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THE INFINITE TIME QUADRATIC COST PROBLEM FOR CERTAIN CLASSES OF INFINITE DIMENSIONAL CONTROL SYSTEMS

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Introduction

The objective of this paper is to study the infinite-time quadratic cost problem for a class of infinite dimensional systems. In undertaking this study we insist on using an approach which clarifies the relationship between the system-theoretic issues of controllability (reachability), stabilizability, L^2-stability and the existence of a positive solution to an operator equation of quadratic type.

Problem Formulation

Let H and U be Hilbert spaces. Let $L^2(0,T;U)$ denote the space ~~equivalence~~ class of all square integrable functions defined on $[0,T]$. Let $\mathcal{L}(U,H)$ denote the space of continuous linear transformations from U into H.

Consider the linear differential system

$$
(1) \qquad \begin{cases} \dfrac{dx}{dt} = Ax(t) + Bv(t) \\ x(0) = x_0 \end{cases}
$$

where $v \in L^2(0,T;U)$, $A: D(A) \to H$ is a closed linear operator with dense domain which is the infinitesimal generator of a one-parameter strongly continuous semi-group $\Phi(t)$, $B \in \mathcal{L}(U,H)$ and $x_0 \in D(A)$.

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We are interested in the following optimal control problems:

A. Find $u \in L^2(0,T;U)$ which minimizes

$$J(v,x) = \int_0^T [(x(t),Qx(t))_H + (v(t),Nv(t))_U]dt \tag{2}$$

where $Q \in \mathcal{L}(H,H)$ is a symmetric positive operator and $N \in \mathcal{L}(U,U)$ is a symmetric operator satisfying $(Nv,v)_U \geq \alpha \|v\|_U^2$, $\alpha > 0$ $\forall v \in U$. One is however interested in obtaining the optimal control in feedback form, that is in the form $u(t) = \varphi(t,x(t))$.

B. Study the same problem on an infinite time interval.

Examples of Systems Covered by the Problem Formulation and Summary of Existing Results

Suppose H and U are finite-dimensional spaces. It is then well known that for the finite-time problem an optimal control exists, is unique and is characterized by

$$u(t) = -N^{-1}B^*\pi(t)x(t;u) \tag{3}$$

where $B^* \in \mathcal{L}(H^*,U^*)$ and $\pi(t)$ satisfies the following ordinary differential equation of Riccati type:

$$\begin{cases} \dfrac{d\pi}{dt} = A^*\pi(t) + \pi(t)A - \pi(t)BN^{-1}B^*\pi(t) + Q \\ \pi(T) = 0. \end{cases} \tag{4}$$

A global solution to equation (4) can be shown to exist. To study the infinite-time problem we first have to ensure that there exists at least one control for which the cost is finite. The concept of controllability plays a key role here.

A system is said to be *controllable* if any initial state can be brought to the origin in a finite time interval $[0,T]$ using a control $u \in L^2(0,T;U)$. It is well known that a system is controllable $\Leftrightarrow$ rank $[B \;\; AB \;\; \ldots \;\; A^{n-1}B] = n$, the dimension of H.

A system is said to be *stabilizable* if there exists a $K \in \mathcal{L}(H,U)$ such that the eigenvalues of $(A + BK)$ have negative real parts. This implies that the differential system

$$\frac{dx}{dt} = (A + BK)x(t) \tag{5}$$

is asymptotically stable.

There is an important theorem which states that controllability $\Rightarrow$ stabilizability.

If one makes the assumption that the system is controllable and $(A,Q^{\frac{1}{2}})$ is an observable pair, that is, rank $[P^* \; A^*P^* \; (A^*)^{n-1}P^*] = n$, where $P = Q^{\frac{1}{2}}$, then it is well known that the infinite time problem has a unique solution and the optimal control is given by

$$u(t) = -N^{-1}B^*\pi x(t;u), \tag{6}$$

where π is the unique, positive definite solution of the matrix quadratic equation

$$A^*\pi + \pi A - \pi BN^{-1}B^*\pi + Q = 0. \tag{7}$$

Moreover the system

$$\frac{dx}{dt} = (A - BN^{-1}B^*\pi)x(t) \tag{8}$$

is asymptotically stable.

