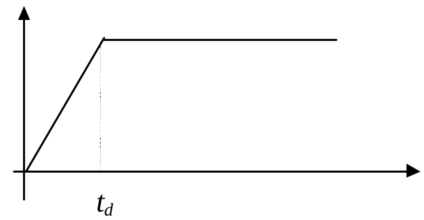
Solutions for Homework #3(Fall 2001)

PROBLEM 1 (<u>P. 10</u> on page 369**)** An *undamped* 1-dof system initially at rest is subjected to a step load with finite rise time t_d. Find the maximum dynamic load factor (i.e. ratio between dynamic and static response) at any time, assuming that t_d=5/4 T, with T being the undamped natural period (you may assume T=1, if you wish).



SOLUTION

The step load with finite rise time can be interpreted as the superposition of the following functions:

$$p(t) = \begin{cases} p_0 \dfrac{t}{t_d} & t > 0 \\ -p_0 \dfrac{(t-t_d)}{t_d} & t > t_d \end{cases}$$

The equation of motion for the forced vibration is:

$$m\ddot{u} + k\,u = p(t) \qquad \{1\}$$

The response of the system is evaluated as follows:

$$u(t) = (u_0 - u_{p0})\cos(w_n t) + \frac{(\dot{u}_0 - \dot{u}_{p0})}{w_n}\sin(w_n t) + u_p(t) \qquad \{2\}$$

$$u_0 = \dot{u}_0 = 0$$

The particular solution will be of the form:

$$u_p(t) = A + Bt$$
$$\dot{u}_p(t) = B \qquad \{3\}$$
$$\ddot{u}_p(t) = 0$$

We substitute {3} into {1} to evaluate the coefficients A, B as follows:

$$k\,(A + B\,t) = p_0 \frac{t}{t_d} \qquad \{4\}$$

Therefore, $A = 0$ and $B = \frac{p_0}{k}\frac{1}{t_d} = \frac{u_{st}}{t_d}$. The particular solution is: $u_p(t) = \frac{u_{st}}{t_d}\,t$.

The initial values of the particular solution are: $u_p(0) = 0$, $\dot{u}_p(0) = \frac{u_{st}}{t_d}$.

Therefore, the response of the system for t > 0 is the following[1]:

$$u(t) = \begin{cases} \dfrac{u_{st}}{t_d}\left[t - \dfrac{\sin(w_n t)}{w_n}\right] & 0 < t < t_d \\ \dfrac{u_{st}}{t_d}\left[t - \dfrac{\sin(w_n t)}{w_n}\right] - \dfrac{u_{st}}{t_d}\left[(t-t_d) - \dfrac{\sin(w_n (t-t_d))}{w_n}\right] & t \ge t_d \end{cases} \qquad \{5\}$$

---

[1] Note.- The response of the system due to the load applied for t > t_d is delayed t_d sec wrt. the response due to the load applied at t = 0.

where: $t_d = \frac{5}{4} T_n = \frac{5}{2} \frac{p}{w_n}$

We define $DLF = u(t) / u_{st}$ and we seek for the maximum DLF for t > 0. We will assume $T_n = 1$ sec and substitute $t_d = 5/4$ in equation {5}.

$$DLF(t) = \begin{cases} \frac{4}{5}\left[t - \frac{\sin(2p\,t)}{2p}\right] & 0 < t < t_d \\ 1 - \frac{4}{5}\,\frac{\cos(2p\,t) + \sin(2p\,t)}{2p} & t \geq t_d \end{cases} \qquad \{6\}$$

