Bashing Geometry with Complex Numbers

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This is a (quick) English translation of the complex numbers note I wrote for Taiwan IMO 2014 training. Incidentally I was also working on an airplane.

1 The Complex Plane

Let $\mathbb{C}$ and $\mathbb{R}$ denote the set of complex and real numbers, respectively.

Each $z \in \mathbb{C}$ can be expressed as

$$z = a + bi = r\left(\cos\theta + i\sin\theta\right) = re^{i\theta}$$

where $a, b, r, \theta \in \mathbb{R}$ and $0 \le \theta < 2\pi$. We write $|z| = r = \sqrt{a^2+b^2}$ and $\arg z = \theta$.

More importantly, each z is associated with a conjugate $\overline{z} = a - bi$. It satisfies the properties

$$\begin{aligned} \overline{w \pm z} &= \overline{w} \pm \overline{z} \\ \overline{w \cdot z} &= \overline{w} \cdot \overline{z} \\ \overline{w/z} &= \overline{w}/\overline{z} \\ |z|^2 &= z \cdot \overline{z} \end{aligned}$$

Note that $z \in \mathbb{R} \iff z = \overline{z}$ and $z \in i\mathbb{R} \iff z + \overline{z} = 0$.

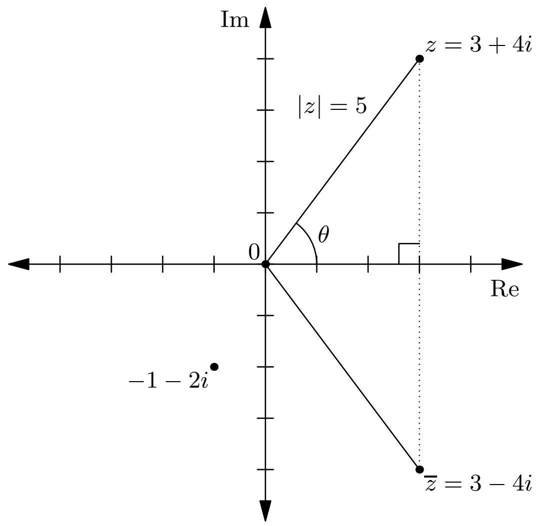


Figure 1: Points $z = 3+4i$ and $-1-2i$; $\overline{z} = 3-4i$ is the conjugate.

We represent every point in the plane by a complex number. In particular, we'll use a capital letter (like Z) to denote the point associated to a complex number (like z).

Complex numbers add in the same way as vectors. The multiplication is more interesting: for each $z_1, z_2 \in \mathbb{C}$ we have

$$|z_1 z_2| = |z_1|\,|z_2| \text{ and } \arg z_1 z_2 = \arg z_1 + \arg z_2.$$

This multiplication lets us capture a geometric structure. For example, for any points Z and W we can express rotation of Z at W by 90° as

$$z \mapsto i(z-w)+w.$$

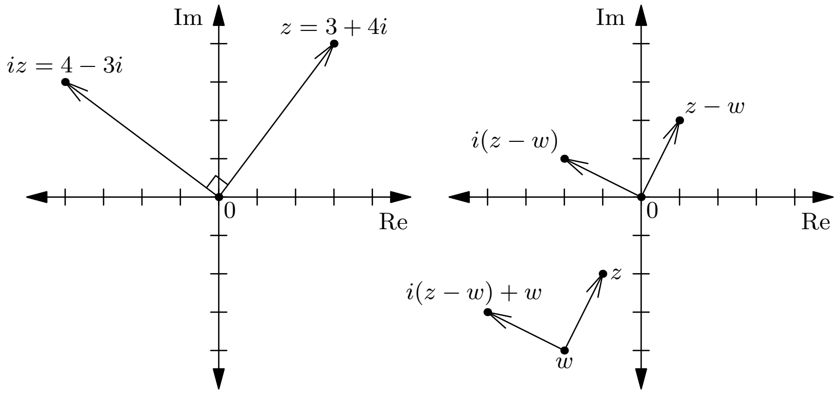


Figure 2: $z \mapsto i(z-w)+w$.

