SECTION 1.1: AFFINE VARIETIES

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1.1.

(a) Let \( Y \) be the plane curve \( y = x^2 \) (i.e., \( Y \) is the zero set of the polynomial \( f = y - x^2 \)). Show that \( A(Y) \) is isomorphic to a polynomial ring in one variable over \( k \).

(b) Let \( Z \) be the plane curve \( xy = 1 \). Show that \( A(Z) \) is not isomorphic to a polynomial ring in one variable over \( k \).

(c) Let \( f \) be any irreducible quadratic polynomial in \( k[x, y] \), and let \( W \) be the conic defined by \( f \). Show that \( A(W) \) is isomorphic to \( A(Y) \) or \( A(Z) \). Which one is it when?

Solution.

(a) We know \( A(Y) = k[x, y]/I(Y) \), so our first step is to determine the ideal \( I(Y) \). As the function \( f(x, y) = y - x^2 \) is irreducible in \( k[x, y] \), and as \( k[x, y] \) is a UFD (so that irreducible elements are prime), the ideal \( \langle y - x^2 \rangle \) is prime. Therefore \( I(Y) = \sqrt{\langle y - x^2 \rangle} = \langle y - x^2 \rangle \), and so \( A(Y) = k[x, y]/\langle y - x^2 \rangle \).

We now show \( A(Y) \) is isomorphic to the polynomial ring \( k[t] \). Define a \( k \)-algebra homomorphism \( \phi : k[x, y] \to k[t] \) by \( \phi(x) = t, \phi(y) = t^2 \). Then \( \phi \) is surjective with kernel precisely the ideal \( \langle y - x^2 \rangle \), and hence induces an isomorphism

\[
A(Y) = k[x, y]/\langle y - x^2 \rangle \cong k[t].
\]

This isomorphism is equivalent to the statement that the plane curve \( Y \) is isomorphic (as an affine variety) to the affine line \( \mathbb{A}_k^1 \).

(b) As the function \( f(x, y) = xy - 1 \) is irreducible in \( k[x, y] \), the ideal \( \langle xy - 1 \rangle \) is prime, and hence \( I(Z) = \sqrt{\langle xy - 1 \rangle} = \langle xy - 1 \rangle \). Thus, \( A(Z) = k[x, y]/\langle xy - 1 \rangle \). Now suppose \( \phi : k[x, y] \to k[t] \) is any \( k \)-algebra homomorphism with \( xy - 1 \in \ker \phi \). Let \( \phi(x) = p(t), \phi(y) = q(t) \). Then

\[
0 = \phi(xy - 1) = \phi(x)\phi(y) - 1 = p(t)q(t) - 1,
\]

and so \( p(t)q(t) = 1 \) in \( k[t] \). As \( k \) is algebraically closed (and hence every non-constant polynomial has a root), this implies \( p(t), q(t) \) are constants. Therefore, \( \text{im } \phi = k \), and so \( \phi \) is not surjective. Thus, \( k[x, y]/\langle xy - 1 \rangle \) cannot be isomorphic to \( k[t] \).

(c) First let us sketch a geometric argument. We will omit many of the details that would need checking to make this argument rigorous. The basic idea is to consider the projective closure of the given affine conic, \( \overline{W} \subset \mathbb{P}_k^2 \). As \( W \) is irreducible, one can check that \( \overline{W} \) must be nonsingular. One then recalls the fact that all nonsingular conics in \( \mathbb{P}_k^2 \) are isomorphic to \( V_{\mathbb{P}_k^2}(xy + yz + xz) \), which is

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in turn isomorphic to \( \mathbb{P}_k^1 \). (An outline of the argument: choose any three distinct points on the given conic, and choose coordinates on \( \mathbb{P}_k^2 \) so that these points are \([1:0:0],[1:1:0],[0:0:1]\). The conic must then have the equation \( Ax + By + Cxz = 0 \), with \( A,B,C \) all nonzero (as the conic is nonsingular). Rescaling \( x,y,z \) by \( B,C,A \), respectively, we may thus assume the conic has equation \( xy + yz + zx = 0 \). It follows that all nonsingular conics are isomorphic, and as it is easy to find one isomorphic to \( \mathbb{P}_k^1 \), they all are.) In any case, we know that \( \overline{W} \) is isomorphic to \( \mathbb{P}_k^1 \).