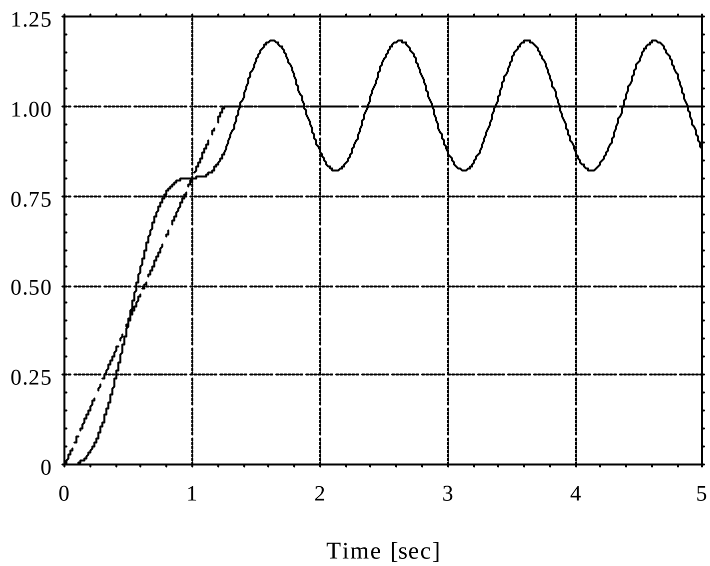
We now evaluate the maximum DLF for t > 0:

$$\frac{d}{dt}DLF(t) = \begin{cases} \frac{4}{5}\left[1 - \cos(2p\,t)\right] = 0 & \Rightarrow \quad t = k & 0 < t < t_d \\ \frac{4}{5}\left[\sin(2p\,t) - \cos(2p\,t)\right] = 0 & \Rightarrow \quad t = \frac{1}{8} + \frac{k}{2} & t \geq t_d \end{cases}$$

$k = 0,1...$

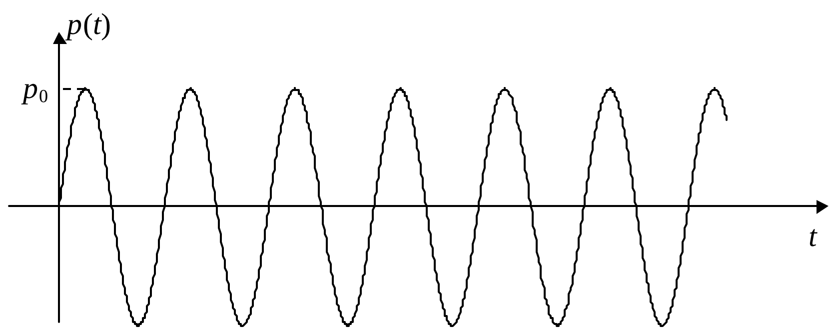
Therefore, the overall maximum DLF happens for $t = 0.125 + 0.5\,k$, $k \geq 3$ and $DLF_{max} = 1.18$ (substitute in equation {6} for $t \geq t_d$)

The variation of the dynamic load factor (DLF) with time is shown in the Figure below.



PROBLEM 2 (<u>P. 11</u> on page 370)

A 1-dof system is subjected to a sinusoidal load that starts at t=0 (i.e. there is no load for negative times). If the system is initially at rest, find an expression for the true transient response. Sketch the response (say, some ten cycles) when the excitation frequency equals the natural frequency (i.e. at resonance), assuming that the fraction of damping is x=0.05. Interpret your result in terms of the “number of cycles to 50%” rule, in particular, the number of cycles required to attain steady-state (for practical purposes, you could say that you have achieved steady-state when the transient part has decayed by some two orders of magnitude, say 1/128.



SOLUTION

The equation of motion for the system subjected to a sinusoidal load is $m\ddot{u} + c\,\dot{u} + k\,u = p_0 \sin(wt)$.

The response of the system is evaluated as follows:

$$u(t) = e^{-x\,w_n\,t}\left[\left(u_0 - u_{p0}\right)\cos\left(w_d\,t\right) + \frac{\left(\dot{u}_0 - \dot{u}_{p0}\right) + x\,w_n\left(u_0 - u_{p0}\right)}{w_d}\sin\left(w_d\,t\right)\right] + u_p(t) \qquad \{1\}$$

$$w_d = w_n\sqrt{1 - x^2}$$

We try a particular solution of the form:

$$u_p(t) = A\cos(wt) + B\sin(wt)$$
$$\dot{u}_p(t) = -Aw\sin(wt) + B\,w\cos(wt) \qquad \{2\}$$
$$u_p(t) = -Aw^2\cos(wt) - Bw^2\sin(wt)$$

We substitute equation {2} in the equation of motion, to evaluate the coefficients A, B.

