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# Approximation of Ideal Compensators for Delay Systems

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We present, for the single–input/single–output $\mathscr{H}^\infty$ minimal sensitivity problem, proofs for the construction of a sequence of proper, finite dimensional compensators for which the closed loop sensitivity function approaches the generally unachievable infimal value.

## I. INTRODUCTION AND PROBLEM FORMULATION

In Flamm and Mitter [1] we announced results on the single–input/single output $\mathscr{H}^\infty$ weighted sensitivity minimization control problem formulated in Zames [2], but with transfer functions of the form $P(s) = e^{-s\Delta}P_0(s)$, where $P_0(s)$ is a minimum phase and stable rational function, and $\Delta > 0$. In this paper we present proofs for the results relating to realization of the computed ideal compensators. Although we refer specifically to plants with delays because we have calculated the explicit compensators, our approximation procedure applies to the case of plants with general inner factors.

The block diagram in Figure 1 shows the feedback system models we are considering. The closed loop sensitivity $S(s)$ is the transfer function from $d(s)$ to $y(s)$. The weighted sensitivity $X(s)$ for the weighting function $W(s)$ is given by $X(s) = W(s)S(s) = W(s)[1+P(s)C(s)]^{-1}$. The problem is to minimize the $\mathscr{H}^\infty$ norm of $X(s)$ over all stabilizing proper feedbacks $C(s)$, that is, to solve


Fig. 1. Feedback System Considered

$$\inf_{C(s)} \|W(s)[1+P(s)C(s)]^{-1}\|_\infty \tag{1.1}$$

where $C(s)$ ranges over all proper compensators for which the feedback system in Figure 1 is internally stable.

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As in the rational plant case, we restrict our attention to the modified problem

$$\inf_{H \in \mathcal{H}^\infty} \|W(s) - e^{-s\Delta}H(s)\|_\infty \tag{1.2}$$

since the infimum in (1.1) is generally not attainable when $P(s)$ is strictly proper, and the solution to (1.2) is not greater than that of (1.1). In this paper we shall use a compensator arising from a solution to (1.2) to find a sequence of compensators for which the closed loop weighted sensitivity approaches the infimum in (1.1). As in Flamm and Mitter [1], we assume the weighting function is given by $W(s) = (s+1)/(s+\beta)$ with $0 < \beta$.

For later convenience we define the "$Q$-parameter" as $Q = C/(1+PC)$, and we note that $C = Q/(1-PQ)$. For stable plants $P$, there is a one-to-one correspondence between stable proper $Q$ (*i.e.*, $Q \in \mathcal{H}^\infty$) and proper $C$ for which the system in Figure 1 is internally stable. See Zames [2]. Note that in (1.2) $Q$ may be identified as $P_0(s)^{-1}H(s)$.

## II. RESULTS FROM FLAMM AND MITTER [1] AND OVERVIEW

Our starting point is the following result of Flamm [4], Flamm and Mitter [1]:

THEOREM 1. For $\beta < 1$ the unique infimal sensitivity $\bar{X}(s)$ corresponding to problem (1.2) is given by

$$\bar{X}(s) = \lambda \frac{s+1 - e^{-s\Delta}\lambda(s-\beta)}{\lambda(s+\beta) - e^{-s\Delta}(s-1)}, \tag{2.1}$$

where $\lambda = [(\omega_0^2+1)/(\omega_0^2+\beta^2)]^{\frac{1}{2}}$, and $\omega_0$ is the smallest positive solution of $\cot(\omega\Delta) = [\omega^2-\beta]/[\omega(1+\beta)]$. ▮

Remark: A version of this result for $\beta \geq 1$ appears in Flamm and Mitter [1, p. 21].

Using the optimal sensitivity (2.1) for problem (1.2), one can compute a corresponding compensator for (1.1) as if (2.1) actually resulted from a compensator satisfying the restrictions of (1.1). Taking $\zeta = \lambda(1-\beta^2)/(\omega_0^2+1) = (\lambda^2-1)/\lambda$, we get

$$\bar{C} = \zeta \cdot P_0^{-1} \cdot \frac{s^2+\omega_0^2}{s^2+(\beta+1)s+\beta} \cdot \frac{1}{1 + e^{-s\Delta}\cdot\lambda\cdot\frac{\beta-s}{s+1}},$$

which can be realized as shown in Figure 2.

