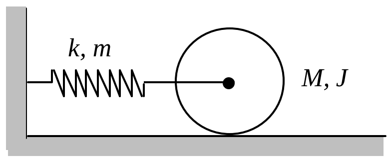
Solutions for Homework #11

PROBLEM 1. (**P. 38** on page 382 in the note) A homogeneous, rigid cylinder with radius R and length L, rolls without sliding over a flat surface. The axis of the cylinder is attached to the wall via a spring of stiffness k and mass m.



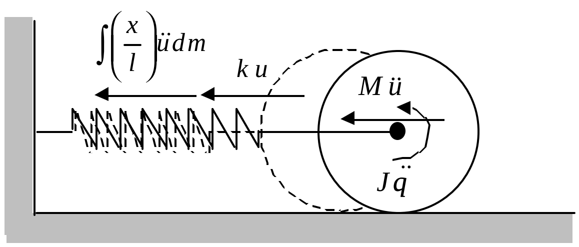
$m =$ total mass of the spring
$k =$ stiffness of the spring
$M = r\,p\,R^2\,L = 5m =$ total mass of the cylinder
$J = \dfrac{MR^2}{2} =$ mass moment of inertia about the axis

- Estimate the frequency of oscillation.

Hint: You may assume that the spring stretches linearly

SOLUTION

To estimate the frequency of oscillation, we evaluate the kinetic and strain energy of the system. We choose the translation of the center of mass of the cylinder, u, as the generalized coordinate of the system. Given that the cylinder rolls without sliding, the rotation q of the cylinder as a function of the displacement of the center is $q = u / R$.



Assuming that the spring deforms linearly, the displacement along the spring is:

$$u_{spring}(x) = \left(\frac{x}{l}\right)u$$

where l is the total length of spring.

Kinetic Energy:

$$K = \frac{1}{2}M\,\dot{u}^2 + \frac{1}{2}J\,\dot{q}^2 + \frac{1}{2}\int\left(\frac{x}{l}\right)^2\dot{u}^2\,dm = \frac{1}{2}\left(M + \frac{J}{R^2}\right)\dot{u}^2 + \frac{1}{2}\left(\frac{\dot{u}}{l}\right)^2\frac{l^3}{3}\,\frac{m}{l} = \frac{1}{2}\left(M + \frac{J}{R^2} + \frac{m}{3}\right)\dot{u}^2$$

Potential (Elastic) Energy:

$$V = \frac{1}{2}k\,u^2$$

For $u = u_0 \sin(w\,t)$, we have:

$$K = \frac{1}{2}\left(M + \frac{J}{R^2} + \frac{m}{3}\right)u_0^{\,2}\,w^2\cos^2(w\,t) \qquad V = \frac{1}{2}k\,u_0^{\,2}\sin^2(wt)$$

From the energy conservation principle, the maximum kinetic energy is equal to the maximum potential energy, therefore:

$$K_{\max} = \frac{1}{2}\left(M + \frac{J}{R^2} + \frac{m}{3}\right)u_0^{\,2}\,w^2 = V_{\max} = \frac{1}{2}k\,u_0^{\,2}$$

$$w = \sqrt{\frac{k}{\left(M + \frac{J}{R^2} + \frac{m}{3}\right)}}$$

<u>PROBLEM 2.</u> (**<u>P. 39</u>** on page 382 in the note) Design an optimal TMD for a square building with 60 stories, a 5:1 aspect ratio, and an inter-story height of 3.5 m. Use heuristic methods to model the building (mass, period, etc.).

SOLUTION

Assuming the interstory height to be h = 3.50 m, the total height of the building is $H = 60\ h\ =\ 210.00$m. From the aspect ratio H/a = 5, we therefore evaluate the width of the building, namely $a\ =\ 42$ m. Assuming that the density of the building is approximately 10% of the density of the solid concrete, namely:

$$r_{\text{b}} \cong 10\%\ r_{\text{solid concrete}} = 10\%\ 2500\ [\text{kg/m}^3] = 250\ [\text{kg/m}^3]$$

we evaluate the mass of the building as follows:

$$m_{\text{b}} = V\ r_{\text{b}} = H\,a^2\ r_{\text{b}} = 92.61\text{x}10^6\ \text{kg} \qquad \{1\}$$

To a first approximation, the fundamental period of the structure is T_n = 0.10 x (No. of stories).

