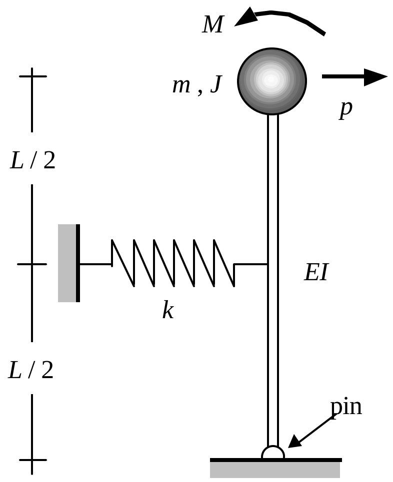
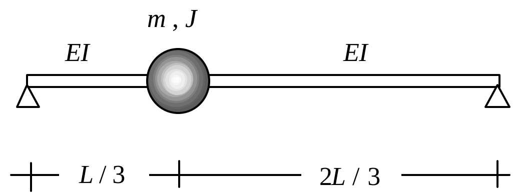
Solutions for Homework #10

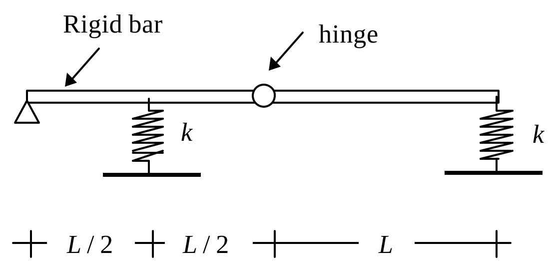
PROBLEM 1. (**P. 34** on page 381 in the note) Find the equations of motion for the structures shown in the figures below:



(b)

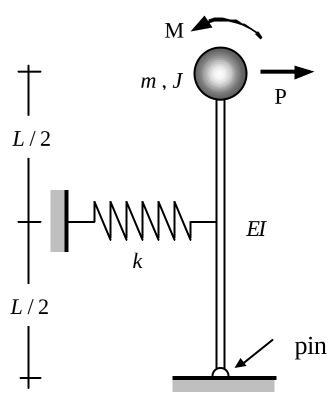


The rigid bar has mass / lenght m / L. The rotational inertia of the bar is the same as that of a cylindrical bar with negligible radius, $L >> 2R$.



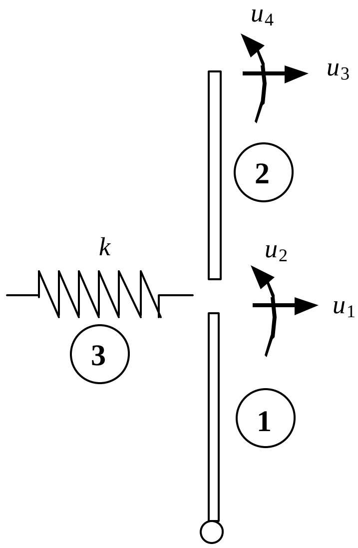
SOLUTION

(a) For the structure shown in the Figure below, we first formulate the stiffness matrix, following the direct stiffness approach:



This is a 2 degree of freedom system, with the horizontal displacement at the top, u, and the counterclockwise rotation at the top, q, being the independent degrees of freedom.

We initially formulate the stiffness matrices for the individual elements shown in the Figure below, successively formulate the stiffness matrix of the whole assembly.



$$\boldsymbol{K}_2 = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} \dfrac{12EI}{(L/2)^3} & \dfrac{6EI}{(L/2)^2} & -\dfrac{12EI}{(L/2)^3} & \dfrac{6EI}{(L/2)^2} \\ \dfrac{6EI}{(L/2)^2} & \dfrac{4EI}{(L/2)} & -\dfrac{6EI}{(L/2)^2} & \dfrac{2EI}{(L/2)} \\ -\dfrac{12EI}{(L/2)^3} & -\dfrac{6EI}{(L/2)^2} & \dfrac{12EI}{(L/2)^3} & -\dfrac{6EI}{(L/2)^2} \\ \dfrac{6EI}{(L/2)^2} & \dfrac{2EI}{(L/2)} & -\dfrac{6EI}{(L/2)^2} & \dfrac{4EI}{(L/2)} \end{bmatrix} \end{array}$$

$$\boldsymbol{K}_1 = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} \dfrac{3EI}{(L/2)^3} & -\dfrac{3EI}{(L/2)^2} & 0 & 0 \\ -\dfrac{3EI}{(L/2)^2} & \dfrac{3EI}{(L/2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \boldsymbol{K}_3 = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The stiffness matrix of the whole assembly, is now formulated by adding the stiffness coefficients of the corresponding degrees of freedom as follows:

