1) by computing $\frac{P_\epsilon}{1+P_\epsilon\overline{C}}$ and checking the location of its poles. A straightforward computation gives

$$\frac{P_\epsilon}{1+P_\epsilon C} = P_0 \cdot \frac{e^{s\epsilon}D(c-s)}{e^{s\epsilon}D(c-s) + [W(s)(c+s)-D(c-s)]}.$$

We now show that this is unstable by showing that the denominator, call it $m(s)$, in the above expression has infinitely many right half plane zeros.

For this we evaluate $m(s)$ on the imaginary axis and separate it into real and imaginary parts, $m(i\omega) = f(\omega) + i \cdot g(\omega)$. Now we show that $f(\omega)$ has only finitely many real zeros, and conclude by appealing to [Pontryagin 1955], Theorems 3 and 6,[11] that $m(s)$ has zeros in the right half plane.

A computation gives

$$f(\omega) = \omega^2 D[\cos(\epsilon\omega) - \frac{D+1}{D}] + \omega D(\beta-c)\sin(\epsilon\omega) + \beta c D[\cos(\epsilon\omega)-1] + c.$$

Since $D > 0$, the equation $f(\omega) = 0$ cannot be satisfied for sufficiently large ω, and we conclude that $f(\omega)$ has only finitely many real zeros. Therefore, $m(s)$ has zeros in the right half plane.

A simpler and more insightful, though informal, proof is as follows. As $s \to \infty$, $m(s) \to s(1+D-e^{s\epsilon})$. Therefore the zeros of $m(s)$ approach the vertical line $\mathcal{Re}(s) = \frac{1}{\epsilon}\ln(1+D)$, which is in the right half plane since $D > 0$. Furthermore, as $\epsilon \to 0$, the poles move infinitely far

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[11]For the content of these theorems, see footnote in Chapter 3.

into the right half plane.

2. Significance of the Example

In this simple example the $\mathcal{H}^\infty$-minimal sensitivity feedback system does not remain stable when a small delay is added to the feedback loop. This in itself is not different from the case of systems with a delay in the plant, and is not surprising inasmuch as we do not expect the input-output behaviour of the system to be continuous in the uniform operator topology.

What is significant is that the input-output behaviour is discontinuous in the strong operator topology (of linear operators on the space L^2_e)[12]. To see this, let G_0 be the time domain operator on L^2_e which is convolution with the inverse Laplace transform of $\frac{P}{1+PC}$, and let G_ϵ be the similar operator corresponding to $\frac{P_\epsilon}{1+P_\epsilon C}$. We note from the discussion of the asymptotic pole locations above, that for small ϵ, the closed loop operator $\frac{P_\epsilon}{1+P_\epsilon C}$ has a pole very close to $\frac{1}{\epsilon}\ln(1+D)$. As a result, for practically any fixed $x \in L^2_e$ and for any fixed $T > 0$, $\lim_{\epsilon\to 0} \|\mathcal{P}_T(G_\epsilon x - G_0 x)\| \neq 0$, since $G_\epsilon x$ "blows up" arbitrarily fast as $\epsilon \to 0$. ($\mathcal{P}_T$ is the truncation operator defined in the appendix.)

This means that there is something non-physical about the $\mathcal{H}^\infty$ sensitivity problem as commonly expressed. It is to avoid such problems with mathematical models that the concept of *well-posed* feedback system

[12] See appendix at end of this chapter.

is defined.

B. Definition of Well-posed

The definition of well-posed we use is from [Willems 1971]. There are four elements to the definition, which we repeat here in an informal manner:

A feedback system is *well-posed* if

WP 1. An input in L^2_e produces a unique solution in L^2_e.

WP 2. The system is causal.

WP 3. On finite time intervals, the outputs depend Lipschitz continuously on the outputs.

WP 4. The various closed loop transfer functions are continuous in the strong operator topology with respect to the addition of a small delay and with respect to parameter variations.

(See the [Willems 1971, pp. 90-91] for precise definitions.)

