Adaptive Control and the Definition of Exponential Stability

Travis E. Gibson† and Anuradha M. Annaswamy‡

† HARVARD MEDICAL SCHOOL

‡ Massachusetts Institute of Technology

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Objectives

Prove that the following statement is **incorrect**

- "If the reference model is persistently exciting then the adaptive system is globally exponentially stable"

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Main insights

- Indeed if the reference model is PE then after some time the plant will be PE, **but after exactly how much time**?
- We will show how a PE condition on the reference model implies a **weak** PE condition on the plant state.

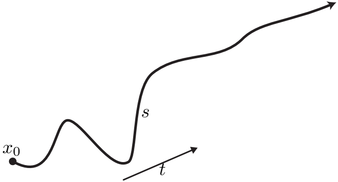
Outline

- Definitions
  - Stability
  - Exponential Stability
  - Persistent Excitation (PE)
  - **weak** Persistent Excitation (PE*)
- Link between PE and Exponential Stability
- Link between PE* and Uniform Asymptotic Stability
- Simulation Studies

Uniform Stability in the Large (Global)

$$\dot{x}(t) = f(x(t), t)$$
$$x_0 \triangleq x(t_0)$$

Solution $s(t; x_0, t_0)$



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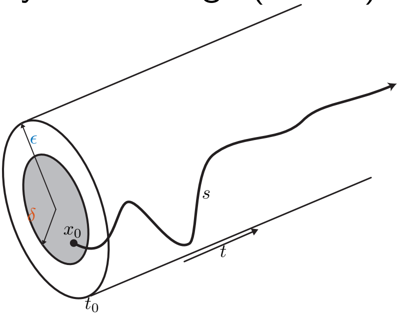

Definition: Uniform Stability in the Large (Massera, 1956)

(i) **Uniformly Stable**: $\forall\ \epsilon > 0\ \exists\ \delta(\epsilon) > 0$ s.t.

$$\|x_0\| \le \delta \implies \|s(t; x_0, t_0)\| \le \epsilon.$$

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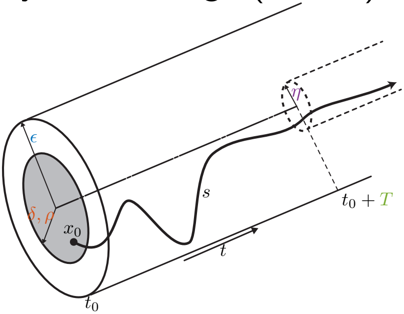

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(ii) **Uniformly Attracting in the Large**: **For all** $\boldsymbol{\rho}, \boldsymbol{\eta}\ \exists\ \boldsymbol{T}(\boldsymbol{\eta}, \boldsymbol{\rho})$
$$\|x_0\| \leq \boldsymbol{\rho} \implies \|s(t; x_0, t_0)\| \leq \boldsymbol{\eta} \quad \forall\ t \geq t_0 + \boldsymbol{T}.$$

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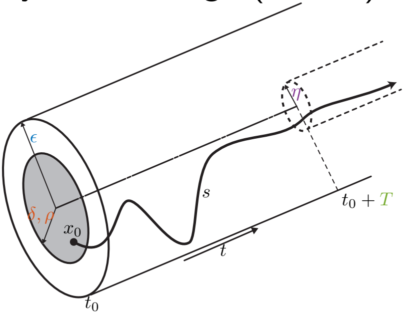

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(iii) **Uniformly Asymptotically Stable in the Large** (UASL)
= **uniformly stable** + **uniformly bounded** +
uniformly attracting in the large.