Suppose now that we are in a parabolic equation situation and $-A$ is a coercive operator. A typical example is the heat equation. This case has been extensively studied by Lions. The results here are analogous to the finite dimensional case with the optimal control being given by (3) and (4). Equation (4) must now however be interpreted appropriately since A is an unbounded operator. The infinite time problem has also been studied by Lions. The crucial point here is that parabolic equations are essentially stable, that is for $v \in L^2(0,\infty;U)$, the solution $x(v) \in L^2(0,\infty;H)$. Hence a theorem of the type controllability $\Rightarrow$ stabilizability is not required and the infinite time problem can be solved by studying limiting versions of the finite-time problem.

Suppose now that the operator $-A$ is maximal dissipative (in the sense of Phillips) and hence the infinitesimal generator of a contraction semi-group. This situation would arise if the wave equation were to be considered in an abstract setting. The finite-time quadratic cost optimal control problem for this class of systems has been solved by Lions. The stabilizability problem for this class of systems is somewhat open although some progress has recently been made by Slemrod. As a result the infinite-time problem is still an open problem.

Optimal Control of Linear Hereditary Differential Systems with a Quadratic Cost: The Finite-Time Case

We now show how hereditary differential systems fit in the general model (1).

Notation and Basic Definitions. Let $N \geqslant 1$ be an integer, let $a > 0$, $0 = \theta_0 > \theta_1 > \quad > \theta_N = -a$ be real numbers and $b \in [a,\infty]$. Let $I(\alpha,\beta) = R \cap [\alpha,\beta]$ for any $\alpha < \beta$ in $[-\infty,\infty]$. Let $|\ |_H$ (resp. $|\ |_U$) and $(\ |\ )_H$ (resp. $(\ |\ )_U$) denote the norm and inner products on H (resp. U).

Space of Initial Data and Space of Solutions. Consider the space $\mathcal{L}^2(-b,0;H)$ (not to be confused with $L^2(-b,0;H)$) of all maps $I(-b,0) \to H$ which are square integrable in $I(-b,0)$ endowed with the seminorm

$$\|y\|_{M^2} = \left[|y(0)|_H^2 + \int_{-b}^{0} |y(\theta)|_H^2 \, d\theta\right]^{1/2}$$

The quotient space of $\mathcal{L}^2(-b,0;H)$ by the linear subspace of all y such that $\|y\|_{M^2} = 0$ is a Hilbert space which is isometrically isomorphic to the product space $H \times L^2(-b,0;H)$. It will be denoted by $M^2(-b,0;H)$ and the isomorphism between $H \times L^2$ and $M^2(-b,0;X)$ is denoted by K.

In order to discuss the Cauchy problem we must also describe the space in which solutions will be sought. Let $1 \leqslant p < \infty$, $t_0 \in R$. For all $t \in]t_0, \infty[$ we denote by $AC^p(t_0,t;H)$ the vector space of all absolutely continuous maps $[t_0,t] \to H$ with a derivative in $L^p(t_0,t;H)$. When $AC^p(t_0,t;H)$ is endowed with the norm

$$\|x\|_{AC^p} = \left[|x(t_0)|_H^p + \int_{t_0}^{t} \left|\frac{dx}{ds}(s)\right|_H^p ds\right]^{1/p},$$

it is a Banach space isometrically isomorphic to $H \times L^p(t_0,t;H)$. In particular, $AC^2(t_0,t;H)$ is a Hilbert space. We shall also need $C(t_0,t;H)$, the Banach space of all continuous maps $[t_0,t] \mapsto H$ endowed with the sup norm $\| \ \|_C$.

When we consider the evolution of a system in an infinite-time interval it is useful and quite natural to introduce the following spaces. Let $\pi_t(x)$ be the restriction of the map $x:[t_0,\infty[\to H$ to the interval $[t_0,t]$, $t \in]t_0,\infty[$. Denote by $L^p_{loc}(t_0,\infty;H)$, $AC^p_{loc}(t_0,\infty;H)$ and $C_{loc}(t_0,\infty;H)$ the vector space of all maps $x:[t_0,\infty[\to H$ such that for all $t \in]t_0,\infty[, \pi_t(x)$ is in $L^p(t_0,t;H)$, $AC^p(t_0,t;H)$ and $C(t_0,t;H)$, respectively. They are Frechet spaces (cf. Delfour) when their respective topologies are defined by the saturated family of seminorms $q_t(x) = \|\pi_t(x)\|_F$, $t \in]t_0,\infty[$, where F is either L^p, AC^p or C.

