HOMEWORK #1(2001)

PROBLEM 1

An irregularly shaped rigid object is hinged at one point and is supported by several springs. The measured natural frequency of this object is 5Hz. When a static force F=10 N is applied at 0.4m to the right of the pivot, the object rotates through an angle of q = 0.005 rad. What is the mass moment of inertia of the object around the pivot?

SOLUTION

First of all, we establish the equation of motion in term of moment of equilibrium with respect to the pivot.

$$J\ddot{q} + k_q q = 0 \qquad (1)$$

where q is the rotational angle around the pivot, J is the mass moment of inertia around the pivot, and k_q is the associated stiffness. Then, we obtain the natural frequency w_n of the form.

$$w_n = \sqrt{\frac{k_q}{J}} \qquad (2)$$

If we knew the values of k_q and w_n, we can find out J. As given in this problem, w_n=2p*f_n=2p*5=10p. Also, with the information of the static loading experiment of F, k_q can be obtained as follows.

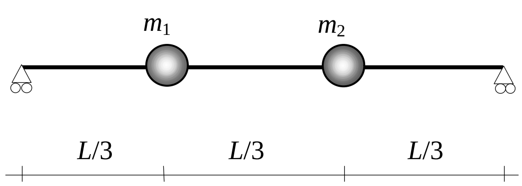
$$k_q = \frac{M}{q} = \frac{10*0.4}{0.005} = 800 \qquad \left(\frac{N*m}{rad}\right)$$

From Eq. (2), we obtain

$$J = \frac{k_q}{w_n^2} = \frac{800}{(10p)^2} = 0.8106 \qquad \left(\frac{Nm}{rad}\frac{\sec^2}{rad^2}\right)$$

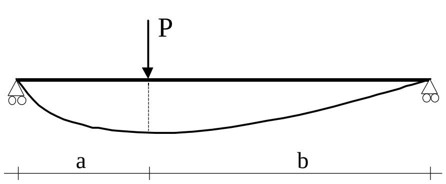
PROBLEM 2

Obtain the equations of motion for the following system:



SOLUTION

The deflection shape of the beam is given by the following expressions:



$$u(x) = \frac{PL^3}{6EI}\frac{b}{L}\frac{x}{L}\left[1-\left(\frac{b}{L}\right)^2-\left(\frac{x}{L}\right)^2\right] \qquad x \le a$$

$$= \frac{PL^3}{6EI}\frac{a}{L}\left(1-\frac{x}{L}\right)\left[1-\left(\frac{a}{L}\right)^2-\left(1-\frac{x}{L}\right)^2\right] \qquad x \ge a$$

{1}

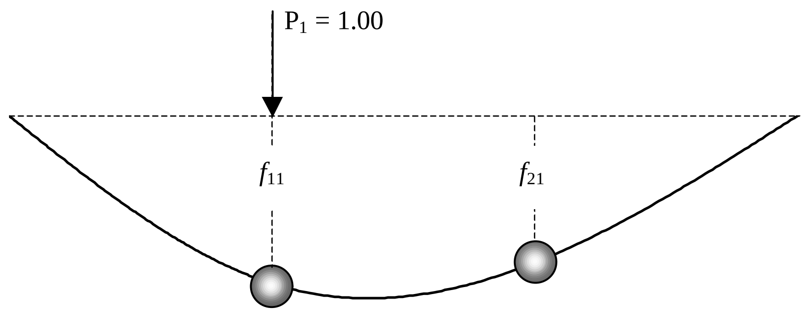
The structure under consideration has 2 dynamic dof's, namely u_1 and u_2.

The *mass* matrix of the structure is simply:

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad \{2\}$$

To evaluate the *stiffness* matrix, we first construct the *flexibility* matrix as follows:

- We apply a unit load $P_1 = 1.00$ at the 1^{rst} degree of freedom, and calculate the deflection at the 1^{rst} degree of freedom, f_{11}, and at the 2^{nd} degree of freedom, f_{21}.