This definition in the context of feedback systems seems to be due to Zames (see [Zames 1964]). The continuity requirement of (WP 4) is not universally adopted. See [Vidyasagar 1980, p. 414], for example. It is (WP 4), however, that makes the feedback system resulting from the ideal solution to the $\mathcal{H}^\infty$ problem with a finite dimensional plant ill-posed.

It is therefore natural to ask why one would want to require continuity with respect to delay in the feedback loop. Part of the answer is that without (WP 4) the definition would allow certain non-physical examples to be considered well-posed systems. We next look at such an example.

C. Motivational Example

Perhaps the simplest example is mentioned in [Zames 1964, p. 186] as being due to Nyquist, and this analyzed further in [Willems 1971, pp. 95-96]. In the feedback system of Figure 9, algebra tells us that the

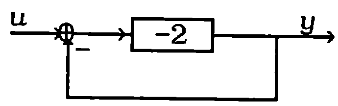

Figure 8. Ill-posed example.

input-output relation is given by $y(s) = 2 \cdot u(s)$. However this feedback system cannot be constructed physically since the inevitable delay present in the loop will cause instability which gets worse as the delay gets smaller. For practically no input on a finite interval will the output approximate the output from the system in Figure 8, as the delay goes to zero.

This is essentially the motivation for the continuity requirement. What goes wrong with the simple example of Figure 8 is essentially what goes wrong with the ideal $\mathcal{H}^{\infty}$-optimal finite dimensional system.

The ill-posedness for these feedback systems can be thought of as a result of the memoryless part of the opened feedback loop having gain which is too large. See [Willems 1971] p. 100.

A strictly proper approximation to the optimal compensator for an $\mathcal{H}^{\infty}$ problem for a rational transfer function plant will result in a well-posed system, since the memoryless part will have gain less than 1. Nonetheless, it is troublesome to use a design technique for which the goal is to approximate an ill-posed system.

D. Effect of Plant Time Delay.

For plants with a delay in the input the optimal feedback systems will be well-posed, using [Willems 1971] Corollary 4.1.1, precisely because of the delay in the plant. One consequence of this is that the mathematical model of our ideal closed loop system is continuous for small variations in plant parameters, including the delay time.

Unfortunately, it seems that the ideal $\mathcal{H}^{\infty}$-optimal feedback system will still be unstable for delay perturbations, even when the nominal plant contains a delay.[13] The consolation must be that since the system is well-posed, if the error in the delay is small, the real system we would build based on our model is a good approximation to the ideal system on finite time intervals.

E. Strict Properness

In much published work, well-posedness is taken to be equivalent to

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[13] The instability of the perturbed system can possibly be seen from the Nyquist plot. Let γ be the magnitude of the optimal sensitivity. As $\omega \to \infty$, the Nyquist plot approaches the circle about the point -1 with radius γ. When we introduce a pure delay into the loop, this circle revolves about the origin, and if $|\gamma| < 2$, the plot eventually circles the origin without encircling -1. This happens arbitrarily many times as $\omega \to \infty$. We conjecture that this "missing" of encirclements implies instability in the same way as the count of encirclements does for finitely many right half plane poles of the loop transfer function. This argument is not complete, because the underlying Nyquist stability result for a function with infinitely many encirclements has not been demonstrated.

the invertibility of (1+PC). As we saw in section (C), this is not the case for our definition of well-posedness. However, when this condition is augmented by the assumption that the loop transfer function is strictly proper, (WP 4) is then satisfied. It is for this reason that the mention of well-posedness is commonly correct, given the strict properness assumption (usually made separately).

We hasten to point out that well-posedness does not require a strictly proper loop transfer function. Thus the ideal $\mathcal{H}^{\infty}$ closed loop for the minimum weighted sensitivity criterion is well-posed when the plant model has a delay in the input, as is any finite dimensional system with loop transfer function having memoryless part with magnitude less than 1.

F. Alternate Optimality Criteria

It is know that the introduction of other criteria in the $\mathcal{H}^{\infty}$ minimization set-up can result in control of the high frequency behaviour of the closed loop. This can serve to make the optimal compensator proper, and therefore give non-zero delay margin. This is an area for future work.