System Description. Consider the affine hereditary differential system defined on $[0,\infty[$:

$$(9) \quad \frac{dx}{dt}(t) = A_{00}(t)x(t) + \sum_{i=1}^{N} A_i(t) \begin{Bmatrix} x(t+\theta_i), & t+\theta_i \geq 0 \\ h(t+\theta_i), & t+\theta_i < 0 \end{Bmatrix}$$

$$+ \int_{-b}^{0} A_{01}(t,\theta) \begin{Bmatrix} x(t+\theta), & t+\theta \geq 0 \\ h(t+\theta), & t+\theta < 0 \end{Bmatrix} d\theta$$

$$+ B(t)v(t) + f(t) \quad \text{a.e. in } [0,\infty),$$

$$x(0) = h(0), \quad h \in M^2(-b,0;H),$$

where A_{00} and $A_i (i = 1,2,\ldots,N)$ are in $L^\infty_{loc}(0,\infty;\mathcal{L}(H))$, $A_{01} \in L^\infty_{loc}(0,\infty;-b,0;\mathcal{L}(h))$, $B \in L^\infty_{loc}(0,\infty;\mathcal{L}(U,H))$, $v \in L^2_{loc}(0,\infty;U)$ and $f \in L^2_{loc}(0,\infty;H)$.

v is to be thought of as the control to be applied to the system and f is a known external input to the system. Under the above hypotheses, (9) has a unique solution $\varphi(\cdot;h,v)$ in $AC^2_{loc}(0,\infty;H)$ and the map

$$(10) \quad (h,v) \mapsto \varphi(\cdot;h,v): M^2(-b,0;H) \times L^2_{loc}(0,\infty;U) \to AC^2_{loc}(0,\infty;H)$$

is affine and continuous (cf. Delfour and Mitter [1], [2] and Delfour [1]). We also have the variation of constants formula

$$(11)\quad \varphi(t;h,v) = \Phi(t,0)h + \int_0^t \Phi^0(t,s)B(s)v(s)ds + \int_0^t \Phi^0(t,s)f(s)ds,$$

where

$$\Phi(t,s)h = \Phi^0(t,s)h(0) + \int_{-b}^0 \Phi^1(t,s,\alpha)h(\alpha)d\alpha,$$

and $\Phi^0(t,s) \in \mathcal{L}(H)$ is the unique solution in $AC^2_{loc}(s,\infty;\mathcal{L}(H))$ of the system

$$(12)\quad \frac{\partial \Phi^0}{\partial t}(t,s) = A_{00}(t)\Phi^0(t,s) + \sum_{i=1}^N A_i(t)\begin{cases}\Phi^0(t+\theta_i,s), & t+\theta_i \geq s\\ 0, & \text{otherwise}\end{cases}$$

$$+ \int_{-b}^0 A_{01}(t,\theta)\begin{cases}\Phi^0(t+\theta,s), & t+\theta \geq s\\ 0, & \text{otherwise}\end{cases} d\theta$$

$$\text{a.e. in } [s,\infty[$$

and

$$(13)\quad \Phi^1(t,s,\alpha) = \sum_{i=1}^N \begin{cases}\Phi^0(t,s+\alpha-\theta_i)A_i(s+\alpha-\theta_i), & \alpha+s-t<\theta_i\leq\alpha\\ 0, & \text{otherwise}\end{cases}$$

$$\begin{cases}\int_{-b}^{\alpha}\Phi^0(t,s+\alpha-\theta)A_{01}(s+\alpha-\theta,\theta)d\theta, & s+\alpha\leq t-b\\ \int_{\alpha-t+s}^{\alpha}\Phi^0(t,s+\alpha-\theta)A_{01}(s+\alpha-\theta,\theta)d\theta, & s+\alpha>t-b\end{cases}$$

State Equation of the System.