Now, by Bézout’s theorem the original affine conic \( W \) (considered as embedded in \( \mathbb{P}_k^2 \)) intersects the hyperplane at infinity in either two distinct points, or one point of multiplicity two. If \( W \) is defined by \( f(x,y) = ax^2 + bxy + cy^2 + \) lower order terms, then these two cases are distinguished precisely by the discriminant. In other words, the conic intersects in two points when \( b^2 \neq 4ac \), and in one point when \( b^2 = 4ac \). In the first case, \( W \) is isomorphic to \( \mathbb{P}_k^1 \setminus \{p\} \cong \mathbb{A}_k^1 \), and so \( A(W) \cong k[t] \cong A(Y) \). In the second case, \( W \) is isomorphic to \( \mathbb{P}_k^1 \setminus \{p,q\} \cong \mathbb{A}_k^1 \setminus \{0\} \cong Z, \) and so \( A(W) \cong A(Z) \).

For a more algebraic (but less intuitive) solution that does not rely on projective geometry, one could follow the classification of affine conics, say as presented in Gibson’s Elementary Geometry of Algebraic Curves (see Chapter 5). Assuming our field \( k \) is algebraically closed and characteristic zero, one can follow a series of simple changes of coordinates to reduce every affine plane conic to one of the following five types (i.e., isomorphism classes): the parabola \( y = x^2 \); the general conic \( x^2 − y^2 = 1 \); the line-pair \( x^2 = y^2 \); the parallel lines \( y^2 = 1 \); and the repeated line \( y^2 = 0 \). Only the first two classes are irreducible. The first is precisely the conic \( Y \) in part (a). The second is clearly isomorphic to the conic \( Z \) in part (b).

\[ \square \]

1.2. The twisted cubic curve. Let \( Y \subseteq \mathbb{A}^3 \) be the set \( Y = \{ (t, t^2, t^3) \mid t \in k \} \). Show that \( Y \) is an affine variety of dimension 1. Find generators for the ideal \( I(Y) \). Show that \( A(Y) \) is isomorphic to a polynomial ring in one variable over \( k \). We say that \( Y \) is given by the parametric representation \( x = t, y = t^2, z = t^3 \).

Solution. First observe that the two simplest obvious relations that hold for any point \((a_1, a_2, a_3) \in Y\) are \( a_1^2 = a_2 \) and \( a_1^3 = a_3 \). We therefore certainly have \( Y \subseteq V(x^2 − y, x^3 − z) \). On the other hand, if \((a_1, a_2, a_3) \in V(x^2 − y, x^3 − z)\), then \( a_1^2 = a_2, a_1^3 = a_3 \), and so \((a_1, a_2, a_3) = (a_1, a_1^2, a_1^3) \in Y \). Thus, \( V(x^2 − y, x^3 − z) \subseteq Y \), and equality follows.

We next claim that \( I(Y) = \langle x^2 − y, x^3 − z \rangle \). To prove this, we follow an argument similar to the Division Algorithm. (See Cox, Little, and O’Shea’s Ideals, Varieties, and Algorithms for additional details and insights.) Observe that any monomial in \( k[x,y,z] \) can be rewritten as

\[
x^ay^bz^c = x^a(x^2 + (y - x^2))^b(y^2 + (z - x^3))^c
= x^{a+2b+3c} + p_1(x,y,z)(y - x^2) + p_2(x,y,z)(z - x^3),
\]
and so every polynomial \( f(x, y, z) \in k[x, y, z] \) can be expressed as
\[
f(x, y, z) = p(x) + p_1(x, y, z)(y - x^2) + p_2(x, y, z)(z - x^3).
\]
Suppose now \( f \in I(Y) \). Then \( f(t, t^2, t^3) = 0 \) for every \( t \in k \), and hence (from the above identity) \( p(t) = 0 \) for every \( t \in k \). It follows that \( p \) must be the zero polynomial in \( k[x] \). Thus, \( f(x, y, z) \in \langle x^2 - y, x^3 - z \rangle \), and so \( I(Y) \subseteq \langle x^2 - y, x^3 - z \rangle \). Since the other inclusion is clear (and was implicit in our first sentence of this solution), equality follows.

