ENTIRE FUNCTIONS MAPPING ARBITRARY COUNTABLE DENSE SETS AND THEIR COMPLEMENTS ONTO EACH OTHER

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1. Introduction

In W. K. Hayman's function theory problem book [2] the following problem (attributed to P. Erdős) is stated [2; p. 17, Problem 2.31]:

Let A, B be two countable dense sets in the plane. Does there exist an integral function f(z) so that f(z)∈B if and only if z∈A? If the answer is negative, it would be desirable to have conditions on A, B when this is so.

The following theorem answers the first part of the problem affirmatively for arbitrary A and B and in addition, of course, voids the second part of the problem.

THEOREM 1. Let A, B be two countable dense sets in the plane. Then there exists an entire function w = F(z) such that F(z)∈B if and only if z∈A.

If F(z) is not a polynomial, then, except possibly for one value w, F(z) must be infinitely many-to-one because of Picard's theorem. The construction of F(z) is based partially on techniques first used in [1]. Finally, we might add that a transcendental entire function which maps A onto B (but not the complement of A onto the complement of B) was constructed a few years ago by W. D. Maurer [3].

2. Proof of Theorem

First we shall need the following technical lemma, which though easy to prove is unfortunately somewhat complicated to state.

LEMMA 1 (adjustment lemma for finitely many values in the domain of the function). Let ε > 0 be given and let G(z) be an arbitrary polynomial. In addition,

(i) let s₁, s₂, ..., sₙ be distinct elements of the complex plane C,
(ii) let {pₖ}ₖ=₁ be any sequence of positive numbers which satisfy p₁ < p₂ < ... < pₙ and

\[ (\bigcup_{j=1}^{p} G^{-1}(s_j)) \cap \{|z| = \rho_k\} = \emptyset \] for k = 1, 2, ..., n,
(iii) let \{G⁻¹(sₖ)\} \cap \{|z| < ρₙ\} = \{zₖ,₁, zₖ,₂, ..., zₖ,nₖ\} = M_k for k = 1, 2, ..., n,
(iv) let κ be an arbitrary element of \bigcup_{k=1}^{n} M_k and let τ = G(κ).
(v) Finally, suppose G′(κ) ≠ 0.

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Then there exists a $\delta_0 > 0$ such that the following simultaneous interpolation and approximation problem can be solved:

**Problem:** For any $z^* \in \Delta (= \{|z - \kappa| < \delta_0\})$ find a polynomial $f(z)$ with the following properties:

(i) $|f(z)| < \varepsilon$ in $\{|z| < \rho_j\},$

(ii) $f(z) = 0$ for all $z$ in $\bigcup_{k=1}^{n} M_k$ except for $z = \kappa,$

(iii) $f(z^*) + G(z^*) = \tau,$ and if $\exists \in (\Delta - \{z^*\}),$ then $f(\exists) + G(\exists) \neq \tau,$

(iv) for each fixed $k$ the functions $[G(z) + f(z) - s_k]$ and $[G(z) - s_k]$ have the same number of zeros in $\{|z| < \rho_j\}$ for $j = k, k+1, \ldots, n.$

**Proof of Lemma 1.** Let

$$f(z) = \zeta \cdot \prod_{k=1}^{n} \left[ \prod_{j=1}^{l_k} \frac{1}{z - z_{k,j}} \right].$$

Pick small disjoint disks $D_{k,j}$ about the $z_{k,j} (k = 1, \ldots, n; j = 1, 2, \ldots, l_k)$ which do not intersect $\bigcup_{k=1}^{n} \{|z| = \rho_k\}.$ Then for all complex $\zeta$ in absolute value less than some fixed positive number $\sigma_0,$ by Rouché's theorem, we have:

(i) $[G(z) - s_k]$ and $[f(z) + G(z) - s_k]$ have exactly the same number of zeros in $D_{k,j} (k = 1, 2, \ldots, n; j = 1, 2, \ldots, l_k),$

(ii) $[G(z) - s_k]$ and $[f(z) + G(z) - s_k]$ have exactly the same number of zeros in $\{|z| < \rho_j\} (i = k, k+1, \ldots, n).$