2 Elementary Propositions

First, some fundamental formulas:

Proposition 1. *Let A, B, C, D be pairwise distinct points. Then $\overline{AB} \perp \overline{CD}$ if and only if $\frac{d-c}{b-a} \in i\mathbb{R}$; i.e.*

$$\frac{d-c}{b-a} + \overline{\left(\frac{d-c}{b-a}\right)} = 0.$$

Proof. It's equivalent to $\frac{d-c}{b-a} \in i\mathbb{R} \iff \arg\left(\frac{d-c}{b-a}\right) \equiv \pm 90^\circ \iff \overline{AB} \perp \overline{CD}$ □

Proposition 2. *Let A, B, C be pairwise distinct points. Then A, B, C are collinear if and only if $\frac{c-a}{c-b} \in \mathbb{R}$; i.e.*

$$\frac{c-a}{c-b} = \overline{\left(\frac{c-a}{c-b}\right)}.$$

Proof. Similar to the previous one. □

Proposition 3. *Let A, B, C, D be pairwise distinct points. Then A, B, C, D are concyclic if and only if*

$$\frac{c-a}{c-b} : \frac{d-a}{d-b} \in \mathbb{R}.$$

Proof. It's not hard to see that $\arg\left(\frac{c-a}{c-b}\right) = \angle ACB$ and $\arg\left(\frac{d-a}{d-b}\right) = \angle ADB$. (Here angles are directed). □

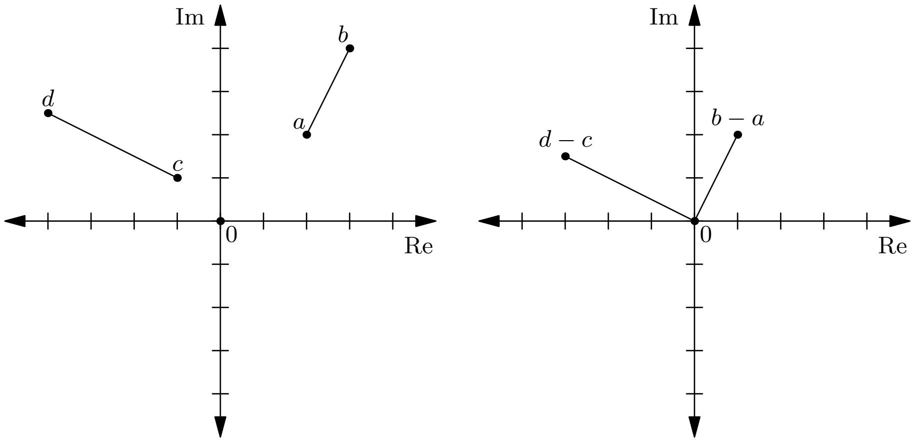


Figure 3: $\overline{AB} \perp \overline{CD} \iff \frac{d-c}{b-a} \in i\mathbb{R}$.

Now, let's state a more commonly used formula.

Lemma 4 (Reflection About a Segment)**.** *Let W be the reflection of Z across $\overline{AB}$. Then*

$$w = \frac{(a-b)\overline{z} + \overline{a}b - a\overline{b}}{\overline{a} - \overline{b}}.$$

Of course, it then follows that the foot from Z to $\overline{AB}$ is exactly $\frac{1}{2}(w+z)$.

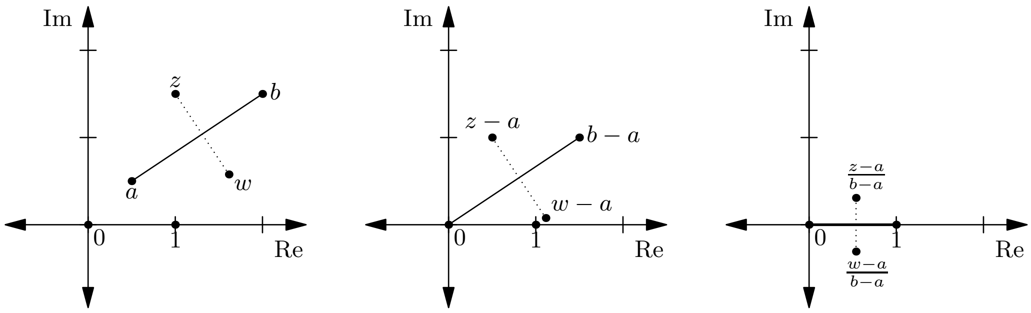


Figure 4: The reflection of Z across $\overline{AB}$.

