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Hermitian and Normal Completions

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The problems of finding the minimal and maximal possible negative and positive inertia for partial Hermitian banded matrices has been solved without any non singularity restrictions on the given blocks. The solution has striking similarities with minimal rank completion results. Applications to Hermitian matrix (in)equalities are presented. In addition, the minimal normal completion problem is introduced.

Keywords: Partial matrices; Hermitian completions; normal completions; matrix inequalities; inertia; subnormal

AMS Subject Classification: 15A57, 15A24, 15A42

1. INTRODUCTION

Hermitian completion problems concern partial Hermitian matrices for which completions are to be found that have certain requirements on the eigenvalues. Recall that a partial $n \times m$ (complex) matrix is an $n \times m$ array in which some entries are specified complex numbers while the other entries are free variables ranging over the complex plane $\mathbb{C}$ ("the unspecified entries"). These unspecified entries are usually denoted by ?. A completion of a partial matrix is the complex

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A partial matrix is called Hermitian if entry \((i, j)\) is a specified complex number \(a_{ij}\) if and only if entry \((j, i)\) is a specified complex number \(a_{ji}\) and \(a_{ji} = \overline{a_{ij}}\). The above partial matrix is an example of a Hermitian partial matrix. In this paper we are interested in the possible inactivity of Hermitian completions of partial Hermitian matrices. Recall that the inertia of a \(n \times n\) Hermitian matrix \(M\) is a triple \((i_{<0}(M), i_{=0}(M), i_{>0}(M))\) of natural numbers in which \(i_{<0}(M), i_{=0}(M), i_{>0}(M)\) denotes the number of negative, zero, positive eigenvalues of \(M\), respectively. We shall also use

\[
i_{\leq 0}(M) = i_{<0}(M) + i_{=0}(M), \quad i_{\geq 0}(M) = i_{=0}(M) + i_{>0}(M),
\]

which we shall refer to as the nonpositive and nonnegative inertia of \(M\), respectively. Note that for a \(n \times n\) Hermitian matrix \(M\) we have that

\[
i_{<0}(M) + i_{=0}(M) + i_{>0}(M) = n.
\]

The study of possible inertia of Hermitian completions of partial matrices goes back at least to [4] and [3] in which the authors deal with \(2 \times 2\) and \(n \times n\) partial block diagonal Hermitian matrices, respectively. Around the same time the paper [11] appeared which considered the positive definite completion problem for partial banded matrices, motivated as a matrix analogue of the classical Carathéodory-Toeplitz problem. Since then, especially triggered by [11], many papers have
followed treating different aspects of the positive definite and Hermitian completion problems. Most pertinent to the present paper are [21], [10], [15], [7] and [5].

In this paper we solve the problem of finding the minimal negative and positive inertia among all Hermitian completions of a partial banded Hermitian matrix. With this we extend the results of [15] (see also [10]) and [7] where the $3 \times 3$ tridiagonal block matrix and the $n \times n$ tridiagonal block matrix case were treated. Earlier, in [21], the case was treated where the maximal given principal blocks are nonsingular. The minimal negative inertia result here (Theorem 3.4) was triggered by a discussion with T. Constantinescu in which he conjectured the validity of the current result. The proof uses ideas from [5] and [31]. In addition, we solve the problem of minimal nonpositive and non-negative inertia among all completions (Theorem 3.5). The $3 \times 3$ block case may be found in [5], where the authors give in fact all possible inertia for the $3 \times 3$ block case. Other related papers are [18], [19], [12] and [25].

As observed before (see e.g., [31], [2] and [5]) results concerning matrix completions may often be applied to matrix equations or inequalities. We shall illustrate in Section 4 how the new Hermitian completion results may be applied to Hermitian matrix equalities and inequalities, such as Riccati-type equalities.

We shall end the paper with the normal completion problem. Since normality of a matrix $A$ is described via the matrix equation $AA^* = A^*A$, one may use the results in Section 4 to view the normal completion as a special Hermitian completion problem. The notion of subnormality may be viewed as an extended normal completion problem. This connection will be the inspiration for some of the questions raised and partially answered.

2. SYMMETRIC BANDED AND TRIANGULAR PATTERNS

In this section we shall develop results for banded patterns that are needed to state and prove our main result.

For $n \in \mathbb{N}$ we let $n$ denote the set $n = \{1, \ldots, n\}$. A $(n \times m)$ pattern $J$ is a subset $J$ of $\hat{n} \times \hat{m}$. If $J \subseteq \hat{I}$ we call $J$ a subpattern of $\hat{I}$. For $I \subseteq \mathbb{N}$ we will denote $I \times I$ by $I^2$. A pattern $J \subseteq \hat{n}^2$ is called symmetric if $(i, j) \in J$. 


Conversely, if a pattern $J \subseteq \mathbb{N}^2$ can be written in the form $\text{(2.1)}$ for some choice of $s$, $n_1, \ldots, n_s$, $m_1, \ldots, m_s$, and $J_k \subseteq n_k \times m_k$ ($k = 1, \ldots, s$), we say that $J$ decomposes as the direct sum

$$J_1 \oplus \cdots \oplus J_s \subseteq N \times M$$

with $N = \sum_{i=1}^s n_p$ and $M = \sum_{i=1}^s m_p$, as the pattern

$$J_1 \oplus \cdots \oplus J_s = \bigcup_{p=1, \ldots, s} \left\{ \left( i + \sum_{k=1}^{i-1} n_k, j + \sum_{k=1}^{j-1} m_k \right) : (i, j) \in J_s \right\}. \tag{2.1}$$

Conversely, if a pattern $J$ can be written in the form (2.1) for some choice of $s, n_1, \ldots, n_s, m_1, \ldots, m_s$, and $J_k \subseteq n_k \times m_k$ ($k = 1, \ldots, s$), we say that $J$ decomposes as the direct sum $J_1 \oplus \cdots \oplus J_s$, and refer to the patterns $J_1, \ldots, J_s$ as the summands.

**Lemma 2.1** Each symmetric banded pattern $J \subseteq \mathbb{N}^2$ can be decomposed as a direct sum $J_1 \oplus \cdots \oplus J_s$, where each summand $J_k$, $k = 1, \ldots, s$, is a symmetric banded pattern in $\mathbb{N}^2$ which is either the empty set or contains the diagonal $D_{n_k}$.

This result is easy to verify, and is left to the reader. Based on this lemma we may often assume that a symmetric banded pattern contains the diagonal.

Similar as for the minimal rank completion problem (see [31]) triangular subpatterns of a banded pattern will play an important role here as well. Recall that a pattern $T$ is called triangular if $(i, j) \in T$ and $(k, l) \in T$ imply that $(i, l) \in T$ or $(k, j) \in T$. 

The following lemma describes triangular patterns perhaps somewhat more intuitively.

**Lemma 2.2** Let \( T \subseteq \mathbb{N} \times m \) be a pattern. Denote \( \text{col}_j(T) = \{i : (i, j) \in T\} \). Then \( T \) is triangular if and only if for all \( j_1, j_2 \in m \) we have \( \text{col}_{j_1}(T) \subseteq \text{col}_{j_2}(T) \) or \( \text{col}_{j_2}(T) \subseteq \text{col}_{j_1}(T) \). In other words, the pattern \( T \) is triangular if and only if the sets \( \text{col}_j(T), j \in m, \) may be linearly ordered by inclusion.

**Proof** Suppose there exist \( i_1 \in \text{col}_{j_1}(T) \setminus \text{col}_{j_2}(T) \) and \( i_2 \in \text{col}_{j_2}(T) \setminus \text{col}_{j_1}(T) \). Then \( (i_1, j_1) \in T \) and \( (i_2, j_2) \in T \) but \( (i_1, j_2) \notin T \) and \( (i_2, j_1) \notin T \). Thus \( T \) is not triangular. The converse follows just as easily.

Clearly, an analogous result holds for the “rows” of a triangular pattern \( T \).

A pattern \( R \subseteq \mathbb{N} \times m \) is called rectangular if it is of the form \( R = K \times L \) with \( K \subseteq \mathbb{N} \) and \( L \subseteq m \); equivalently, \( R \) is rectangular if \( (i, j) \in R \) and \( (k, l) \in R \) imply that \( (i, l) \in R \) (and \( (k, j) \in R \). The sets \( K \) and \( L \) will be referred to as the rows and columns covered by \( R \), respectively. Note that \( R \) is symmetric rectangular (that is, symmetric and rectangular) if and only if it is of the form \( K \times K \). A (symmetric) rectangular pattern will also be referred to as a (symmetric) rectangle.

Lastly, we introduce the notion of maximality in a pattern with respect to a property “special”. For “special” one should think of rectangular, symmetric rectangular, triangular, or symmetric triangular. Given two patterns \( T \) and \( J \), we say that \( T \) is maximal special in \( J \) if

1. \( T \subseteq J \)
2. \( T \) is special
3. if \( T \subseteq S \subseteq J \) and \( S \) is special, then \( T = S \).

**Lemma 2.3** Let \( T \) be a symmetric triangular pattern. If \( P \times P \subseteq T \) and \( Q \times Q \subseteq T \), then \( (P \cup Q) \times (P \cup Q) \subseteq T \).

**Proof** Let \( p, q \in P \cup Q \). Then \( (p, p) \) and \( (q, q) \) belong to \( (P \times P) \cup (Q \times Q) \subseteq T \). Since \( T \) is triangular either \( (p, q) \in T \) or \( (q, p) \in T \). Since \( T \) is symmetric, we get in fact that both \( (p, q) \) and \( (q, p) \) belong to \( T \).

**Corollary 2.4** Each symmetric triangular pattern \( T \) has a unique maximal symmetric rectangle.
The unique maximal symmetric rectangle of a symmetric triangular pattern \( T \) will be denoted by \( R_{\text{sym}}(T) \). The rows/columns covered by \( R_{\text{sym}}(T) \) are denoted by \( rs(T) \). Thus

\[
R_{\text{sym}}(T) = rs(T) \times rs(T) = rs(T)^2.
\]

**Lemma 2.5** If \( T \) is a symmetric triangular pattern, then for all \((r, s) \in T \) we have \( r \in rs(T) \) or \( s \in rs(T) \) (i.e., \( T \subseteq (rs(T) \times n) \cup (n \times rs(T)) \)).