We now know that \( A(Y) = k[x, y, z]/\langle x^2 - y, x^3 - z \rangle \). Define a \( k \)-algebra homomorphism \( \phi : k[x, y, z] \to k[t] \) by \( \phi(x) = t, \phi(y) = t^2, \phi(z) = t^3 \). This morphism is clearly surjective, with kernel exactly the ideal \( \langle x^2 - y, x^3 - z \rangle \). It thus induces an isomorphism \( A(Y) \cong k[t] \).

1.3. Let \( Y \) be the algebraic set in \( \mathbb{A}^3 \) defined by the two polynomials \( x^2 - yz \) and \( xz - x \). Show that \( Y \) is the union of three irreducible components. Describe them and find their prime ideals.

**Solution.** As the second polynomial factors as \( x(z - 1) \), we have
\[
Y = V(x^2 - yz, xz - x) \\
= V(x^2 - yz, x(z - 1)) \\
= V(x^2 - yz, x) \cup V(x^2 - yz, z - 1) \\
= V(yz, x) \cup V(x^2 - y, z - 1) \\
= V(y, x) \cup V(z, x) \cup V(x^2 - y, z - 1).
\]
The ideals \( \langle y, x \rangle, \langle z, x \rangle \) are clearly prime, and hence \( V(y, x), V(z, x) \) are irreducible components of \( Y \). (Indeed, they are simply the \( z \)- and \( y \)-axes, respectively.) The ideal \( \langle x^2 - y, z - 1 \rangle \) is also prime, as can be seen by considering the surjective \( k \)-algebra homomorphism \( \phi : k[x, y, z] \to k[t] \) defined by \( \phi(x) = t, \phi(y) = t^2, \phi(z) = t^3 \); the kernel of this morphism is precisely the ideal \( \langle x^2 - y, z - 1 \rangle \). As \( k[t] \) is an integral domain, this implies the ideal \( \langle x^2 - y, z - 1 \rangle \) is prime. It follows that \( V(x^2 - y, z - 1) \) is isomorphic to the affine line (which gives another way to see the variety is irreducible).

Geometrically, the first two components are the \( z \)- and \( y \)-axes, while the third is the planar parabola \( y = x^2 \) sitting in the plane \( z = 1 \).

1.4. If we identify \( \mathbb{A}^2 \) with \( \mathbb{A}^1 \times \mathbb{A}^1 \) in the natural way, show that the Zariski topology on \( \mathbb{A}^2 \) is not the product topology of the Zariski topologies on the two copies of \( \mathbb{A}^1 \).

**Solution.** Suppose \( k \) is algebraically closed. Then the only closed sets in \( \mathbb{A}^1 \times \mathbb{A}^1 \) in the product topology are the empty set, products of pairs of finite sets, and products of finite sets with \( \mathbb{A}^1 \). Identifying the space with \( \mathbb{A}^2 \), this would mean that the only closed sets are the empty set, finite collections of points, and finite collections of horizontal and vertical lines.
On the other hand, in the Zariski topology on \( A^2 \) there is a closed subset for every irreducible polynomial \( f(x, y) \in k[x, y] \); i.e., every irreducible curve is a closed subset. Thus, there are more closed subsets in the Zariski topology on \( A^2 \) than there are in the product topology on \( A^1 \times A^1 \).

1.5. Show that a \( k \)-algebra \( B \) is isomorphic to the affine coordinate ring of some algebraic set in \( A^n \), for some \( n \), if and only if \( B \) is a finitely generated \( k \)-algebra with no nilpotent elements.

Solution. If \( B \) is a \( k \)-algebra isomorphic to \( A(S) \) for some algebraic set \( S \subseteq A^n \), then \( B \) fits into an exact sequence of \( k \)-algebras

\[
0 \to \mathfrak{I}(S) \to k[x_1, \ldots, x_n] \to B \to 0.
\]

This implies \( B \) is finitely generated, and as \( \mathfrak{I}(S) \) is always a radical ideal, this also implies \( B \) is nilpotent-free.

Conversely, if \( B \) is a nilpotent-free, finitely-generated \( k \)-algebra, then there is an exact sequence

\[
0 \to I \to k[x_1, \ldots, x_n] \to B \to 0
\]

with \( I \) a radical ideal. Let \( S = V(I) \subseteq A^n \). Then \( \mathfrak{I}(S) = \sqrt{I} = I \), and so \( A(Y) = k[x_1, \ldots, x_n]/I \cong B \).