Exponential Stability

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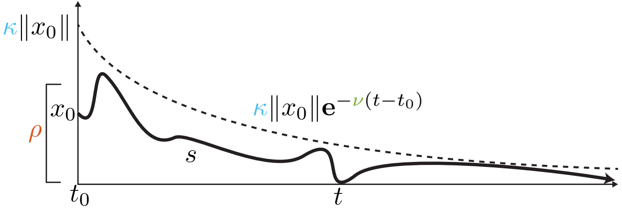
Solution $s(t; x_0, t_0)$



Definition: (Malkin, 1935; Kalman and Bertram, 1960)

(i) **Exponentially Stable** (ES): $\forall\ \boldsymbol{\rho} > 0\ \exists\ \boldsymbol{\nu}(\boldsymbol{\rho}), \boldsymbol{\kappa}(\boldsymbol{\rho})$ s.t.

$$\|x_0\| \leq \boldsymbol{\rho} \implies \|s(t; x_0, t_0)\| \leq \boldsymbol{\kappa}\|x_0\|\mathbf{e}^{-\boldsymbol{\nu}(t-t_0)}$$

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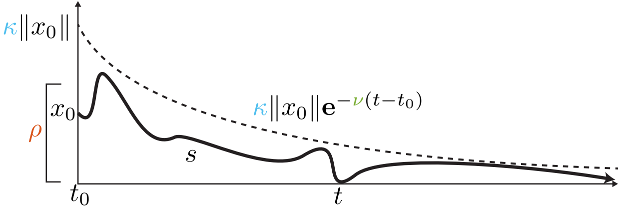

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Persistent Excitation

"Exogenous Signal" : $\omega : [t_0, \infty) \to \mathbb{R}^p$

Initial Condition : $\omega_0 = \omega(t_0)$

Parameterized Function : $y(t, \omega) : [t_0, \infty) \times \mathbb{R}^p \to \mathbb{R}^m$

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$\exists\ \boldsymbol{T}, \boldsymbol{\alpha}$ s.t.

$$\int_t^{t+\boldsymbol{T}} y(\tau, \omega) y^{\mathsf{T}}(\tau, \omega) d\tau \geq \boldsymbol{\alpha} I$$

for all $t \geq t_0$ and $\omega_0 \in \mathbb{R}^p$.

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(ii) **weakly Persistently Exciting** ($\mathrm{PE}^*(\omega, \Omega)$):
$\exists$ a compact set $\Omega \subset \mathbb{R}^p$, $\boldsymbol{T}(\Omega) > 0, \boldsymbol{\alpha}(\Omega)$ s.t.

$$\int_t^{t+\boldsymbol{T}} y(\tau, \omega) y^{\mathsf{T}}(\tau, \omega) d\tau \geq \boldsymbol{\alpha} I$$

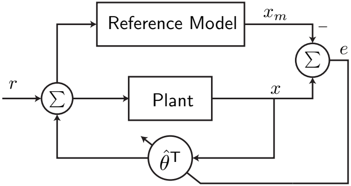
for all $\omega_0 \in \Omega$ and $t \geq t_0$.

properties of adaptive control

Adaptive Control

Plant $\quad \dot{x} = Ax - B\theta^{\mathsf{T}}x + Bu$

Reference Model $\quad \dot{x}_m = Ax_m + Br$



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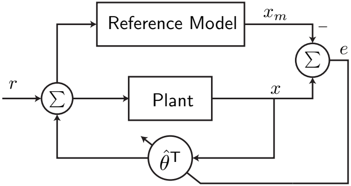

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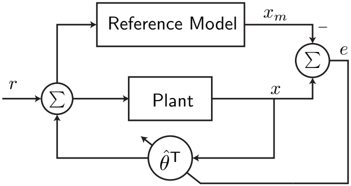

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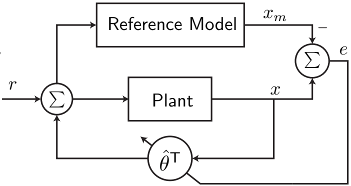

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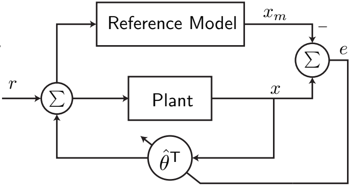

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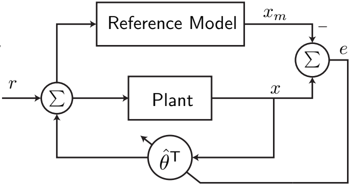

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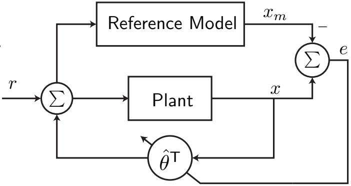

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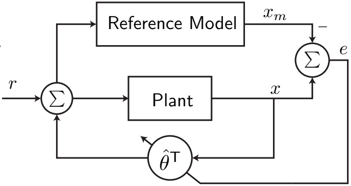

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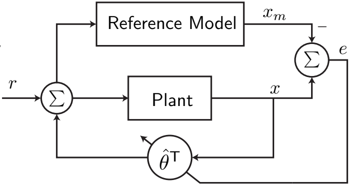

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The L-norms of e are initial condition dependent!!

Exponential Stability and Adaptive Control

$$\dot{z}(t) = \begin{bmatrix} A & Bx^{\mathsf{T}}(t) \\ -x(t)B^{\mathsf{T}}P & 0 \end{bmatrix} z(t), \qquad z(t) \triangleq \begin{bmatrix} e(t) \\ \tilde{\theta}(t) \end{bmatrix}$$

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Theorem: (Morgan and Narendra, 1977)

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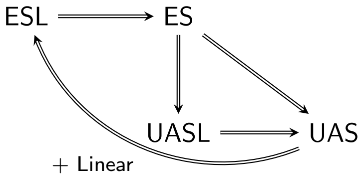

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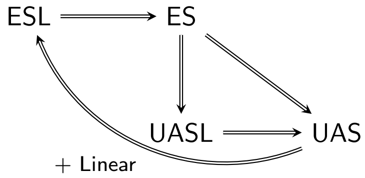

- Therefore, when $x \in \mathrm{PE}$ the dynamics $z(t)$ are globally exponentially stable (Anderson, 1977).

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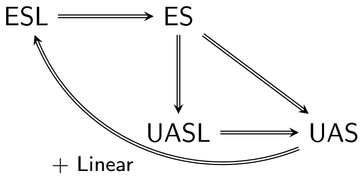

- Therefore, when $x \in \mathrm{PE}$ the dynamics $z(t)$ are globally exponentially stable (Anderson, 1977).
- The condition of PE for $x(t)$ however does not follow from $x_m(t) \in \mathrm{PE}$.

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Recall that $e = x - x_m$, then for any fixed unitary vector h

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$$(x_m^\top h)^2 - (x^\top h)^2 \leq \|e\| \left(\sqrt{\frac{V(z_0)}{P_{\min}}} + 2 x_m^{\max} \right)$$

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Move x_m to the RHS, multiply by -1, and integrate to pT

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$x_m \in \mathrm{PE}$

$$\int_{t_0}^{t_0+T} x_m x_m^\top \geq \alpha I$$

Move x_m to the RHS, multiply by -1, and integrate to pT

$$\int_t^{t+pT} (x^\top(\tau)h)^2 d\tau \geq$$

$$p\alpha - \left(\sqrt{\frac{V(z_0)}{P_{\min}}} + 2x_m^{\max} \right) \sqrt{pT \int_t^{t+pT} \|e(\tau)\|^2 d\tau}.$$

If $x_m \in \mathrm{PE}$ then $x \in$?

Recall that $e = x - x_m$, then for any fixed unitary vector h

$$(x_m^\top h)^2 - (x^\top h)^2 = \underbrace{-(x^\top h - x_m^\top h)}_{\leq \|e\|}\underbrace{(x^\top h + x_m^\top h)}_{= e^\top h + 2x_m^\top h}$$

$$(x_m^\top h)^2 - (x^\top h)^2 \leq \|e\| \left(\sqrt{\frac{V(z_0)}{P_{\min}}} + 2x_m^{\max}\right)$$

$$\|e\|_{L_\infty} \leq \sqrt{\frac{V(z_0)}{P_{\min}}}$$

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Recall that $e = x - x_m$, then for any fixed unitary vector h

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Clean the notation

$$\int_t^{t+pT} \|x\|^2 d\tau \geq p\alpha - \left(\sqrt{\frac{V(z_0)}{P_{\min}}} + 2 x_m^{\max} \right) \sqrt{pT \frac{V(z_0)}{Q_{\min}}}.$$

$x \in \text{PE}^*$ $\underline{x \notin \text{PE}}$

$$\int_t^{t+T} x_m(\tau)x_m^{\mathsf{T}}(\tau)d\tau \geq \alpha I$$

$$\int_t^{t+pT} x^{\mathsf{T}}(\tau,z)x(\tau,z)d\tau \geq p\alpha - \left(\sqrt{\frac{V(z_0)}{P_{\min}}} + 2x_m^{\mathsf{max}}\right)\sqrt{pT\frac{V(z_0)}{Q_{\min}}}.$$

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Fixed T, α

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Fixed T, α Free p Initial Condition z_0

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If the initial condition $\|z(t_0)\|$ increases ($V(z_0)$ increases), then p must increase, and thus the time (pT) must increase to keep α' constant.

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Revisit the definitions for PE

(i) **Persistently Exciting** (PE): $\exists\, T, \alpha$ s.t.

$$\int_t^{t+T} x(\tau,\omega)x^\mathsf{T}(\tau,\omega)d\tau \geq \alpha I$$

for all $t \geq t_0$ and $\omega_0 \in \mathbb{R}^p$.

(ii) **weakly Persistently Exciting** ($\mathrm{PE}^*(\omega,\Omega)$): $\exists$ a compact set $\Omega \subset \mathbb{R}^p$, $T(\Omega) > 0, \alpha(\Omega)$ s.t.

$$\int_t^{t+T} x(\tau,\omega)x^\mathsf{T}(\tau,\omega)d\tau \geq \alpha I$$

for all $\omega_0 \in \Omega$ and $t \geq t_0$.

Adaptive Control and UASL

Revisit the adaptive control problem

$$\dot{z}(t) = \begin{bmatrix} A & Bx^{\mathsf{T}}(t) \\ -x(t)B^{\mathsf{T}}P & 0 \end{bmatrix} z(t), \qquad z(t) \triangleq \begin{bmatrix} e(t) \\ \tilde{\theta}(t) \end{bmatrix}$$

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Define the following compact set

$$\Omega(\zeta) \triangleq \{z : V(z) \leq \zeta\}$$

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If $x_m \in \mathrm{PE}$ then $x \in \mathrm{PE}^*(z, \Omega(\zeta))$, for any $\zeta > 0$, and it follows that the dynamics above are UASL.

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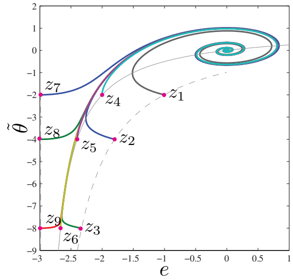
- Next we prove (by counter example) $x_m \in \mathrm{PE}$ does not imply ESL.

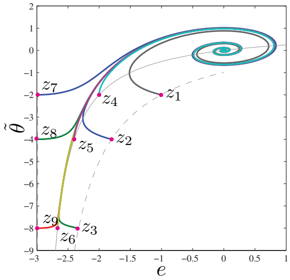
Simulation Example

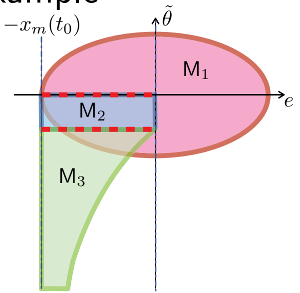


Plant	$\dot{x} = Ax - B\theta^{\mathsf{T}}x + Bu$
Reference	$\dot{x}_m = Ax_m + Br$
Control	$u = \hat{\theta}^{\mathsf{T}}(t)x + r$

$$A = -1$$
$$B = 1$$
$$r = 3$$
$$x_m(t_0) = 3$$

Simulation Example





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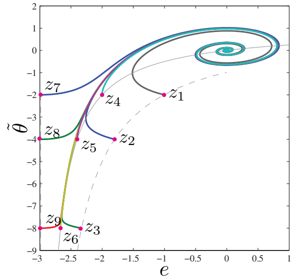
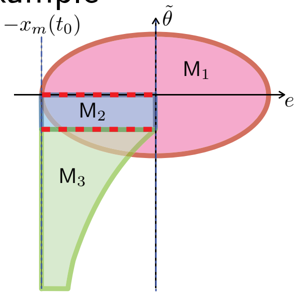
(Jenkins et al., 2013a; 2013b)