**Proof** Let \((r, s) \in T \) and consequently either \((r, r) \subseteq T \) or \((s, s) \subseteq T \). Let us say that \((r, r) \in T \). By Lemma 2.3 (with \( P = \{r\} \) and \( Q = rs(T) \)) we get that \((\{r\} \times rs(T))^2 \subseteq T \). Since \( rs(T)^2 \) is maximal symmetric rectangular in \( T \) it follows that \( r \in rs(T) \).

It follows from Lemma 2.5 that if \( T \neq \emptyset \) then \( rs(T) \neq \emptyset \) as well.

**Lemma 2.6** If \( T \) is a symmetric triangular pattern and \( rs(T) \subseteq P \), then so is \( T \cup P^2 \).

**Proof** Since both \( T \) and \( P^2 \) are triangular and symmetric, it suffices to show that \((r, s) \in T \) and \((p, q) \in P^2 \) imply that \((r, q) \in T \cup P^2 \) or \((p, s) \in T \cup P^2 \). So suppose that \((r, s) \in T \) and \((p, q) \in P^2 \). By Lemma 2.5 we have that \( r \in rs(T) \) or \( s \in rs(T) \). But then either \((r, q) \in rs(T) \times P \subseteq P^2 \subseteq T \cup P^2 \) or \((p, s) \in T \cup P^2 \).

For a symmetric banded pattern \( J \) we let \( T(J) \) denote the set of maximal symmetric triangular patterns in \( J \).

**Corollary 2.7** Let \( J \) be a symmetric banded pattern and \( T \in T(J) \). Then \( rs(T)^2 \) is a maximal symmetric rectangle in \( J \).

**Proof** Choose a maximal symmetric rectangle \( R \) in \( J \) containing \( R_{\text{sym}}(T) \). By Lemma 2.6 the pattern \( T \cup R \) is also a symmetric triangular pattern in \( J \). By maximality of \( T \) in \( J \), we must have that \( T \cup R = T \), thus \( R \subseteq T \). But then it follows from Corollary 2.4 that \( R = R_{\text{sym}}(T) \).

For a symmetric triangular pattern \( T \subseteq n \times n \) we let \( \mathcal{R}(T) \) denote the set of maximal rectangles in \( T \cap (n \times rs(T)) \) together with \( rs(T)^2 \) (if not already present). Note that \( \mathcal{R}(T) \) does not contain all maximal rectangles in \( T \). The following lemma shows that \( rs(T)^2 \) is always
present among the maximal rectangles in $T \cap (n \times rs(T))$ when $T$ is a maximal triangular pattern in a symmetric banded one.

**Lemma 2.8** For any nonempty symmetric banded pattern $J \subseteq n \times n$ and any $T \in T(J)$ we have that $R_{\text{symm}}(T)$ is a maximal rectangle in $T \cap (n \times rs(T))$.

**Proof** Suppose

$$R_{\text{symm}}(T) \subseteq R \subseteq T \cap (n \times rs(T)),$$

with $R$ rectangular. Then $R - P \times rs(T)$ with $rs(T) \subseteq P$. Suppose there exists a $p \in P_{rs(T)}$. Then $((p) \cup rs(T)) \times rs(T) \subseteq T$, and since $T$ is symmetric also $rs(T) \times \{(p) \cup rs(T)\} \subseteq T$.

We claim that $((p, p)) \cup T$ is still triangular. Indeed, if $(r, s) \in T$, we get by Lemma 2.5 that $r \in rs(T)$ or $s \in rs(T)$. But then $(r, p) \in rs(T) \times (\{(p) \cup rs(T)\}) \subseteq T \cup \{(p, p)\}$ or $(p, s) \in T \cup \{(p, p)\}$, proving that $T \cup \{(p, p)\}$ is triangular. Since $p \notin rs(T) \neq \emptyset$, there exists an $r \in rs(T)$ with $r < p$ or $r > p$. Without loss of generality we assume that $r < p$. Then $(r, p) \in T \subseteq J$ and $(p, r) \in T \subseteq J$, thus, since $J$ is banded,

$$\{r, \ldots, p\} \times \{r, \ldots, p\} \subseteq J.$$

In particular, $(p, p) \in J$. But then, $T \cup \{(p, p)\}$ is a symmetric triangular pattern in $J$, and thus by maximality of $T$ it follows that $(p, p) \in T$. Lemma 2.5 now implies that $p \in rs(T)$. Contradiction. 

Not for every symmetric triangular pattern $T$ we have that $rs(T)^2$ is a maximal rectangle in $T \cap (n \times rs(T))$. E.g., $T = \{(1, 1), (1, 2), (2, 1)\} \subseteq 2^2$.

The following proposition supplies us with the main tools to state our results. For a set $X$ we let $|X|$ denote its cardinality.

**Proposition 2.9** Let $T$ be a symmetric triangular pattern. Then the members of $R(T)$ can be ordered uniquely as $R_1, \ldots, R_m$ (with $m = |R(T)|$) such that the number of rows that $R_i$ covers is strictly smaller than the number of rows that $R_{i+1}$ covers, $i = 1, \ldots, m-1$. Moreover, $R_1 = R_{\text{symm}}(T)$. 

Proof  Since $T \cap (n \times rs(T))$ is a triangular pattern, Lemma 2.2 yields that we can order the members of $rs(T)$ as $\{j_1, \ldots, j_s\}$ such that
\[
\text{col}_{j_1}(T) \subseteq \text{col}_{j_2}(T) \subseteq \cdots \subseteq \text{col}_{j_s}(T).
\] (2.2)

Note that from Lemma 2.8 it follows that $rs(T) \subseteq \text{col}_{j_s}(T)$. If $rs(T) = \text{col}_{j_s}(T)$, put $\tilde{m} = |R(T)|$. Otherwise put $\tilde{m} = |R(T)| - 1$. Write now
\[
rs(T) = \{j_1, \ldots, j_{n_1}\} \cup \{j_{n_1+1}, \ldots, j_{n_2}\} \cup \cdots \cup \{j_{n_{s-1}+1}, \ldots, j_{n_s}\}
\]
\[
=: P_1 \cup \cdots \cup P_{\tilde{m}},
\]
where $n_1, \ldots, n_{\tilde{m}} = s$ are such that
\[
\text{col}_{j_{n_i}}(T) \neq \text{col}_{j_{n_{i+1}}}(T).
\]
Choose now $R_i = R_{\text{symm}}$, and
\[
R_{i+\varepsilon} = \text{col}_{j_{n_i}} \times (P_1 \cup \cdots \cup P_i), \quad i = 1, \ldots, \tilde{m},
\]
where $\varepsilon = |R(T)| - \tilde{m}$. It is easy to verify that this choice of $R_i$ has the required properties.

An example is useful at this point.

Example 2.10  Let $J = \{(i, j) \in Z \times Z; |i-j| \leq 2\}$ and $T = J \setminus (2 \times 2) \in T(J)$. This may be depicted as
\[
\begin{array}{ccccccc}
\cdot & \cdot & \cdot & x \\
\cdot & \cdot & x & x \\
x & x & x & x & x \\
x & x & x & x \\
x & x & x & x \\
\end{array}
\]

Then
\[
R(T) = \left\{ (3, 4, 5)^2, (2, 3, 4, 5) \times (3, 4), Z \times \{3\} \right\}.
\]

We need to develop some further properties regarding banded patterns and their maximal symmetric triangular subpatterns, which will be needed in the proof.
Lemma 2.11 Let $J$ be a symmetric handed pattern in $\mathbb{R}^2$, and suppose that for $i_1, i_2 \in \mathbb{N}$ we have that

$$i_2 \in \{j : (i_1, j) \in J\} \subseteq \{j : (i_2, j) \in J\}.$$

Then $(i_1, j) \in T \in T(J)$ implies $(i_2, j) \in T$. In other words, for every $T \in T(J)$ we have that

$$\{j : (i_1, j) \in T\} \subseteq \{j : (i_2, j) \in T\}.$$

Proof Let $J, i_1, i_2$ and $T$ be as above, and denote $K = \{j : (i_1, j) \in J\}$. Then $i_2 \in K$. We will show that

$$T \cup (\{i_2\} \times K) \cup (K \times \{i_2\})$$

is still triangular. Since $(\{i_2\} \times K) \cup (K \times \{i_2\})$ is triangular (since $i_2 \in K$), it suffices to show that $(p, q) \in T$ and $k \in K$ imply that $(p, k)$ or $(i_2, q)$ belongs to (2.3). To this end let $p, q,$ and $k$ be as above. Since $(i_1, k) \in T$ either $(i_1, q) \in T$ or $(p, k) \in T$. In the latter case we are done, so assume $(i_1, q) \in T \subseteq J$. But then $q \in K$, and thus $(i_2, q)$ belongs to (2.3).

Lemma 2.12 Let $\Delta$ be a triangular pattern in $\mathbb{R}^2$ and let $i_1, i_2 \in \mathbb{N}$ be such that

$$\{j : (i_1, j) \in \Delta\} \subseteq \{j : (i_2, j) \in \Delta\}.$$ 

If $(i_1, j)$ belongs to a maximal rectangle $R$ in $\Delta$, then $(i_2, j)$ belongs also to $R$. In other words, for every maximal rectangle $R$ of $\Delta$ we have that

$$\{j : (i_1, j) \in R\} \subseteq \{j : (i_2, j) \in R\}.$$ 

Proof Let $\Delta, i_1, i_2$, and $R = K \times L$ be as above. If the set on the left hand side of (2.5) is empty, then there is nothing to prove. So assume that $\{j : (i_1, j) \in R\}$ is nonempty, which then implies that this set equals $L$. Thus $L$ is a subset of the left hand side of (2.4), and thus also of the right hand side of (2.4). Consequently,

$$\{i_2\} \cup K \times L \subseteq \Delta.$$
Since $R$ is a maximal rectangle in $\Delta$, we must have that $i_2 \in K$. Thus \( \{i_2\} \times L \subseteq R \), proving the claim.

Note that in Lemma 2.12 we did not require $\Delta$ to be symmetric. We shall apply the result to $\Delta = T \cap (\pi \times rs(T))$, where $T$ is a symmetric triangular pattern.