1.6. [Prove the following:] Any nonempty open subset of an irreducible topological space is dense and irreducible. If \( Y \) is a subset of a topological space \( X \), which is irreducible in its induced topology, then the closure \( \overline{Y} \) is also irreducible.

Solution. Suppose \( X \) is an irreducible topological space and \( U \subseteq X \) is a nonempty open subset. Then \( X = \overline{U} \cup (X \setminus U) \), and as \( X \) is irreducible this implies either \( \overline{U} = X \) or \( X \setminus U = X \). As \( U \) is assumed to be nonempty, the latter cannot be true, and hence we must have \( \overline{U} = X \); i.e., \( U \) is dense in \( X \).

We next show \( U \) is irreducible. Suppose we can write \( U = V_1 \cup V_2 \) for some nonempty subsets \( V_1, V_2 \subseteq U \) that are closed in \( U \). For each \( i \), we can write \( V_i = F_i \cap U \) for some closed set \( F_i \subseteq X \). Then observe that \( U = (F_1 \cup F_2) \cap U \), hence \( X = \overline{U} = F_1 \cup F_2 \). As \( X \) is irreducible, this implies either \( F_1 = X \) (and hence \( V_1 = U \)) or \( F_2 = X \) (and hence \( V_2 = U \)). Thus, \( U \) is irreducible.

Now let us drop the irreducibility assumption on \( X \), and suppose \( Y \subseteq X \) is irreducible in its induced topology. We claim \( \overline{Y} \) is also irreducible. Suppose we can write \( \overline{Y} = V_1 \cup V_2 \) for some nonempty subsets \( V_1, V_2 \) that are closed in \( \overline{Y} \) (and hence also closed in \( X \)). Then we have \( \overline{Y} = (V_1 \cap \overline{Y}) \cup (V_2 \cap \overline{Y}) \). As \( \overline{Y} \) is assumed to be irreducible, this implies \( V_i \cap \overline{Y} = Y \) for some \( i \). It then follows that \( Y \subseteq V_i \), hence \( \overline{Y} \subseteq \overline{V_i} \), from which it follows that \( \overline{Y} = V_i \). Thus, \( \overline{Y} \) is indeed irreducible.
1.7.
(a) Show that the following conditions are equivalent for a topological space $X$: 
(i) $X$ is noetherian; (ii) every nonempty family of closed subsets has a minimal element; (iii) $X$ satisfies the ascending chain condition for open subsets; (iv) every nonempty family of open subsets has a maximal element.

(b) [Prove:] A noetherian topological space is \textit{quasi-compact}, i.e., every open cover has a finite subcover.

(c) [Prove:] Any subset of a noetherian topological space is noetherian in its induced topology.

(d) [Prove:] A noetherian topological space that is also Hausdorff must be a finite set with the discrete topology.

Solution.

(a) We first prove (i) is equivalent to (ii). Assume (i), and suppose $\{F_\alpha\}_\alpha$ is any nonempty family of closed subsets of $X$. We claim the family must have a minimal member. Choose any member $F_\alpha_1$. If $F_\alpha_1$ is minimal, we are done. If not, then there exists some family member $F_\alpha_2$ such that $F_\alpha_2 \subsetneq F_\alpha_1$. If $F_\alpha_2$ is minimal, we are done. If not, we continue inductively. As $X$ is noetherian, the process must terminate, for otherwise we would construct a strictly descending chain of closed subsets $F_\alpha_1 \supsetneq F_\alpha_2 \supsetneq \cdots$. Letting $n$ be the step at which the process terminates, we have found our minimal element, $F_\alpha_n$. Conversely, assume (ii), and suppose $F_1 \supseteq F_2 \supseteq \cdots$ is a descending chain of closed subsets of $X$. The set $\{F_i\}_i$ must have a minimal element, say $F_n$. It follows that our descending chain stabilizes at step $n$. Thus, $X$ is noetherian.

To prove (i) is equivalent to (iii), observe that a strictly decreasing chain of closed subsets of $X$ gives rise to a strictly increasing chain of open subsets of $X$ (by complementation), and conversely.