$$\boldsymbol{K} = \boldsymbol{K}_1 + \boldsymbol{K}_2 + \boldsymbol{K}_3 = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \left[\begin{array}{cc:cc} \dfrac{15EI}{(L/2)^3}+k & \dfrac{3EI}{(L/2)^2} & -\dfrac{12EI}{(L/2)^3} & \dfrac{6EI}{(L/2)^2} \\ \dfrac{3EI}{(L/2)^2} & \dfrac{7EI}{(L/2)} & -\dfrac{6EI}{(L/2)^2} & \dfrac{2EI}{(L/2)} \\ \hdashline -\dfrac{12EI}{(L/2)^3} & -\dfrac{6EI}{(L/2)^2} & \dfrac{12EI}{(L/2)^3} & -\dfrac{6EI}{(L/2)^2} \\ \dfrac{6EI}{(L/2)^2} & \dfrac{2EI}{(L/2)} & -\dfrac{6EI}{(L/2)^2} & \dfrac{4EI}{(L/2)} \end{array}\right] \end{array} = \left[\begin{array}{c:c} [S_{11}] & [S_{12}] \\ \hdashline [S_{21}] & [S_{22}] \end{array}\right]$$

The degrees of freedom u_1, u_2 are internal, therefore, to evaluate the stiffness matrix of the assembly, we perform static condensation, as follows:

$$\boldsymbol{K}^{*} = [S_{22}] - [S_{21}][S_{11}]^{-1}[S_{12}] = \begin{bmatrix} \frac{12EI}{(L/2)^3} & -\frac{6EI}{(L/2)^2} \\ -\frac{6EI}{(L/2)^2} & \frac{4EI}{(L/2)} \end{bmatrix} -$$

$$- \begin{bmatrix} -\frac{12EI}{(L/2)^3} & -\frac{6EI}{(L/2)^2} \\ \frac{6EI}{(L/2)^2} & \frac{2EI}{(L/2)} \end{bmatrix} \begin{bmatrix} \frac{15EI}{(L/2)^3}+k & \frac{3EI}{(L/2)^2} \\ \frac{3EI}{(L/2)^2} & \frac{7EI}{(L/2)} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{12EI}{(L/2)^3} & \frac{6EI}{(L/2)^2} \\ -\frac{6EI}{(L/2)^2} & \frac{2EI}{(L/2)} \end{bmatrix}$$

Performing the matrix operations, the stiffness matrix is evaluated and the equation of motion for the structure is therefore formulated as follows:

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{u} \\ \ddot{q} \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u \\ q \end{Bmatrix} = \begin{Bmatrix} P \\ M \end{Bmatrix}$$

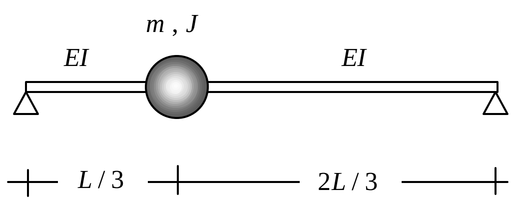
where

$$k_{11} = \frac{3EI}{L^3}\frac{768EI+128kL^3}{768EI+7kL^3}$$

$$k_{12} = k_{21} = -\frac{EI}{L^2}\frac{2304EI+120kL}{768EI+7kL^3}$$

$$k_{22} = \frac{3EI}{L}\frac{(48EI+kL^3)}{768EI+7kL^3}$$

(b) For the structure shown in the Figure below, we first formulate the stiffness matrix, following the direct stiffness approach:



This is a 2 degree of freedom system, with the displacement, u, and the counterclockwise rotation,q, being the independent degrees of freedom.

- We calculate the force required to apply unit displacement $u = 1$, k_{11} and the moment to prevent the rotation $q = 0$, k_{21}.

$$k_{11} = \frac{3EI}{\left(\frac{L}{3}\right)^3} + \frac{3EI}{\left(\frac{2L}{3}\right)^3} = 91.125\,\frac{EI}{L^3}$$

$$k_{21} = \frac{3EI}{\left(\frac{L}{3}\right)^2} - \frac{3EI}{\left(\frac{2L}{3}\right)^2} = 20.25\,\frac{EI}{L^2}$$

- We now calculate the moment required to apply unit rotation $q = 1$, k_{22} and the force to prevent the displacement $u = 0$, k_{12}.

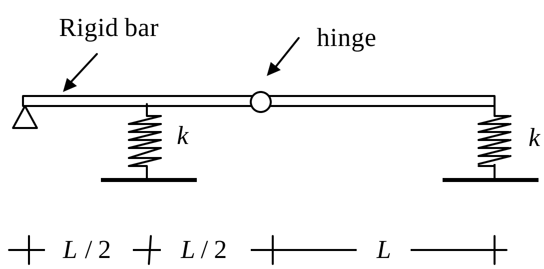
$$k_{12} = \frac{3EI}{\left(\frac{L}{3}\right)^2} - \frac{3EI}{\left(\frac{2L}{3}\right)^2} = 20.25\ \frac{EI}{L^2}$$

$$k_{22} = \frac{3EI}{\left(\frac{L}{3}\right)} + \frac{3EI}{\left(\frac{2L}{3}\right)} = 13.50\ \frac{EI}{L}$$