The last results culminate in the following auxiliary proposition which is easy to comprehend when explained with pictures. The version that follows is in words and symbols. For a symmetric pattern $J$ we denote

$$\gamma_J(i) = \min\{k : (i, k) \in J\},$$
$$\zeta_J(i) = \max\{k : (i, k) \in J\}.$$

**Proposition 2.13** Let $J$ be a symmetric banded pattern in $\pi$ containing the diagonal $D_n$, and let $T \in T(J)$ be so that $T$ contains an element $(i, j)$ with $j \geq \gamma_J(n)$ and $i \geq \zeta_J(\gamma_J(n) - 1)$. Denote

$$Q = \{\zeta_J(\gamma_J(n) - 1) + 1, \ldots, n\} \times \{j\}.$$  

Then $Q \subseteq T$, and for any maximal rectangle $R$ in $T \cap (\pi \times rs(T))$ we have that

$$Q \cap R \neq \emptyset \implies \{j + 1, \ldots, \zeta_J(\gamma_J(n) - 1)\} \times \{j\} \subseteq R.$$  

Consequently, if $R_1, R_2 \in \mathcal{R}(T)$ then either $R_1 \cap R_2 \cap Q = \emptyset$ or $\{j + 1, \ldots, \zeta_J(\gamma_J(n) - 1)\} \times \{j\} \subseteq R_1 \cap R_2$.

**Proof** First note that $\{(\gamma_J(n), \ldots, n)\} \subseteq J$ (since $(\gamma_J(n), n)$ and $(n, \gamma_J(n))$ belong to $J$ and $J$ is banded). Thus for $k \geq \gamma(n)$ we have

$$\{p : (k, p) \in J\} \supseteq \{\gamma_J(n), \ldots, n\}.$$  

Furthermore, since $i \geq \zeta_J(\gamma_J(n) - 1)$,

$$\{p : (i, p) \in J\} = \{\gamma_J(n), \ldots, n\}.$$  

But then it follows from Lemma 2.11 (with $i_1 = i$ and $i_2 = k$) that \{\(p : (i, p) \in T\} \subseteq \{p : (k, p) \in T\}$. Now, $(i, j) \in T$ yields that $(k, j) \in T$ for
all \( k \geq \gamma(n) \). In particular, \( Q \subseteq T \). In addition, it follows that
\[
\{ p : (i, p) \in T \cap (n \times rs(T)) \} \subseteq \{ p : (k, p) \in T \cap (n \times rs(T)) \}.
\]

Lemma 2.12 now yields that \( (i, p) \in R \in \mathcal{R}(T) \) implies that \( \{ \gamma_J(n), \ldots, n \} \times \{ p \} \in R \). In fact, this argument extends exactly the same to the case when \( i \) is replaced by another integer \( i_1 \), with \( \zeta_J(\gamma_J(n) - 1) < i_1 \leq n \). But then the second part of the proposition also follows.

We end this section with a few results that are not needed for the proof, but which might further the understanding of symmetric banded patterns and their symmetric triangular subpatterns.

**Lemma 2.14** Let \( J \) be a symmetric banded pattern in \( n^2 \) and \( T \in T(J) \). Then \( rs(T)^2 \) is a maximal rectangle in \( J \).

**Proof** Suppose \( rs(T)^2 \subseteq rs(T) \times K \subseteq J \). Then also \( K \times rs(T) \subseteq J \). Thus
\[
(\min K, \max rs(T)) \in J, \\
(\max rs(T), \min K) \in J.
\]

Therefore
\[
rs(T)^2 \subseteq \{ \min K, \ldots, rs(T) \}^2 \subseteq J.
\]

By Corollary 2.7 it follows that
\[
rs(T) = \{ \min K, \ldots, \max rs(T) \}.
\]

So \( \min K \in rs(T) \). Similarly, \( \max K \in rs(T) \). Thus
\[
K \subseteq \{ \min rs(T), \ldots, \max rs(T) \} = rs(T).
\]

The last equality follows from
\[
(\min rs(T), \max rs(T)) \in J
\]
and \( J \) being symmetric and banded.
LEMMA 2.15  Let $J$ be a symmetric banded pattern, $T \in T(J)$, and $R$ a maximal rectangle in $T \cap (n \times rs(T))$. Then $R$ is a maximal rectangle in $T$.

Proof  Suppose $R = K \times L \subseteq \tilde{K} \times \tilde{L} \subseteq T$. If $\tilde{L} \subseteq rs(T)$ we are done since $R = K \times L \subseteq \tilde{K} \times \tilde{L} \subseteq T \cap (n \times rs(T))$ implies that $R = \tilde{K} \times \tilde{L}$. Suppose $\tilde{L} \nsubseteq rs(T)$. By Lemma 2.5 we must have that $\tilde{K} \subseteq rs(T)$. Thus also $\tilde{K} \nsubseteq rs(T)$. But then $K = L = rs(T)$, since $K \times L \subseteq rs(T)^2 \subseteq T \cap (n \times rs(T))$ and $K \times L$ is a maximal rectangle in $T \cap (n \times rs(T))$. But then

$$R = rs(T)^2 \subseteq \tilde{K} \times \tilde{L} \subseteq T \subseteq J.$$

And so by Lemma 2.14 we get that $R = \tilde{K} \times \tilde{L}$.

3. HERMITIAN COMPLETIONS OF PARTIAL MATRICES

As before, by a partial (complex) matrix we mean a matrix with some entries specified complex numbers and the remaining entries to be chosen freely from $\mathbb{C}$. We denote a $n \times m$ partial matrix by $A = \{(i, j), a_{ij} \mid (i, j) \in J\}$ where $J \subseteq n \times m$ denotes the set containing all indices corresponding to specified entries, and $a_{ij} \in \mathbb{C}$ the specifications. When $n = m$ and $J$ is symmetric, we say that $A$ is a Hermitian partial matrix when $a_{ij} = \overline{a_{ji}}$ for $(i, j) \in J$. A matrix $B = (b_{ij})_{n \times m}$ with $b_{ij} \in \mathbb{C}$ is called a completion of the partial matrix $A$ above if $b_{ij} = a_{ij}$ for all $(i, j) \in J$. A Hermitian completion is a completion that is Hermitian. Similarly, one may introduce other types of completions, such as normal completions, positive definite completions, etc. We shall denote partial matrices with script letters, and usual matrices in Roman font. In most cases, a completion of a partial matrix shall be denoted by the same letter (in a different font). So, typically $A$ is a completion of $A$.

Let $A = \{(i, j), a_{ij} \mid (i, j) \in J\}$. For a subpattern $\bar{J}$ of $J$ we let $A|\bar{J}$ denote the partial matrix

$$\{(i, j), a_{ij} \mid (i, j) \in \bar{J}\}.$$

In other words, $A|\bar{J}$ is obtained from $A$ by replacing the specified entries $J \setminus \bar{J}$ in $A$ by free variables. When $R = P \times Q$ is a rectangle in $J$
we may view $A|R$ as the usual matrix

$$(a_{ij})_{(i,j)\in R} = (a_{ij})_{i\in P, j \in Q}.$$  

As such we may define

$$\text{rank}(A|R) = \text{rank}\{(a_{ij})_{(i,j)\in R}\}.$$  

When $R$ is in addition symmetric, and $A$ is a Hermitian partial matrix, we may introduce

$$i_l(A|R) = i_l\{(a_{ij})_{(i,j)\in R}\}$$

for $l \in \{ < 0, \leq 0, \geq 0, > 0 \}$.

We are now ready to introduce the main inertial notions for partial matrices. Let $A = \{(i,j), a_{ij} : (i,j) \in J\}$ be a partial Hermitian matrix. For a symmetric triangular subpattern $T$ of $J$ we define the triangular negative/positive inertia of $A|T$ by

$$i_l(A|T) = i_l(A|R_1) + \sum_{j=1}^{m-1} \left( \text{rank}(A|R_{j+1}) - \text{rank}(A|R_{j+1} \cap R_j) \right),$$

where $l \in \{ < 0, > 0 \}$, and

$$R(T) = \{R_1, \ldots, R_m\}$$

is ordered increasingly according to the number of rows covered (see Proposition 2.9).

**Example 3.1** For $J$ and $T$ as in Example 2.10, we have for $A = \{(i,j), a_{ij} : (i,j) \in J\}$ that

$$i_{<0}(A|T) = i_{<0} \begin{pmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{pmatrix} + \text{rank} \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \\ a_{53} & a_{54} \end{pmatrix}$$
Let $J$ be a symmetric pattern, $A = \{(i, j), a_{ij} : (i, j) \in J\}$ a Hermitian partial matrix, and $T$ a symmetric triangular subpattern. Then for any Hermitian completion $\overline{A}$ of $A$ we have

$$i_l(A) \geq i_l(\overline{A}[T]), \quad l \in \{<, >\}. \quad (3.1)$$

**Proof** Let $\mathcal{R}(T) = \{R_1, \ldots, R_m\}$ be ordered as in Proposition 2.9. Perform, if necessary, a permutation similarity so that

$$R_i = \{1, \ldots, n_i\} \times \{1, \ldots, m_i\}, \quad i = 1, \ldots, m,$$

with $n_1 < n_2 < \cdots < n_m$ and $(n_1 =) m_1 > m_2 > \cdots > m_m$. Decompose $A$ as

$$
\begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{12}^* & A_{22} & A_{23} \\
A_{13}^* & A_{23}^* & A_{33}
\end{bmatrix}
$$

with

$$A_{11} = (a_{ij})_{i, j=1}^{m_1}, \quad A_{22} = (a_{ij})_{i, j=m_1+1}^{m_1}, \quad A_{33} = (a_{ij})_{i, j=m_1+1}^{m_m},$$

and $A_{12}$, $A_{13}$ and $A_{22}$ defined accordingly. By the results in [15] (see also [7], [5]) it follows that

$$i_{<0}(A) \geq i_{<0} \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} \right) + \text{rank} \left( \begin{bmatrix} A_{11} \\ A_{12}^* \\ A_{13} \end{bmatrix} \right) - \text{rank} \left( \begin{bmatrix} A_{11} \\ A_{12}^* \end{bmatrix} \right)
$$

$$= i_{<0}(A[R_{\text{symm}}(T)]) + \text{rank} \left( \begin{bmatrix} A_{11} \\ A_{12}^* \\ A_{13} \end{bmatrix} \right) - \text{rank}(A[R_1 \cap R_2]). \quad (3.2)$$
By the minimal rank formula [29] it further follows that
\[ \text{rank} \begin{pmatrix} A_{ij} \\ A_{ij}^{*} \end{pmatrix} \geq \sum_{j=2}^{m} \text{rank}(A|R_j) - \sum_{j=1}^{m} \text{rank}(A|R_{j-1} \cap R_j). \]  
Combining (3.2) and (3.3) gives (3.1).