To prove (ii) is equivalent to (iv), observe that a minimal element of a nonempty family of closed subsets gives rise to a maximal element of a nonempty family of open subsets (by complementation), and conversely.

(b) Let $U \subseteq X$ be any open subset, and let $\{U_i \subseteq U\}_i$ be any (countable) open covering of $U$. For each natural number $k$, define $V_k = \bigcup_{i=1}^k U_i$. Then we have an increasing sequence of open subsets of $X$, $V_1 \subseteq V_2 \subseteq \cdots$, and as $X$ is noetherian, this sequence must stabilize. This implies there exists some number $N$ such that $V_k = V_N$ for all $k \geq N$, and hence

$$U = \bigcup_{i=1}^\infty U_i = \bigcup_{i=1}^N U_i.$$  

Thus, there is a finite subcover of $U$.

(c) Suppose $X$ is noetherian and $Y \subseteq X$ is any subset. We claim $Y$ is then noetherian in its induced topology. To prove this, suppose $W_1 \supseteq W_2 \supseteq \cdots$ is any descending chain of closed subsets of $Y$. Write $W_1 = F_1 \cap Y$ for a closed subset $F_1$ of $X$. Next write $W_2 = F_2 \cap Y$ for a closed subset $F_2 \subseteq F_1$ of $X$. (Note that if $F_2$ were not contained in $F_1$, then we could replace $F_2$ with the closed set $F_2' = F_2 \cap F_1$.) Continuing this process inductively, we may write $W_i = F_i \cap Y$ for a closed subset $F_i \subseteq F_{i-1}$ of $X$. We thus obtain a descending sequence of closed subsets of $X$. 


$F_1 \supseteq F_2 \supseteq \cdots$. As $X$ is noetherian, this sequence must stabilize, and hence there must exist an integer $N$ such that $F_i = F_N$ for all $i \geq N$. It then follows that $W_i = W_N$ for all $i \geq N$, and hence $Y$ is noetherian.

(d) Suppose $X$ is a noetherian and Hausdorff topological space. We claim every subset $Y \subseteq X$ is closed. To see this, suppose $Y \not\subseteq X$ is a nonempty proper subset. We’ll show $X \setminus Y$ is open. Choose any $x \in X \setminus Y$. As $X$ is Hausdorff, for each $y \in Y$ there exist disjoint open subsets $V_1(y), V_2(y) \subset X$ with $y \in V_1(y)$ and $x \in V_2(y)$. We thus obtain an open cover $\{V_1(y) \subseteq Y\}_{y \in Y}$ of $Y$. By the result of part (c), above, $Y$ is noetherian and hence (by part (b)) quasi-compact. As such, there exists a finite subcover $\{V_1(y_i) \subseteq Y\}_{i=1}^n$. Let $V_2 = \bigcap_{i=1}^n V_2(y_i)$. Then $V_2$ is an open subset of $X$ disjoint from $Y$ and containing $x$. Thus, $X \setminus Y$ is open, and hence $Y$ is closed.

We have shown every subset of $X$ is closed, and hence every subset is also open. Thus, $X$ has the discrete topology. It follows that $\{\{x\} \subset X\}_{x \in X}$ is a disjoint open cover of $X$. Since $X$ is noetherian– and hence quasi-compact (by part (b)) –there must exist a finite subcover. As the cover is disjoint, this is only possible if the original cover is finite, i.e., $X$ is a finite set.

\[ \square \]

1.8. Let $Y$ be an affine variety of dimension $r$ in $A^n$. Let $H$ be a hypersurface in $A^n$, and assume that $Y \not\subseteq H$. [Prove:] Then every irreducible component of $Y \cap H$ has dimension $r-1$. (See (7.1) for a generalization.)

**Solution.** Implicit in the statement of the problem is the assumption that $Y \cap H \neq \emptyset$. Write $H = V(f)$ for some irreducible polynomial $f \in k[x_1, \ldots, x_n]$, and let $\overline{f} \in A(Y)$ be the image of $f$ under the natural projection $k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/I(Y) = A(Y)$. Note that $\overline{f}$ is neither a unit (as $Y \cap H \neq \emptyset$) nor a zero divisor (as $Y \not\subseteq H$). The irreducible components of $Y \cap H$ therefore correspond to the minimal prime ideals of $A(Y)$ that contain $\overline{f}$. Let $p$ be any such prime ideal. By Theorem 1.11A, $p$ has height 1. It then follows from Theorem 1.8A, part (b), that

$$\dim A(Y)/p = \dim A(Y) - \text{height } p = \dim Y - 1.$$ 

Thus, the irreducible component of $Y \cap H$ corresponding to the prime ideal $p$ has dimension $r-1$. \[ \square \]

1.9. Let $a \subseteq A = k[x_1, \ldots, x_n]$ be an ideal that can be generated by $r$ elements. [Prove:] Then every irreducible component of $V(a)$ has dimension $\geq n-r$. 