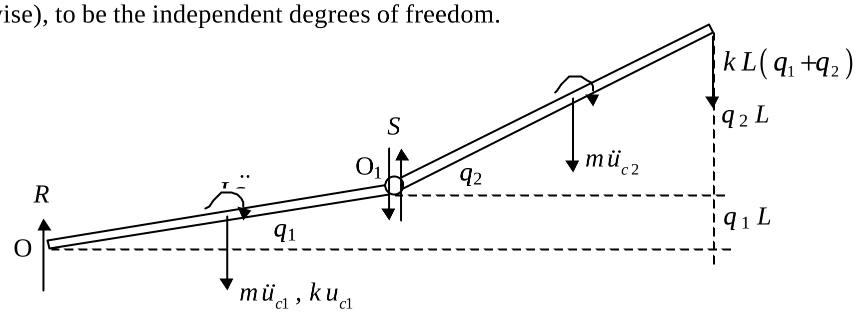
The equation of motion for the structure is therefore formulated as follows:

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{u} \\ \ddot{q} \end{Bmatrix} + \begin{bmatrix} 91.125\ \frac{EI}{L^3} & 20.25\ \frac{EI}{L^2} \\ 20.25\ \frac{EI}{L^2} & 13.50\ \frac{EI}{L} \end{bmatrix} \begin{Bmatrix} u \\ q \end{Bmatrix} = \begin{Bmatrix} P \\ M \end{Bmatrix}$$

(c) The rigid bar has mass / lenght m / L. The rotational inertia of the bar is the same as that of a cylindrical bar with negligable radius ($L >> 2R$).



This is a 2-degree of freedom system, and we choose the rotation of the bars q_1 and q_2 (positive counterclockwise), to be the independent degrees of freedom.



The equations of motion are formulated by applying moment equilibrium of the structure around the fixed support, O, as well as moment equilibrium of the right part of the structure around the hinge, O_1.

Therefore, we have:

$$\Sigma M_O = 0$$

$$\left(\frac{1}{2}\, m\, L\, \ddot{q}_1\right) \frac{L}{2} + m\, \frac{L^2}{12}\, \ddot{q}_1 + \left(\frac{1}{2}\, k\, L\, q_1\right) \frac{L}{2} + m\left(L\, \ddot{q}_1 + \frac{L}{2}\, \ddot{q}_2\right) \frac{3L}{2} + m\, \frac{L^2}{12}\, \ddot{q}_2 + k\, \left(L\, q_1 + L\, q_2\right) 2L = 0$$

$$\frac{11}{6}\, m L^2\, \ddot{q}_1 + \frac{5}{6}\, m L^2\, \ddot{q}_2 + \frac{9}{4}\, k\, L^2\, q_1 + 2\, k\, L^2\, q_2 = 0$$

{1}

$$\Sigma M_{O_1} = 0$$

$$m\left(L\,\ddot{q}_1 + \frac{L}{2}\ddot{q}_2\right)\frac{L}{2} + m\,\frac{L^2}{12}\,\ddot{q}_2 + k\,\left(L\,q_1 + L\,q_2\right)L = 0 \qquad \{2\}$$

$$\frac{1}{2}\,m\,L^2\,\ddot{q}_1 + \frac{1}{3}\,m\,L^2\,\ddot{q}_2 + k\,L^2\,q_1 + k\,L^2\,q_2 = 0$$

From equations {1} and {2}, we formulate the equation of motion for the structure:

$$\begin{bmatrix} \frac{11}{6} & \frac{5}{6} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} m\,L^2 \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} \frac{9}{4} & 2 \\ 1 & 1 \end{bmatrix} k\,L^2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Simply subtracting the second row from the first row, we obtain the following symmetric form.

$$\begin{bmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} m\,L^2 \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} \frac{5}{4} & 1 \\ 1 & 1 \end{bmatrix} k\,L^2 \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

PROBLEM 2. (**P. 35** on page 381 in the note) Consider a lumped mass N-dof system, with *known* stiffness matrix **K** and mass matrix **M**. If you also know N-1 modes (both the frequencies and the modal shapes), what are the frequency and the modal shape of the N^{th} mode?

SOLUTION

$$trace\left(\boldsymbol{K}^{-1}\boldsymbol{M}\right)=\frac{1}{w_1^2}+\frac{1}{w_2^2}+\ldots+\frac{1}{w_N^2}$$

$$w_N^2=\frac{1}{trace\left(\boldsymbol{K}^{-1}\boldsymbol{M}\right)-\left(\frac{1}{w_1^2}+\frac{1}{w_2^2}+\ldots+\frac{1}{w_{N-1}^2}\right)} \qquad \{1\}$$

The N^{th} modal shape is directly calculated from the eigenvalue problem, as follows:

$$\boldsymbol{K}\ \boldsymbol{f}_N=w_N^2\ \boldsymbol{M}\ \boldsymbol{f}_N \qquad \Rightarrow\ \left(\boldsymbol{K}-w_N^2\ \boldsymbol{M}\right)\boldsymbol{f}_N=\boldsymbol{0} \qquad \{2\}$$

which is a system with N-1 unknowns, as the first component of the eigenvector is arbitrarily assigned unit value.

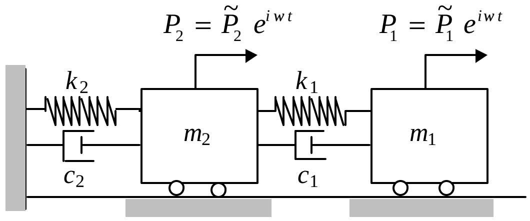
Alternatively, we could use the orthogonality properties of the eigenvectors with respect to the mass matrix:

$$\boldsymbol{f}_N{}^T\boldsymbol{M}\ \boldsymbol{f}_j=0 \qquad \forall\ j\neq N \qquad \{3\}$$

Therefore, this is a system with N-1 unknowns, as the first component of the eigenvector is arbitrarily assigned unit value. Successively, the N^{th} eigenvalue would be calculated using the Rayleigh quotient, as follows:

$$R=w_N^2=\frac{\boldsymbol{f}_N{}^T\boldsymbol{K}\ \boldsymbol{f}_N}{\boldsymbol{f}_N{}^T\boldsymbol{M}\ \boldsymbol{f}_N} \qquad \{4\}$$

PROBLEM 3. (**P. 36** on page 381 in the note) Consider a 2-degree freedom system subjected to harmonic loads P_1 and P_2.



If $P_1 \neq 0$ and $P_2 = 0$, is there any non–zero frequency for which the responses are in phase?

Same as above, with $P_2 \neq 0$ and $P_1 = 0$.

SOLUTION

The dynamic equilibrium equation of the system is:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1+c_2 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} \qquad \{1\}$$

where:

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} \tilde{p}_1\, e^{iwt} \\ \tilde{p}_2\, e^{iwt} \end{Bmatrix} \text{ and } \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \tilde{u}_1\, e^{iwt} \\ \tilde{u}_2\, e^{iwt} \end{Bmatrix} \qquad \{2\}$$

Substituting equation {2} in equation {1} and canceling out the common exponential factor, we have:

$$-w^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{Bmatrix} + i\,w \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1+c_2 \end{bmatrix} \begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix} \begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{Bmatrix} = \begin{Bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{Bmatrix}$$

$$\{3\}$$

$$\underbrace{\begin{bmatrix} k_1 - w^2 m_1 + i\,w\,c_1 & -k_1 - i\,w\,c_1 \\ -k_1 - i\,w\,c_1 & (k_1+k_2) - w^2 m_2 + i w\,(c_1+c_2) \end{bmatrix}}_{\text{DynamicStiffnessorImpedanceMatrix } \mathbf{K}_{dyn}(w)} \begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{Bmatrix} = \begin{Bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{Bmatrix}$$

Therefore, the response of the system is evaluated as follows:

$$\begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{Bmatrix} = \frac{1}{\det\left[\mathbf{K}_{dyn}(w)\right]} \begin{bmatrix} (k_1+k_2) - w^2 m_2 + i w\,(c_1+c_2) & k_1 + i\,w\,c_1 \\ k_1 + i\,w\,c_1 & k_1 - w^2 m_1 + i\,w\,c_1 \end{bmatrix} \begin{Bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{Bmatrix} \qquad \{4\}$$

a. If $P_1 \neq 0$ and $P_2 = 0$, is there any non – zero frequency for which the responses are in phase?

The response of the system for this case is:

$$\tilde{u}_1 = \frac{\tilde{p}_1}{\det\left[\mathbf{K}_{dyn}(w)\right]}\left\{(k_1+k_2)-w^2 m_2+i w\,(c_1+c_2)\right\}$$

$$\tilde{u}_2 = \frac{\tilde{p}_1}{\det\left[\mathbf{K}_{dyn}(w)\right]}\left\{k_1+i\,w\,c_1\right\} \qquad \{5\}$$

We seek for the non – zero frequency, for which the phase angle of the degrees of freedom of the system will be the same, namely:

$$\left.\begin{array}{l} \tan(f_1) = \dfrac{\dfrac{\tilde{p}_1}{\det\left[\mathbf{K}_{dyn}(w)\right]}w\,(c_1+c_2)}{\dfrac{\tilde{p}_1}{\det\left[\mathbf{K}_{dyn}(w)\right]}\left\{(k_1+k_2)-w^2 m_2\right\}} = \dfrac{w\,(c_1+c_2)}{\left\{(k_1+k_2)-w^2 m_2\right\}} \\ \\ \tan(f_2) = \dfrac{\dfrac{\tilde{p}_1}{\det\left[\mathbf{K}_{dyn}(w)\right]}\,w\,c_1}{\dfrac{\tilde{p}_1}{\det\left[\mathbf{K}_{dyn}(w)\right]}\,k_1} = \dfrac{w\,c_1}{k_1} \end{array}\right\} \quad \frac{w\,(c_1+c_2)}{\left\{(k_1+k_2)-w^2 m_2\right\}} = \frac{w\,c_1}{k_1} \quad \{6\}$$

Solving equation {6} for *w*, we find: $\dfrac{(c_1+c_2)}{\left\{(k_1+k_2)-w^2 m_2\right\}} = \dfrac{c_1}{k_1} \Rightarrow w^2\dfrac{m_2}{k_1} = \dfrac{k_2}{k_1} - \dfrac{c_2}{c_1}$ {7}

For the frequency *w* to be real, we demand $\dfrac{k_2}{k_1} > \dfrac{c_2}{c_1}$, and the frequency for which the responses are in phase is : $w = \sqrt{\dfrac{k_1}{m_2}\left(\dfrac{k_2}{k_1} - \dfrac{c_2}{c_1}\right)}$

b. Same as above, with $P_2 \neq 0$ and $P_1 = 0$.

The response of the system for this case is:

$$\tilde{u}_1 = \frac{\tilde{p}_2}{\det\left[\mathbf{K}_{dyn}(w)\right]}\left\{k_1+i\,w\,c_1\right\}$$

$$\tilde{u}_2 = \frac{\tilde{p}_2}{\det\left[\mathbf{K}_{dyn}(w)\right]}\left\{k_1 - m_1 w^2 + i\,w\,c_1\right\} \qquad \{8\}$$

We seek for the non – zero frequency, for which the phase angle of the degrees of freedom of the system will be the same, namely:

$$\left.\begin{aligned}\tan(f_1) &= \frac{\dfrac{\tilde{p}_2}{\det\left[\mathbf{K}_{dyn}(w)\right]} w\,c_1}{\dfrac{\tilde{p}_2}{\det\left[\mathbf{K}_{dyn}(w)\right]} k_1} = \frac{w\,c_1}{k_1} \\ \tan(f_2) &= \frac{\dfrac{\tilde{p}_2}{\det\left[\mathbf{K}_{dyn}(w)\right]} w\,c_1}{\dfrac{\tilde{p}_2}{\det\left[\mathbf{K}_{dyn}(w)\right]} \left(k_1 - w^2\,m_1\right)} = \frac{w\,c_1}{k_1 - w^2\,m_1}\end{aligned}\right\} \frac{w\,c_1}{k_1} = \frac{w\,c_1}{k_1 - w^2\,m_1} \qquad \{9\}$$

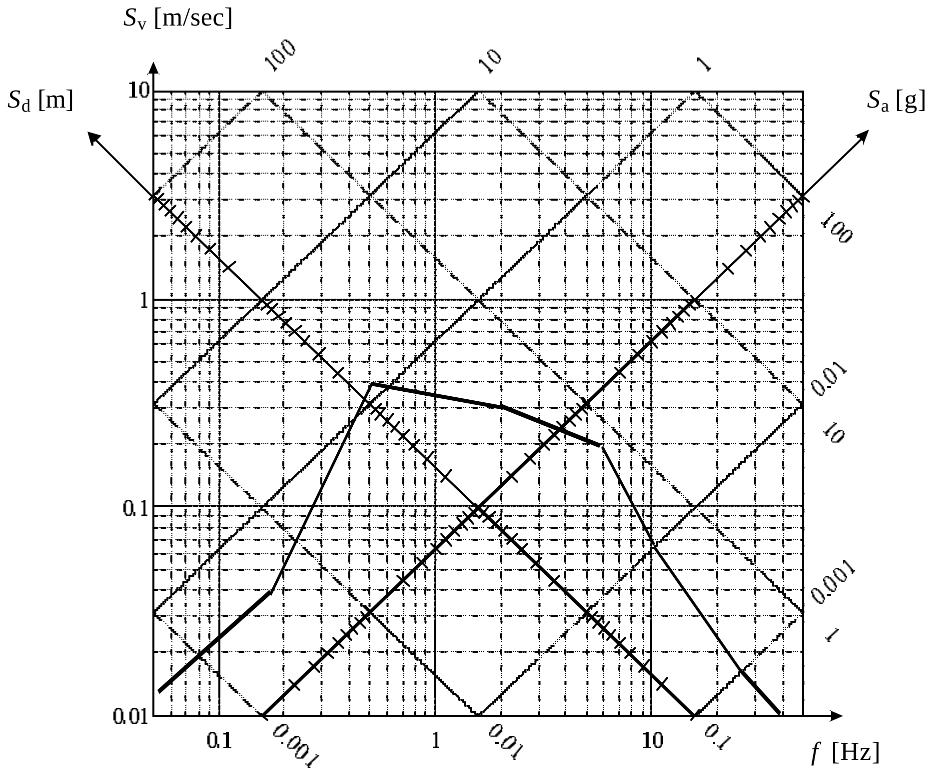
Solving equation {9} for *w*, we find: $\frac{1}{k_1} = \frac{1}{k_1 - w^2\,m_1} \Rightarrow w^2\,m_1 = 0$ {10}

Therefore, for $P_2 \neq 0$ and $P_1 = 0$, there is no non – zero frequency for which the responses are in phase.

PROBLEM 4. (**P. 37** on page 381 in the note) A 5-story building can be idealized as the 5-degree of freedom system shown, with m = 1000 kg, $k = 4\times10^5$ N/m and interstory height h = 4.00 m. The system is subjected to an earthquake, with the tripartite response spectrum shown below.