Let \( J \) be a symmetric pattern and \( A = \{(i, j), a_{ij} : (i, j) \in J \} \) a partial Hermitian matrix. Define the \textit{negative/positive triangular inertia} of \( A \) by
\[ i_l(A) = \max_{T \in T(J)} i_l(A[T]), \]
with \( l \in \{-, 0, +\} \).

\textbf{Corollary 3.3} \ Let \( J \) be a symmetric pattern and \( A \) a partial Hermitian matrix with pattern \( J \). Then for any Hermitian completion \( \tilde{A} \) of \( A \) we have that
\[ i_l(A) \geq i_l(\tilde{A}), \quad l \in \{-, 0, +\}. \]

We now get to our first main result.

\textbf{Theorem 3.4} \ Let \( J \) be a symmetric banded pattern in \( \mathbb{Z}_3 \), and \( A = \{(i, j), a_{ij} : (i, j) \in J \} \) a partial Hermitian matrix. Then
\[ \min \{i_l(A) : A \text{ is a Hermitian completion of } A\} = i_l(\tilde{A}), \quad l \in \{-, 0, +\}. \]
Moreover, \( A \) has a Hermitian completion \( \bar{A} = (a_{ij})_{j=1}^{n} \) such that for each \( k \in \{1, \ldots, n\} \) the matrix \( (a_{ij})_{k \in k} \) is a completion of \( A|J \cap \{k, \ldots, n\} \) with minimal possible negative/positive inertia. If \( A \) is a real symmetric partial matrix, then the above completion may be chosen to be real symmetric as well.

Before proving this result, let us also address the easier question regarding the minimal possible nonpositive/nonnegative inertia.

\textbf{Theorem 3.5} \ Let \( J \) be a symmetric banded pattern and \( A \) a partial Hermitian matrix. Then
\[ \min \{i_l(A) : A \text{ Hermitian completion of } A\} = \max_{K^2 \subseteq J} i_l(A|K^2), \quad l \in \{-, 0, +\}. \]
**Proof.** Because of the main result in [3] and Lemma 2.1 we may assume that $D_n \subseteq J$. Let $p$ be the number of maximal symmetric rectangles in $J$. When $p = 2$ the result follows from Theorem 5.1 in [5]. A simple induction argument on $p$ provides the general case.

Since for a $n \times n$ Hermitian matrix $A$ we have that

\[
i_{20}(A) = n - i_{>0}(A),
\]
\[
i_{20}(A) = n - i_{<0}(A),
\]

we may interpret Theorem 3.5 also as a result on the maximal negative/positive inertia of a Hermitian completion of a partial Hermitian matrix with a banded pattern. The results in [6] suggest that Theorem 3.5 may in fact hold for all symmetric patterns. This possible generalization has not been pursued here.

The remainder of this section is devoted to the proof of the negative inertia part of Theorem 3.4. The positive counterpart follows by replacing $A$ by $-A$. As the proof of Proposition 3.2 shows, minimal rank completions play an important role. We will therefore start with recalling some definitions and results from [29], [30], and [31].

Given a partial matrix $A$, we call $A$ a **minimal rank completion** of $A$ if among all completions of $A$ the completion $A$ has the lowest possible rank. This lowest possible rank is denoted by $mr(A)$. Thus

\[mr(A) = \min \{ \text{rank } A : A \text{ completion of } A \}.\]

In [29] the following formula for the minimal rank of a triangular partial matrix was proven:

\[
\begin{align*}
\begin{bmatrix}
A_{11} & \cdots & \cdots & \cdots \\
A_{21} & A_{22} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{bmatrix}
= \sum_{j=1}^{n} \text{rank } \begin{bmatrix}
A_{j1} & \cdots & A_{ji}
\end{bmatrix}
- \sum_{j=1}^{n-1} \text{rank } \begin{bmatrix}
A_{j+1,1} & \cdots & A_{j+1,i}
\end{bmatrix}
\end{align*}
\]

We will make use of the following observation.
Lemma 3.6 Let $A_{ij}$, $1 \leq j \leq i \leq n$, be complex matrices of size $1 \times p_j$ (with possibly $p_j = 0$). If

$$
\begin{bmatrix}
A_{11} & & \\
A_{21} & A_{22} & \\
\vdots & \vdots & \ddots \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{bmatrix}
= mr
\begin{bmatrix}
A_{21} & A_{22} & \\
\vdots & \vdots & \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{bmatrix}
$$

Then $A_{11}, A_{n1} \notin \text{rowspace}\begin{bmatrix} A_{21} \\ \vdots \\ A_{n1} \end{bmatrix}$, and there exist $s_1, \ldots, s_n \in \mathbb{C}$ with $s_1 \neq 0$ and $s_n \neq 0$ so that

$$
(s_1 \cdots s_n) \begin{bmatrix}
A_{11} \\
\vdots \\
A_{n1}
\end{bmatrix} = 0.
$$

If $A_{ij}$, $1 \leq j \leq i \leq n$, are real matrices then $s_1, \ldots, s_n$ can be chosen to be real as well.

Proof The first equality in (3.4) shows that

$$
A_{11} \in \text{rowspace} \begin{bmatrix} A_{21} \\ \vdots \\ A_{n1} \end{bmatrix}
$$
(use the minimal rank formula). By the same token, the third equality in (3.4) shows that

\[ A_{11} \notin \text{rowspace} \begin{pmatrix} A_{21} \\ \vdots \\ A_{n-1,1} \end{pmatrix}. \]

These two observations imply that

\[ A_{11} = s_2 A_{21} + \cdots + s_n A_{n,1} \]

for some \( s_2, \ldots, s_n \in \mathbb{C} \) with \( s_n \neq 0 \). In addition, we get that

\[ A_{n,1} \notin \text{rowspace} \begin{pmatrix} A_{21} \\ \vdots \\ A_{n-1,1} \end{pmatrix} \]

(otherwise

\[ \text{rowspace} \begin{pmatrix} A_{21} \\ \vdots \\ A_{n,1} \end{pmatrix} = \text{rowspace} \begin{pmatrix} A_{21} \\ \vdots \\ A_{n-1,1} \end{pmatrix}. \]

Clearly, \( s_2, \ldots, s_n \) can be chosen to be real, when \( A_{11}, \ldots, A_{n,1} \) are real.

**Lemma 3.7** Let \( A = A^* \in \mathbb{C}^{(n-1) \times (n-1)}, b \in \mathbb{C}^{n-1}, \) and \( c \in \mathbb{R} \). Then

\[
i_{<0} \begin{pmatrix} A \\ b^* \\ c \end{pmatrix} = i_{<0}(A)
\]

if and only if there exists a \( \mu \geq 0 \) and \( s \in \mathbb{C}^{n-1} \) such that

\[
s^*(A \ b) = (b^* \ c - \mu).
\]

**Proof** If (3.6) holds then

\[
\begin{pmatrix} A \\ b^* \\ c \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ 0 \\ \mu \end{pmatrix} \begin{pmatrix} I \\ s \end{pmatrix}.
\]
Thus
\[ i_{<0} \begin{pmatrix} A & b \\ b^* & c \end{pmatrix} = i_{<0}(A) + i_{<0}(\mu) = i_{<0}(\lambda). \]

Conversely, if (3.5) holds, decompose \( A \) as
\[
A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} \text{Im}A \\ \text{Ker}A \end{pmatrix}
\]
and
\[
b^* = \begin{pmatrix} b_1^* & b_2^* \end{pmatrix} \oplus \text{Ker}A
\]
accordingly. Then, using a Schur complement,
\[
i_{<0} \begin{pmatrix} A_{11} & 0 & b_1 \\ 0 & 0 & b_2 \\ b_1^* & b_2^* & c \end{pmatrix} = i_{<0} \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2^* & c - b_1^* A_{11}^{-1} b_1 \end{pmatrix}
\]
\[
= i_{<0}(A_{11}) + i_{<0} \begin{pmatrix} 0 & b_2 \\ b_1^* & c - b_1^* A_{11}^{-1} b_1 \end{pmatrix}
\]
\[
= i_{<0}(A) + i_{<0} \begin{pmatrix} 0 & b_2 \\ b_2^* & c - b_1^* A_{11}^{-1} b_1 \end{pmatrix}
\]

Since (3.5) holds we must have that
\[
i_{<0} \begin{pmatrix} 0 & b_2 \\ b_2^* & c - b_1^* A_{11}^{-1} b_1 \end{pmatrix} = 0,
\]
so \( b_2 = 0 \) and \( c - b_1^* A_{11}^{-1} b_1 \geq 0 \). Put now \( \mu = c - b_1^* A_{11}^{-1} b_1 \) and \( s = A_{11}^{-1} b_1 \). Then (3.6) holds.

Recall that
\[
\gamma_i(j) = \min \{ i : (i, j) \in J \},
\]
\[
\zeta_i(j) = \max \{ i : (i, j) \in J \}.
\]
Proof of Theorem 3.4  Because of the main result in [3] and Lemma 2.1 we may assume that $D_n \subseteq J$. We prove the result by induction on the size of $n$. When $n=1$ the theorem is trivially true. So suppose that the theorem holds for partial Hermitian banded matrices of size $(n-1)\times (n-1)$.

Let now $A = \{(i,j), a_{ij} : (i,j) \in J\}$ be a $n \times n$ partial Hermitian banded matrix, and let $k = \tilde{i}c_{\sigma}(A)$ and

$$\tilde{J} = J \cap (n-1)^2.$$

Denote $\tilde{A} = \{((i,j), a_{ij}) : (i,j) \in \tilde{J}\}$ By the induction hypothesis there exists a Hermitian completion $\tilde{A} = (\tilde{a}_{ij})_{i,j=1}^{n-1}$ of $\tilde{A}$ such that $\tilde{i}c_{\sigma}(\tilde{A}) = \tilde{i}c_{\sigma}(A)$, and for each $p \in \{1, \ldots, n-1\}$ the matrix

$$(\tilde{a}_{ij})_{i,j=1}^{n-1}$$

is a Hermitian completion of

$$\{((i,j), a_{ij}) : (i,j) \in \tilde{J} \cap \{p, \ldots, n-1\} \times \{p, \ldots, n-1\}\}$$

with minimal possible negative eigenvalues.

We distinguish between 3 cases.