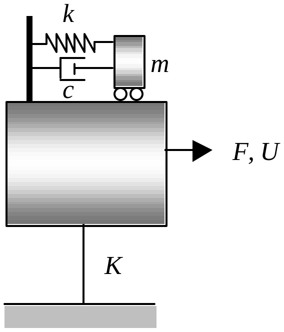
Therefore, we have: T_n = 0.10 x 60 = 6.0 [sec]

$\Omega = \pi / 3$ [rad/sec]

Approximating the structure as a 1-dof system, we lump half the building mass at the top, and we evaluate the stiffness of the system as follows:

$$\Omega = \sqrt{\frac{K}{(m_b / 2)}} \Rightarrow K = (m_b / 2)\,\Omega^2 = 50.8\,10^6\ N/m \qquad \{2\}$$

We neglect the damping of the structure, and choose the mass ratio of the system $m = 1\%$, namely:



$$m = \frac{m}{M} = 0.01$$

Theoptimaltuningconditionis:

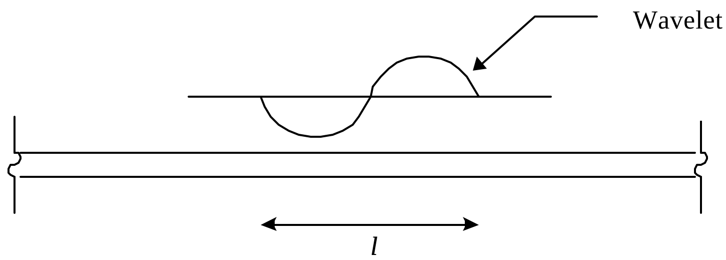
$$\frac{w_0}{\Omega} = \frac{\sqrt{k/m}}{\sqrt{K/M}} = \frac{1}{m+1} = \frac{1}{1.01}$$

$$k = \left(\frac{1}{1.01}\right)^2 \Omega^2\, m = 5\times10^5\ N/m$$

Theoptimaldampingis:

$$x = \sqrt{\frac{3m}{8(1+m)^3}}\ \frac{\Omega}{w_0} = \sqrt{\frac{3\cdot 0.01}{8(1+0.01)^3}}\ 1.01 = 0.06$$

PROBLEM 3. (**P. 42** on page 384 in the note) An infinitely long shear beam has cross–section A = 0.01 m^2, shear wave velocity C_s = 100 m/sec and mass density r = 2000 kg/m^3. This beam is subjected to a wavelet consisting of a *single* sine wave pulse:



$$u(x,t) = u_0 \sin w \left(t - \frac{x}{c} \right) \qquad \frac{w}{2p} = 1\,\text{Hz}$$

- Determine the stresses at a point x as a function of time.
- Determine the constant D of the dashpot that would completely absorb the wavelet.
- What is the wavelength l of the wavelet?

SOLUTION

To determine the stress, we begin by expressing the given sine pulse more precisely in mathematical sense as the following:

$$u(x,t) = u_0 \sin w \left(t - x/C_S\right)\left[H\left(x - C_S t\right) - H\left(x - C_S t - l\right)\right] \tag{1}$$

where $l(=2\pi/k=2\pi C_S/w)$ is the wavelength of the wavelet as shown above, and $H(\cdot)$ is the Heaviside step function whose value is 1 for positive argument, and 0 for negative argument. Note also that

$$\frac{\partial H(x)}{\partial x} = d(x) \tag{2}$$

where $d(\cdot)$ is the Dirac delta function. Then, the stress in the shear beam is given as:

$$\begin{aligned} s = G\frac{\partial u}{\partial x} = Gu_0 \Big\{ -w/C_S \cos w\left(t - w/C_S\right)\left[H\left(x - C_S t\right) - H\left(x - C_S t - l\right)\right] + \\ \sin w\left(t - x/C_S\right)\left[d\left(x - C_S t\right) - d\left(x - C_S t - l\right)\right]\Big\} \end{aligned} \tag{3}$$

As seen in equation (3), the stress s shows singularities when $x=C_S t$ and $x=C_S t+l$.

The constant D of the dashpot that would completely absorb the wavelet is given as:

$$D = r C_S A = 2000 \times 100 \times 0.01 = 2000(kg\,/\text{sec}) \tag{4}$$

The wavelength $l(=2\pi/k=2\pi C_S/w)$ is given as:

$$l = 2pC_S/w = 2p \times 100 \div 2p = 100(\text{m}) \tag{5}$$