$$-m\,w^2\left[A\cos(wt) + B\sin(wt)\right] + c\,w\left[-A\sin(wt) + B\cos(wt)\right] + k\left[A\cos(wt) + B\sin(wt)\right] = p_0\sin(wt)$$

We equate the coefficients of the sin and cos terms and we obtain:

$$-Aw^2 - 2x\,w_n\,w\,B + w_n^2\,A = 0$$
$$\{3\}$$
$$-Bw^2 + 2x\,w_n\,w\,A + w_n^2\,B = \frac{p_0}{m}$$

Equation {3} is a system of 2 equations with unknowns the coefficients A, B. Solving {3} we have:

$$A = \frac{-2\,x\,w\,w_n}{w_n^4 - 2w^2 w_n^2 + w^4 + 4\,x^2 w^2 w_n^2}\,\frac{p_0}{m}$$
$$\{4\}$$
$$B = \frac{-(w^2 - w_n^2)}{w_n^4 - 2w^2 w_n^2 + w^4 + 4\,x^2 w^2 w_n^2}\,\frac{p_0}{m}$$

At resonance, the frequency of the applied load, ω coincides with the natural frequency of the system and equation {4} reduces to $B = 0$ and $A = -\frac{p_0}{k}\frac{1}{2x} = -\frac{u_{st}}{2x}$.

Therefore, the particular solution is:

$$\begin{aligned} u_p(t) &= -\frac{u_{st}}{2x}\cos(w\,t) \qquad u_p(0) = -\frac{u_{st}}{2x} \\ u_p(t) &= \frac{u_{st}\,w}{2x}\sin(w\,t) \qquad u_p(0) = 0 \end{aligned} \tag{5}$$

We substitute equation {5} in {1} to obtain the response of the system:

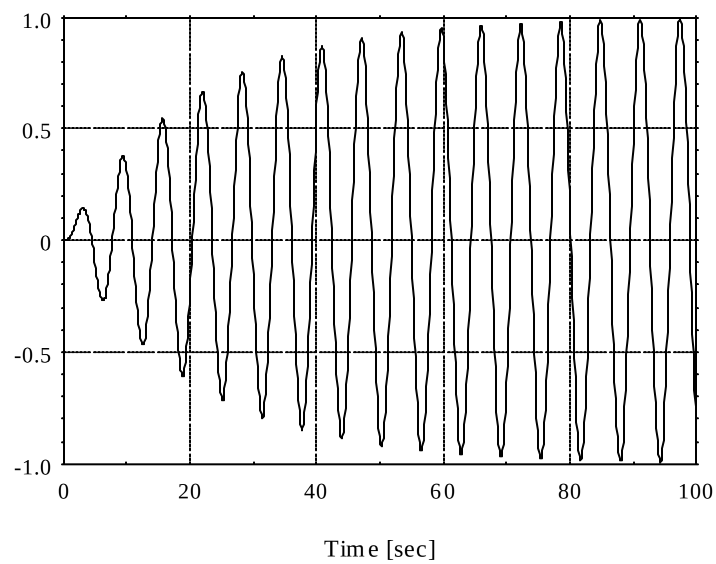
$$u(t) = \underbrace{e^{-x\,w_n\,t}\left[\frac{u_{st}}{2x}\cos(w_d\,t) + \frac{u_{st}}{2\sqrt{1-x^2}}\sin(w_d\,t)\right]}_{\text{Transient Response}} + \underbrace{\frac{u_{st}}{2x}\cos(w\,t)}_{\text{Steady-State Response}}$$

For $u_{st} = 0.1$ m, $x = 0.05$ and $w_n = 1$ rad/sec we plot the total and transient response for 16 cycles.