**Solution.** Write $a = \langle f_1, \ldots, f_r \rangle$, where each $f_i$ is neither a unit nor a zero divisor. Then $V(a) = V(f_1) \cap \cdots \cap V(f_r)$. For each $i$, let $f_i = \prod_{j} f_{ij}$ be a factorization of $f_i$ into irreducibles. Then $V(f_i) = \bigcup_{j} V(f_{ij})$ is a decomposition of $V(f_i)$ into irreducible components, and $V(a) = \bigcup_{(j_1, \ldots, j_r)} (V(f_{1j_1}) \cap \cdots \cap V(f_{rj_r}))$. Fix any index $(j_1, \ldots, j_r)$. By Proposition 1.13, we have $\dim V(f_{1j_1}) = n-1$. By Exercise 1.8, with $Y = V(f_{1j_1})$ and $H = V(f_{2j_2})$, we then have $\dim (V(f_{1j_1}) \cap V(f_{2j_2})) \geq (n-1)-1 = n-2$. (Note that in this case we might
have $Y \subseteq H$, hence the inequality on dimensions.) Using induction and Exercise 1.8, we then see that $\dim(V(f_{j_1}) \cap \ldots \cap V(f_{j_r})) \geq n - r$, and hence $\dim V(a) \geq n - r$. 

1.10.

(a) [Prove:] If $Y$ is any subset of a topological space $X$, then $\dim Y \leq \dim X$.

(b) [Prove:] If $X$ is a topological space that is covered by a family of open subsets $\{U_i\}$, then $\dim X = \sup_i \dim U_i$.

(c) Give an example of a topological space $X$ and a dense open subset $U$ with $\dim U < \dim X$.

(d) [Prove:] If $Y$ is a closed subset of an irreducible finite-dimensional topological space $X$, and if $\dim Y = \dim X$, then $Y = X$.

(e) Give an example of a noetherian topological space of infinite dimension.

Solution.

(a) Let $W_0 \subset W_1 \subset \cdots \subset W_n$ be a chain of distinct sets, each closed and irreducible in $Y$. Let $\overline{W}_i$ be the closure of $W_i$ in $X$, for each $i$. By Exercise 1.6, the sets $\overline{W}_i$ are irreducible. Also notice that $\overline{W}_{i-1} \subset \overline{W}_i$, for otherwise we would have $\overline{W}_{i-1} = \overline{W}_i$, hence $W_{i-1} = \overline{W}_{i-1} \cap Y \supseteq W_i \cap Y = W_i$, contrary to hypothesis. We thus have a chain of distinct sets, $\overline{W}_0 \subset \overline{W}_1 \subset \cdots \subset \overline{W}_n$, each closed and irreducible in $X$. It follows that $\dim X \geq \dim Y$.

(b) By (a), for any subset $U \subseteq X$ we have $\dim U \leq \dim X$. So, if $\{U_i \subseteq X\}$ is any collection of subsets, then $\sup_i \dim U_i \leq \dim X$. Now suppose the collection is an open cover of $X$ (consisting of nonempty, open sets), and let $n = \dim X$. We will show $\sup_i \dim U_i \geq \dim X$, and hence equality holds. We separate consider three cases, according to the dimension of $X$.

Case 1 ($n = 0$): In this case, since every $U_i$ is assumed to be nonempty, we have $0 \leq \dim U_i \leq \dim X = 0$, hence $\sup_i \dim U_i = 0 = \dim X$.