- Find the SRSS relative displacement at the top and bottom of the structure.
- Find the SRSS overturning moment.

Hint: Find the natural modes and frequencies of the system using the closed form solution for a discrete shear beam



SOLUTION

We first calculate the modal frequencies and modal shapes, using the closed form solution. The modal frequencies are calculated as follows:

$$w_j = 2\sqrt{\frac{k}{m}}\sin\left(\frac{p}{4N}(2j-1)\right) \qquad \{1\}$$

Therefore, for N = 5, substituting in {1} j = 1 – 5, we calculate:

$$w_1 = 2\sqrt{\frac{k}{m}}\sin\left(\frac{p}{20}(2-1)\right) = 40\sin\left(\frac{p}{20}\right) = 6.257\ [rad/\sec]$$

$$w_2 = 2\sqrt{\frac{k}{m}}\sin\left(\frac{p}{20}(4-1)\right) = 40\sin\left(\frac{3p}{20}\right) = 18.160[rad/\sec]$$

$$w_3 = 2\sqrt{\frac{k}{m}}\sin\left(\frac{p}{20}(6-1)\right) = 40\sin\left(\frac{p}{4}\right) = 28.284[rad/\sec]$$

$$w_4 = 2\sqrt{\frac{k}{m}}\sin\left(\frac{p}{20}(8-1)\right) = 40\ \sin\left(\frac{7p}{20}\right) = 35.640[rad/\sec]$$

$$w_5 = 2\sqrt{\frac{k}{m}}\sin\left(\frac{p}{20}(10-1)\right) = 40\ \sin\left(\frac{9p}{20}\right) = 39.508[rad/\sec]$$

The modal shapes are calculated as follows:

$$f_{ij} = \cos\left(\frac{p}{2N}(i-1)(2j-1)\right) \qquad \{2\}$$

Therefore, we evaluate equation {2} for $j = 1 - 5$ and $i = 1 - 5$ (for the modal components) as follows:

$$f_{11} = \cos\left(\frac{p}{10}(1-1)(2-1)\right) = 1.00 \qquad f_{12} = \cos\left(\frac{p}{10}(1-1)(4-1)\right) = 1.00$$

$$f_{21} = \cos\left(\frac{p}{10}(2-1)(2-1)\right) = 0.951 \qquad f_{22} = \cos\left(\frac{p}{10}(2-1)(4-1)\right) = 0.588$$

$$f_{31} = \cos\left(\frac{p}{10}(3-1)(2-1)\right) = 0.809 \qquad f_{32} = \cos\left(\frac{p}{10}(3-1)(4-1)\right) = -0.309$$

$$f_{41} = \cos\left(\frac{p}{10}(4-1)(2-1)\right) = 0.588 \qquad f_{42} = \cos\left(\frac{p}{10}(4-1)(4-1)\right) = -0.951$$

$$f_{51} = \cos\left(\frac{p}{10}(5-1)(2-1)\right) = 0.309 \qquad f_{52} = \cos\left(\frac{p}{10}(5-1)(4-1)\right) = -0.809$$

$$f_{13} = \cos\left(\frac{p}{10}(1-1)(6-1)\right) = 1.00 \qquad f_{14} = \cos\left(\frac{p}{10}(1-1)(8-1)\right) = 1.00$$

$$f_{23} = \cos\left(\frac{p}{10}(2-1)(6-1)\right) = 0 \qquad f_{24} = \cos\left(\frac{p}{10}(2-1)(8-1)\right) = -0.588$$

$$f_{33} = \cos\left(\frac{p}{10}(3-1)(6-1)\right) = -1.00 \qquad f_{34} = \cos\left(\frac{p}{10}(3-1)(8-1)\right) = -0.309$$

$$f_{43} = \cos\left(\frac{p}{10}(4-1)(6-1)\right) = 0 \qquad f_{44} = \cos\left(\frac{p}{10}(4-1)(8-1)\right) = 0.951$$

$$f_{53} = \cos\left(\frac{p}{10}(5-1)(6-1)\right) = 1.00 \qquad f_{54} = \cos\left(\frac{p}{10}(5-1)(8-1)\right) = -0.809$$

$$f_{15} = \cos\left(\frac{p}{10}(1-1)(10-1)\right) = 1.00$$

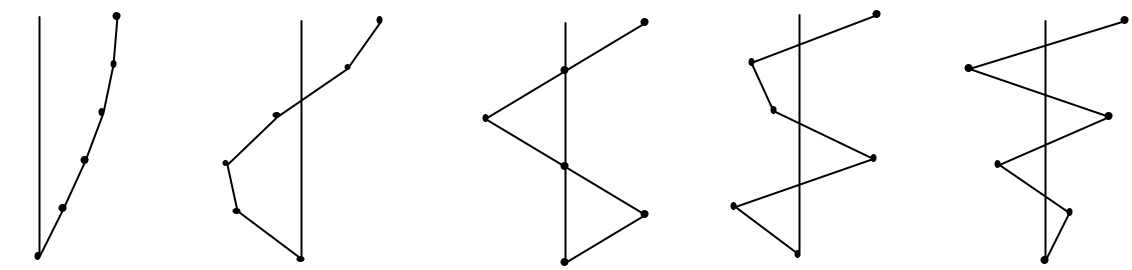
$$f_{25} = \cos\left(\frac{p}{10}(2-1)(10-1)\right) = -0.951$$

$$f_{35} = \cos\left(\frac{p}{10}(3-1)(10-1)\right) = 0.809$$

$$f_{45} = \cos\left(\frac{p}{10}(4-1)(10-1)\right) = -0.588$$

$$f_{55} = \cos\left(\frac{p}{10}(5-1)(10-1)\right) = 0.309$$

The modal shapes are schematically shown below:



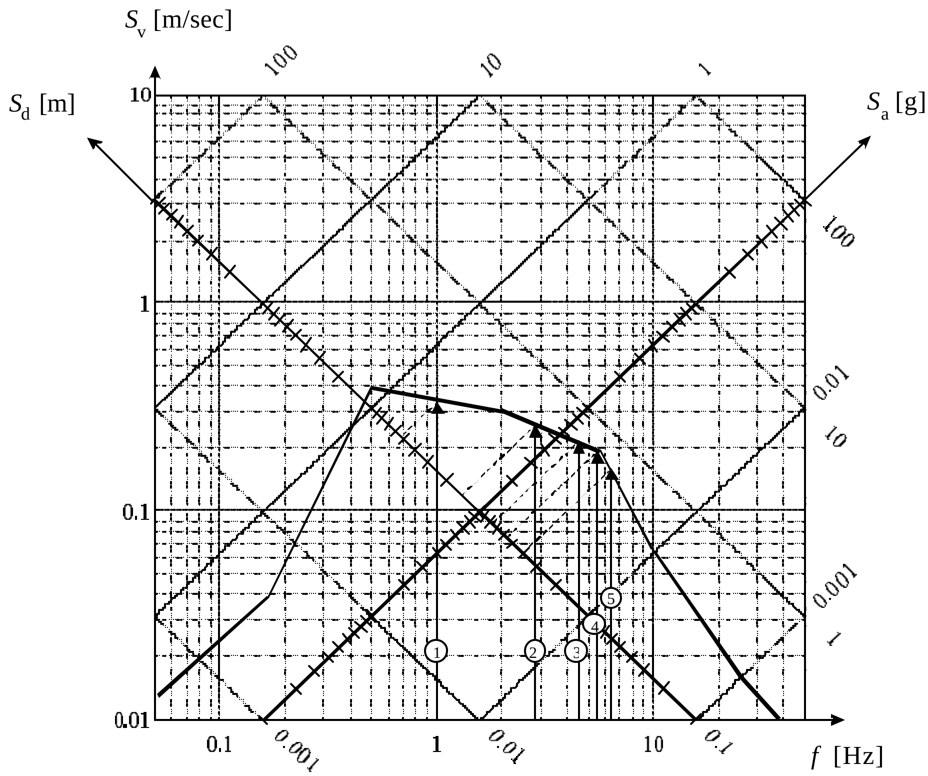
Mode 1 Mode 2 Mode 3 Mode 4 Mode 5

a. Find the SRSS relative displacement at the top and bottom of the structure.

We calculate the modal participation factors as follows: $g_j = \dfrac{\boldsymbol{f}_j^T \ \boldsymbol{M} \ \boldsymbol{e}}{\boldsymbol{f}_j^T \ \boldsymbol{M} \ \boldsymbol{f}_j}$ where:

$$\boldsymbol{M} = \begin{bmatrix} \frac{m}{2} & & & & \\ & m & & & \\ & & m & & \\ & & & m & \\ & & & & m \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{e} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \text{ (the rigid body vector)}$$

From the tripartite spectrum, we find the maximum relative displacement S_{d} for each modal frequency, as follows:



$f_1 = 1.00$ Hz $\quad S_{d1} = 0.52$ m

$f_2 = 2.89$ Hz $\quad S_{d2} = 0.15$ m

$f_3 = 4.50$ Hz $\quad S_{d3} = 0.072$ m

$f_4 = 5.67$ Hz $\quad S_{d4} = 0.055$ m

$f_5 = 6.29$ Hz $\quad S_{d5} = 0.038$ m

The relative displacement of the i^{th} degree of freedom for the j^{th} mode, is evaluated as follows:

$$v_{ij} = f_{ij}\ q_j(t)$$

The maximum modal relative displacement for the j^{th} mode estimated by means of the response spectrum is:

$$\left| q_j \right|^{\max} = \left| g_j \right| S_{dj}$$

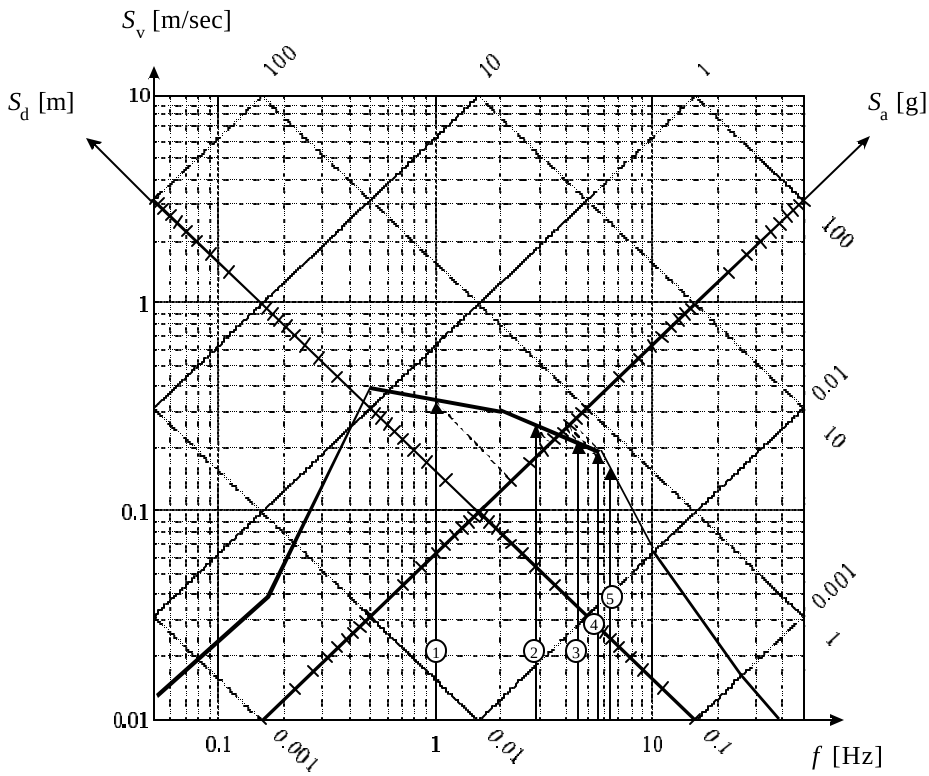
Therefore, the SRSS relative displacement of the top (dof 1) and bottom (dof 5) of the structure are:

$$\left| v_1 \right|^{\max} = \sqrt{\sum_{j=1}^{5} \left(f_{1j} \cdot g_j \cdot S_{dj} \right)^2} \qquad \left| v_5 \right|^{\max} = \sqrt{\sum_{j=1}^{5} \left(f_{5j} \cdot g_j \cdot S_{dj} \right)^2} \qquad \{3\}$$

Substituting in {3} the values obtained above, we calculate: $\left| v_1 \right|^{\max} = 0.661\ m$, $\left| v_5 \right|^{\max} = 0.212\ m$

b. Find the SRSS overturning moment.

From the tripartite spectrum, we find the maximum acceleration S_a for each modal frequency, as follows:



f_1 = 1.00 Hz $\quad$ S_{a1} = 0.22 g

f_2 = 2.89 Hz $\quad$ S_{a2} = 0.45 g

f_3 = 4.50 Hz $\quad$ S_{a3} = 0.62 g

f_4 = 5.67 Hz $\quad$ S_{a4} = 0.65 g

f_5 = 6.29 Hz $\quad$ S_{a5} = 0.63 g

We define $\boldsymbol{h}$ the vector that stores the height of each degree of freedom, namely:

$$\boldsymbol{h}^T = h\ \{5 \quad 4 \quad 3 \quad 2 \quad 1\} = 4.00 \cdot \{5 \quad 4 \quad 3 \quad 2 \quad 1\}$$

The overturning moment for the j^{th} mode, is evaluated as follows:

$$M_j = \sum_{i=1}^{5} f_{ij}\ \ddot{q}_j(t)\ m_i\, h_i = \boldsymbol{f}_j^T\, \boldsymbol{M}\ \boldsymbol{h}\ \ddot{q}_j(t)$$

The maximum modal acceleration for the j^{th} mode estimated by means of the response spectrum is:

$$\left| \ddot{q}_j \right|^{\max} = \left| g_j \right| S_{aj} = \left| g_j \right| w_j^2\ S_{dj}$$

Therefore, the SRSS overturning moment for the structure is:

$$\left|M\right|^{\max} = \sqrt{\sum_{j=1}^{5}\left(\sum_{i=1}^{5} f_{ij}\; g_{j} \cdot S_{aj}\; m_i\, h_i\right)^2} = \sqrt{\sum_{j=1}^{5}\left(g_{j} \cdot S_{aj} \cdot \sum_{i=1}^{5} f_{ij}\; m_i\, h_i\right)^2}$$

$$= \sqrt{\sum_{j=1}^{5}\left(g_{j} \cdot S_{aj} \cdot \boldsymbol{f}_j^T\, \boldsymbol{M}\;\boldsymbol{h}\right)^2} = \sqrt{\sum_{j=1}^{5}\left(g_{j} \cdot w_j^2 \cdot S_{dj} \cdot \boldsymbol{f}_j^T\, \boldsymbol{M}\;\boldsymbol{h}\right)^2} \qquad \{4\}$$

Substituting in {4} the values obtained above, we calculate: $\left|M\right|^{\max} = 172744.79\ Nm = 172.75\ kNm$