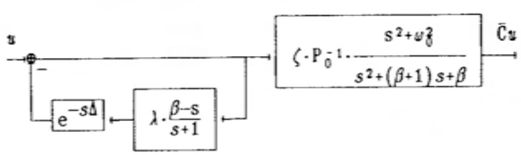


Figure 2. Realization of Optimal Compensator



The result of these calculations show that the resulting compensator is generally improper, and therefore (2.1) cannot be obtained as a solution to (1.1). We showed in Flamm [4] that the optimal improper compensator is also unstable, and that the resulting closed loop system has zero "delay margin."

Here we show how to approximate the sensitivity resulting from this improper compensator by proper ones which result in stable closed loop systems Since the infimal value for (1.1) is not generally attainable, this sequence of compensators solves the problem (1.1). Finally, since the compensators in our sequence contain delays, we show how to modify these compensators to eliminate the delays, yet preserve stability and yield sensitivities with norms approaching the infimum. Since the infimal weighted sensitivity is unique (when it has norm greater than 1), and the corresponding compensator is improper when the plant is strictly proper, there can be no proper compensator which achieves the infimum. Thus we obtain a sequence of proper and finite dimensional compensators which solve (1.1).

## III. APPROXIMATION BY PROPER FINITE DIMENSIONAL COMPENSATORS

The proof of the following theorem describes how to construct an approximating sequence of compensators which *does* produce sensitivities which approach the optimum.

THEOREM 2. There exists a sequence of rational proper feedback compensators which result in weighted sensitivities of norm approaching the optimal value $\lambda$ of Theorem 1. ▮

Remark: Theorem 2 solves (1.1) for $\beta < 1$. A similar results holds for $\beta \geq 1$. For $\beta \geq 1$, the infimal sensitivity is not unique, and there is a proper compensator which achieves the infimum, namely the zero compensator $C(s) = 0$. See Theorem 2 in Flamm and Mitter [1].

There are three elements to the solution. First, we control the excursions of the approximate optimal sensitivity from that circle which is the ideal sensitivity at high frequency by controlling the maximum phase of $h(j\omega)$. Then, so long as $h(j\omega)$ rolls off and $\sup \arg(h(j\omega)) \to 0$, we shall have $\|X(s)\| \to \|\bar{X}(s)\|$. The obvious way to do this is to use an irrational function for $h(j\omega)$, say, get $1/n$ slope from $[\gamma/(s+\gamma)]^{1/n}$. Then when we want to implement it, we can approximate it by a lead/lag filter.

Assume $\bar{Q}$ is the optimal (improper) $Q$ parameter resulting in the optimal weighted sensitivity $\bar{X}(s) = W(s)[1-e^{-s\Delta}P_0(s)\bar{Q}(s)]$. Define $\lambda = \|\bar{X}\|$, and note that $\lambda \geq 1$. We can write $\bar{X}(j\omega) = \lambda e^{j\alpha(\omega)}$, where $\alpha(\omega)$ is real.

The proof proceeds via three propositions which show how to modify $\bar{C}(s)$ in three steps: first to make $C(s)P(s)$ strictly proper, next to make $C(s)$ proper, and finally to make $C(s)$ finite dimensional.

PROPOSITION 1. Let $\bar{X}(s)$ be the infimal sensitivity given in (2.1), with norm $\|\bar{X}(s)\|_\infty = \lambda$. Let $h_n(s) = [\gamma/(s+\gamma)]^{1/n}$ with $\gamma>0$. Then the compensator given by

$$C_n'(s) = \frac{h_n(s)[W(s) - \bar{X}(s)]}{P(s)[(1-h_n(s))W(s) + h_n(s)\bar{X}(s)]}$$

results in a stable closed loop for which the sensitivity approaches $\lambda$ as $n \to \infty$. Furthermore, the loop gain $|P(j\omega)C_n'(j\omega)| \to 0$ as $|\omega| \to \infty$. ▮

Remarks: This formula amounts to using a different roll–off function in the Vidyasagar approach. Also, note that $\lambda > 1$ in Theorem 1.

Proof: The resulting compensator in Proposition 1 is that arising from the $Q$–parameter $Q_n' = h_n\bar{Q}$: $C_n' = (h_n\bar{Q})/(1-Ph_n\bar{Q})$. We note that using the sequence of $Q$–parameters $Q_n'$ preserves a stable closed loop since this is a stable $Q$–parameter, and that the resulting loop gains $P(j\omega)C_n'(j\omega)$ go to zero because $P(j\omega)C_n'(j\omega)$ is strictly proper.

We now consider the magnitude squared of the sensitivity, $|X_n(j\omega)|^2 = |W(j\omega) + h_n(j\omega)[\lambda e^{j\alpha(\omega)} - W(j\omega)]|^2$. Suppose $\omega_n$ is the frequency at which $|X_n(j\omega_n)| = \|X_n\|$ ($\omega_n$ is finite for any given $n$ since the sensitivity function is 1 at $\infty$), and define $h = |h_n(j\omega_n)|$, $\delta = arg[h_n(j\omega_n)]$, $W = W(j\omega_n)$ and $\alpha = \alpha(\omega_n)$. Note that these are functions of $n$, as is $\omega_n$. We also note for later use that $\delta$ satisfies $0 \le \delta \le 2\pi/n$, i.e, $\delta \sim \mathcal{O}(1/n)$.

Now we show that the norm of this sensitivity approaches the infimal value $\lambda$ as $n$ increases.

$$|X_n(j\omega)|^2 \le |W + h\cdot e^{j\delta}(\lambda e^{j\alpha} - W)|^2 = |W|^2 - 2h|W|^2\cos\delta + h^2|W|^2 + h^2\lambda^2 + 2h\lambda\Big\{ \mathcal{R}e(W)\Big[\cos(\alpha+\delta) - h\cdot\cos(\alpha)\Big] + \mathcal{I}m(W)\Big[\sin(\alpha+\delta) - h\cdot\sin(\alpha)\Big]\Big\}$$

Assume we have taken $n$ large enough so that $cos(\delta) \approx 1$ and $sin(\delta) \approx \delta$. Then we write

$$|X_n(j\omega)|^2 \le |W|^2 - 2h|W|^2 + h^2|W|^2 + h^2\lambda^2 + 2h\lambda\Big\{\mathcal{R}e(W)\Big[\cos(\alpha) - h\cos(\alpha) - \delta\sin(\alpha)\Big] + \mathcal{I}m(W)\Big[\sin(\alpha)[1-h] + \delta\cdot\cos(\alpha)\Big]\Big\} + \mathcal{O}(\delta^2) \le [|W|(1-h) + h\lambda]^2 + 2h\lambda|W|\delta + \mathcal{O}(1/n^2)$$

Given $n$, there are two possibilities: **case (i). ($\omega_n > n$)** In this case we shall use the fact that $|W| \to 1$ as $\omega_n \to \infty$. $|W(j\omega_n)|^2 < (n^2+1)/(n^2+\beta^2) = 1 + (1-\beta^2)/(n^2+\beta^2) < 1 + 1/n^2$, since $W(j\omega) = (j\omega+1)/(j\omega+\beta)$ and $\beta < 1$. Therefore $|X_n(j\omega)|^2 \le [(1+1/n)(1-h) + h\lambda]^2 + 2h\lambda\delta(1+1/n) + \mathcal{O}(1/n^2) \le \lambda^2 + \mathcal{O}(1/n)$. We conclude that $(\|\bar{X}\| - \|X_n\|) \to 0$ as $n \to \infty$. **case (ii). ($\omega_n \le n$)** The idea in this case is that $h \to 1$ since $\omega_n \le n$. Since $|W| \ge 1$ and $h \le 1$, $|X(j\omega_n)|^2 \le [|W|(1-h) + h\lambda]^2 + 2h\lambda\delta|W| + \mathcal{O}(1/n^2) \le \lambda^2 + \mathcal{O}(1-h) + \mathcal{O}(1/n)$. Now since $\omega_n \le n$ in this case, $h^2 = |h(j\omega_n)|^2 = [\gamma^2/(\gamma^2+\omega_n^2)]^{1/n} > [\gamma^2/(\gamma^2+n^2)]^{1/n}$ $\therefore h^2 \to 1$ as $n \to \infty$, and therefore $(1-h) \to 0$. Once again we conclude that $(\|\bar{X}\| - \|X_n\|) \to 0$ as $n \to \infty$. ▮

In Proposition 1 a roll–off function could be substituted for $h_n(j\omega)$ which would roll off faster after the loop gain has decreased sufficiently: That is, we can modify $h_n(j\omega)$ (call the modification $g_n(j\omega)$) so that after some frequency $\omega_n$ $|g_n(j\omega)|$ decreases arbitrarily fast without increasing the $\mathscr{H}^\infty$–norm of the resulting sensitivity. This is because $|h_n(j\omega)| \to 0$ as $\omega \to \infty$ and $|e^{-j\omega\Delta}P_0(j\omega)\bar{Q}(j\omega)|$ is bounded. It suffices to take $\omega_n$ large enough so that for $\omega > \omega_n$ we have both

$$|h_n(j\omega)| < \frac{\lambda - |W(j\omega)| - \epsilon}{\lambda + |W(j\omega)|} \tag{3.1}$$

and

$$|W(j\omega)| < 1+\epsilon \tag{3.2}$$



These conditions are motivated by graphical considerations: They ensure that the graphs of the sensitivity $X_n(j\omega)$ and the weighting $W(j\omega)$ are close enough to the point $(1+j0)$ so that rotation of the graph of $X_n(j\omega)$ about any point in the locus $W(j\omega)$ will not cause it to leave the disk $|s| = \lambda$. Such rotation will be an effect of increased phase due to rolling off $h_n(j\omega)$ more rapidly.

In order to demonstrate a specific $g_n(j\omega)$, we must must estimate the smallest value of $\omega_n$ which allows (3.1) to hold. Then we must choose $g_n(j\omega)$ so that for $\omega < \omega_n$ it differs very little from $h_n(j\omega)$ both in magnitude and phase.

(3.2) will hold for $\omega_n$ sufficiently large independently of $n$, and some simple calculations give a sufficient condition of

$$\omega_n \ge \gamma\left[\left[\frac{\lambda + |W(j\omega)|}{\lambda - |W(j\omega)| - \epsilon}\right]^n - 1\right] \tag{3.3}$$

for (3.1) to hold. Now we pick $g_n(j\omega) = f_n(j\omega)\cdot h_n(j\omega)$ where $f_n(s)$ is a stable rational function of $s$ which satisfies $|1-f_n(j\omega)| < 1/n^2$ for $\omega \le \omega_n$, $|f_n(j\omega)|$ is strictly decreasing for $|\omega| > \omega_n$, and which eventually rolls off at least as fast as $P(s)$.

The argument works with a stable finite dimensional approximation to $h_n$. We also note that roll–off of the parameter $\bar{Q}$ can ensure a proper compensator: Let $Q_n = \bar{Q}h_n$, the rolled–off $Q$ parameter. Then if $C_n$ is the resulting compensator, $C_n = Q_n/(1-PQ_n)$. Thus if $Q_n$ is proper, so is $C_n$ since $P$ is proper.

PROPOSITION 2. Let $h_n(s)$ be as in Proposition 1, and take $\epsilon$ such that $0 < \epsilon < \lambda - 1$. For each $n$ take $\omega_n$ to satisfy (3.2) and (3.3). Take $f_n(s)$ to be a stable minimum phase rational function of $s$ which satisfies

$$\|f_n\|_\infty \le 1 \tag{3.4}$$

and

$$|1-f_n(j\omega)| < 1/n^2 \text{ for } \omega \le \omega_n \tag{3.5}$$

and which eventually rolls off at least as fast as $P(s)$. Take $\omega_r$ to be the least frequency above $\omega_n$ at which

$$|f_n(j\omega)| \le |h_n(j\omega)| \text{ for } |\omega| \ge \omega_r. \tag{3.6}$$

Let $\tilde{h}_n(s)$ be any stable rational minimum phase function which satisfies

$$\|\tilde{h}_n\|_\infty \le 1 \tag{3.7}$$

and

$$|h_n(j\omega) - \tilde{h}_n(j\omega)| < \frac{1}{n^2} \text{ for } |\omega| \le \omega_r. \tag{3.8}$$

Define

$$g_n(s) = f_n(j\omega)\cdot\tilde{h}_n(s) \tag{3.9}$$

and

$$C_n(s) = \frac{g_n(s)[W(s) - \bar{X}(s)]}{P(s)[(1-g_n(s))W(s) + g_n(s)\bar{X}(s)]}. \tag{3.10}$$

Then the closed loop feedback system using $C_n(s)$ as the compensator is stable, and the closed loop weighted sensitivity $X_n(s) = W(s)[1+P(s)C_n(s)]^{-1}$ has $\infty$–norm which approaches the infimal value $\lambda$ as $n \to \infty$. Furthermore $C_n(s)$ is a proper function. ▮

Proof: The expression (3.10) results from using the $Q$–parameter $g_n(j\omega)\bar{Q}(j\omega)$ in the formula for $C(s)$. The closed loop is stable because $g_n(j\omega)\bar{Q}(j\omega)$ is a stable $Q$–parameter.

We show that the $\infty$–norm of the sensitivity approaches the optimal value by showing first that the magnitude of the sensitivity function is bounded above by $\lambda$ on the range $|\omega| \in (\omega_n,\infty)$, and second that on the range $[0,\omega_n]$ it can be made arbitrarily close to $\lambda$ by increasing $n$.

Remark: From (3.3) we see $\omega_n \to \infty$ as $n \to \infty$.

For $\omega > \omega_n$ we have $\lambda-|W(j\omega)|-\epsilon > |h_n(j\omega)|\cdot(\lambda + |W(j\omega)|) \geq |h_n(j\omega)\cdot(X(j\omega) - W(j\omega))|$. For $\omega > \omega_r$ we have from (3.6) that $|f_n(j\omega)| \leq |h_n(j\omega)|$, and $|\tilde{h}_n(j\omega)| \leq 1$ from (3.7), so $|g_n(j\omega)\cdot(\lambda e^{j\alpha(j\omega)} - W(j\omega))| < \lambda-|W(j\omega)|-\epsilon$. But $X_n(j\omega) = W(j\omega) + g_n(j\omega)\cdot(\lambda e^{j\alpha(j\omega)} - W(j\omega))$, so using (3.2) we conclude that $|X_n(j\omega)| < \lambda$ for $\omega \geq \omega_r$. On $(\omega_n,\omega_r)$, $|g_n(j\omega)| = |f_n(j\omega)\tilde{h}_n(j\omega)| \leq |h_n(j\omega)| + 1/n^2$ using (3.4) and (3.8). Further calculation using (3.1) shows that the condition $n^2 > (\lambda+\|W\|)/\epsilon$ will ensure that $|g_n(j\omega)(\lambda e^{j\alpha(j\omega)} - W(j\omega))| \leq \lambda-|W(j\omega)|$, and it follows that $|X_n(j\omega)| < \lambda$ on $(\omega_n,\omega_r)$.

For $\omega \leq \omega_n$, we repeat the argument in Proposition 1. Take $|X_n|^2 = |W(j\omega) + g_n(j\omega)[\lambda e^{j\alpha(\omega)} - W(j\omega)]|^2$ and set $\omega_m$ to the frequency at which $|X_n(j\omega_n)| = sup_{\omega\in[0,\omega_n]}|X_n(j\omega)|$. Now define $g = |g_n(j\omega_m)|$, $\delta = arg[g_n(j\omega_m)]$, $W = W(j\omega_m)$, and $\alpha = \alpha(\omega_m)$. These are functions of $n$, as is $\omega_m$. One can see from (3.3), (3.5), (3.8) and the definition of $h_n(s)$ that $\delta \sim \mathcal{O}(1/n)$.