Case 2 ($0 < n < \infty$): In this case, we proceed by induction on $n$. As $\dim X = n$, there exists a chain $Z_0 \subset \cdots \subset Z_n$ of distinct, irreducible, closed subsets of $X$. Observe that $\dim Z_{n-1} = n - 1$, and that $\{U_i \cap Z_{n-1}\}_i$ is a cover of $Z_{n-1}$, so by induction it follows that $\sup_i \dim(U_i \cap Z_{n-1}) = n - 1$. As the set $\{\dim(U_i \cap Z_{n-1})\}_i$ is a finite, discrete set, its supremum is actually a maximum and is attained by some index $i$. Without loss of generality, we may assume this supremum is achieved by $i = 1$, i.e., $\dim(U_1 \cap Z_{n-1}) = n - 1$. There therefore exists a chain $W_0 \subset \cdots \subset W_{n-2} \subset U_1 \cap Z_{n-1}$ of distinct, irreducible, closed subsets of $U_1 \cap Z_{n-1}$. (Note that $U_1 \cap Z_{n-1}$ is irreducible by Exercise 1.6.) Notice that this same chain is also a chain of distinct, irreducible, closed subsets of $U_1$.

Now observe that $Z_n$ is irreducible, hence $U_1 \cap Z_n$ is irreducible and dense in $Z_n$. By assumption $Z_{n-1} \subset Z_n$, so $Z_n \setminus Z_{n-1}$ is a nonempty open subset of $Z_n$, hence also irreducible and dense in $Z_n$. It follows that $(U_1 \cap Z_n) \cap (Z_n \setminus Z_{n-1})$ is nonempty, and hence $U_1 \cap Z_n \supseteq U_1 \cap Z_{n-1}$. But then $W_0 \subset \cdots \subset W_{n-2} \subset U_1 \cap Z_{n-1} \subset U_1 \cap Z_n$ is a chain of distinct, irreducible closed subsets of $U_1$, and hence $\dim U_1 \geq n = \dim X$.

Case 3 ($n = \infty$): In this case, for every positive integer $m$ there exists an irreducible closed subset $Z_m \subset X$ of dimension $m$. By the argument in Case 2, for each
Then there is an index $i_m$ such that $\dim U_{i_m} = \dim Z_m = m$. It follows that $\sup_i \dim U_i \geq \sup_m \dim U_{i_m} = \sup_m m = \infty = \dim X$.

(c) Let $X$ be the two-point set $X = \{p, q\}$, and endow $X$ with the topology in which the only nonempty proper open subset is the singleton set $U = \{q\}$. Then $U$ is open and dense (as the only closed set containing $U$ is $X$), $U$ is irreducible (as it consists of a single point), and $\dim U = 0$ (as the only nonempty closed subset of $U$ is $U$ itself). On the other hand, $\dim X = 1$, as $\emptyset \nsubseteq \{p\} \subseteq X$ is a chain of distinct, irreducible, closed subsets of $X$.

(d) Let $n = \dim Y = \dim X < \infty$. Then there exists a chain $Z_0 \subsetneq \cdots \subsetneq Z_n$ of distinct, closed, irreducible subsets of $Y$. As $Y$ is closed in $X$, each $Z_i$ is also closed in $X$ (and still irreducible). As $X$ is irreducible, we therefore have a chain $Z_0 \subsetneq \cdots \subsetneq Z_n \subseteq X$ of closed, irreducible subsets of $X$. If $Z_n$ were properly contained in $X$, then this would imply $\dim X \geq n + 1$, contrary to our initial hypothesis. As such, we must have $Z_n = X$. But $Z_n \subseteq Y$, so this implies $Y = Z_n = X$.

(e) A topological space is noetherian if there does not exist an infinite descending chain of distinct closed sets, but an infinite ascending chain is not excluded (and will imply the space is infinite-dimensional). This observation suggests the following simple example. As a set take $X = \mathbb{N}$, and topologize $X$ by declaring the (nonempty, proper) closed sets to be those sets of the form $\{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. It is clear such sets are closed under finite union and arbitrary intersection, and hence do indeed define a topology on $X$. This space is noetherian, since any descending chain of distinct closed sets $F_0 \supsetneq F_1 \supsetneq \cdots$ has length at most equal to the (finite) number of elements in $F_0$. On the other hand, we have an infinite ascending chain $\{1\} \subseteq \{1, 2\} \subseteq \cdots$ of distinct, closed, irreducible subsets of $X$, and hence $X$ must be infinite-dimensional.