Proof. According to Figure 4 we obtain

$$\frac{w-a}{b-a} = \overline{\left(\frac{z-a}{b-a}\right)} = \frac{\overline{z}-\overline{a}}{\overline{b}-\overline{a}}.$$

From this we derive $w = \frac{(a-b)\overline{z}+\overline{a}b-a\overline{b}}{\overline{a}-\overline{b}}$. □

Here are two more formulas.

Theorem 5 (Complex Shoelace)**.** *Let A, B, C be points. Then $\triangle ABC$ has signed area*

$$\frac{i}{4}\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}.$$

In particular, A, B, C are collinear if and only if this determinant vanishes.

Proof. Cartesian coordinates. □

Often, Theorem 5 is easier to use than Theorem 2.

Actually, we can even write down the formula for an arbitrary intersection of lines.

Proposition 6. *Let A, B, C, D be points. Then lines AB and CD intersect at*

$$\frac{(\bar{a}b - a\bar{b})(c-d) - (a-b)(\bar{c}d - c\bar{d})}{(\bar{a}-\bar{b})(c-d) - (a-b)(\bar{c}-\bar{d})}.$$

But unless $d = 0$ or a, b, c, d are on the unit circle, this formula is often too messy to use.

3 The Unit Circle, and Triangle Centers

On the complex plane, the **unit circle** is of critical importance. Indeed if $|z| = 1$ we have

$$\overline{z} = \frac{1}{z}.$$

Using the above, we can derive the following lemmas.

Lemma 7. *If $|a| = |b| = 1$ and $z \in \mathbb{C}$, then the reflection of Z across $\overline{AB}$ is $a + b - ab\overline{z}$, and the foot from Z to $\overline{AB}$ is*

$$\frac{1}{2}\left(z + a + b - ab\overline{z}\right).$$

Lemma 8. *If A, B, C, D lie on the unit circle then the intersection of $\overline{AB}$ and $\overline{CD}$ is given by*

$$\frac{ab(c+d) - cd(a+b)}{ab - cd}.$$

These are much easier to work with than the corresponding formulas in general. We can also obtain the triangle centers immediately:

Theorem 9. *Let ABC be a triangle center, and assume that the circumcircle of ABC coincides with the unit circle of the complex plane. Then the circumcenter, centroid, and orthocenter of ABC are given by 0, $\frac{1}{3}(a+b+c)$, $a+b+c$, respectively.*

Observe that the Euler line follows from this.

Proof. The results for the circumcenter and centroid are immediate. Let $h = a + b + c$. By symmetry it suffices to prove $\overline{AH} \perp \overline{BC}$. We may set

$$z = \frac{h-a}{b-c} = \frac{b+c}{b-c}.$$

Then

$$\overline{z} = \overline{\left(\frac{b+c}{b-c}\right)} = \frac{\overline{b}+\overline{c}}{\overline{b}-\overline{c}} = \frac{\frac{1}{b}+\frac{1}{c}}{\frac{1}{b}-\frac{1}{c}} = \frac{c+b}{c-b} = -z$$

so $z \in i\mathbb{R}$ as desired. □

We can actually even get the formula for the incenter.

Theorem 10. *Let triangle ABC have incenter I and circumcircle Γ. Lines AI, BI, CI meet Γ again at D, E, F. If Γ is the unit circle of the complex plane then there exists $x, y, z \in \mathbb{C}$ satisfying*

$$a = x^2, b = y^2, c = z^2 \text{ and } d = -yz, e = -zx, f = -xy.$$

Note that $|x| = |y| = |z| = 1$. Moreover, the incenter I is given by $-(xy + yz + zx)$.