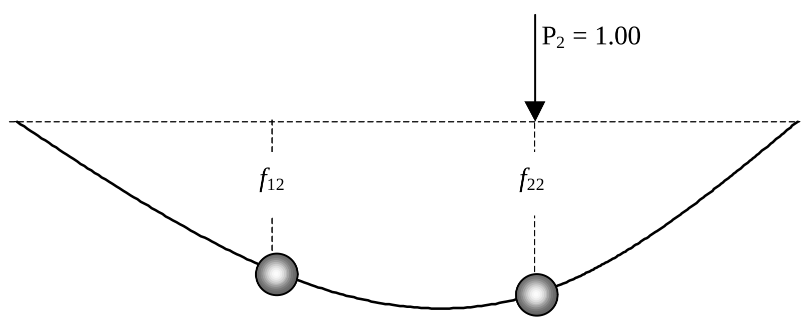


For a = L/3 and b = 2L/3, we evaluate the flexibility coefficients by substituting in equation {1}:

$$f_{11} = u\left(L/3\right) = \frac{L^3}{6EI}\frac{2L/3}{L}\frac{L/3}{L}\left[1-\left(\frac{2L/3}{L}\right)^2-\left(\frac{L/3}{L}\right)^2\right] = \frac{4}{243}\frac{L^3}{EI}$$

$$f_{21} = u\left(2L/3\right) = \frac{L^3}{6EI}\frac{L/3}{L}\left(1-\frac{2L/3}{L}\right)\left[1-\left(\frac{L/3}{L}\right)^2-\left(1-\frac{2L/3}{L}\right)^2\right] = \frac{7}{486}\frac{L^3}{EI}$$

- We apply a unit load $P_2 = 1.00$ at the 2^{nd} degree of freedom, and calculate the deflection at the 1^{rst} degree of freedom, f_{12}, and at the 2^{nd} degree of freedom, f_{22}.



For a = 2L/3 and b = L/3, we evaluate the flexibility coefficients by substituting in equation {1}:

$$f_{21} = u(L/3) = \frac{L^3}{6EI}\frac{L/3}{L}\frac{L/3}{L}\left[1-\left(\frac{L/3}{L}\right)^2-\left(\frac{L/3}{L}\right)^2\right] = \frac{7}{486}\frac{L^3}{EI}$$

$$f_{22} = u(2L/3) = \frac{L^3}{6EI}\frac{2L/3}{L}\left(1-\frac{2L/3}{L}\right)\left[1-\left(\frac{2L/3}{L}\right)^2-\left(1-\frac{2L/3}{L}\right)^2\right] = \frac{4}{243}\frac{L^3}{EI}$$

The flexibility matrix of the structure is formulated as follows:

$$\boldsymbol{F} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \frac{L^3}{EI}\begin{bmatrix} \frac{4}{243} & \frac{7}{486} \\ \frac{7}{486} & \frac{4}{243} \end{bmatrix} \qquad \{4\}$$

Therefore, the stiffness matrix is:

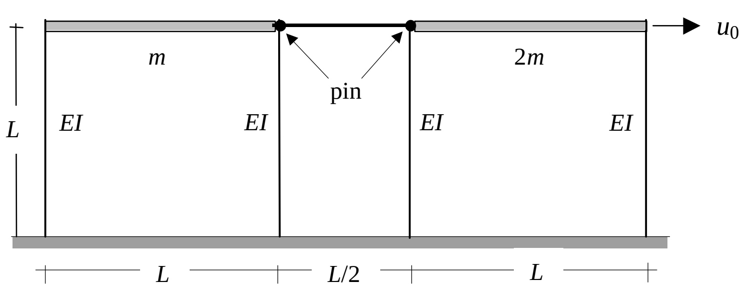
$$\boldsymbol{F}^{-1} = \boldsymbol{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \frac{64.8\,EI}{L^3}\begin{bmatrix} 4 & -3.5 \\ -3.5 & 4 \end{bmatrix} \qquad \{5\}$$

Therefore, the equation of motion for the following system is:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{64.8EI}{L^3}\begin{bmatrix} 4 & -3.5 \\ -3.5 & 4 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

PROBLEM 3

Two frames are connected by means of a rigid, massless strut (or link). The girders (i.e. beams) in each frame are infinitely rigid, and have the masses indicated in the drawing. On the other hand, the columns are massless. The system is subjected to an initial lateral deflection u_0, released in that position, and allowed to vibrate freely. The system has no damping.



SOLUTION

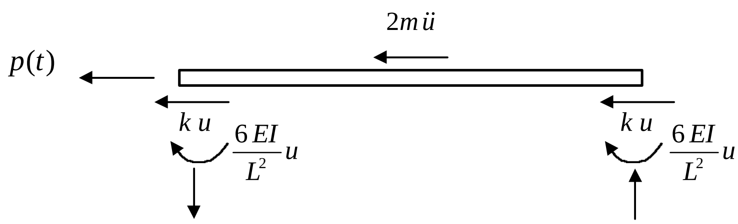
The system has only 1 degree of freedom, which is the common horizontal translation of the frames, due to the presence of the rigid link.

The total mass of the assembly is: $m_{tat} = m+2m = 3m$

We define $k = \dfrac{12\,EI}{L^3}$ the stiffness of a single column, therefore the stiffness of the assembly is: $k_{tot} = 4k$

The natural frequency of the structure is: $w_n = \sqrt{\dfrac{k_{tot}}{m_{tot}}} = \sqrt{\dfrac{4k}{3m}}$

To evaluate the force in the strut as a function of time, we draw the free body diagram of the right beam[1]:



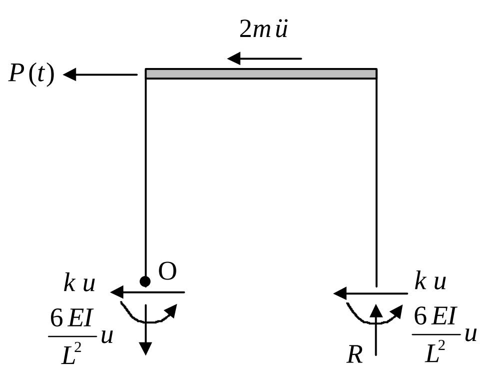
From equilibrium of horizontal forces we have: $2m\ddot{u} + 2k\,u + P(t) = 0$ {1}

The equation of motion for the assembly is given by:
$$\begin{gathered} 3m\ddot{u} + 4k\,u = 0 \\ u(0) = u_0 \qquad \dot{u}(0) = 0 \end{gathered} \qquad \{2\}$$

The solution of {2} is: $u(t) = u_0 \cos\left(w_n t\right)$. Therefore, the force in the strut is evaluated as follows:

$$P(t) = -\,2m\ddot{u} - 2k\,u = \left(2mw_n^2 - 2k\right)u_0 \cos\left(w_n t\right) = \frac{2}{3}k\,u_0 \cos\left(w_n t\right)$$

To evaluate the vertical reaction at the rightmost support, we draw the free body diagram of the right frame, and evaluate moment equilibrium around point O as follows:



$$\Sigma M_O = 0$$

$$P(t)\,L + 2m\ddot{u}\;L + 2\underbrace{\frac{6\,EI}{L^2}}_{k\,L}\,u + R(t)\;L = 0$$

$$R(t) = 2mw_n^2\;u_0\cos\left(w_n t\right) - \frac{2}{3}k\,u_0\cos\left(w_n t\right) - k\;u_0\cos\left(w_n t\right)$$

$$= \;k\;u_0\cos\left(w_n t\right)$$

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[1] Note.- Had we chosen the left beam, the equilibrium of the horizontal forces would be: $m\ddot{u} + 2k\,u - P(t) = 0$. Substituting the response $u(t) = u_0\cos\left(w_n t\right)$, we would obtain the same result for *P*(t).