Case 1  $\tilde{i}c_{\sigma}(a_{ij})_{i,j=\gamma(n)}^{\sigma} = \tilde{i}c_{\sigma}(a_{ij})_{i,j=\gamma(n)}^{\sigma-1}$. By Lemma 3.7 there exists a $\mu \geq 0$ and $S_{\gamma(n)}, \ldots, S_{n-1} \in \mathbb{C}$ such that

$$(a_{\mu}, a_{n-1} \cdot a_{\mu} - \mu) = (s_{\gamma(n)} \cdots s_{n-1})$$

Put now

$$(a_{n1} \cdots a_{n,\gamma(n)-1}) = (s_{\gamma(n)} \cdots s_{n-1}) \cdot \begin{pmatrix}
(a_{\gamma(n)}, a_{\gamma(n)} \cdots a_{\gamma(n)}) \\
\vdots \\
(a_{n-1,\gamma(n)} \cdots a_{n-1,n})
\end{pmatrix}.$$
Let \( n \) be as large as possible so that there exists a \( p > 0 \) such that
\[
ir(\begin{pmatrix}
a_{mn} & \cdots & a_{m1} \\
\vdots & & \vdots \\
a_{an} & \cdots & a_{an}
\end{pmatrix}) = ir(\begin{pmatrix}
a_{mn} & \cdots & a_{m,n-1} \\
\vdots & & \vdots \\
a_{n-1,m} & \cdots & a_{n-1,n-1}
\end{pmatrix}),
\]
and thus this completion has the required properties.

For the remaining cases we will need the following triangular subpattern of \( J \):
\[
T_1 := J \cap (n \times \gamma_J(n)).
\]

**Case 2** \( i_{\mathbb{C}}(a_{ji}^{n-1}) = i_{\mathbb{C}}(a_{ji}^{n-1}) + 1 \) and there is a \( \mu \geq 0 \) such that
\[
\text{mr}\{((i, j), a_j) : (i, j) \in T_1 \setminus \{(n, n)\} \cup \{(n, n), a_{nn} - \mu\}\}
= \text{mr}\{((i, j), a_j) : (i, j) \in T_1 \cap (n-1 \times n)\}.
\]

Let \( m \) be as large as possible so that there exists a \( \mu \geq 0 \) such that
\[
\text{mr}\{((i, j), a_j) : (i, j) \in (T_1 \cap \{m, \ldots, n\} \times n) \setminus \{(n, n), a_{nn} - \mu\}\}
= \text{mr}\{((i, j), a_j) : (i, j) \in T_1 \cap \{m, \ldots, n-1\} \times n)\},
\]
and choose \( \mu \geq 0 \) accordingly. Note that by the assumptions in this case, \( m \) is well defined and
\[
\gamma_J(\gamma_J(n)) \leq m \leq \gamma_J(n) - 1.
\]

Because of the choice of \( m \), by Lemma 3.6 there exist \( s_m, \ldots, s_n \) with \( s_m \neq 0 \) and \( s_n \neq 0 \) such that
\[
\begin{pmatrix}
s_1 \\
\vdots \\
s_n
\end{pmatrix}
\begin{pmatrix}
a_{m,\gamma_J(n)} & \cdots & a_{m,\zeta(m)} \\
\vdots & & \vdots \\
a_{n,\gamma_J(n)} & \cdots & a_{n,\zeta(m)}
\end{pmatrix}
= \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}.
\]

Let now
\[
(a_{n, m+1} \ldots a_{n, \gamma_J(n)-1}) = \begin{pmatrix}
\frac{s_m}{s_n} & \cdots & \frac{s_{n-1}}{s_n} \\
\vdots & & \vdots \\
\frac{s_{n-1}}{s_n} & \cdots & \frac{s_{n-1}}{s_n}
\end{pmatrix}
\begin{pmatrix}
a_{m, m+1} & \cdots & a_{m, \gamma_J(n)-1} \\
\vdots & & \vdots \\
a_{n-1, m+1} & \cdots & a_{n-1, \gamma_J(n)-1}
\end{pmatrix}.
\]
Consider the $3 \times 3$ block matrix

$$
\begin{pmatrix}
\alpha & \beta & \chi \\
\beta^* & \gamma & \delta \\
\chi^* & \delta^* & \varepsilon
\end{pmatrix}
$$

where

$$
\alpha = a_{m,m}, \quad \beta = (a_{m,m+1} \cdots a_{m,\zeta(m)}), \\
\gamma = (a_{\gamma i,m+1}^1)_{i \neq m+1}, \quad \delta = (a_{\gamma i,m+1}^n)_{i \neq \zeta(i)+1}, \\
\varepsilon = (a_{\varepsilon i,j}^m)_{i \neq \zeta(i)+1},
$$

and $\chi$ is considered as unspecified. We claim that

$$
\min_{i < 0} \begin{pmatrix}
\alpha & \beta & \chi \\
\beta^* & \gamma & \delta \\
\chi^* & \delta^* & \varepsilon
\end{pmatrix} = \min_{i < 0} (a_{\gamma i,j}^{n-1}) = i_{<0} + 1.
$$

Indeed, by [15] (see also [7] or [5]),

$$
\min_{i < 0} \begin{pmatrix}
\alpha & \beta & \chi \\
\beta^* & \gamma & \delta \\
\chi^* & \delta^* & \varepsilon
\end{pmatrix} = \max \left\{ i_{<0} \begin{pmatrix}
\alpha & \beta \\
\beta^* & \gamma \\
\chi^* & \delta^* \\
\varepsilon
\end{pmatrix} + \text{rank} \begin{pmatrix}
\beta \\
\gamma \\
\delta \\
\varepsilon
\end{pmatrix} - \text{rank} \begin{pmatrix}
\beta \\
\gamma \\
\delta \\
\varepsilon
\end{pmatrix}, \right. \\
\left. i_{<0} \begin{pmatrix}
\gamma & \delta \\
\delta^* & \varepsilon
\end{pmatrix} + \text{rank} \begin{pmatrix}
\beta \\
\gamma \\
\delta \\
\varepsilon
\end{pmatrix} - \text{rank} \begin{pmatrix}
\gamma \\
\delta \\
\varepsilon
\end{pmatrix} \right\}.
$$

Since $\beta \in \text{rowspace} \begin{pmatrix}
\gamma \\
\delta
\end{pmatrix}$ we have

$$
\text{rank} \begin{pmatrix}
\beta \\
\gamma \\
\delta
\end{pmatrix} = \text{rank} \begin{pmatrix}
\gamma \\
\delta
\end{pmatrix}.$$
Since \( \beta \notin \text{rowspace } \gamma \) (use Lemma 3.6 and that \( m < \gamma(n) \)), we have

\[
\text{rank}
\begin{pmatrix}
\beta \\
\gamma
\end{pmatrix}
= \text{rank } \gamma + 1
\]

and

\[
i_{<0}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}
= i_{<0}(\gamma) + 1.
\]

Since \( \beta \notin \text{rowspace } (a_{ij})_{i=m+1, j=m+1}^{n-1, m(m)} \) (again we use Lemma 3.6 and that \( m < \gamma(n) \)), and \( \beta \in \text{rowspace } \left( \begin{array}{c}
\gamma \\
\delta^*
\end{array} \right) \) we get that

\[
\text{rank}
\begin{pmatrix}
\beta \\
\gamma \\
\delta^*
\end{pmatrix}
= \text{rank }
\begin{pmatrix}
a_{m+1,m+1} & \cdots & a_{m+1,m(m)} \\
\vdots & & \vdots \\
a_{n-1,m+1} & \cdots & a_{n-1,m(m)}
\end{pmatrix}
+ 1.
\]

So the right hand side of (3.7) equals

\[
\max\left\{ i_{<0}(\gamma) + 1 + \text{rank}(a_{ij})_{i=m+1, j=m+1}^{n-1, m(m)} - \text{rank } \gamma, i_{<0}
\begin{pmatrix}
\gamma \\
\delta^*
\end{pmatrix}
\right\}
= i_{<0}(a_{ij})_{i,j=m+1}^{n-1} + 1.
\]

Here we used that

\[
(a_{n,m+1} \cdots a_{n,n-1}) \notin \text{rowspace } (a_{ij})_{i,j=m+1}^{n-1},
\]

so that

\[
b_y
\begin{pmatrix}
\gamma \\
\delta^*
\end{pmatrix}
= i_{<0}(a_{ij})_{i,j=m+1}^{n-1} + 1.
\]

Choose completion \( (b_y)_{i,j=m}^{n, n} \) of

\[
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta^*
\end{pmatrix}
\]

with minimal negative inertia. Note that \( b_y = a_y \) for \( i,j \geq m \) with \( (i,j) \notin (\{m\} \times \{\xi_f(m) + 1, \ldots, n\} \cup \{\xi_f(m) + 1, \ldots, n\} \times \{m\}) \). Now for \( p = m, \ldots, \gamma(n) \) we have that \( (b_y)_{i,j=p}^{n, n} \) is a completion of

\[
\{(i,j), (i,j) \in J \cap \{p, \ldots, n\}^2\}
\]
In order to prove (3.8) we will show that the inequalities in (3.9) are trivial. In order to prove the first equality in (3.9) first note that

\[
(b_{n, \gamma_j(n)} \ldots b_{n, \xi_j(m)}) \notin \operatorname{rowspace} \begin{pmatrix}
(b_{m+1, \gamma_j(n)} & \cdots & b_{m+1, \xi_j(m)} \\
\vdots & & \vdots \\
b_{n-1, \gamma_j(n)} & \cdots & b_{n-1, \xi_j(m)}
\end{pmatrix}.
\]

For \(p = m\) this follows from the observations that

\[
i_{<c}(b_j)^n_{i, j=1} = i_{<c}(b_j)^{n-1}_{i, j=m+1} + 1 = i_{<c} \left\{ \{(i, j), b_j\} : J \cap \{m \cup \{m + 1, \ldots, n-1\}\} \right\} + 1 = i_{<c} \left\{ \{(i, j), b_j\} : J \cap \{m + 1, \ldots, n\} \right\}.
\]

For \(i < m\) or \(j < m\) let \(b_j = a_{ij}\). Furthermore,

\[
i_{<c} \left\{ \{(i, j), b_j\} : (i, j) \in J \cup \{m, \ldots, n\} \right\} = i_{<c} \left\{ \{(i, j), a_{ij}\} : (i, j) \in J \right\} (= i_{<c} \left\{ \{(i, j), a_{ii}\} : (i, j) \in J \right\}).
\]

In order to prove (3.8) we will show that

\[
i_{<c} \left\{ \{(i, j), a_{ij}\} : (i, j) \in J \right\} \geq i_{<c} \left\{ \{(i, j), a_{ij}\} : (i, j) \in (J \cap (n-1)^2) \cup \{m, \ldots, n-1\} \right\} \]

\[
= i_{<c} \left\{ \{(i, j), b_j\} : (i, j) \in (J \cap (n-1)^2) \cup \{m, \ldots, n-1\} \right\}
= i_{<c} \left\{ \{(i, j), b_j\} : (i, j) \in J \cup \{m, \ldots, n\} \right\} \geq i_{<c} \left\{ \{(i, j), a_{ij}\} : (i, j) \in J \right\}.
\]

The inequalities in (3.9) are trivial.

