■ Fifth Proof. After analysis it’s topology now! Consider the following curious topology on the set \( Z \) of integers. For \( a, b \in Z, b > 0 \), we set

\[
N_{a,b} = \{ a + nb : n \in Z \}.
\]

Each set \( N_{a,b} \) is a two-way infinite arithmetic progression. Now call a set \( O \subseteq Z \) open if either \( O \) is empty, or if to every \( a \in O \) there exists some \( b > 0 \) with \( N_{a,b} \subseteq O \). Clearly, the union of open sets is open again. If \( O_1, O_2 \) are open, and \( a \in O_1 \cap O_2 \) with \( N_{a,b_1} \subseteq O_1 \) and \( N_{a,b_2} \subseteq O_2 \), then \( a \in N_{a,b_1,b_2} \subseteq O_1 \cap O_2 \). So we conclude that any finite intersection of open sets is again open. So, this family of open sets induces a bona fide topology on \( Z \).

Let us note two facts:

(A) Any nonempty open set is infinite.

(B) Any set \( N_{a,b} \) is closed as well.

Indeed, the first fact follows from the definition. For the second we observe

\[
N_{a,b} = Z \setminus \bigcup_{i=1}^{b-1} N_{a+i,b},
\]

which proves that \( N_{a,b} \) is the complement of an open set and hence closed.

So far the primes have not yet entered the picture — but here they come. Since any number \( n \neq 1, -1 \) has a prime divisor \( p \), and hence is contained in \( N_{0,p} \), we conclude

\[
Z \setminus \{1, -1\} = \bigcup_{p \in \mathbb{P}} N_{0,p}.
\]

Now if \( \mathbb{P} \) were finite, then \( \bigcup_{p \in \mathbb{P}} N_{0,p} \) would be a finite union of closed sets (by (B)), and hence closed. Consequently, \( \{1, -1\} \) would be an open set, in violation of (A).

■ Sixth Proof. Our final proof goes a considerable step further and demonstrates not only that there are infinitely many primes, but also that the series \( \sum_{p \in \mathbb{P}} \frac{1}{p} \) diverges. The first proof of this important result was given by Euler (and is interesting in its own right), but our proof, devised by Erdős, is of compelling beauty.

Let \( p_1, p_2, p_3, \ldots \) be the sequence of primes in increasing order, and assume that \( \sum_{p \in \mathbb{P}} \frac{1}{p} \) converges. Then there must be a natural number \( k \) such that \( \sum_{i \geq k+1} \frac{1}{p_i} < \frac{1}{2} \). Let us call \( p_1, \ldots, p_k \) the small primes, and \( p_{k+1}, p_{k+2}, \ldots \) the big primes. For an arbitrary natural number \( N \) we therefore find

\[
\sum_{i \geq k+1} \frac{N}{p_i} < \frac{N}{2}. \tag{1}
\]
Let $N_b$ be the number of positive integers $n \leq N$ which are divisible by at least one big prime, and $N_s$ the number of positive integers $n \leq N$ which have only small prime divisors. We are going to show that for a suitable $N$

$$N_b + N_s < N,$$

which will be our desired contradiction, since by definition $N_b + N_s$ would have to be equal to $N$.

To estimate $N_b$ note that $\left\lfloor \frac{N}{p_i} \right\rfloor$ counts the positive integers $n \leq N$ which are multiples of $p_i$. Hence by (1) we obtain

$$N_b \leq \sum_{i \geq k+1} \left\lfloor \frac{N}{p_i} \right\rfloor < \frac{N}{2}. \quad (2)$$

Let us now look at $N_s$. We write every $n \leq N$ which has only small prime divisors in the form $n = a_n b_n^2$, where $a_n$ is the square-free part. Every $a_n$ is thus a product of different small primes, and we conclude that there are precisely $2^k$ different square-free parts. Furthermore, as $b_n \leq \sqrt{n} \leq \sqrt{N}$, we find that there are at most $\sqrt{N}$ different square parts, and so

$$N_s \leq 2^k \sqrt{N}.$$

Since (2) holds for any $N$, it remains to find a number $N$ with $2^k \sqrt{N} \leq \frac{N}{2}$ or $2^{k+1} \leq \sqrt{N}$, and for this $N = 2^{2k+2}$ will do. \qed

**Appendix: Infinitely many more proofs**

Our collection of proofs for the infinitude of primes contains several other old and new treasures, but there is one of very recent vintage that is quite different and deserves special mention. Let us try to identify sequences $S$ of integers such that the set of primes $\mathbb{P}_S$ that divide some member of $S$ is infinite. Every such sequence would then provide its own proof for the infinity of primes. The Fermat numbers $F_n$ studied in the second proof form such a sequence, while the powers of 2 don’t. Many more examples are provided by a theorem of Issai Schur, who showed in 1912 that for every nonconstant polynomial $p(x)$ with integer coefficients the set of all nonzero values $\{p(n) \neq 0 : n \in \mathbb{N}\}$ is such a sequence. For the polynomial $p(x) = x$, Schur’s result gives us Euclid’s theorem. As another example, for $p(x) = x^2 + 1$ we get that the “squares plus one” contain infinitely many different prime factors.

The following result due to Christian Elsholtz is a real gem: It generalizes Schur’s theorem, the proof is just clever counting, and it is in a certain sense best possible.