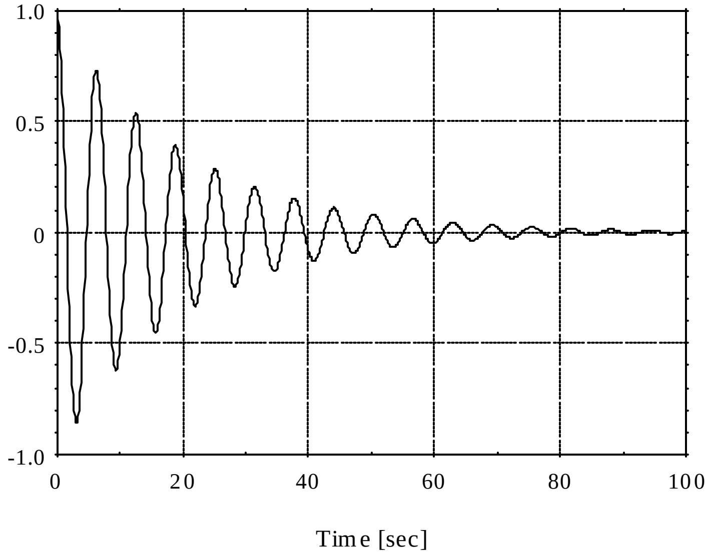
The number of cycles for 50% amplitude is: $N_{50\%} \simeq \frac{1}{10x} = 2$ cycles .

Assuming that steady – state is achieved for $\frac{u(t)}{u(t+N\,T_d)} = 128 \quad \therefore\ N_{\text{steady-state}} = \frac{\ln(128)}{2p\,0.05} = 16$ cycles

TOTAL RESPONSE OF A 1-DOF SUBJECTED TO A SINUSOIDAL LOAD AT RESONANCE



TRANSIENT RESPONSE OF A 1-DOF SUBJECTED TO A SINUSOIDAL LOAD AT RESONANCE

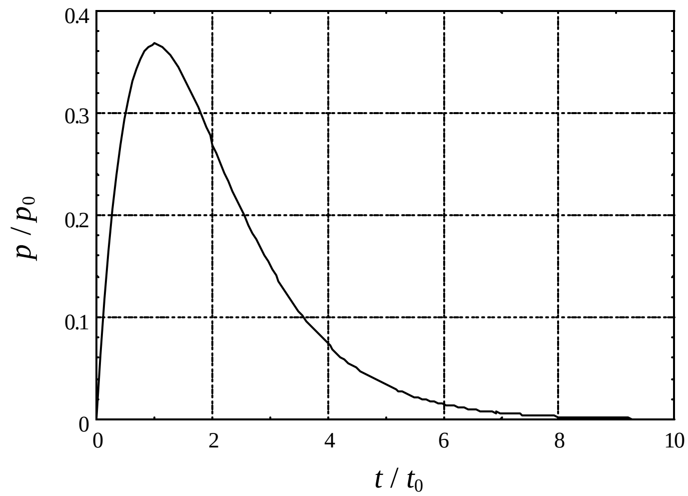


PROBLEM 3 (P. 12 on page 370)

An *undamped* 1-dof system has a natural period T_n, and is subjected to the bell-shaped forcing function

$$p = p_0 \frac{t}{t_0} e^{-\frac{t}{t_0}}$$

shown in the figure. If the system is at rest at t=0, and if the force time parameter satisfies $t_0 = T_n$, what is the expression for the transient response for t>0?



SOLUTION

The equation of motion for the undamped system is:

$$m\ddot{u} + k\,u = p_0 \frac{t}{t_0} e^{-\frac{t}{t_0}} \qquad \{1\}$$

The forcing function is the product of a polynomial and an exponential term. Therefore, we try a particular solution:

$$u_p(t) = (A + Bt)\, e^{-\frac{t}{t_0}}$$

$$\dot{u}_p(t) = -\frac{1}{t_0}\left(A + B(t - t_0)\right) e^{-\frac{t}{t_0}} \qquad \{2\}$$

$$\ddot{u}_p(t) = \frac{1}{t_0^2}\left(A + B(t - 2t_0)\right) e^{-\frac{t}{t_0}}$$

We substitute {2} into {1} and we have:

$$m\left[\frac{1}{t_0^2}\left(A + B(t - 2t_0)\right) e^{-\frac{t}{t_0}}\right] + k\left[(A + Bt)\, e^{-\frac{t}{t_0}}\right] = p_0 \frac{t}{t_0} e^{-\frac{t}{t_0}} \qquad \{3\}$$