In order to prove the first equality in (3.9), first note that

\[
(a_{m, m+1} \ldots a_{m, \xi_j(m)}) \notin \operatorname{rowspace} \begin{pmatrix}
a_{m+1, m+1} & \cdots & a_{m+1, \xi_j(m)} \\
\vdots & & \vdots \\
a_{n-1, m+1} & \cdots & a_{n-1, \xi_j(m)}
\end{pmatrix}.
\]
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Thus for any \( Y \in \mathbb{C}^{1 \times q} \) there exists an \( S \in \mathbb{C}^{(r(m)-m)\times q} \) such that

\[
(a_{m,m+1} \cdots a_{m,(m)})S = Y,
\]

\[
(a_{ij})^{(m)}_{i=m+1, j=m+1} S = 0.
\]

Let \( Y = (b_{m,\zeta(m)+1} \cdots b_{m,n-1}) - (a_{m,\zeta(m)+1} \cdots a_{m,n-1}) \), and choose \( S = (S_{ij})^{(m)}_{i=m+1, j=\zeta(m)+1} \) accordingly. Then for all \( s \leq m+1, r \geq \zeta'(m) + 1 \) and \( m+1 \leq l \leq \zeta(m) \), we have that

\[
[(a_{ij})^{(m)}_{i=m+1, j=m+1}]
\begin{bmatrix}
I_{m+1-s} & 0 & 0 \\
0 & I_{(m)-m} & S \\
0 & 0 & I_{r-\zeta(m)}
\end{bmatrix}
= [(b_{ij})^{(m)}_{i=m+1, j=m+1}]
\]

where \( S = (S_{ij})^{(m)}_{i=m+1, j=\zeta(m)+1} \). Thus

\[
\text{rank}(a_{ij})^{(m)}_{i=m+1, j=m+1} = \text{rank}(b_{ij})^{(m)}_{i=m+1, j=m+1}.
\] (3.10)

Furthermore, the only rectangle appearing in \( \mathcal{R}(T) \) for some \( T \in \mathcal{T} ((J \cap (n-1)^2) \cup \{m, \ldots, n-1\})^2 \) which contains both elements from \( \{m \times \{\zeta(m) + 1, \ldots, n-1\} \) and \( \{(m) + 1, \ldots, n-1\} \times \{m, \ldots, n-1\}, \) and here we have

\[
i_{\zeta 0}(a_{ij})^{(m)}_{i=\zeta=m} = i_{\zeta 0}(b_{ij})^{(m)}_{i=\zeta=m}.
\] (3.11)

By Proposition 2.13 (applied to the pattern \((J \cap (n-1)^2) \cup \{m, \ldots, n-1\}^2\)) and the above observation, the only possible difference in the terms of the expressions for

\[
i_{\zeta 0}\{(i, j), (i, j) \in (J \cap (n-1)^2) \cup \{m, \ldots, n-1\}^2\}
\] (3.12)

and

\[
i_{\zeta 0}\{(i, j), (i, j) \in (J \cap (n-1)^2) \cup \{m, \ldots, n-1\}^2\}
\] (3.13)

might be the difference between left hand sides and right hand sides of (3.10) and (3.11). Consequently, it follows that (3.12) and (3.13) are equal.
To see that the second equality in (3.9) holds, note that if for $T \in T(J)$ a member of $R(T)$ contains a part of the $m$th row, it will also contain the corresponding part (i.e., the part within the same columns) of rows $m, \ldots, n-1$. Since

$$i_{=0}(b_y)_{i,m} = i_{=0}(b_y)_{i,m}$$

and for $K \subseteq \{m, \ldots, n-1\}$,

$$\text{rank}(b_y)_{i=m, j \in K} = \text{rank}(b_y)_{i=m, j \in K},$$

the second equality in (3.9) follows.

We may thus conclude that (3.8) is true. Now we may apply Case 1 to the partial matrix

$$\{(i, j, b_y): (i, j) \in J \cup \{m, \ldots, n\}^2\}$$

and obtain the desired completion. This finishes Case 2.

**Case 3** For all $\mu \geq 0$ we have that

$$\text{mr}\{(i, j, a_y): (i, j) \in T_1 \setminus \{(n, n)\} \cup \{(n, n), (n, a_{\mu} - \mu)\}\}$$

$$\geq \text{mr}\{(i, j, a_y): (i, j) \in T_1 \cap (n-1 \times n)\}.$$ 

Let $\tilde{q}$ be as large as possible so that

$$\text{mr}\{(i, j, a_y): (i, j) \in T_1 \setminus \{(n, \tilde{q} + 1), \ldots, (n, n)\}\}$$

$$= \text{mr}\{(i, j, a_y): (i, j) \in T_1 \cap (n-1 \times n)\}.$$ 

Denote $\tilde{p} = \tilde{q} + 1$. Note that $\gamma_{\tilde{p}}(n) \leq \tilde{p} \leq n$. Furthermore, as in the proof of Lemma 1.3 in [31] (see (1.6)),

$$\text{rank}(a_y)_{i=\gamma_{\tilde{p}}(n), j=\gamma_{\tilde{q}}(n)} = \text{rank}(a_y)_{i=\gamma_{\tilde{p}}(n), j=\gamma_{\tilde{q}}(n)} - 1.$$ 

(3.14)

Consider the pattern

$$J_1 = \{\gamma_{\tilde{p}}(\tilde{p}), \ldots, n\}^2 \cap J.$$
By Lemma 1.2 in [31] any maximal triangular pattern in $J_1$ contains 
$(\gamma_J(\bar{p}), \ldots, n) \times (\gamma_J(n), \ldots, \bar{p})$. But then it follows from (3.14) that

$t_{c_0}(\{(i, f), a_{ij}: (i, f) \in J_1\}) = t_{c_0}(\{(i, f), a_{ij}: (i, f) \in (\gamma_{\bar{p}}, \ldots, n-1)^2 \cap J_1\}) + 1. \quad (3.15)$

Recall from the induction assumption that $\hat{A} = (a_{ij})_{i,j=1}^{n-1}$ is a Hermitian completion of $\hat{A}$ so that for each $p \in \{1, \ldots, n-1\}$ the matrix $(a_{ij})_{i,j=p}^{n-1}$

is a Hermitian completion of

\[ \{(i, f), a_{ij}: (i, f) \in J \cap \{p, \ldots, n-1\}^2\} \]

with minimal possible negative eigenvalues. Moreover, $i_{c_0}(\hat{A}) = t_{c_0}(\hat{A}) = l$. Let

\[ j_s = \max \{ \nu: i_{c_0}(a_{ij})_{i,j=\nu}^{n-1} = l - s + 1 \}, \quad s = 1, \ldots, l. \]

In case $j_1 \geq \gamma_J(\bar{p})$, we obtain that

$t_{c_0}(\{(i, f), a_{ij}: (i, f) \in (\gamma_{\bar{p}}, \ldots, n-1)^2 \cap J_1\}) = l,$

and thus by (3.15) that

$t_{c_0}(\hat{A}) \geq l + 1.$

On the other hand,

$t_{c_0}(\hat{A}) \leq t_{c_0}(\hat{A}) + 1 = l + 1.$

Thus

$l + 1 = t_{c_0}(\hat{A}) = t_{c_0}(\hat{A}) + 1.$

But then any choice for $(a_{ij})_{i,j=1}^{n, \gamma_J(\bar{p})-1}$ will yield a desired completion for $\hat{A}$. In case $j_1 < \gamma_J(\bar{p})$, choose $q$ as large as possible such that

(i) $j_q < \gamma_J(\bar{p})$

(ii) $\text{rank}(a_{ij})_{i,j=1}^{n, \gamma_J(\bar{p})-1} = \text{rank}(a_{ij})_{i,j=1}^{n, \gamma_J(\bar{p})-1}$.
Because of the choice of $\rho$ such a $q$ must exist. Change now $\{a_{i, (\mathbb{C}^{(\mathbb{C})})^+}, \ldots, a_{i, n-1}\}$ to $\{b_{i, (\mathbb{C}^{(\mathbb{C})})^+} \ldots, b_{i, n-1}\}$ so that
\[
\text{rank}(b_{i})^{n-1, n-1} = \text{rank}(b_{i})^{n-1, n-1}_i
\]
where for $(i, j) \notin \{J_1\} \times \{\mathbb{C}^{(\mathbb{C})} + 1, \ldots, n-1\}$ we let $b_{ij} = a_{ij}$. Note that $b_{ij} = a_{ij}$ for $(i, j) \in J$. This is possible, because of (ii). Let $J_2 = J \cup \{j_2, \ldots, n-1\}$. Now, with $B = \{((i, j), b_{ij}); (i, j) \in J_2\}$ (which has $\mathbb{C}^{(\mathbb{C})} = \mathbb{C} \angle (A)$) we can apply Case 2 to get the desired result. (Or to reduce it to Case 1, choose $b_{i, j}^{n-1, n-1}$ so that one obtains a Hermitian completion with minimal negative inertia of
\[
\{(i, j), b_{ij}; (i, j) \in \{j_2, \ldots, n-1\} \uplus \{n-1, \ldots, n\}^2\}
\]
Then one can apply Case 1 to
\[
\{(i, j), b_{ij}; (i, j) \in J \cup \{j_2, \ldots, n\}^2\}
\]
This finishes the proof for the complex case. For the real case observe that all the arguments remain the same when the underlying field is the real line.

It should be noted that the above proof followed closely ideas from the proof of Theorem 1.1 in [31].

4. HERMITIAN MATRIX INEQUALITIES

Several Hermitian matrix expressions can be viewed as Schur complements in a block matrix. This observation allows one to obtain results for matrix (in)equalities from the statements in the previous section.

As an example we shall illustrate this phenomena to Riccati inequalities. Recall from [5] the following observation.

**Lemma 4.1** Let $A = A^* \in \mathbb{C}^{m \times m}$, $B, Z \in \mathbb{C}^{m \times n}$ and $C = C^* \in \mathbb{C}^{n \times n}$. The Riccati expression $A + BZ + ZB^* + ZCZ^*$ appears as the Schur
Complement of \( \begin{pmatrix} C & -I_n \\ -I_n & 0 \end{pmatrix} \) in
\[
\begin{pmatrix} A & B & Z \\ B^* & C & -I_n \\ Z^* & -I_n & 0 \end{pmatrix}
\] (4.1)

Proof

\[
A - (B \quad Z) \begin{pmatrix} 0 & -I \\ -I & -C \end{pmatrix} \begin{pmatrix} B^* \\ Z^* \end{pmatrix} = A + BZ^* + ZB^* + ZCZ^*.
\]

The following consequence was observed in [20].

**Corollary 4.2** Let \( A = A^* \in \mathbb{C}^{m \times m} \), \( B, Z \in \mathbb{C}^{m \times n} \) and \( C = C^* \in \mathbb{C}^{n \times n} \). Then the following are equivalent:

1. \( A + BZ^* + ZB^* + ZCZ^* \geq 0 \),

2. \[
\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} Z \\ Z^* \end{pmatrix} = n,
\]

3. \[
\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} Z \\ Z^* \end{pmatrix} \leq n.
\]

Proof This follows immediately from Lemma 4.1 and the observation that

\[
\begin{pmatrix} C & -I_n \\ -I_n & 0 \end{pmatrix} = n.
\]
One may now reduce the question whether partial matrices \( A, B, C, \) and \( Z \) can be completed to matrices \( A (= A^*), B, C (= C^*), \) and \( Z \) such that

\[
A + BZ^* + ZB^* + ZCZ^* \geq 0
\]

(4.2)
to the question whether there exists a Hermitian completion of

\[
\begin{pmatrix}
A & B & Z \\
B^* & C^* & -I_n \\
-Z^* & -I_n & 0
\end{pmatrix}
\]

(4.3)

with negative inertia \( \leq n \). When the specified entries in \( A, B, C, \) and \( Z \) are located in such a way that the pattern of specified entries in (4.1) is permutation similar to a banded pattern, one can apply Theorem 3.4 to obtain necessary and sufficient conditions on the specified parts of \( A, B, C, \) and \( Z \) so that completions \( A, B, C, \) and \( Z \) exist satisfying (4.2). Several such examples based on the \( 3 \times 3 \) block matrix version of Theorem 3.4 ([15], see also [7] and [5]), were presented in [20]. It should be noted, though, that of most interest regarding Riccati equations are solutions \( Z \) that are Hermitian. Our results do not handle this extra requirement. A complete exposition of Riccati equations and their Hermitian solutions may be found in the recent book [26].

Let us next present an example where a combination of Theorem 3.4 and Corollary 4.2 is used to obtain a result regarding the Riccati equation. We note that many other variations may also be treated in this way.

**Theorem 4.3** Let \( A_{11} = A_{11}^* \in \mathbb{C}^{p \times p}, A_{12} = A_{12}^* \in \mathbb{C}^{p \times q}, A_{22} = A_{22}^* \in \mathbb{C}^{q \times q}, B_{11} \in \mathbb{C}^{p \times k}, B_{21} \in \mathbb{C}^{q \times k}, B_{22} \in \mathbb{C}^{q \times l}, C_{11} = C_{11}^* \in \mathbb{C}^{k \times k}, C_{12} \in \mathbb{C}^{k \times l}, \) and \( C_{22} \in \mathbb{C}^{l \times l} \) be given. Then there exist \( B_{12} \in \mathbb{C}^{p \times l} \), and \( Z \in \mathbb{C}^{(p+q) \times (k+l)} \) so that

\[
A + BZ^* + ZB^* + ZCZ^* \geq 0
\]

if and only if

\[
\begin{bmatrix}
A_{11} & A_{12} & B_{11} \\
A_{12}^* & A_{22} & B_{21} \\
B_{11}^* & B_{21} & C_{11}
\end{bmatrix}
\]

+ rank

\[
\begin{bmatrix}
A_{12} & B_{11} \\
A_{22} & B_{21} \\
B_{22} & C_{12}
\end{bmatrix}
\]
Proof. By Theorem 3.4 and the observations above, $B_{12}$ and $Z$ with the desired properties exist if and only if

$$i_{0} \begin{bmatrix} A_{22} & B_{21} & B_{22} \\ B_{21}^{*} & C_{11} & C_{12} \\ B_{22}^{*} & C_{12}^{*} & C_{22} \end{bmatrix} + \text{rank} \begin{bmatrix} A_{12} & B_{11} \\ A_{22} & B_{21} \\ B_{21}^{*} & C_{11} \\ B_{22}^{*} & C_{12}^{*} \end{bmatrix}$$

and

$$i_{0} \begin{bmatrix} A_{22} & B_{21} & B_{22} \\ B_{21}^{*} & C_{11} & C_{12} \\ B_{22}^{*} & C_{12}^{*} & C_{22} \end{bmatrix} \leq \text{rank} \begin{bmatrix} B_{21} & B_{22} \\ C_{11} & C_{12} \\ C_{12}^{*} & C_{22} \end{bmatrix}.$$

Observe that in this $6 \times 6$ (block) pattern the 4 maximal symmetric triangular patterns are

$$T_1 = (3 \times 2) \cup \{(2, 4), (4, 2)\} \cup \{(3) \times \{4, 5, 6\}\} \cup \{(4, 5, 6) \times \{3\}\},$$

$$T_2 = ((2, 3) \times 4) \cup (4 \times \{2, 3\}) \cup \{(3) \times 6\} \cup \{6 \times \{3\}\} \cup \{(4, 4)\},$$

$$T_3 = ((3, 4) \times \{2, 3, 4, 5, 6\}\} \cup \{(2, 3, 4, 5, 6) \times \{3, 4\}\} \cup \{(1, 3), (2, 2), (3, 1), (3, 2), (4, 2)\},$$

$$T_4 = \{3, 4, 5, 6\}^2 \cup \{(1, 3), (2, 3), (2, 4), (3, 1), (3, 2), (4, 2)\}.$$
Rewriting the left hand side of (4.4) yields (after reworking) the desired result. \[ \text{□} \]

When in the Riccati equation one chooses \( A = C = 0 \) one obtains the Lyapunov type inequality

\[ BZ^* + ZB^* > 0, \]

which was studied in [27]. When \( B = 0 \) we get the Stein type inequality

\[ A + ZCZ^* \geq 0. \]

Aside from these classical inequalities one can also introduce new ones such as

\[ A - BD^* - CC^* + CF^*D^* - DB^* + DFC^* + DFF^*D^* - DED^* \geq 0, \]

the left hand side of which appears as the Schur complement of

\[
\begin{pmatrix}
E & F & I \\
F^* & I & 0 \\
I & 0 & 0
\end{pmatrix}
\]

in

\[
\begin{pmatrix}
A & B & C & D \\
B^* & E & F & I \\
C^* & F^* & I & 0 \\
D^* & I & 0 & 0
\end{pmatrix}.
\]

Using the scheme outlined above one easily derives the following corollary.

\begin{corollary}
Given are \( n \times n \) complex matrices \( A, B, C, D, E \) of which \( A \) and \( E \) are Hermitian. There exists an \( n \times n \) matrix \( F \) such that (4.5) holds if and only if

\[ A - BD^* - DB^* + DED^* \geq 0, \quad \text{rank} \ [A - BD^* C D] = \text{rank} \ [A - BD^* D], \]
\end{corollary}
The two maximal symmetric triangular in this pattern are

\[ T_1 = 3 \times 3 \cup \{(4, 2), (4, 3), (3, 4), (2, 4)\} \]

and

\[ T_2 = \{2, 3, 4\} \cup \{(1, 2), (1, 3), (3, 1), (2, 1)\}. \]

Rewriting (4.6) yields the conditions in the corollary.

5. NORMAL COMPLETIONS

As a variation of the Hermitian completion problem, we consider here the normal completion problem, i.e., given a partial matrix, find a completion that is normal. In the special case of a partial matrix with some columns prescribed and the others unknown, the normal completion problem was introduced in [17], and pursued in [13] and [14]. While it is trivial to check whether a partial matrix has a Hermitian completion or not, it is in general non-trivial to check whether a normal completion exists. For this reason we omit any restrictions on the spectrum which are usual in the Hermitian case. As an example, consider the partial matrix

\[
\begin{bmatrix}
a & ? \\
b & c
\end{bmatrix}
\]  

(5.1)
One obtains all normal completions of (5.1) by choosing

$$
? = \begin{cases} \frac{a - e \xi}{\xi}, & \text{when } a \neq c \\ \xi b e^{\xi}, & \text{when } a = c, \end{cases}
$$

where $\xi \in [0, 2\pi]$ is to be chosen freely. In this case, a normal completion always exists. Going up one dimension, the reader can check immediately that the normal completion problem gets already quite involved.

Another thing that is not hard to observe but may initially be somewhat surprising is that the question of parameterizing all solutions has substantially different answers in the normal and Hermitian case. Consider for instance the $2 \times 2$ block diagonal case

$$
\begin{bmatrix} A & ? \\ ? & B \end{bmatrix}
$$

When $A$ and $B$ are normal with disjoint spectra, all normal completions of (5.2) are described as follows. We use the sign function

$$
\text{sign}(\varepsilon) = \begin{cases} \frac{\varepsilon}{|\varepsilon|}, & \text{when } \varepsilon \neq 0 \\ 1, & \text{when } \varepsilon = 0. \end{cases}
$$

**Proposition 5.1** Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{m \times m}$ be normal matrices, with disjoint spectra $\sigma(A) = \{\alpha_1, \ldots, \alpha_n\}$ and $\sigma(B) = \{\beta_1, \ldots, \beta_m\}$, respectively. The set of normal completions of (5.2) is in one-to-one correspondence with the set

$$
\left\{ Y \in \mathbb{C}^{m \times n} : \sum_{k=1}^{m} \left(1 - \frac{\sigma_i^2}{\sigma_k^2}\right) y_{ik} y_{kj} = 0, \sum_{k=1}^{n} \left(1 - \frac{\sigma_j^2}{\sigma_k^2}\right) y_{ik} y_{jk} = 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \right\},
$$

where $\sigma_{ij} = \text{sign}(\alpha_i - \beta_j)$. 
The set in (5.3) is in general much smaller than $C^{n \times m}$, which would be the parameterizing set in case $A = A^*$ and $B = B^*$, and one was looking for Hermitian completions.