In (1), below, the function $\zeta(z^*)$ is defined for each $z^* \in \Delta.$ Also $\zeta(z^*)$ is in absolute value less than $\sigma_0$ if $\delta_0$ is sufficiently small (this follows since $\zeta(z^*)$ is a continuous function of $z^*$).

$$\zeta(z^*) = \frac{\tau - G(z^*)}{\prod_{k=1}^{n} \left[ \prod_{j=1}^{l_k} \frac{1}{z - z_{k,j}} \right]}.$$

We can choose $\delta_0$ such that the remaining condition in Property (iii) of the Problem holds since $\zeta'(\kappa) \neq 0.$ This follows from simply computing $\zeta'(\kappa)$ and using the fact that $G'(\kappa) \neq 0.$

We also need another technical lemma which is similar to Lemma 1.

**Lemma 2** (adjustment lemma for finitely many values in the range of the function). Let $\varepsilon > 0$ be given and let $F(z)$ be an arbitrary polynomial. Moreover,

(i) let $s_1, s_2, \ldots, s_n$ be distinct elements of $C,$

(ii) let $\{\rho_k\}_{k=1}^{n}$ be any sequence of positive numbers which satisfy $\rho_k < \rho_{k+1}$ ($k = 1, 2, \ldots, n-1$) and $(\bigcup_{j=1}^{n} \{F^{-1}(s_j)\}) \cap \{|z| = \rho_k\} = \emptyset$ for $k = 1, 2, \ldots, n.$

(iii) Let $\{F^{-1}(s_k)\} \cap \{|z| < \rho_n\} = \{z_{k,1}, z_{k,2}, \ldots, z_{k,l_k}\} = M_k,$ for $k = 1, 2, \ldots, n.$

(iv) Suppose $\kappa$ is any element of $\{|z| < \rho_n\}$ which is not in $\bigcup_{k=1}^{n} \{F^{-1}(s_k)\}$ and let $\tau = F(\kappa).$
Then there is a $\delta_0 > 0$ such that for all $b_0 \ (|b_0| < \delta_0)$ it is possible to solve the following problem:

**Problem.** Find a polynomial $g(z)$ with the following properties:

(i) $|g(z)| < \varepsilon$ in $\{|z| < \rho_0\}$,

(ii) $g(z) = 0$ for all $z$ in $\bigcup_{k=1}^n M_k$,

(iii) $g(\kappa) = f(\kappa) + b_0$,

(iv) for each fixed $k$ the functions $[F(z) + g(z) - s_k]$ and $[F(z) - s_k]$ have the same number of zeros in $\{|z| < \rho_j\}$ for $j = k, k+1, \ldots, n$.

**Proof of Lemma 2.** This follows by the same type of argument as in Lemma 1, but is much easier.

**Proof of theorem.** We shall prove Theorem 1 in the special case where $A = B$ since this avoids many exceedingly cumbersome (but technically straightforward) details. Since Lemmas 1 and 2 were proved for arbitrary finite sets of points, there is no essential difficulty in extending the proof to the case where $A \neq B$.

Let $\{\alpha_n \}_{n=1}^\infty$ be a sequence of positive real numbers satisfying $\alpha_n \uparrow \infty$ and another compatible side condition to be added later. Also, let $A = \{a_1, a_2, \ldots\}$ and choose a non-linear polynomial $g_1(z)$ such that $g_1(a_1) \in A$. Choose $\alpha_i$ such that

$$\{g_i^{-1}(a_i)\} \cap \{|z| = \alpha_i\} = \emptyset \text{ and } |a_i| < \alpha_i.$$

**First step**

**Remark 1.** We can assume that $g_1'(z) \neq 0$ at any point of

$$S = (\{g_1^{-1}(a_i)\} \cap \{|z| < \alpha_i\}) \cup \{a_i\}.$$

First we note that the only restriction on $g_1(z)$ is that it be a polynomial function with $g_1(a_1) \in A$. Let $K$ be the sum of the multiplicities of the zeros of $g_1'(z)$ at all of the points of $S$. It is sufficient to show that we can transform $g_1(z)$ into another function $\tilde{g}_1(z)$ which has all the properties of $g_1(z)$ mentioned above but such that the sum of the multiplicities of the zeros of $\tilde{g}_1'(z)$ at all of the points of $S$ is less than $K$. This can be taken care of by adding