Simulation Example





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- $M_1 \cup M_2 \cup M_3$ is invariant

(Jenkins et al., 2013a; 2013b)

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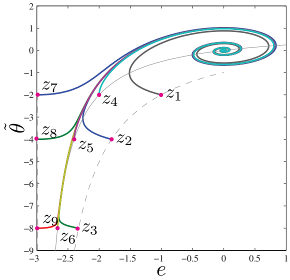

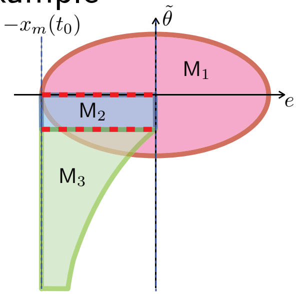

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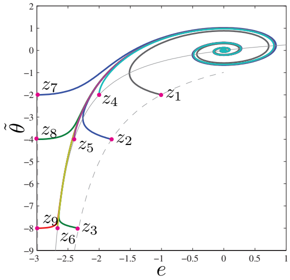
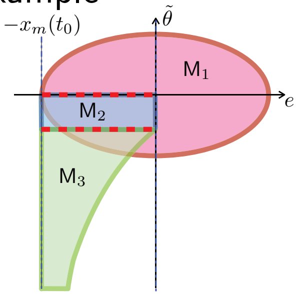
- $M_1 \cup M_2 \cup M_3$ is invariant
- M_3 extends down in an unbounded fashion

(Jenkins et al., 2013a; 2013b)

Simulation Example





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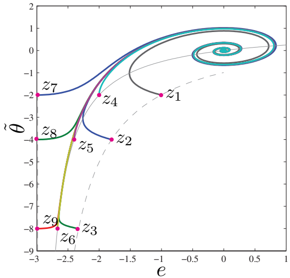
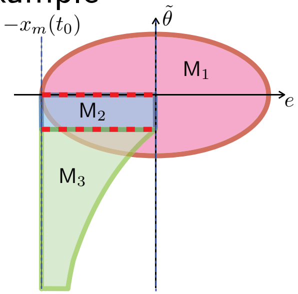
- $M_1 \cup M_2 \cup M_3$ is invariant
- M_3 extends down in an unbounded fashion
- maximum rate of change in M_3 is bounded

(Jenkins et al., 2013a; 2013b)

Simulation Example





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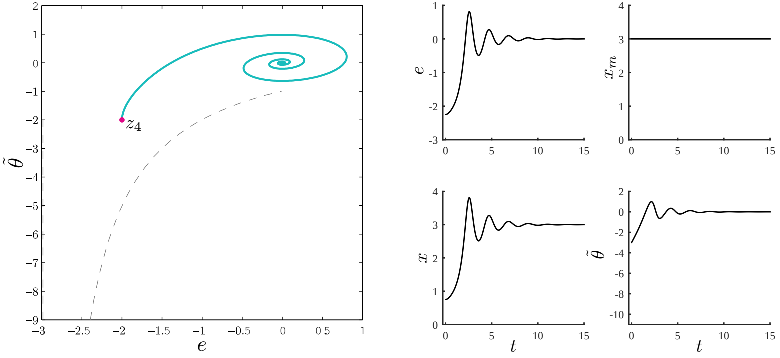
- $M_1 \cup M_2 \cup M_3$ is invariant
- M_3 extends down in an unbounded fashion
- maximum rate of change in M_3 is bounded
- The fixed rate regardless of initial condition implies that ESL is impossible

(Jenkins et al., 2013a; 2013b)

Simulation Example Continued



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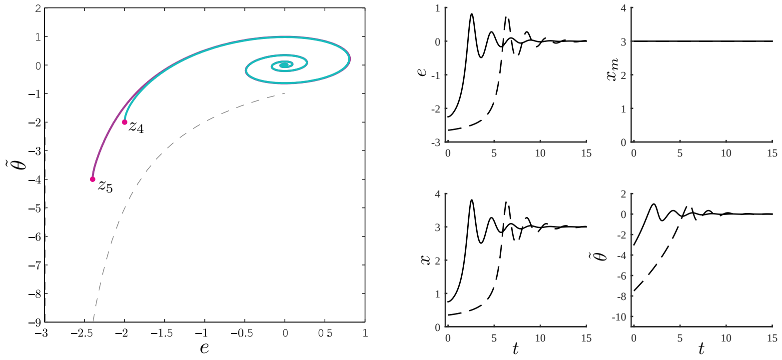
Simulation Example Continued



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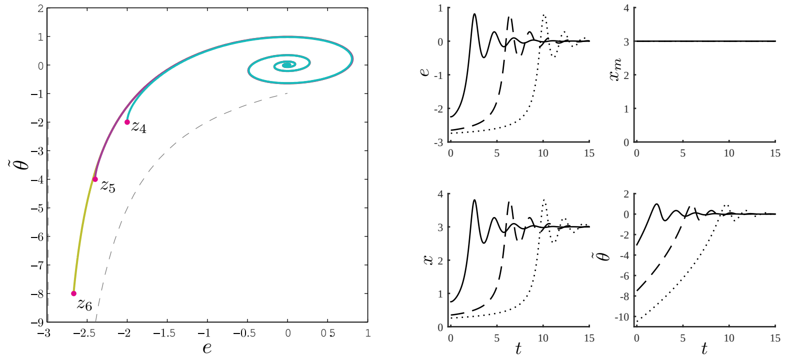

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Summary

- PE of the reference model does not imply PE for the state vector
- Adaptive control in general can not be guaranteed to be ESL

loria@lss.supelec.fr Bibliography

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backup slides

Example or recent literature making this mistake

53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

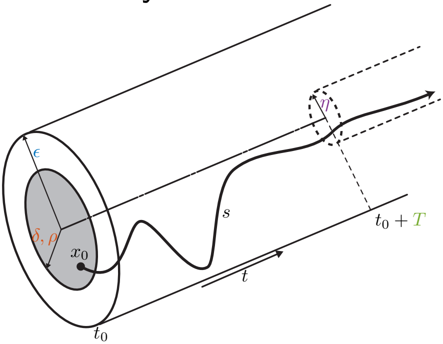
Concurrent Learning Adaptive Control for Systems with Unknown Sign of Control Effectiveness

***Theorem** 1:* Consider the system in (1), the control law of (3), and let $p \geq n$ be the number of recorded data points. Let $X_k = [x_1, x_2, \ldots, x_p]$ be the history stack matrix containing recorded states, and $R_k = [r_1, r_2, \ldots, r_p]$ be the history stack matrix containing recorded reference signals. Assume that over a finite interval $[0, T]$ the exogenous reference input $r(t)$ is exciting, the history stack matrices are empty at $t = 0$, and are consequently updated using Algorithm 1 of [20]. Then, the concurrent learning weight update laws of (7) and (8) guarantee that the zero solution $(e(t), \widetilde{K}(t), \widetilde{K_r}(t)) \equiv 0$ is globally exponentially stable.

Stability

$$\dot{x}(t) = f(x(t), t)$$
$$x_0 \triangleq x(t_0)$$

Solution $s(t; x_0, t_0)$



Definition: Stability (Massera, 1956)

(i) **Stable**: $\forall\ \epsilon > 0\ \exists\ \delta(\epsilon, x_0, t_0) > 0$ s.t.
$$\|x_0\| \leq \delta \implies \|s(t; x_0, t_0)\| \leq \epsilon.$$

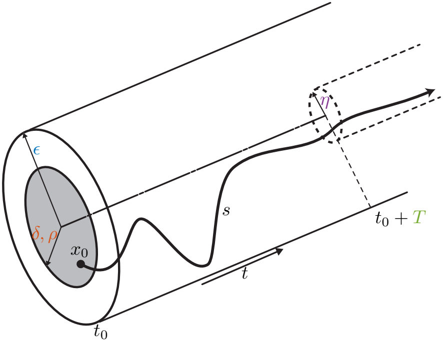
(ii) **Attracting**: $\exists\ \rho(t_0) > 0$ s.t. $\forall\ \eta > 0\ \exists$ an *attraction time* $T(\eta, x_0, t_0)$ s.t.
$$\|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \eta \quad \forall\ t \geq t_0 + T.$$

(iii) **Asymptotically Stable** = **stable** + **attracting**.

Uniform Stability

$$\dot{x}(t) = f(x(t), t)$$
$$x_0 \triangleq x(t_0)$$

Solution $s(t; x_0, t_0)$



Definition: Uniform Stability (Massera, 1956)

(iv) **Uniformly Stable**: $\boldsymbol{\delta}(\boldsymbol{\epsilon})$ in (i) is uniform in t_0 and x_0.

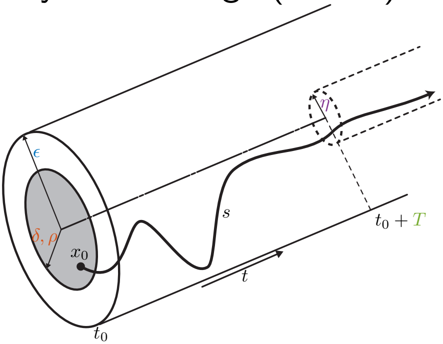
(v) **Uniformly Attracting**: $\boldsymbol{\rho}$ and $\boldsymbol{T}$ do not depend on t_0 or x_0 and thus the attracting times take the form $\boldsymbol{T}(\boldsymbol{\eta}, \boldsymbol{\rho})$.

(vi) **Uniformly Asymptotically Stable**, (UAS) $=$ **uniformly stable** $+$ **uniformly attracting**.

Uniform Stability in the Large (Global)

$$\dot{x}(t) = f(x(t), t)$$
$$x_0 \triangleq x(t_0)$$

Solution $s(t; x_0, t_0)$



Definition: Uniform Stability in the Large (Massera, 1956)

(vii) **Uniformly Attracting in the Large**: For all ρ, η $\exists$ $T(\eta, \rho)$

$$\|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \eta \quad \forall\, t \geq t_0 + T.$$

(viii) **Uniformly Asymptotically Stable in the Large** (UASL)

= **uniformly stable** +

uniformly bounded +

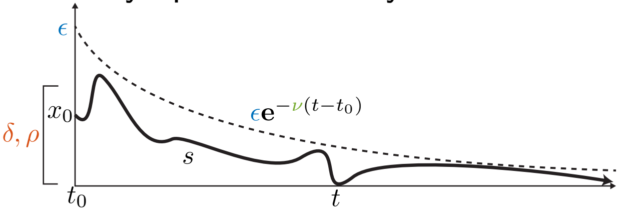
uniformly attracting in the large.

Exponential Asymptotic Stability

$$\dot{x}(t) = f(x(t), t)$$

$$x_0 \triangleq x(t_0)$$

Solution $s(t; x_0, t_0)$



Definition: (Malkin, 1935; Kalman and Bertram, 1960)

(i) **Exponentially Asymptotically Stable** (EAS):
$\forall\ \epsilon > 0\ \exists\ \delta(\epsilon), \nu(\epsilon)$ s.t.

$$\|x_0\| \leq \delta \implies \|s(t; x_0, t_0)\| \leq \epsilon \mathbf{e}^{-\nu(t-t_0)}$$

(ii) **Exponentially Asymptotically Stable in the Large** (EASL): $\forall\ \rho > 0\ \exists\ \epsilon(\rho), \nu(\rho)$ s.t.

$$\|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \epsilon \mathbf{e}^{-\nu(t-t_0)}$$

(iii) **Exponentially Stable** (ES): $\forall\ \rho > 0\ \exists\ \nu(\rho), \kappa(\rho)$ s.t.

$$\|x_0\| \leq \rho \implies \|s(t; x_0, t_0)\| \leq \kappa \|x_0\| \mathbf{e}^{-\nu(t-t_0)}$$

(iv) **Exponentially Stable in the Large** (ESL): $\exists\ \nu, \kappa$ s.t.

$$\|s(t; x_0, t_0)\| \leq \kappa \|x_0\| \mathbf{e}^{-\nu(t-t_0)}$$

Rant about "uniform transients"