CHAPTER 9 Appendix

"Extended" L^2 Spaces and Induced Topologies

We stated in Chapter 2 section A that $\mathcal{H}^\infty$ is the space of transfer functions of linear systems which in the time domain are L^2-stable, causal and time-invariant, and that the $\mathcal{H}^\infty$-norm is the induced L^2 norm. The topology on $\mathcal{H}^\infty$ arising from this norm is the uniform operator topology. For an explanation of the various operator topologies see [Dunford and Schwartz 1958, pp. 475-477].

In the definition of well-posedness one would like to allow unstable as well as stable systems, so it is not sufficient to consider systems whose transfer functions lie in $\mathcal{H}^\infty$. To handle the case of unstable systems and their interconnections, we must allow inputs and outputs to be unbounded functions of time. For this purpose one defines the "extended" L^2 space L^2_e as follows. Let X be the space of real-valued functions on $(0,\infty)$. The operator $\mathcal{P}_T$ on X is defined for each T > 0 by

$$\mathcal{P}_T(f) = \begin{cases} f(t) & \text{for } t \in (0,T] \\ 0 & \text{for } t \in (T,\infty). \end{cases}$$

L^2_e is the subspace of X consisting of functions f for which $\mathcal{P}_T(f) \in L^2(0,\infty)$ for all T > 0. (See [Zames 1966, p. 230].) Let $\mathcal{O}$ denote the set of causal linear time-invariant operators on L^2_e. We define a topology on $\mathcal{O}$ with the base

$$n_\epsilon(F) = \{G \in \mathcal{O}: (F-G) \in \mathcal{H}^\infty \text{ and } \|F-G\|_\infty < \epsilon\}.$$

We call this the *uniform operator topology* of operators on L^2_e since it is induced by the "extended norm" on L^2_e defined by

$$\|f\|_{2_e} = \begin{cases} \|f\|_2 & \text{if } f \in L^2(0,\infty) \\ \infty & \text{otherwise.} \end{cases}$$

Thus F_λ approximates F in the uniform operator topology if $\|\mathcal{P}_T \circ (F-F_\lambda)\|_\infty$ is small for all $T \geq 0$. (See [Willems 1971, p. 93].)

Similarly, we define the *strong operator topology* so that F_λ approximates F if for all $x \in L^2_e$, $\|\mathcal{P}_T((F-F_\lambda)(x))\|_2$ is small for all $T \geq 0$. A base for this topology is

$$m_\epsilon(F) = \{G \in \mathcal{O}: \text{for all } x \in L^2_e,\ (F-G)(x) \in L^2(0,\infty) \text{ and } \|(F-G)(x)\|_2 < \epsilon\}.$$

CHAPTER 10

CONCLUSIONS AND FUTURE WORK

For certain choices of weighting functions we have found explicit expressions for the optimal weighted sensitivity and feedback compensator when our plant design model is a stable minimum phase rational transfer function in cascade with a delay. These weighting functions are: (Weighting functions are assumed normalized to 1 at ∞.) those whose magnitude approaches 1 from above at high frequency, and those whose magnitude does not exceed 1 (high pass functions). For the first case the optimal sensitivity is an infinite Blaschke product. The optimal compensator is unstable, and improper when the plant is strictly proper. It also contains an ideal delay. Computation of the solution involves the solution of simultaneous nonlinear equations. For the case of high pass weighting functions, we have found two minimum sensitivities, the weighting function itself and an all-pass function (the latter results from a conjecture which remains to be proven).

The completeness of our results is marred by the fact that the most general weighting functions we might wish to consider do not allow us to guarantee that we can compute the optimal sensitivity when there is no maximum eigenvalue. This is not a major limitation, however, since we can always slightly modify the weighting function so as to guarantee the existence of an infimal sensitivity we can compute. Future investigation will examine whether it is generally possible to find an optimal weighted sensitivity for the no maximal eigenvalue case by taking the limit of the expression (5.24) as $\lambda \to 1$.

We have shown how to approximate the compensator with a finite dimensional compensator, in such a manner as to achieve a closed loop sensitivity close to the infimal one. The lead characteristic of the compensator must be very gradually tapered off, until the loop gain is small enough, in order to preserve approximation to the optimal sensitivity and preserve stability.