Proof. Show that I is the orthocenter of $\triangle DEF$. □

4 Some Other Lemmas

Lemma 11. *Let A, B be on the unit circle and select P so that $\overline{PA}$, $\overline{PB}$ are tangents. Then*

$$p = \frac{2}{\overline{a} + \overline{b}} = \frac{2ab}{a+b}.$$

Proof. Let M be the midpoint of $\overline{AB}$ and set $O = 0$. One can show $OM \cdot OP = 1$ and that O, M, P are collinear; the result follows from this. □

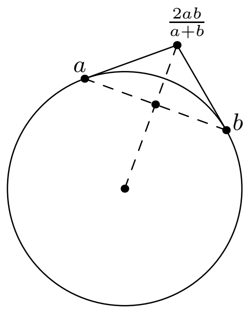


Figure 5: Two tangents. $p = \frac{2}{\overline{a}+\overline{b}}$.

Lemma 12. *For any x, y, z, the circumcenter of $\triangle XYZ$ is given by*

$$\begin{vmatrix} x & x\bar{x} & 1 \\ y & y\bar{y} & 1 \\ z & z\bar{z} & 1 \end{vmatrix} \div \begin{vmatrix} x & \bar{x} & 1 \\ y & \bar{y} & 1 \\ z & \bar{z} & 1 \end{vmatrix}.$$

This formula is often easier to apply if we shift z to the point 0 first, then shift back afterwards.

5 Examples

Example 13 (MOP 2006)**.** Let H be the orthocenter of triangle ABC. Let D, E, F lie on the circumcircle of ABC such that $\overline{AD} \parallel \overline{BE} \parallel \overline{CF}$. Let S, T, U respectively denote the reflections of D, E, F across $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Prove that points S, T, U, H are concyclic.

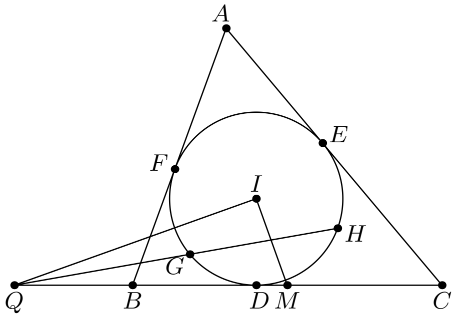
Proof. Let (ABC) be the unit circle and $h = a + b + c$. WLOG, $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are perpendicular to the real axis (rotate appropriately); thus $d = \overline{a}$ and so on. Thus $s = b + c - bc\overline{d} = b + c - abc$ and so on; we now have

$$\frac{s-t}{s-u} = \frac{b-a}{c-a} \quad \text{and} \quad \frac{h-t}{h-u} = \frac{b+abc}{c+abc}.$$

Compute

$$\frac{s-t}{s-u} : \frac{h-t}{h-u} = \frac{(b-a)(c+abc)}{(c-a)(b+abc)} = \frac{\left(\frac{1}{b} - \frac{1}{a}\right)\left(\frac{1}{c} + \frac{1}{abc}\right)}{\left(\frac{1}{c} - \frac{1}{a}\right)\left(\frac{1}{b} + \frac{1}{abc}\right)} \implies \frac{s-t}{s-u} : \frac{h-t}{h-u} \in \mathbb{R}$$

as desired. □



Example 14 (Taiwan TST 2014)**.** In $\triangle ABC$ with incenter I, the incircle is tangent to $\overline{CA}$, $\overline{AB}$ at E, F. The reflections of E, F across I are G, H. Let Q be the intersection of $\overline{GH}$ and $\overline{BC}$, and let M be the midpoint of $\overline{BC}$. Prove that $\overline{IQ}$ and $\overline{IM}$ are perpendicular.

Solution. Let D be the foot from I to $\overline{BC}$, and set (DEF) as the unit circle. (This lets us exploit the results of Section 3.) Thus $|d| = |e| = |f| = 1$, and moreover $g = -e$, $h = -f$. Let $x = \overline{d} = \frac{1}{d}$ and define y, z similarly. Then

$$b = \frac{2}{\overline{d} + \overline{f}} = \frac{2}{x+z}.$$

Similarly, $c = \frac{2}{x+y}$, so

$$m = \frac{1}{2}(b+c) = \frac{1}{x+y} + \frac{1}{x+z} = \frac{2x+y+z}{(x+y)(x+z)}.$$

Next, we have $Q = DD \cap GH$, which implies

$$q = \frac{dd(g+h) - gh(d+d)}{d^2 - gh} = \frac{\frac{1}{x^2}\left(-\frac{1}{y} - \frac{1}{z}\right) - \frac{1}{yz}\frac{2}{x}}{\frac{1}{x^2} - \frac{1}{yz}} = \frac{2x+y+z}{x^2 - yz}.$$

so

$$m/q = \frac{x^2 - yz}{(x+y)(x+z)}.$$

Now,

$$\overline{m/q} = \frac{\frac{1}{x^2} - \frac{1}{yz}}{\left(\frac{1}{x} + \frac{1}{y}\right)\left(\frac{1}{x} + \frac{1}{z}\right)} = \frac{yz - x^2}{(x+y)(x+z)} = -m/q$$

thus $m/q \in i\mathbb{R}$, as desired. □

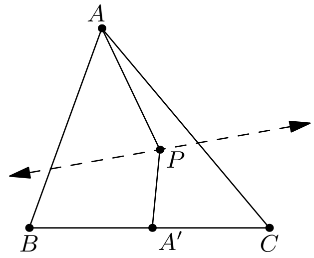
Example 15 (USAMO 2012)**.** Let P be a point in the plane of $\triangle ABC$, and γ a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A', B', C' are collinear.

Solution. Let $p = 0$ and set γ as the real line. Then A' is the intersection of bc and $p\bar{a}$. So, using Theorem 6 we get

$$a' = \frac{\bar{a}(\bar{b}c - b\bar{c})}{(\bar{b} - \bar{c})\bar{a} - (b-c)a}.$$

Note that

$$\bar{a}' = \frac{a(b\bar{c} - \bar{b}c)}{(b-c)a - (\bar{b} - \bar{c})\bar{a}}.$$



Thus by Theorem 5, it suffices to prove

$$0=\begin{vmatrix} \frac{\bar a(\bar bc-b\bar c)}{(\bar b-\bar c)\bar a-(b-c)a} & \frac{a(b\bar c-\bar bc)}{(b-c)a-(\bar b-\bar c)\bar a} & 1 \\ \frac{\bar b(\bar ca-c\bar a)}{(\bar c-\bar a)\bar b-(c-a)b} & \frac{b(c\bar a-\bar ca)}{(c-a)b-(\bar c-\bar a)\bar b} & 1 \\ \frac{\bar c(\bar ab-a\bar b)}{(\bar a-\bar b)\bar c-(a-b)c} & \frac{c(a\bar b-\bar ab)}{(a-b)c-(\bar a-\bar b)\bar c} & 1 \end{vmatrix}.$$

This is equivalent to

$$0=\begin{vmatrix} \bar a(\bar bc-b\bar c) & a(\bar bc-b\bar c) & (\bar b-\bar c)\bar a-(b-c)a \\ \bar b(\bar ca-c\bar a) & b(\bar ca-c\bar a) & (\bar c-\bar a)\bar b-(c-a)b \\ \bar c(\bar ab-a\bar b) & c(\bar ab-a\bar b) & (\bar a-\bar b)\bar c-(a-b)c \end{vmatrix}.$$

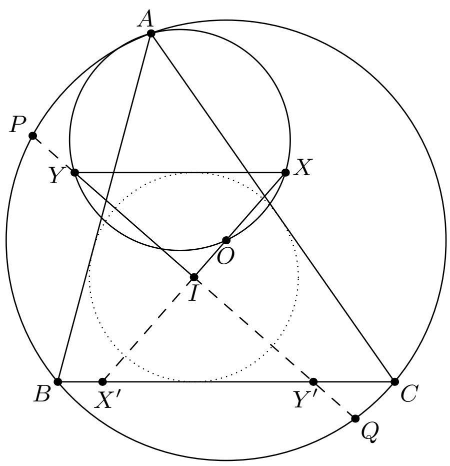
Evaluating the determinant gives

$$\sum_{\text{cyc}}((\bar b-\bar c)\bar a-(b-c)a)\cdot-\begin{vmatrix} b & \bar b \\ c & \bar c \end{vmatrix}\cdot(\bar ca-c\bar a)\left(\bar ab-a\bar b\right)$$

or, noting the determinant is $b\bar c-\bar bc$ and factoring it out,

$$(\bar bc-c\bar b)(\bar ca-c\bar a)(\bar ab-a\bar b)\sum_{\text{cyc}}\left(ab-ac+\bar c\bar a-\bar b\bar a\right)=0. \qquad \square$$

Example 16 (Taiwan TST Quiz 2014)**.** Let I and O be the incenter and circumcenter of ABC. A line ℓ is drawn parallel to $\overline{BC}$ and tangent to the incircle of ABC. Let X, Y be on ℓ so that I, O, X are collinear and $\angle XIY=90^\circ$. Show that A, X, O, Y are concyclic.



Solution. Let X' and Y' respectively denote the reflections of X and Y across I. Note that X, Y lie on $\overline{BC}$. Also, let P, Q be the intersections of $\overline{IY}$ with the circumcircle.

Of course, (ABC) is the unit circle. Let j be the complex number corresponding to I (to avoid confusion with $i = \sqrt{-1}$). Thus,

$$x' = \frac{\left(\overline{b}c - b\overline{c}\right)(j-0) - \left(\overline{j}0 - j\overline{0}\right)(b-c)}{(\overline{b}-\overline{c})(j-0) - (b-c)(\overline{j}-\overline{0})} = \frac{j \cdot \frac{c^2-b^2}{bc}}{j \cdot \frac{c-b}{bc} - (b-c)\overline{j}} = \frac{j(b+c)}{j+bc\overline{j}}.$$

We seek y' now. Consider the quadratic equation in z given by

$$\frac{z-j}{j} + \frac{\frac{1}{z} - \overline{j}}{\overline{j}} = 0 \iff z^2 - 2jz + j/\overline{j} = 0.$$

Its zeros in z are p and q, which implies that $p+q = 2j$ and $pq = j/\overline{j}$ (by Vieta!). From this we can compute

$$y' = \frac{pq(b+c) - bc(p+q)}{pq - bc} = \frac{j(b+c) - 2bcj\overline{j}}{j - bc\overline{j}} = \frac{j(b+c) - 2bcj\overline{j}}{j - bc\overline{j}}.$$

which gives

$$x = 2j - x' = \frac{j(2j-b-c+2bc\overline{j})}{j+bc\overline{j}} \quad \text{and} \quad y = 2j - y' = \frac{j(2j-b-c)}{j-bc\overline{j}}.$$

From this we can obtain

$$\begin{aligned} y - x &= j \cdot \frac{(2j-b-c)(j+bc\overline{j}) - (2j-b-c+2bc\overline{j})(j-bc\overline{j})}{(j-bc\overline{j})(j+bc\overline{j})} \\ &= j \cdot \frac{2bc\overline{j}(2j-b-c) - 2bc\overline{j}(j-bc\overline{j})}{(j-bc\overline{j})(j+bc\overline{j})} \\ &= j \cdot \frac{2bc\overline{j}\left(j-b-c+bc\overline{j}\right)}{(j-bc\overline{j})(j+bc\overline{j})} \\ X = \frac{y-x}{x} &= \frac{2bc\overline{j}\left(j-b-c+bc\overline{j}\right)}{(j-bc\overline{j})(2j-b-c+2bc\overline{j})} \\ A = \frac{y-a}{a} &= \frac{j(2j-b-c-a) + abc\overline{j}}{a(j-bc\overline{j})} \end{aligned}$$

We need to prove $X/A = \overline{X/A}$. Now set $a = x^2$, $b = y^2$, $c = z^2$, $j = -(xy+yz+zx)$, $\overline{j} = -\frac{x+y+z}{xyz}$ (this is a different x, y than the points X and Y.) So, the above rewrites as

$$\begin{aligned} X &= \frac{2\frac{yz}{x}(x+y+z)(\frac{yz}{x}(x+y+z) + y^2 + z^2 + xy + yz + zx)}{\left(-\frac{yz}{x}(x+y+z) + xy + yz + zx\right)\left(y^2 + z^2 + 2(xy+yz+zx) + 2\frac{yz}{x}(x+y+z)\right)} \\ &= \frac{2yz(x+y+z)\left(2xyz + \sum_{\text{sym}} x^2y\right)}{(y+z)(x^2-yz)\left(x(y+z)(2x+y+z) + 2yz(x+y+z)\right)} \\ &= \frac{2yz(x+y+z)(x+y)(x+z)}{(x^2-yz)\left((x^2+yz)(y+z) + (xy+yz+zx)(x+y+z)\right)} \end{aligned}$$

and

$$A = \frac{(xy+yz+zx)(x+y+z)^2 - xyz(x+y+z)}{x^2(-(xy+yz+zx) + \frac{yz}{x}(x+y+z))} = \frac{(x+y+z)(x+y)(y+z)(z+x)}{x(yz-x^2)(y+z)}$$

thus

$$X/A = \frac{-2xyz}{(x^2+yz)(y+z)+(x+y+z)(xy+yz+zx)}$$
$$= \frac{-\frac{2}{xyz}}{(\frac{1}{x^2}+\frac{1}{yz})(\frac{1}{y}+\frac{1}{z})+(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx})} = \overline{X/A}. \qquad \square$$

6 Practice Problems

1. Let $ABCD$ be cyclic. Let H_A, H_B, H_C, H_D denote the orthocenters of BCD, CDA, DAB, ABC. Show that $\overline{AH_A}$, $\overline{BH_B}$, $\overline{CH_C}$, $\overline{DH_D}$ are concurrent.

2. (China TST 2011) Let Γ be the circumcircle of a triangle ABC. Assume $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ are diameters of Γ. Let P be a point inside ABC and let D, E, F be the feet from P to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Let X be the reflection of A' across D; define Y and Z similarly. Prove that $\triangle XYZ \sim \triangle ABC$.

3. In circumscribed quadrilateral $ABCD$ with incircle ω, Prove that the midpoint of $\overline{AC}$ and the midpoint of $\overline{BD}$ are collinear with the center of ω.

4. (Simson Line) Let ABC be a triangle and P a point on its circumcircle.
   (a) Let D, E, F be the feet from P to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Show that D, E, F are collinear.
   (b) Moreover, prove that the line through these points bisects $\overline{PH}$, where H is the orthocenter of ABC.

5. (PUMaC Finals) Let γ and I be the incircle and incenter of triangle ABC. Let D, E, F be the tangency points of γ to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and let D' be the reflection of D about I. Assume EF intersects the tangents to γ at D and D' at points P and Q. Show that $\angle DAD' + \angle PIQ = 180^\circ$.

6. (Schiffler Point) Let triangle ABC have incenter I. Prove that the Euler lines of $\triangle AIB$, $\triangle BIC$, $\triangle CIA$, $\triangle ABC$ are concurrent.

7. (USA TST 2014) Let $ABCD$ be a cyclic quadrilateral and let E, F, G, H be the midpoinst of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$. Call W, X, Y, Z the orthocenters of AHE, BEF, CFG, DGH. Prove that $ABCD$ and $WXYZ$ have the same area.

8. Let O be the circumcenter of ABC. A line ℓ through O cuts $\overline{AB}$ and $\overline{AC}$ at points X and Y. Let M and N be the midpoints of $\overline{BY}$, $\overline{CX}$. Show that $\angle MON = \angle BAC$.

9. (APMO 2010) Let ABC be an acute triangle, where $AB > BC$ and $AC > BC$. Denote by O and H the circumcenter and orthocenter. The circumcircle of AHC intersects AB again at M; the circumcircle of AHB intersects AC again at N. Prove that the circumcenter of triangle MNH lies on line OH.

10. (Iran 2013) Let ABC be acute, and M the midpoint of minor arc $\widehat{BC}$. Let N be on the circumcircle of ABC such that $\overline{AN} \perp \overline{BC}$, and let K, L lie on AB, AC so that $\overline{OK} \parallel \overline{MB}$, $\overline{OL} \parallel \overline{MC}$. (Here O is the circumcenter of ABC). Prove that $NK = NL$.

11. (MOP 2006) Cyclic quadrilateral $ABCD$ has circumcenter O. Let P be a point in the plane and let O_1, O_2, O_3, O_4 be the circumcenters of PAB, PBC, PCD, PDA. Show that $\overline{O_1O_3}$, $\overline{O_2O_4}$, $\overline{OP}$ are concurrent.