Definition 1. Let $f = 0$, $v = 0$ in (9). The evolution of the state of the homogeneous system is given by the map

$$(14)\quad t \to \tilde{\varphi}(t;h): [0,\infty[\to M^2(-b,0;H)$$

defined as

$$(15)\qquad \tilde{\varphi}(t;h)(\theta) = \begin{cases} \varphi(t+\theta;h), & t+\theta \geq 0, \\ h(t+\theta), & t+\theta < 0. \end{cases}$$

It is easy to verify the following theorem.

Theorem 1. *Consider* (9) *with* $f = 0$, $v = 0$ *on* $[s,\infty[$ *with initial datum* h *at time* s. *Let* $\tilde{\varphi}_s(t;h)$ *denote the solution of this system in* $AC^2_{loc}(s,\infty;H)$. *The map* $(t,s) \mapsto \tilde{\varphi}_s(t;h)$ *generates a two-parameter semigroup* $\tilde{\Phi}(t,s)$ *satisfying the following properties:*

(i) $\tilde{\Phi}(t,s) \in \mathcal{L}(M^2)$, $t \geq s \geq 0$;

(ii) $\tilde{\Phi}(t,r) = \tilde{\Phi}(t,s)\tilde{\Phi}(s,r)$, $t \geq s \geq r \geq 0$;

(iii) $t \mapsto \tilde{\Phi}(t,s)h: [s,\infty[\to M^2$ *is continuous for all* $h \in M^2$ *and* $s \in [0,\infty[$;

(iv) $\tilde{\Phi}(s,s) = I$, *where* I *is the identity operator in* $\mathcal{L}(M^2)$;

(v) *for* $t - s \geq b$, $\tilde{\Phi}(t,s): M^2 \to M^2$ *is compact (i.e., maps bounded sets into relatively compact sets);*

(vi) *Let* $\mathcal{D} = AC^2(-b,0;H) \cap M^2(-b,0;H)$. *Then for all* $h \in \mathcal{D}$, $\tilde{\Phi}(t,s)h \in \mathcal{D}$.

Since M^2 is isomorphic to $H \times L^2(-b,0;H)$, $\tilde{\varphi}(t,s)$ can be decomposed into two operators $\tilde{\varphi}^0(t,s) \in \mathcal{L}(H,M^2)$ and $\tilde{\varphi}^1(t,s) \in \mathcal{L}(L^2(-b,0;H),M^2)$ such that

$$\tilde{\Phi}(t,s)h = \tilde{\Phi}^0(t,s)h^0 + \tilde{\Phi}^0(t,s)h^1,$$

where

$$(16)\qquad [\tilde{\Phi}^0(t,s)h^0](\alpha) = \begin{cases} \Phi^0(t+\alpha,s)h^0, & t+\alpha \geq s, \\ 0, & t+\alpha < s, \end{cases}$$

and

$$(17)\qquad [\tilde{\Phi}^1(t,s)h^1](\alpha) = \begin{cases} \int_{-b}^{0} \Phi^1(t+\alpha,s,\eta)h^1(\eta)\,d\eta, & t+\alpha \geq s, \\ h^1(t+\alpha-s), & t+\alpha < s. \end{cases}$$

Finally corresponding to (9) we have the *state equation* in *integral form*

$$(18)\quad \tilde{\varphi}(t;h,v) = \tilde{\Phi}(t,0)h + \int_0^t \tilde{\Phi}^0(t,s)B(s)v(s)ds + \int_0^t \tilde{\Phi}^0(t,s)f(s)ds.$$

We now wish to obtain the state equation in differential form. We first construct an unbounded operator $\tilde{A}(t)$ whose domain is

$$= AC^2(-b,0;H) \cap M^2(-b,0;H).$$

For this purpose define the linear maps

$$\tilde{A}^0(t)\colon \mathcal{D} \to H \quad \text{and} \quad \tilde{A}^1\colon \mathcal{D} \to L^2(-b,0;H)$$

as follows

$$(19)\quad \tilde{A}^0(t)h = A_{00}(t)h(0) + \sum_{i=1}^{N} A_i(t)h(\theta_i) + \int_{-b}^{0} A_{01}(t,\theta)h(\theta)d\theta$$

and

$$(20)\qquad (\tilde{A}^1 h)(\theta) = \frac{dh(\theta)}{d\theta}.$$

From the operators $\tilde{A}^0(t)$ and $\tilde{A}^1$ we construct the unbounded operator $\tilde{A}(t)\colon \mathcal{D} \to M^2(-b,0;H)$ as

$$(21)\qquad [\tilde{A}(t)h](\alpha) = \begin{cases} \tilde{A}^0(t)h, & \alpha = 0, \\ [\tilde{A}^1 h](\alpha), & \alpha \neq 0. \end{cases}$$