Canceling out the non-zero exponential term in {3}, we have:

$$\left[\frac{m}{{t_0}^2}A - \frac{2\,B\,m}{t_0} + kA\right] + \left[\frac{m}{{t_0}^2}B + kB\right]t \;=\; 0 + \frac{p_0}{t_0}t$$

$$t_0 = T_n \;\Rightarrow\; \frac{k}{m}\,t_0^2 = w_n^2\;t_0^2 = w_n^2\;T_n^2 = 4p^2 \qquad \{4\}$$

$$\left.\begin{aligned} &\frac{m}{{t_0}^2}A - \frac{2\,B\,m}{t_0} + kA = 0 \\ &\frac{m}{{t_0}^2}B + kB = \frac{p_0}{t_0} \end{aligned}\right\} \xrightarrow{\{4\}} \begin{aligned} B &= \frac{p_0}{t_0}\,\frac{1}{\frac{m}{t_0^2}\left(1+\frac{k}{m}\,t_0^2\right)} = \frac{p_0}{T_n\;k}\,\frac{1}{\left(1+\frac{1}{4p^2}\right)} \simeq \frac{p_0}{T_n\;k} = \frac{u_{st}}{T_n} \\ A &= \frac{2\,B\,t_0}{1+\frac{k}{m}\,t_0^2} = \frac{p_0}{k}\,\frac{1}{2p^2}\,\frac{1}{\left(1+\frac{1}{4p^2}\right)} \simeq \frac{p_0}{k}\,\frac{1}{2p^2} = \frac{u_{st}}{2p^2} \end{aligned}$$

Substituting the values of A, B in {2}, the particular solution of {1} is formulated as follows:

$$u_p(t) = \left(\frac{u_{st}}{2p^2} + \frac{u_{st}}{T_n}t\right)e^{-\frac{t}{T_n}} = u_{st}\left(\frac{1}{2p^2} + \frac{1}{T_n}t\right)e^{-\frac{t}{T_n}} \qquad \{5\}$$

$$u_p(0) = \frac{u_{st}}{2p^2} \qquad \dot{u}_p(0) = \frac{u_{st}}{T_n} - \frac{1}{2p^2}\,\frac{u_{st}}{T_n} = \frac{u_{st}}{T_n}\,\frac{2p^2-1}{2p^2}$$

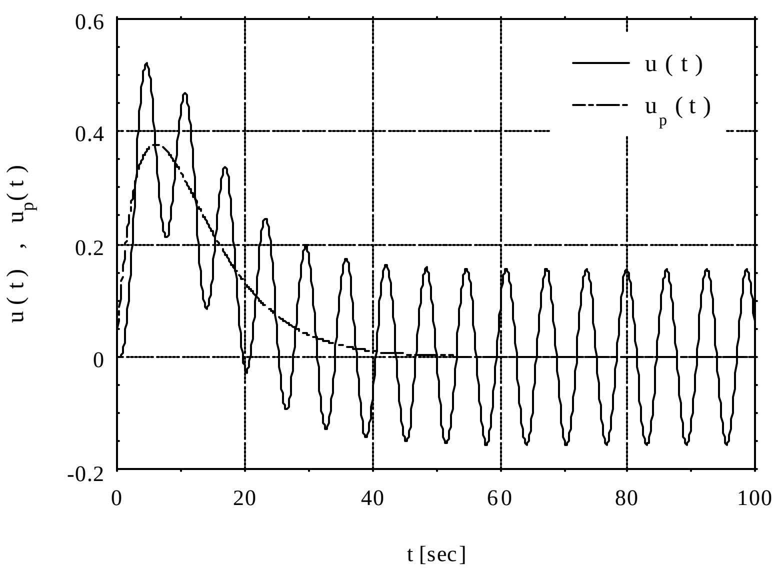
The system is at rest at $t = 0$, therefore:

$$u(0) = \dot{u}(0) = 0$$

$$u(t) = \left[u(0) - u_p(0)\right]\cos(w_n t) + \frac{\left[\dot{u}(0) - \dot{u}_p(0)\right]}{w_n}\sin(w_n t) + u_p(t)$$

$$u(t) = -\frac{u_{st}}{2p^2}\cos(w_n t) - u_{st}\,\frac{2p^2-1}{4p^3}\sin(w_n t) + u_{st}\left(\frac{1}{2p^2} + \frac{1}{T_n}t\right)e^{-\frac{t}{T_n}}$$