\[
\text{\hfill \Box}
\]

*1.11. Let $Y \subseteq \mathbb{A}^3$ be the curve given parametrically by $x = t^3, y = t^4, z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y, z]$ that cannot be generated by 2 elements. We say $Y$ is not a local complete intersection – cf. (Ex. 2.17).

\[\text{Solution.}\] We first claim $Y$ is homeomorphic (i.e., isomorphic as a topological space) to $\mathbb{A}^1_k$. By definition, $Y$ is the image of the map $\phi : \mathbb{A}^1_k \to \mathbb{A}^3_k$ defined by $\phi(a) = (a^3, a^4, a^5)$ for $a \in k$. Observe first that this map is injective. Indeed, suppose $a, b \in k$ and $\phi(a) = \phi(b)$. Then $a^3 = b^3, a^4 = b^4$ and $a^5 = b^5$. If $a = 0$, then the first equation gives $b = 0$, and hence $a = b$. If $a \neq 0$, then the first equation implies we also have $b \neq 0$, and dividing the second equation by the first then gives $a = b$. The map $\phi$ is also continuous, as it corresponds precisely to the $k$-algebra morphism $\phi^* : k[x, y, z] \to k[t]$ defined by $\phi^*(x) = t^3, \phi^*(y) = t^4, \phi^*(z) = t^5$. Thus, $\phi$ induces a homeomorphism from $\mathbb{A}^1_k$ to $Y$. It follows that $Y$ is one-dimensional and irreducible (as $\mathbb{A}^1_k$ is one-dimensional and irreducible), the latter of which implies the ideal $I(Y) = \ker \phi^*$ is prime. By Theorem 1.8a, it follows that $I(Y)$ must be a prime of height 2.

It remains to show $I(Y)$ cannot be generated by two or fewer elements. As $I(Y) = \ker \phi^*$, by a direct calculation one sees that $I(Y)$ is the set of all polynomials $f(x, y, z) =$
\[ \sum_{(a,b,c)} c_{a,b,c} x^a y^b z^c \text{ such that } \sum_{(a,b,c)} c_{a,b,c} = 0 \text{ for every } d \in \mathbb{N}_{\geq 0}. \]

Considering \( d = 0, 3, 4, \) and 5, respectively, we see that \( f \in I(Y) \) only if \( c_{0,0,0} = 0, c_{1,0,0} = 0, c_{0,1,0} = 0, \) and \( c_{0,0,1} = 0, \) respectively. In other words, if \( f \in I(Y) \), then \( f \) cannot contain any monomials of degree 0 or 1. (The cases \( d = 1, 2 \) are vacuous.) From the cases \( d = 6, 7, \) we see that we must have \( c_{2,0,0} = c_{1,1,0} = 0, \) i.e., \( f \) cannot contain the monomials \( x^2, xy \). The remaining degree two monomials are \( xz, y^2, yz, z^2, \) and \( f \) can contain these monomials. Indeed, the cases \( d = 8, 9, 10 \) correspond to the requirements that \( c_{1,0,1} + c_{0,2,0} = 0, c_{2,0,0} + c_{0,1,1} = 0 \) and \( c_{2,1,0} + c_{0,0,2} = 0, \) respectively. One readily checks that the polynomials \( f_1(x, y, z) = xz - y^2, f_2(x, y, z) = x^2 - yz \) and \( f_3(x, y, z) = x^2y - z^2 \) are all contained in \( I(Y) \), and are minimal generators of the ideal they generate, i.e., are algebraically independent. It follows that \( I(Y) \) needs at least these three elements as generators.

1.12. Give an example of an irreducible polynomial \( f \in \mathbb{R}[x, y] \) whose zero set \( V(f) \) in \( \mathbb{A}_\mathbb{R}^2 \) is not irreducible (cf. 1.4.2).

**Solution.** The polynomial \( f(x, y) = x^2 + y^2 + 1 \) is irreducible in \( \mathbb{R}[x, y] \) but has empty zero set (which is not irreducible by definition). For another (arguably less pathological) example, consider the polynomial \( f(x, y) = y^4 + y^2 + x^2(x - 1)^2 \). One can show this polynomial is irreducible in \( \mathbb{R}[x, y] \), and yet has reducible zero set \( \{0, 0\}, \{1, 0\} \).