Define also the operator $\tilde{B}(t)\colon U \to M^2(-b,0;H)$ as

$$(22)\qquad [\tilde{B}(t)u](\alpha) = \begin{cases} B(t)u, & \alpha = 0, \\ 0, & \text{otherwise}, \end{cases}$$

and $\tilde{f}(t) \in M^2(-b,0;H)$ as

$$(23)\qquad [\tilde{f}(t)](\alpha) = \begin{cases} f(t), & \alpha = 0, \\ 0, & \text{otherwise}. \end{cases}$$

We then have the following theorem.

Theorem 2. (i) *For all* $h \in \mathcal{D}$ *and all* $u \in L^2_{loc}(0, ;U)$, *the system*

$$\frac{dy}{dt}(t) = \tilde{A}(t)y(t) + \tilde{B}(t)u(t) + \tilde{f}(t) \quad \text{a.e. in } [0,\infty[, \tag{24}$$

$$y(0) = h$$

has a unique solution in $AC^2_{loc}(0,\infty;M^2)$ *which coincides with* $\tilde{\varphi}(\cdot;h,u)$.

(ii) *The map* $(h,u) \to \Lambda(h,u) = \tilde{\varphi}(\cdot;h,u): \mathcal{D} \times L^2_{loc}(0, ;U) \to AC^2_{loc}(0,\infty;M^2)$ *can be lifted to a unique continuous affine map:* $\tilde{\Lambda}: M^2 \times L^2_{loc}(0,\infty;U) \to C_{loc}(0,\infty;M^2)$ *and for all pairs* (h,u), $\tilde{\Lambda}(h,u)$ *coincides with* $\tilde{\varphi}(\cdot;h,u)$.

Formulation of Optimal Control Problem. Consider the controlled system (9). We fix the final time $T \in]0,\infty[$ and consider the solution of (9) in the interval $[0,T]$. We also consider f to be given. The solution in $[0,T]$ corresponding to $h \in M^2(-b,0;H)$ and $v \in L^2(0,T;U)$ is denoted by $x(\cdot;h,v)$. We associate with v and h the cost function $J(v,h)$ given by

$$\begin{aligned} J(v,h) = {} & (x(T;h,v)|Fx(T;h,v)) \\ & + \int_0^T [(x(s;h,v)|Q(s)x(s;h,v)) + (v(s)|N(s)v(s))]ds \\ & + 2\int_0^T (v(s)|m(s))ds + 2\int_0^T (x(s;h,v)|g(s))ds, \end{aligned} \tag{25}$$

where $g \in L^2(0,T;H)$, $m \in L^2(0,t;U)$, $F \in \mathcal{L}(H)$, $Q \in L^\infty(0,T;\mathcal{L}(H))$, $N \in L^\infty(0,T;\mathcal{L}(U))$, F, $Q(s)$ and $N(s)$ are positive symmetric transformations and there exists a constant $c > 0$ wuch that $(y|N(s)y) \geq c\|y\|^2_U$ for all s in $[0,T]$.

Existence and Characterization of Optimal Controls. For each h, it can be easily shown that there exists a unique control u which minimizes $J(v,h)$ over all v in $L^2(0,T;U)$. In Delfour-Mitter [3] the following results are proved:

Consider the optimal control problem on the interval $[t,T]$, where $t \in [0,T[$ with initial datum h. Assume $f = 0$, $m = 0$, $g = 0$. Then the minimum value of the cost function can be uniquely characterized by a positive symmetric operator $\pi_T(t) \in \mathcal{L}(M^2)$.