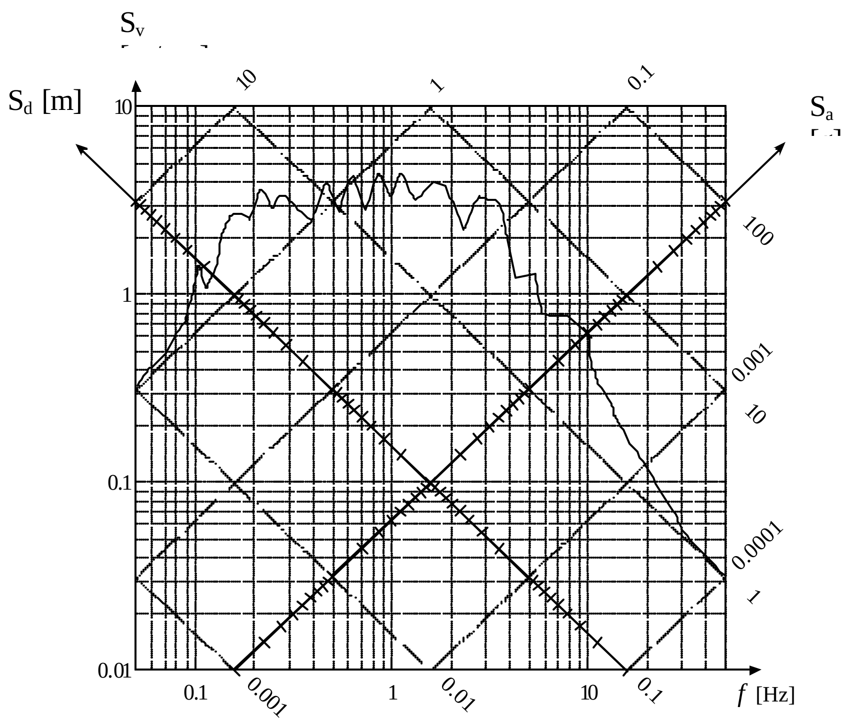
The response for $w_n = 1$ [rad/sec], $u_{st} = 1$ [m] is plotted in the Figure below.



PROBLEM 4 (P. 13 on page 370**)**

A 10-story building is subjected to an earthquake whose response spectrum is shown below. The building is square in plan view, and its aspect ratio (=height/width) is 3:1. Using heuristic methods, estimate

- the building's mass
- the fundamental frequency of the building
- the lateral stiffness, as perceived from the top of the building
- the maximum acceleration at the top of the building
- the maximum drift (relative displacement)



SOLUTION

Assuming the interstory height to be h = 3.50 m, the total height of the building is H=10, h= 35.00(m). From the aspect ratio H/a = 3, we therefore evaluate the width of the building, namely a = 11.67 m. Assuming that the density of the building is approximately 10% of the density of the solid concrete, namely:

$$r_{\text{b}} \cong 10\%\ r_{\text{solid concrete}} = 10\%\ 2500\ [\text{kg/m}^3] = 250\ [\text{kg/m}^3]$$

we evaluate the mass of the building as follows:

$$m_{\text{b}} = V\, r_{\text{b}} = H\, a^2\, r_{\text{b}} = 1190972.22 \text{ kg} \qquad \{1\}$$

To a first approximation, the fundamental period of the structure is T_n = 0.10 x (No. of stories). Therefore, we have:

$$T_n = 0.10 \text{ x } 10 = 1.0 \text{ sec}$$

$$f_n = 1.00 \text{ Hz},\ \omega_n = 2\,\pi \text{ rad/sec}$$

Approximating the structure as a 1-dof system, we lump half the building mass at the top, and we evaluate the stiffness of the system as follows:

$$w_n = \sqrt{\frac{k}{(m_b / 2)}} \Rightarrow k = (m_b / 2)\, w_n^{\,2} = 23508.849\ kN / m \qquad \{2\}$$

For the natural frequency of the system, f_n = 1 Hz, we find from the tripartite response spectrum of the earthquake the maximum absolute acceleration at the top of the building, S_a and the maximum relative displacement S_d: