On De Finetti Coherence and Kolmogorov Probability

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Abstract

This article addresses the problem of existence of a countably additive probability measure in the sense of Kolmogorov that is consistent with a probability assignment to a family of sets which is coherent in the sense of De Finetti.

Key words: probability assignment, coherence condition, subjective probability, countably additive probability

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1 Introduction

In [1], De Finetti develops a subjectivistic theory of probability. In this theory, probabilities are viewed as certain proportions of stakes a player chooses to pay to enter into a lottery in which the stakes are set by a bookie. The probability assignments to various events are seen as being purely subjective since they depend on the player's assessment of the likelihood of various events occurring in the lottery. However, the player has to choose the probabilities in such a way that he always has a chance to win whatever the stakes set by the bookie might be. De Finetti calls such an assignment of probabilities coherent. In this short note we explore the relationship between coherent assignments of probabilities and the modern probability theory in the sense of Kolmogorov.

Modern probability theory works with probability measures on σ-algebras and needs the specification of probabilities of all the events in the σ-algebra. There are, however, situations in which one is interested in working with partial assignments of probabilities. In such cases the collection of all events for which probabilities are known (or believed to be something) need not have any algebraic structure (i.e., do not form an algebra or a ring or a π-system). In such cases, one would like to know if there is a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathcal{F}$ contains all events of interest to us and $\mathbf{P}$ assigns the same probabilities to these as we believe them to be. The purpose of this article is to show that De Finetti's coherence condition, to a very large extent, holds the key to this problem.

The next section considers the simpler case of a finite collection of events where the coherence condition is in fact necessary and sufficient. Section 3 shows that this is so in general if one is willing to settle for finite additivity. Section 4 gives the corresponding result for the countably additive case under additional conditions.

For an extensive discussion of the history of these issues, see [3] and the references therein. [4] contains several original articles of historic interest. [6] addresses some issues similar to those studied in this paper.

2 Finite Probability Spaces

Let Ω be an arbitrary set and $\mathcal{A} = \{A_i\}_{i=1}^N$ a finite collection of nonempty subsets of Ω.

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Definition 1 *A probability assessment on* $(\Omega, \mathcal{A})$ *is a function* $\tilde{P}$ *mapping each set* A *in* $\mathcal{A}$ *to a number* $\tilde{P}(A) \in [0,1]$. *We denote a probability assessment by* $(\Omega, \mathcal{A}, \tilde{P})$.

Definition 2 *A probability assessment* $(\Omega, \mathcal{A}, \tilde{P})$ *is said to be* coherent *if for all* $[c_1, \cdots, c_N] \in \mathcal{R}^N$,

$$\max_{\omega \in \Omega} \sum_{i=1}^{N} c_i [I_{A_i}(\omega) - \tilde{P}(A_i)] \geq 0. \tag{1}$$

This can be given the following convex analytic interpretation: The vector $[\tilde{P}(A_1), \cdots, \tilde{P}(A_N)]$ is in the closed convex hull of the finite set

$$B = \{[I_{A_1}(\omega), \cdots, I_{A_N}(\omega)] : \omega \in \Omega\}.$$

See for example [2].

Therefore, probabilities $p(e)$ can be assigned to each element $e = (e_1, \ldots, e_N)$ of the finite set B such that

$$[\tilde{P}(A_1), \ldots, \tilde{P}(A_N)] = \sum_{e \in B} e p(e).$$

It is easy to see that the following collection of subsets of Ω

$$\left\{ \bigcap_i I_{A_i}^{-1}(e_i); e = (e_1, \ldots, e_N) \in B \right\}.$$

form a partition that generate the same σ-field as the collection $\mathcal{A}$. The probabilities on the set B can be thought of as probabilities of these partitions and therefore define a probability measure on the σ-field generated by the collection $\mathcal{A}$. Therefore we have the following theorem.

Theorem 2.1 *Consider a probability assessment* $(\Omega, \mathcal{A}, \tilde{P})$. *Let* $\mathcal{F}$ *be the finite algebra generated by the collection* $\mathcal{A}$. *Then there exists a probability measure* $\mathbf{P}$ *on* $(\Omega, \mathcal{F})$ *such that* $\mathbf{P}(A) = \tilde{P}(A)$ *for all* A *in* $\mathcal{A}$ *if and only if the probability assessment* $(\Omega, \mathcal{A}, \tilde{P})$ *is coherent.*

3 Finitely Additive Extensions

We first extend the above definitions as follows:

Definition 3 *For an arbitrary collection $\mathcal{A}$ of nonempty subsets of Ω, a probability assessment on $(\Omega, \mathcal{A})$ is defined exactly as in Definition 1, whereas a probability assignment $(\Omega, \mathcal{A}, \tilde{P})$ will be said to be coherent if (1) holds for all finite subcollections $\{A_1, \cdots, A_N\} \subset \mathcal{A}$ and $[c_1, \cdots, c_N] \in \mathcal{R}^N$, $N \geq 1$.*

We first consider a countable $\mathcal{A}$, enumerated as $\{A_1, A_2, \cdots\}$. Let $(\Omega, \mathcal{A}, \tilde{P})$ be a coherent probability assessment. We shall denote by $\sigma(\mathcal{C})$ the σ-algebra generated by a family $\mathcal{C}$ of sets. Let $\mathcal{A}_n = \{A_1, \cdots, A_n\}$ for $n \geq 1$. By Theorem 2.1, the set $\mathcal{P}_n$ of probabilities compatible with $\tilde{P}$ restricted to $\mathcal{A}_n$ is nonempty for each n. Identifying $\mathcal{P}_n$ with a subset of the simplex of probability vectors in $\mathcal{R}^{|\sigma(\mathcal{A}_n)|}$, one easily verifies that it is convex compact. For $m > n$, let $\Pi_{m,n}(P)$ for $P \in \mathcal{P}_m$ denote the element of $\mathcal{P}_n$ obtained by restricting P to $\sigma(\mathcal{A}_n)$.

Lemma 3.1

(i) *For $k > m > n$, $\Pi_{m,n} \circ \Pi_{k,m} = \Pi_{k,n}$.*
(ii) *For $m > n$, $\Pi_{m,n}(\mathcal{P}_m) \subset \mathcal{P}_n$ and is compact nonempty.*
(iii) *For $k > m > n$, $\Pi_{k,n}(\mathcal{P}_k) \subset \Pi_{m,n}(\mathcal{P}_m)$.*
(iv) *$\mathcal{P}_n^* \triangleq \bigcap_{m>n} \Pi_{m,n}(\mathcal{P}_m) \subset \mathcal{P}_n$ is compact nonempty.*
(v) *$\Pi_{m,n}(\mathcal{P}_m^*) = \mathcal{P}_n^*$ for $m > n$.*

Proof. $(i) - (iii)$ are easily verified. (iv) follows from the finite intersection property of families of compact sets. (v) follows from the definition of $\mathcal{P}_n^*$. □

Pick $\mu_n^0 \in \mathcal{P}_n^*$ for $n \geq 1$ and for $m > n$, let $\mu_n^{m-n} = \Pi_{m,n}(\mu_n^0) \in \mathcal{P}_n^*$. Let $\{\mu_1^{n(k)}\}$ denote a subsequence of $\{\mu_1^n\}$ in $\mathcal{P}_1^*$ converging to some $\mu_1^* \in \mathcal{P}_1^*$. Let $\{\mu_2^{n(k(m))}\}$ denote a subsequence of $\{\mu_2^{n(k)}\}$ converging to some $\mu_2^* \in \mathcal{P}_2^*$. Proceeding thus and using a diagonal argument, we can pick $\tilde{\mu}_n \in \mathcal{P}_n^*, n \geq 1$, such that $\Pi_{m,n}(\tilde{\mu}_m) \to \mu_n^*$ as $n < m \to \infty$. Clearly, $\Pi_{m,n}(\mu_m^*) = \mu_n^*$. We have:

Theorem 3.1 *For an arbitrary family $\mathcal{A}$ of nonempty subsets of Ω with $\mathcal{F} =$ the algebra it generates, a probability assessment $(\Omega, \mathcal{A}, \tilde{P})$ is coherent if and only if there exists a finitely additive probability $\mathbf{P}$ on $(\Omega, \mathcal{F})$ that agrees with $\tilde{P}$ on $\mathcal{A}$.*

Proof. The '*only if*' part follows as in Theorem 2.1. For the '*if*' part, consider first a countable $\mathcal{A}$. In the above notation, set $\mathbf{P} = \mu_n^*$ on $\mathcal{A}_n$ for $n \geq 1$. This consistently defines a finitely additive probability on $\mathcal{F} = \bigcup_n \sigma(\mathcal{A}_n)$. For arbitrary $\mathcal{A}$, let $(\mathcal{A}_\alpha, \mu_\alpha), \alpha \in \mathcal{I}$, denote a nested family of countable subsets $\{\mathcal{A}_\alpha\}$ of $\mathcal{A}$ equipped with finitely additive probabilities $\{\mu_\alpha\}$ on the corresponding algebras $\{\mathcal{F}_\alpha\}$ such that the following consistency condition holds: $\mathcal{F}_{\alpha_1} \subset \mathcal{F}_{\alpha_2}$ implies μ_{α_2} restricts to μ_{α_1} on $\mathcal{F}_{\alpha_1}$. Then $\mathcal{F} = \bigcup_\alpha \mathcal{F}_\alpha$ is the algebra generated by $\bar{\mathcal{A}} = \bigcup_\alpha \mathcal{A}_\alpha$ and for $A \in \bar{\mathcal{A}}$, $\mu(A) \triangleq \mu_\alpha(A)$ for any α such that $\mathcal{A}_\alpha$ contains A, defines a finitely additive probability on $(\Omega, \mathcal{F})$

in a consistent way. Consider the family of pairs $(\hat{A}, \hat{\mu})$, where $\hat{\mathcal{A}} \subset \mathcal{A}$ and $\hat{\mu}$ is a finitely additive probability on the algebra generated by $\hat{\mathcal{A}}$. Define a partial order on this family by setting $(\mathcal{B}, \nu) < (\mathcal{D}, \eta)$ if $\mathcal{B} \subset \mathcal{D}$ and η restricts to ν on $\mathcal{B}$. By the foregoing, this family is nonempty. Also, every ordered chain w.r.t. this partial order has a least upper bound: for any ordered family $\{(\mathcal{A}_\alpha, \mu_\alpha), \alpha \in \mathcal{I}\}$, $\bar{\mathcal{A}}, \mu$ defined as above would provide a least upper bound. Thus by Zorn's lemma, there exists a maximal element $(\mathcal{A}^*, \mu^*)$. We are done if $\mathcal{A}^* = \mathcal{A}$. Suppose not. Take $A \in \mathcal{A} - \mathcal{A}^*$. Then the algebra generated by $\mathcal{A}^* \cup \{A\}$ is given by $\tilde{\mathcal{G}} \stackrel{def}{=} \{(A \cap B_1) \cup (A^c \cap B_2) : B_1, B_2 \in \mathcal{G}\}$, where $\mathcal{G}$ is the algebra generated by $\mathcal{A}$. Extend μ^* to a finitely additive probability $\tilde{\mu}$ on $\tilde{\mathcal{G}}$ by setting

$$\tilde{\mu}((A \cap B_1) \cup (A^c \cap B_2)) \stackrel{def}{=} \tilde{\mathcal{P}}(A)\mu^*(B_1) + (1 - \tilde{\mathcal{P}}(A))\mu^*(B_2)$$

for $B_1, B_2 \in \mathcal{G}$. That this does indeed define a finitely additive probability on $\tilde{\mathcal{G}}$ is easily verified. Then $(\mathcal{A} \cup \{A\}, \tilde{\mu})$ contradicts the maximality of $(\mathcal{A}^*, \mu^*)$. It follows that $\mathcal{A}^* = \mathcal{A}$. This completes the proof. □

4 Countably Additive Extensions

As is well known, not all finitely additive probabilities on σ-algebras lead to countably additive extensions. Thus to make a claim akin to the above for countably additive probability measures, we need to impose additional conditions, *stated in terms of our initial collection $\mathcal{A}$ of events.* We give such a condition below. For a set $A \in \mathcal{A}$, let A^i denote A if $i = 0$ and A^c if $i = 1$. The condition is:

(†) If $A_n \in \mathcal{A}, n \geq 1$, satisfies $\bigcap_n A_n^{i(n)} = \phi$ for some choice of $i(n) \in \{0, 1\}, n \geq 1$, then $\bigcap_{n=1}^{N} A_n^{i(n)} = \phi$ for some $1 \leq N < \infty$.

Remark.

(1) This condition is necessary. Consider, for example, a countable $\mathcal{A} = \{A_1, A_2, \cdots\}$ and define $\mathcal{A}_n = \{A_1, A_2, \cdots, A_n\}$ for $n \geq 1$. Let $\mathcal{F}$ denote the Boolean algebra generated by $\mathcal{A}$. Suppose that for some choice of $i(n) \in \{0, 1\}, n \geq 1$, $\bigcap_n A_n^{i(n)} = \phi$, but $\bigcap_{n=1}^{N} A_n^{i(n)} \neq \phi$ for all finite $N \geq 1$. Define a probability μ_n on $(\Omega, \sigma(\mathcal{A}_n))$ by setting $\mu_n(A) = 1$ if $A \in \sigma(\mathcal{A}_n)$ contains $\bigcap_{m=1}^{n} A_m^{i(m)}$ and zero otherwise. Then the finitely additive probability μ defined on $(\Omega, \mathcal{F})$ by $\mu(A) = \mu_n(A)$ for $A \in \sigma(\mathcal{A}_n)$ is well-defined and

corresponds to a coherent probability assignment by Theorem 2. However,

$$\lim_{N\to\infty} \mu(\bigcap_{n=1}^{N} A_n^{i(n)}) = 1 \neq 0 = \mu(\phi).$$

Thus μ does not extend to a countably additive probability on $\sigma(\mathcal{A})$. (This example is adapted from [5], pp. 141-142.)

(2) *As an example of a situation where (†) is satisfied, consider the case when each $A \in \mathcal{A}$ intersects at most finitely many other sets in $\mathcal{A}$. Then $\bigcap_{n=1}^{N} A_n^{i(n)} \neq \phi$ for $1 \leq N < \infty$ would perforce imply that for large N, $\bigcap_{n=1}^{N} A_n^{i(n)}$ equals the intersection of a fixed finite subcollection of sets from $\mathcal{A}$, whence (†) follows.*

Theorem 4.1 *If a probability assignment $(\Omega, \mathcal{A}, \tilde{P})$ is coherent and $\mathcal{A}$ satisfies (†), then there exists a countably additive probability $\boldsymbol{P}$ on $(\Omega, \sigma(\mathcal{A}))$ that agrees with $\tilde{P}$ on $\mathcal{A}$.*

We shall need two preliminary lemmas.

Lemma 4.1 *If $\mathcal{B} = \{A_1, A_2, \cdots\} \subset \mathcal{A}$ is a countable subfamily, then the atoms of $\sigma(\mathcal{B})$ are precisely the nonempty sets of the form $\bigcap_n A_n^{i(n)}$, $i(m) \in \{0, 1\}, n \geq 1$.*

Proof. Consider the collection of sets A with the property: Given any set C of the above form, either $C \subset A$ or $C \subset A^c$. It is easy to see that this is a sigma field that contains $\mathcal{B}$, and therefore contains $\sigma(\mathcal{B})$. Also, the latter contains sets of the form $\bigcap_n A_n^{i(n)}$, $\{i(n)\}$ as above. The claim follows. □

Lemma 4.2 $\sigma(\mathcal{A}) = \bigcup \sigma(\mathcal{B})$ *where the union is over all countable $\mathcal{B} \subset \mathcal{A}$.*

Proof. The r.h.s. is clearly contained in the l.h.s. The claim follows on noting that the r.h.s. is also a σ-field. □

Proof of Theorem 4.1: Let $\mathcal{B} = \{A_1, A_2, \cdots\} \subset \mathcal{A}$ and $\mathcal{A}_n = \{A_1, A_2, \cdots, A_n\}, n \geq 1$. Then $\sigma(\mathcal{A}_n), n \geq 1$, is an increasing family of (finite) σ-fields and $\sigma(\mathcal{B})$ is the smallest σ-field containing $\sigma(\mathcal{A}_n), n \geq 1$. Let μ be the finitely additive probability measure guarranteed by Theorem 3.1. Then by (†), Lemma 4.1 above and Theorem 4.1, pp. 141-143 of [5], it extends to a unique countably additive probability measure on $\sigma(\mathcal{B})$. Since $\mathcal{B}$ was an arbitrary countable subset of $\mathcal{A}$, the claim follows in view of Lemma 4.2 above. □

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