(26) $$(h,\pi_T(t)h) = \min_{v \in L^2(0,T;U)} J(v,h).$$

Moreover the optimal control is given by

(27) $$u_h(s) = -N^{-1} B^* \pi_T^0(s)\tilde{x}_h(s),$$

where $\tilde{x}_h(s)$ is the solution of

(28) $$\frac{d\tilde{x}}{ds} = \widetilde{A}\tilde{x}(s) + \widetilde{B}u(s)$$

$$\tilde{x}(t) = h, \quad \text{on } [t,T]$$

and

(29) $$\pi_T^0(s)\tilde{x}_h(s) = [\pi_T(s)\tilde{x}_h(s)]^0.$$

$\pi_T(s)$ satisfies an operational differential equation of Riccati type which has a global solution.

The Infinite Time Problem

Assume $\widetilde{A}(t) = \widetilde{A}$, $\widetilde{B}(t) = \widetilde{B}$. As we have discussed before, the first question to be settled for the infinite time problem is the question of stabilizability.

Stabilizability. We first need two definitions.

Definition 2. The system

(30) $$\frac{d\tilde{z}}{dt} = \widetilde{A}z(t) \quad \text{on } [0,\infty[,$$
$$\tilde{z}(0) = h \in M^2,$$

where $\widetilde{A}$ is given by (21) is said to be L^2-*stable* if $\forall h \in M^2$

(31) $$\lim_{\to\infty}\int_0^t (\tilde{z}_h(s)|\tilde{z}_h(s))_{M^2}\,ds < \infty.$$

Definition 3. The control system (24) in state-equation form is said to be *stabilizable* if there exists a $G \in \mathcal{L}(M^2(-b,0;H),U)$ such that $\tilde{A} + \tilde{B}G$ defines an L^2-stable system.

Using the spectral properties of $\tilde{A}$ (cf. Hale), necessary and sufficient conditions for stabilizability can be given (cf. Vandevenne). For example, a scalar system with one time delay whose control coefficient is non-zero is stabilizable.

Main Theorems on Optimal Control for Infinite Time Problem.

Theorem 2. *Assume that the pair* $(\tilde{A},\tilde{B})$ *is stabilizable. Then the following statements are true:*

(32) $$\lim_{t<T\to\infty} \pi_T(t) = \pi \ \forall\ t \geq 0.$$

For $\forall h \in M^2$

(33) $$(\pi h,h) = \int_0^\infty ([\tilde{Q} + \pi\tilde{R}\pi]\tilde{x}(s)|\tilde{x}(s))ds,$$

$$\dot{\tilde{x}}(t) = [\tilde{A} - \tilde{R}\pi]\tilde{x}(t), \quad t \in [0,\infty[$$

where

$$\tilde{x}(0) = h,$$

$$R = BN^{-1}B^*, \quad (\tilde{R}h)(\theta) = \begin{cases} Rh(0), & \theta = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$(\tilde{Q}h)(\theta) = \begin{cases} Qh(0), & \theta = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Consider the control law

(34) $$u(t) = N^{-1}B^*\pi^0\tilde{x}_h(t), \quad \textit{where}$$

$\pi^0 h = (\pi h)(0)$ *and* $\tilde{x}_h(t)$ *is the solution of*

$$\frac{d\tilde{x}}{dt} = \tilde{A}\tilde{x}(t) + \tilde{B}u(t)$$

$$\tilde{x}(0) = h.$$

Then the control law (34) *minimizes*

$$J(v,h) = \lim_{T\to\infty} \int_0^T \{(x(t)|Qx(t)) + (v(t)|Nv(t))\}dt \tag{35}$$

over all $v \in L^2_{loc}(0,\infty;U)$ *such that* $J(v,h) \leq C|h|^2_{M^2}$.

Theorem 3. *Let* $Q > 0$. *The pair* $(\tilde{A},\tilde{B})$ *is stabilizable if and only if there exists* $\pi \geq 0$, *symmetric in* $\mathcal{L}(M^2)$ *which is a solution to*

$$\tilde{A}^*\pi + \pi\tilde{A} - \pi\tilde{R}\pi + \tilde{Q} = 0. \tag{36}$$

If a positive solution of (36) *exists, it is unique and given by* (32). *Moreover the operator* $\tilde{A} - \tilde{R}\pi$ *is* L^2*-stable.*

Remark. In the above equation (36) has to be interpreted in an appropriate weak sense since $\tilde{A}$ is an unbounded operator.

The proofs of Theorems 2 and 3 may be found in Delfour-Mitter [4].

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