Now we show that the norm of this sensitivity approaches the infimal value $\lambda$ as $n$ increases. As in the proof of Proposition 1, $|X_n(j\omega)|^2 \leq [|W|(1-g) + g\lambda]^2 + 2g\lambda|W|\delta + \mathcal{O}(1/n^2)$. Given $n$, there are two possibilities: **case (i). ($\omega_m > n$)** Exactly as in the proof of Proposition 1 we find $|X_n(j\omega)|^2 \leq \lambda^2 + \mathcal{O}(1/n)$. We conclude that $(\|\bar{X}\|-\|X_n\|) \to 0$ as $n \to \infty$. **case (ii). ($\omega_m \leq n$)** Let $\eta = \|W(j\omega_m)\|_\infty$. Then since $\eta \geq 1$ and $g \leq 1$, as before $|X_n(j\omega_m)|^2 \leq \lambda^2 + \mathcal{O}(1-g) + \mathcal{O}(1/n)$. Furthermore, $1-g \leq |1-g_n(j\omega_m)| = |1-f_n(j\omega_m)\tilde{h}_n(j\omega_m)| \leq |1-h_n(j\omega_m)| + |h_n(j\omega_m)-\tilde{h}_n(j\omega_m)| + |1-f_n(j\omega_m)||\tilde{h}_n(j\omega_m)|$. Using the estimate of $(1-h)$ from the proof of Proposition 1, along with (3.5), (3.7) and (3.8), we find $1-g \to 0$, and we conclude that $(\|\bar{X}\|-\|X_n\|) \to 0$ as $n \to \infty$. The properness of $C_n$ follows from the definitions (3.9) and (3.10), and the assumed roll–off of $f_n(s)$. ∎

Recall that we are assuming $W(s) = (s+1)/(s+\beta)$. Define $n(s) = s+1$ and $d(s) = s+\beta$. Let $\check{f}(s)$ denote $f(-s)$. The compensator (3.10) can be written

$$C_n = \frac{g_n(n\check{n} - \lambda^2 d\check{d})P_0^{-1}}{\lambda n d + e^{-s\Delta}[(1-g_n)n\check{n} + g_n\lambda^2 d\check{d}]} .$$

A realization of this clearly contains a pure delay. We now examine the effect of approximations to this delay on the closed loop sensitivity in order to further approximate the ideal compensator with one which is finite dimensional. In approximating $e^{-s\Delta}$ with a rational function, we must be concerned with two things: First, we must preserve the stability of the closed loop, and second, we must preserve the approximation of the closed loop sensitivity to the optimal sensitivity. The restrictions these impose on rational approximation of the



delay amount to (1) the delay must be approximated closely enough until $g_n$ and $W$ are sufficiently small, and (2) after that the delay approximation must not exceed 1 in magnitude.

We approximate the delay by replacing $e^{-s\Delta}$ with the rational function $\rho(s)$. (We repeat that approximation of the delay in our sense means only that our closed loop sensitivity approximates the infimal norm of the sensitivity.) The following gives one set of criteria for selecting $\rho(s)$.

Proposition 3. Let $\omega_r$ and $g_n(j\omega)$ be as in Proposition 2. Take $\omega_c \geq \omega_r$ such that, if $\mathcal{R}e(s) \geq 0$ and $|s| \geq \omega_c$, then $|W(s)| < \lambda$, and if $|\omega| > \omega_c$, then

$$|g_n(j\omega)| < (\lambda-|W(j\omega_c)|)/(\lambda+|W(j\omega_c)|) \tag{3.11}$$

Let $\zeta = inf_{\mathcal{R}e(s)\geq 0}|\lambda + e^{-s\Delta}\cdot\check{n}(s)/d(s)|$. Let $\rho(s)$ be a stable rational approximation to $e^{-s\Delta}$ such that $\|\rho\| \leq 1$, and for $\mathcal{R}e(s) \geq 0$ and $|s| < \omega_c$, $|\rho(s)-e^{-s\Delta}| < \sigma < \zeta/(2\|W\|)$. Define $\tilde{C}_n(s)$ as $C_n(s)$ with $\rho(s)$ substituted for $e^{-s\Delta}$. Under these conditions, as $n \to \infty$ and $\sigma \to 0$, the closed loop system with compensator $\tilde{C}_n(s)$ and weighted sensitivity $\tilde{X}_n(s)$ is stable and satisfies $\|\tilde{X}_n(s)\| \to \lambda$. ∎

proof: The $Q$–parameter corresponding to $C_n$ is $Q_n(s) = g_n(s)\cdot P_0^{-1}(s)\cdot[n(s)\check{n}(s) - \lambda^2 d(s)\check{d}(s)]/[n(s)(\lambda d(s) + e^{-s\Delta}\check{n}(s))]$. Let $\tilde{Q}_n$ be $Q_n$ with $\rho(s)$ substituted for $e^{-s\Delta}$, and let $\tilde{C}_n$ be the resulting compensator. Let $\tilde{Q}$ represent the optimal $Q$–parameter $\bar{Q}$ with $\rho(s)$ substituted for $e^{-s\Delta}$.

Since stability of the closed loop is equivalent to stability of the $Q$–parameter, $n(s)$ and $d(s)$ have no zeros in the right half plane, and $Q_n$ is stable, we can show that $|\lambda + e^{-s\Delta}\cdot\check{n}(s)/d(s)| > \zeta$ for some $\zeta > 0$, when $\mathcal{R}e(s) \geq 0$. For stability of $\tilde{Q}_n$ it suffices to show that $[\lambda + \rho(s)\cdot\check{n}(s)/d(s)]$ has no zeros in the right half plane. This is equivalent to showing that $|\lambda + \rho(s)\cdot\check{n}(s)/d(s)| > 0$. The condition $|\rho(s)-e^{-s\Delta}| < \zeta/2\cdot\|W\|$ for $\mathcal{R}e(s) \geq 0$ and $|s| < \omega_c$ suffices to show this. A simple calculation gives, for $\mathcal{R}e(s) \geq 0$ and $|s| < \omega_c$,

$$|\lambda + \rho(s)\cdot\frac{\check{n}(s)}{d(s)}| \geq |\lambda + e^{-s\Delta}\cdot\frac{\check{n}(s)}{d(s)}| - |(\rho(s) - e^{-s\Delta})\cdot\frac{\check{n}(s)}{d(s)}|$$
$$\geq \zeta - |\rho(s) - e^{-s\Delta}|\cdot|W(s)| \geq \zeta - \frac{\zeta|W(s)|}{2\|W\|} \geq \frac{\zeta}{2}$$

For $\mathcal{R}e(s) \geq 0$ and $|s| \geq \omega_c$, $|W(s)| < \lambda$ and $|\rho(s)| \leq 1$, so $|\lambda + \rho(s)\cdot\check{n}(s)/d(s)| > 0$ there as well.

Using the condition (3.11), we now show that $\|\tilde{X}_n\| \to \|X\|$ as $n \to \infty$ and $\sigma \to 0$. The first step is to note that we need only show that $|WP\tilde{Q}_n| < \|X\| - |W|$, since then $|W(1-P\tilde{Q}_n)| < \|X\|$, and therefore $\|\tilde{X}_n\| \leq \|X\|$. Some calculations give

$$|WP\tilde{Q}_n| = \left|g_n(s)\cdot(\lambda^2 - |W(s)|^2)/\left[\lambda\cdot d(s)/\check{d}(s) + \rho(s)\check{W}(s)\right]\right|.$$

Since $W(s) = (s+1)/(s+\beta)$, $|\omega| > \omega_c$ implies that $|W(j\omega)| < |W(j\omega_c)|$. Now we have for $|\omega| > \omega_c$

$$|g_n(j\omega)| < \frac{\lambda-|W(j\omega_c)|}{\lambda+|W(j\omega_c)|} < \frac{\lambda-|W(j\omega)|}{\lambda+|W(j\omega)|} \leq \left[\lambda\cdot(\frac{d(j\omega)}{\check{d}(j\omega)}) + \rho(j\omega)\cdot\check{W}(j\omega)\right]/\left[\lambda + |W(j\omega)|\right].$$

This implies $|g_n(j\omega)| \cdot \left[\lambda^2 - |W(j\omega)|^2\right] / \left[\lambda \cdot \frac{d(j\omega)}{\bar{d}(j\omega)} + \rho(j\omega) \cdot \tilde{W}(j\omega)\right] < \lambda - |W(j\omega)|$.

These arguments apply on the imaginary axis for $|\omega| > \omega_c$. For $|\omega| \leq \omega_c$ taking $\sigma$ sufficiently small and $n$ sufficiently large make $\left| |\tilde{X}_n(j\omega)| - |\bar{X}(j\omega)| \right|$ as small as desired. Since $\tilde{X}_n \in \mathcal{H}^\infty$, this behavior on the imaginary axis guarantees that $\|\tilde{X}_n\| \to \|\bar{X}\|$. ∎

This proposition completes the proof of Theorem 2, since we have now shown a way to construct a sequence of rational proper feedback compensators $\tilde{C}_n(s)$ for which the weighted sensitivities $\tilde{X}_n(s)$ have norms approaching the optimal value $\lambda$. To summarize, we have three ranges of frequency over which the approximation of the optimal compensator takes effect. For $|\omega| < \omega_n$ the approximating compensator is very close in magnitude and phase to the optimal compensator. Over $\omega_n \leq |\omega| \leq \omega_c$ the compensator starts to roll off while maintaining a close approximation to the delay, until by $\omega_c$ (3.11) is satisfied. From then on, $|\omega| \geq \omega_c$, so long as $|\rho(j\omega)| \leq 1$, $\rho(j\omega)$ need not be close to $e^{-j\omega\Delta}$.

## IV. CONCLUSION

We have presented a realizable approximation to the solution to the simplest meaningful $\mathcal{H}^\infty$ minimal weighted sensitivity problem for the case of a plant having a delay at the input. The same technique applies to plants with general inner factors. A longer version of this paper which includes a numerical example motivating the approach and illustrating the steps is available from the authors.

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# SPECTRAL FACTORIZATION OF MATRIX-VALUED FUNCTIONS USING INTERPOLATION THEORY: STATE-SPACE FORMULAE

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In earlier publications [11, 12, 13, 14] we explored the classical Schur algorithm as an algorithmic solution to the problem of spectral factorization. In the present paper we obtain a state-space formulation of the theory and describe an algorithm for spectral factorization using state-space representations of the corresponding matrix-valued functions.

## NOTATION

$\mathbf{D} := \{z : |z| < 1\}$ is the open unit disc in the complex plane $\mathbf{C}$

$\mathbf{T} = \{z : |z| = 1\}$ is the boundary of $\mathbf{D}$

$k^{p \times m}$ are $p \times m$ matrices with elements in $k$

$\|A\|$ is the spectral norm of a $\mathbf{C}^{m \times m}$-matrix

$\mathbf{H}_\infty$ is the Hardy space of functions analytic and of bounded modulus in $\mathbf{D}$

$\mathbf{H}_2$ is the Hardy space of functions analytic in $\mathbf{D}$ and square integrable on $\mathbf{T}$

$\mathbf{RH}_\infty$ are rational $\mathbf{H}_\infty$ functions

$\mathbf{S} := \{f(\lambda)$ *analytic in* $\mathbf{D}$ *with* $\|f(\lambda)\| < 1$ *for all* $\lambda \in \mathbf{D}\}$ is the open ball of $\mathbf{RH}_\infty$ functions

$\tilde{X}$ is the complex conjugate transpose of a matrix $X$ in $\mathbf{C}^{p \times m}$

$X^T$ is the transpose of $X$

$X^{1/2}$ is the Hermitian square root of a Hermitian matrix $X$

$\lambda\ (= z^{-1})$ is a complex variable: $f(\lambda)$ will denote an analytic function in $\mathbf{D}$

$f^{\bar{}}(\lambda)$ denotes complex conjugation of the coefficients of $f(\lambda)$ as opposed to $f(\lambda)^{\bar{}}$ which denotes complex conjugation of $f(\ )$ after it has been evaluated at $\lambda$

The transfer matrix corresponding to the state-space realization $(A, B, C, D)$, following Francis [9], is denoted by $[A, B, C, D]$, i.e., $[A, B, C, D] := D + \lambda C(I - \lambda A)^{-1}B$.

## 1. INTRODUCTION

One of the most powerful results in analytic interpolation theory is undeniably the well known Schur algorithm. This algorithm allows on one hand the iterative solution of the so-called Nevanlinna-Pick interpolation problem, and on the other an iterative characterization of power series that are analytic and of modulus bounded by one in the unit disc. These seemingly purely mathematical problems turned out to be of a profound importance

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