We have indicated how to extend these results to systems with right half plane poles and zeros, having performed the crucial eigenvalue/eigenvector computation for these cases. Future work will derive explicit expressions for the optimal sensitivity and compensator using these results.

We provided an example which points out that the limiting $\mathcal{H}^{\infty}$ optimal compensator for finite dimensional plants produces an ill-posed closed loop system. The limiting compensator for a plant with a delay in the nominal model, on the other hand, produces a well-posed feedback system. We saw the loop to be unstable with the addition of an arbitrarily small delay for either case.

A goal of future work will be to understand if these results can be used to make compensator design robust with respect to the location of right half plane zeros. We believe that by designing in the presence of a delay, which yields unbounded (and possibly fictitious) excess phase, we may be able to design into a compensated system tolerance for unmodelled right half plane zeros. Along with this, we expect to pursue a notion of "delay margin" for feedback systems, which should give us insight into how to include delays and approximations of delays in models of feedback systems.

It will be necessary to consider other criteria for optimality, since the sensitivity criterion by itself has led to systems with zero delay margin.

Future work will also cover the extension of these designs to the case of systems with a delay in the state, and possibly other infinite dimensional systems.

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Symbol	Definition
$\mathbb{C}$	the complex numbers
$\mathbb{C}^+$	the complex numbers with positive real part (the right half plane)
$\mathbb{D}$	the unit circle
δ_t	the Dirac distribution with mass at t=0 (unit impulse)
$\mathcal{H}^p(\cdot)$	$\mathcal{H}^p(\Omega)$ is the p-normed Hardy space of functions analytic in the region Ω
$\mathcal{H}^\infty(\cdot)$	$\mathcal{H}^\infty(\Omega)$ is the supremum-normed Hardy space of functions analytic in the region Ω
$\mathcal{I}m(\cdot)$	$\mathcal{I}m(z)$ is the imaginary part of the complex number z
j	$\sqrt{-1}$
$L^2(\cdot)$	$L^2(0,\Delta)$ is the quadratic-normed Lesbesgue space of functions with support on the interval $(0,\Delta)$
$L^2_e(0,\infty)$	the "extended" L^2-space with support on $(0,\infty)$
$\mathcal{L}$	Laplace transform
$\mathcal{L}^{-1}$	inverse Laplace transform
$\mathcal{P}_T$	the projection operator onto functions with support on (0,T)
Π^+	complex numbers with positive imaginary part (the upper half plane)
Π_K	the projection operator onto the space K
$\mathbb{R}$	the real numbers
$\mathcal{R}e(\cdot)$	$\mathcal{R}e(z)$ is the real part of the complex number z
$\rho(\cdot)$	ρ(T) is the spectral radius of the operator T
$\sigma(\cdot)$	σ(T) is the spectrum of the operator T

sp(•)	sp($\{x_i\}$) is the closed linear span of the set $\{x_i\}$
supp(•)	supp(f) is the support of the function f
T	the unit circle
u(t)	the unit step function (Heaviside function)
• $*$ •	$f * g$ is the convolution of the functions f and g
$\cdot^*$	V^* is the adjoint of the operator V
$\bar{\cdot}$	$\bar{z}$ is the complex conjugate of the complex number z
$\cdot^\perp$	$S^\perp$ is the orthogonal complement of the space S
• $\ominus$ •	$\mathcal{H}^2 \ominus S$ is the orthogonal complement of the subspace S in $\mathcal{H}^2$ (for clarity — the complement *could* be in L^2)
$\Vert\cdot\Vert$	$\Vert V \Vert$ is the norm of the operator V
$\cdot\vert_\cdot$	$X\vert_K$ is the restriction of the operator X to domain K
$\hat{\cdot}$	$\hat{f}$ is the Laplace transform of the function f
$\check{\cdot}$	$\check{W}(s) = W(-s)$
•	multiplication

Notational conventions:

For conciseness the arguments of functions are sometimes omitted. In these cases, the following conventions hold unless an explicit exception is noted: Capital roman letters denote operators or functions in the frequency domain. Capital italic letters denote spaces. Lower case roman letters denote polynomials. Lower case italic letters denote functions in the time domain.

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