$$h(z) = l \cdot \prod_{z \in S} (z - \tilde{z})$$

to $g_1(z)$ and we can find arbitrarily small $l$'s such that $[g_1'(z) + h'(z)] \neq 0$ for $z \in S$. Using Rouché's theorem it is also possible to choose $l$ such that the number of zeros (counting multiplicity) of $[g_1(z) - a_1]$ and $[g_1(z) + h(z) - a_1]$ remains the same. We now set $\tilde{g}_1(z) = g_1(z) + h(z)$, and it follows that the sum of the multiplicities of the zeros of $\tilde{g}_1'(z)$ for

$$z \in (\{\tilde{g}_1^{-1}(a_i)\} \cap \{|z| < \alpha_i\}) \cup \{a_i\}$$

is less than $K$.

Now set:

$$A_1^1 = \{g_1^{-1}(a_i)\} \cap \{|z| < \alpha_i\} \cap A$$

and

$$\bar{A}_1^1 = \{g_1^{-1}(a_i)\} \cap \{|z| < \alpha_i\} \cap (\mathbb{C} - A).$$
By repeated application of Lemma 1, we can choose an $f_1(z)$ such that

(i) $|f_1(z)| < \frac{1}{2}$ for $|z| < \alpha_1$,

(ii) $f_1(z) = 0$ for $z \in \{a_1\} \cup A_1$,

(iii) $\{(f_1 + g_1)^{-1}(a_1)\} \cap \{|z| < \alpha_1\}$ is contained in $A$,

(iv) $(f_1 + g_1)(z) - a_1$ has the same number of zeros in $\{|z| < \alpha_1\}$ as $(f_1(z) - a_1)$, and all the zeros of $(f_1 + g_1)(z) - a_1$ in $\{|z| < \alpha_1\}$ have multiplicity one.

We shall now briefly describe the construction of $f_1$. First, we set $f_1$ equal to

$$\sum_{i=1}^{n} h_i(z)$$

where $h_i$ is the function resulting from the $i$-th application of Lemma 1 and $n$ is the number of applications of Lemma 1 required, i.e., the number of points in $A_1$. The $i$-th time Lemma 1 is applied, the function $G(z)$, in Lemma 1, is equal to

$$g_1(z) + \sum_{j=1}^{i-1} h_j(z)$$

and $\kappa$ is one of the points of $A_1$. Also, we can keep the derivative of $(g_1(z) + \sum_{j=1}^{i} h_j(z))$ non-zero at the points of $A_1 \cup A_1 \cup \{a_1\}$ by choosing the constant $\sigma_0$, in the proof of Lemma 1, sufficiently small. That it is possible to satisfy (iii) depends on the assumption that $A$ is a dense subset of $\mathbb{C}$.

Remark 2. It is important that all the zeros of $(g_1(z) - a_1)$ in $\{|z| < \alpha_1\}$ are of multiplicity one so that no additional points of $\{|z| < \alpha_1\}$ are zeros of

$$(f_1 + g_1)(z) - a_1.$$  

(A careful examination of this situation shows that indeed this can happen if $g_1(z) = 0$ for some $z \in \{|g_1^{-1}(a_1)| \cap \{|z| < \alpha_1\}\}$.)

Second step

Part 1

Let $F_2(z) = g_1(z) + f_1(z)$. Choose $\alpha_2$ so that $\{F_2^{-1}(a_k)\} \cap \{|z| = \alpha_j\} = \emptyset$ $(k = 1, 2; j = k, 2)$ and $|a_2| < \alpha_2$. We can assume that $F_2'(z) \neq 0$ for

$$z \in \left[\{(F_2^{-1}(a_1)) \cap \{|z| < \alpha_1\}\} \cup \{a_1, a_2\}\right].$$

This follows in a manner analogous to Remark 1. If $F_2(a_2) \in A$, we go on to Part 2; if not, by Lemma 2 we can choose $g_2(z)$ such that:

(i) $|g_2(z)| < \left(\frac{1}{2}\right)^2$ for $|z| < \alpha_2$,

(ii) $g_2(z) = 0$ for $z \in \{a_1\} \cup \{F_2^{-1}(a_1)\} \cap \{|z| < \alpha_2\}$,

(iii) $[F_2 + g_2](a_2) \in A$,

(iv) $(F_2 + g_2)(z) - a_1$ has no more zeros in $\{|z| < \alpha_1\}$ than $(F_2(z) - a_1)$ does, and all the zeros of $(F_2 + g_2)(z) - a_1$ in $\{|z| < \alpha_1\}$ have multiplicity one.
Part 2

Now set:

\[ A_2^2 = \{ (F_2 + g_2)^{-1}(a_2) \} \cap \{ |z| < \alpha_2 \} \cap A, \]
\[ \bar{A}_2^2 = \{ (F_2 + g_2)^{-1}(a_2) \} \cap \{ |z| < \alpha_2 \} \cap (C - A), \]

(note there exists no \( A_2^1 \))

\[ A_1^2 = \{ (F_2 + g_2)^{-1}(a_1) \} \cap \{ |z| < \alpha_2 \} \cap A, \]
\[ \bar{A}_1^2 = \{ (F_2 + g_2)^{-1}(a_1) \} \cap \{ |z| < \alpha_2 \} \cap (C - A). \]

Assuming \( F_2'(z) \neq 0 \) for \( z \in (A_1^2 \cup A_2^2 \cup \bar{A}_1^2 \cup \bar{A}_2^2) \cup \{ a_1, a_2 \} \) (this follows as in Remark 1), by repeated application of Lemma 1 we can choose \( f_2(z) \) such that:

(i) \( |f_2(z)| < (\ell)^2 \) for \( |z| < \alpha_2 \),
(ii) \( f_2(z) = 0 \) for \( z \in \{ a_1, a_2 \} \cup (A_1^2 \cup A_2^2) \),
(iii) \( \{ (F_2 + g_2 + f_2)^{-1}(a_k) \} \cap \{ |z| < \alpha_2 \} \) is contained in \( A \) for \( k = 1, 2 \),
(iv) \( \{ (F_2 + g_2 + f_2)(z) - a_k \} \) \( (k = 1, 2) \) has no more zeros in \( \{ |z| < \alpha_j \} \) \( (j = k, 2) \) than \( \{ (F_2 + g_2)(z) - a_k \} \) does.

Third step

Let \( F_3(z) = [F_2 + g_2 + f_2](z) \). Note that we can choose \( \alpha_3 \) so that Lemmas 1 and 2 can be applied.

Part 1

If \( F_3(a_3) \notin A \), we go on to Part 2. If not, by Lemma 2, assuming, as usual, that \( F_3'(z) \neq 0 \), at the appropriate points, we can choose (noting that \( g_2(z) \) has already been chosen) \( g_3(z) \) such that:

(i) \( |g_3(z)| < (\ell)^3 \) for \( |z| < \alpha_3 \),
(ii) \( g_3(z) = 0 \) for \( z \in \{ a_1, a_2 \} \cup \left( \bigcup_{k=1}^2 \{ (F_3)^{-1}(a_k) \} \cap \{ |z| < \alpha_3 \} \right) \),
(iii) \( [F_3 + g_3](a_3) \in A \),
(iv) \( \{ (F_3 + g_3)(z) - a_k \} \) \( (k = 1, 2) \) has no more zeros (all of which are of multiplicity one) in \( \{ |z| < \alpha_j \} \) \( (j = k, 2) \) than \( (F_3(z) - a_k) \) does.

Part 2

Now set:

\[ A_3^3 = \{ (F_3 + g_3)^{-1}(a_3) \} \cap \{ |z| < \alpha_3 \} \cap A, \]
\[ \bar{A}_3^3 = \{ (F_3 + g_3)^{-1}(a_3) \} \cap \{ |z| < \alpha_3 \} \cap (C - A), \]

(note there exists no \( A_3^2 \) or \( A_3^1 \))

\[ A_2^3 = \{ (F_3 + g_3)^{-1}(a_2) \} \cap \{ |z| < \alpha_3 \} \cap A, \]
\[ \bar{A}_2^3 = \{ (F_3 + g_3)^{-1}(a_2) \} \cap \{ |z| < \alpha_3 \} \cap (C - A), \]
\[ A_1^3 = \{ (F_3 + g_3)^{-1}(a_1) \} \cap \{ |z| < \alpha_3 \} \cap A, \]
\[ \bar{A}_1^3 = \{ (F_3 + g_3)^{-1}(a_1) \} \cap \{ |z| < \alpha_3 \} \cap (C - A). \]
By repeated application of Lemma 1, assuming, as usual, that \([F_3 + g_3]'(z) \neq 0\) at the appropriate points, we can choose \(f_3(z)\) such that

1. \(|f_3(z)| < \frac{1}{4}\) for \(|z| < \alpha_3\),
2. \(f_3(z) = 0\) for \(z \in \{a_1, a_2, a_3\} \cup (\bigcup_{k=1}^{3} A_k^3)\),
3. \([\{F_3 + g_3 + f_3\}^{-1}(0)] \cap \{|z| < \alpha_k\}\) is contained in \(A\) \((k = 1, 2, 3)\),
4. \([F_3 + g_3 + f_3](z) - a_k\) \((k = 1, 2, 3)\) has no more zeros (all of which are of multiplicity one) in \(|z| < \alpha_j\) \((j = k, k+1, 3)\) than \([F_3 + g_3](z) - a_k\) does.

In general, we define \(f_n\), \(g_n\) and \(F_n\) inductively in the obvious fashion. Then

\[
F(z) = \sum_{n=1}^{\infty} (g_n(z) + f_n(z))
\]
is an entire function by the Weierstrass \(M\)-test.

We shall now verify that \(F(z)\) has the following properties:

1. \(F(z) \in B\) if \(z \in A\),
2. \(F(z) \in (C - B)\) if \(z \in (C - A)\).

Let \(a_{n_0}\) be an arbitrary element of \(A\). Since, from \(n_0\) on, the \(g_n\)'s and \(f_n\)'s are zero at \(a_{n_0}\) and \(F_n(a_{n_0}) \in A\), we have \(F(a_{n_0}) \in A\).

In order to verify that \(F^{-1}(a_{n_0}) \in A\), suppose there exists a point \(z_0\) such that \(F(z_0) = a_{n_0}\) and \(z_0 \notin A\). Let \(m_0\) be an integer such that \(|z_0| < m_0\). Set \(k_0 = n_0 + m_0\); then any element which is in \(\{F_k^{-1}(a_{n_0})\}\), but not in \(A\), is outside of the disk of radius \(k_0\) (this is taken care of by the (iv)s). Contradiction. Hence the proof is complete.

**Remark 3.** As a corollary to the above method of proof, we could have obtained a function which had the properties in the theorem and which also had a non-zero derivative at each point of \(A\). This follows by also requiring in the \(n\)-th step that

\[
|g_n'(z)| < \frac{1}{2^{n+1}} \min_{\{F_n'(z)\}} \{|F_n'(z)|\}
\]
for \(z \in \bigcup_{n=1}^{\infty} a_k\) and an analogous inequality for \(|f_n'(z)|\).

**Corollary 1.** Let \(\Omega\) be the Gaussian rationals. There exists a non-linear entire function such that \(f(z) \in \Omega\) if and only if \(z \in \Omega\).

**Proof.** All that is needed is a slight alteration of the proof of Theorem 1. In this case we have \(A = B = \Omega\), and we choose the notation so that \(a_1 = 0\) and \(a_2 = 1\). We then choose \(g_1(z)\) so that \(g_1(0) = 0\). In Part I of the Second Step we alter the construction so that, in addition, we have that \([F_2 + g_2](1)\) is a non-zero Gaussian rational. In Part I of the Third Step we pick \(g_3(z)\) so that we have the additional property that there is no polynomial of first degree which maps 0 to 0, 1 to \([F_2 + g_2](1)\), and \(a_3\) to \([F_3 + g_3](a_3)\). The rest of the proof remains unchanged and it follows that \(F(z)\) cannot be a polynomial of first degree.

It is clear that a variation of the proof of Corollary 1 yields

**Corollary 2.** Let \(A, B\) be two countable dense sets in the plane. Then there exists a transcendental entire function \(w = F(z)\) such that \(F(z) \in B\) if and only if \(z \in A\).
In conclusion, we might also remark that a natural question to ask is whether the function constructed in [1] (an entire function which maps two arbitrary countable dense subsets of the real line onto each other monotonically) can be obtained as a special case of the above theorem. This does not appear to be the case since it is not possible in the above construction to assure that the reals go onto the reals.

References


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