**Proof.** By performing a block diagonal unitary similarity we may assume that

$$A = \text{diag}(\alpha_i)_{i=1}^n, \quad B = \text{diag}(\beta_i)_{i=1}^m.$$ 

Suppose that

$$\begin{bmatrix} A & Y \\ Y & B \end{bmatrix}$$

is a normal completion of (5.2). Then

$$AA^* + XX^* = A^* A + Y^* Y, \quad (5.4)$$

$$AY^* + XB^* = A^* X + Y^* B, \quad (5.5)$$

$$YY^* + BB^* = X^* X + B^* B. \quad (5.6)$$

Rewriting (5.5) we get that

$$\left(\alpha_i - \beta_j\right)_{i=1, j=1}^{n, m} \circ X = Y^* \circ \left(\alpha_i - \beta_j\right)_{i=1, j=1}^{n, m},$$

where $\circ$ denotes the Schur product (or Hadamard product). Thus $x_{ij} = \bar{y}_{k} \sigma_{ij}^2$.

From (5.4) we then get that

$$\sum_{k=1}^m \tilde{y}_{ki} y_{kj} = \sum_{k=1}^m x_{jk} \bar{x}_{jk} = \sum_{k=1}^m \tilde{y}_{ki} \sigma_{ik}^2 y_{kj} \sigma_{jk}^2,$$

thus

$$\sum_{k=1}^m \left(1 - \sigma_{jk}^2 / \sigma_{ik}^2\right) \tilde{y}_{ki} y_{kj} = 0.$$
Similarly, (5.6) may be rewritten as

$$\sum_{k=1}^{n} \left(1 - \frac{\sigma_k^2}{\lambda_k^2}\right)\sigma_k \lambda_k = 0$$

The proposition now follows.

In order to relate the normal completion problem to the previous sections we make the following observations.

**Lemma 5.2** Let $A$ be an $n \times n$ matrix. Then $A$ is normal if and only if

$$\begin{bmatrix} -I_n & A & 0 \\ A^* & 0 & A \\ 0 & A^* & I_n \end{bmatrix} = n.$$  

**Proof** Note that

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & -A \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} -I & A & 0 \\ A^* & 0 & A \\ 0 & A^* & I \end{bmatrix},$$

so that

$$\begin{bmatrix} -I & A & 0 \\ A^* & 0 & A \\ 0 & A^* & I \end{bmatrix} = n$$

if and only if $A^* A - A A^* \geq 0$ (i.e., $A$ is hyponormal). But since in finite dimensions hyponormality implies normality (see, e.g., [8], Corollary II.4.9) we are done.

Using this lemma we can reduce the normal completion problem to a structured Hermitian minimal negative inertia problem, as follows.
Proposition 5.3 Let $A$ be a $n \times n$ partial matrix over $\mathbb{C}$. Then $A$ has a normal completion if and only if the partial matrix

$$
\begin{bmatrix}
    -I_n & A & 0 \\
    A^* & 0 & A \\
    0 & A^* & I_n
\end{bmatrix}
$$

has a Hermitian completion with block entries $(1, 2)$ and $(2, 3)$ equal to one another that has negative inertia (smaller than or equal to $n$).

Proof Apply Lemma 5.2.

It is probably the most natural to formulate a normal completion problem inspired by the theory of subnormal operators, which was done in [17] as well. In fact, the problem of subnormality may be viewed as an “extended” normal completion problem. Indeed, the Hilbert space operator $A : \mathcal{H} \to \mathcal{H}$ is subnormal if and only if

$$
\begin{bmatrix}
    A & ? \\
    0 & ?
\end{bmatrix} : \mathcal{H} \to \mathcal{H}
$$

has a normal completion. The difference with a usual completion problem is that also the underlying space needs to be determined. We therefore propose the following minimal normal completion (MNC) problem:

(MNC). Given $A \in \mathbb{C}^{n \times n}$. Find a smallest possible normal matrix with $A$ as a principal submatrix.

In other words, find a normal completion of

$$
\begin{bmatrix}
    A & ? \\
    ? & ?
\end{bmatrix} : \mathbb{C}^n \to \mathbb{C}^n
$$

of smallest possible size (thus smallest possible $k$). We shall call this smallest number $k$ the normal defect of $A$, and denote it by $\text{nd}(A)$. Clearly, $\text{nd}(A) = 0$ if and only if $A$ is normal. As observed in [17], the matrix

$$
\begin{bmatrix}
    A & A^* \\
    A^* & A
\end{bmatrix}
$$
is normal, so it follows that for a $n \times n$ matrix $A$ we have that $\text{nd}(A) \leq n$.

Using what is known about unitary completions one can actually easily give a stronger bound. For this we introduce the number

$$\text{ud}(A) := \text{rank}(\|A\|^2 - A^*A),$$

where $\|\cdot\|$ denotes the spectral norm. The abbreviation ud stands for "unitary defect".

**Proposition 5.4** Let $A \in \mathbb{C}^{n \times n}$. Then

$$\text{nd}(A) \leq \text{ud}(A).$$

**Proof** Write the singular value decomposition of $A$ as

$$A = (U_1 \ U_2) \left( \begin{array}{cc} \|A\|I & 0 \\ 0 & \Sigma \end{array} \right) \left( \begin{array}{c} V_1 \\ V_2 \end{array} \right),$$

where $U= (U_1 \ U_2)$ and $V= (V_1 \ V_2)$ are unitary and $\Sigma$ is diagonal with $\|\Sigma\| < \|A\|$. Note that $\Sigma$ has size $\text{ud}(A)$. Then

$$E(A) := \left( \begin{array}{cc} A & U_2 \left( \|A\|^2 - \Sigma^2 \right)^{1/2} \\ \left( \|A\|^2 - \Sigma^2 \right)^{1/2} V_2 & -\Sigma \end{array} \right)$$

is $\|A\|$ times a unitary matrix. Thus (5.8) is normal.

The operator $\frac{1}{\|A\|} E(A)$ is in fact the so-called Julia operator for $T := \frac{1}{\|A\|} A$ or rotation operator for $T$. The introduction of this operator goes back to G. Julia [22]–[24] or [16].

As direct sums of normal matrices are normal, it is not hard to show that

$$\text{nd} \left( \begin{array}{ccc} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ \vdots & \ddots & \ddots \end{array} \right) \leq \sum_{i=1}^{m} \text{nd}(A_i) \leq \sum_{i=1}^{m} \text{ud}(A_i).$$

We therefore introduce the block unitary defect of $A$ (denoted by $\text{bud}(A)$) as follows. We call $B \in \mathbb{C}^{p \times p}$ unitarily irreducible if

$$B = U \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} U^*.$$
with $U$ unitary, implies that $B_1$ or $B_2$ is of trivial size (or equivalently, $B_1 \in \mathbb{C}^{p \times p}$ or $B_2 \in \mathbb{C}^{q \times q}$). Given $A \in \mathbb{C}^{n \times n}$, write $A$ as

$$A = U (\text{diag}(A_i)_{i=1}^m) U^*, \quad (5.10)$$

with $U$ unitary and $A_1, \ldots, A_m$ unitarily irreducible. Define

$$\text{bud}(A) = \sum_{i=1}^m \text{ud}(A_i).$$

That this is a well-defined notion follows from the following lemma.

**Lemma 5.5** Let $A \in \mathbb{C}^{n \times n}$. Then there exist a unitary $U$, unitarily irreducible matrices $B_1, \ldots, B_k$, and natural numbers $m_1, \ldots, m_k$, so that

$$A = U \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B_k \end{pmatrix} U^*,$$

where $B_i$ appears $m_i$ times, and $B_i$ is not unitarily similar to $B_j$ for $i \neq j$. Moreover, if $A$ is unitarily equivalent with $\text{diag}(C_i)_{i=1}^l$ with $C_1, \ldots, C_l$ unitarily irreducible then for each $i = 1, \ldots, k$ there exist exactly $m_i$ elements in $\{C_1, \ldots, C_l\}$ that are unitarily similar to $B_i$ (in particular, $l = m_1 + \cdots + m_k$).

**Proof** See Section 1.4 in [1], or apply Theorem 1.21.16 in [28] to the $C^*$-algebra generated by $A$.

We now have the following result.

**Theorem 5.6** Let $A \in \mathbb{C}^{n \times n}$. Then

$$\text{nd}(A) \leq \text{bud}(A) \quad (5.11)$$
Proof Suppose \( A \) is decomposed as in (5.10) with \( A_i \in \mathbb{C}^{n_i \times n_i} \). By Proposition 5.4, for each \( i \) there exist \( B_i, C_i, D_i \) such that

\[
\begin{bmatrix}
A_i & B_i \\
C_i & D_i
\end{bmatrix} : \mathbb{C}^{n_i} \oplus \mathbb{C}^{n_i} \rightarrow \mathbb{C}^{n_i} \oplus \mathbb{C}^{n_i} \to \mathbb{C}^{n_i}
\]

is normal. Then

\[
\begin{pmatrix}
U & 0 \\
0 & I
\end{pmatrix} \begin{pmatrix}
\text{diag}(A_i) & \text{diag}(B_i) \\
\text{diag}(C_i) & \text{diag}(D_i)
\end{pmatrix} \begin{pmatrix}
U^* & 0 \\
0 & I
\end{pmatrix}
\]

is normal with \( A \) as a principal submatrix.

Observe that when \( A \) is normal we have \( \text{nd}(A) = \text{bud}(A) = 0 \). In addition, when \( A \in \mathbb{C}^{2 \times 2} \) we also get equality in (5.11). Indeed, when \( A \in \mathbb{C}^{2 \times 2} \) is not normal then \( 0 < \text{nd}(A) \leq \text{bud}(A) \leq 1 \), so equality follows.

Let us end this section with some open problems. First of all, there is the problem of strengthening Theorem 5.6 (does equality hold? If not, how does one determine \( \text{nd}(A) \)?) And related, is it even true that \( \text{nd}((\text{diag}(A_i))_{i=1}^{m}) = \sum_{i=1}^{m} \text{nd}(A_i) \)? Next, are the minimal normal completions unique in some sense? When

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

is a minimal normal completion of \( A \) then so is

\[
\begin{bmatrix}
I & 0 \\
0 & U
\end{bmatrix} \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \begin{bmatrix}
I & 0 \\
0 & U^*
\end{bmatrix},
\]

for \( U \) unitary. Is this all the freedom?

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