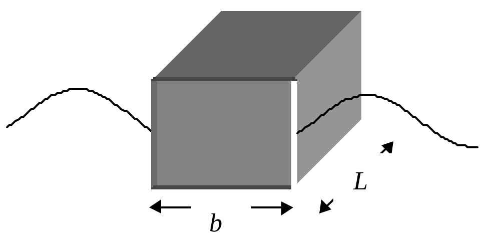
Solutions for Homework #4

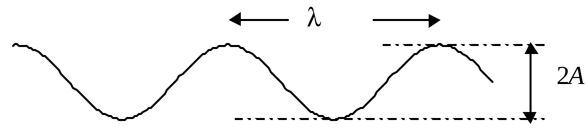
PROBLEM 1 (P. 15 on page 372) A ship that weights 40,000 ton (40×10^6 kg) can be idealized, to a crude approximation, as a rectangular block of length L = 200 m and width b = 20 m. On stormy days, the ocean develops waves, whose wavelengths l and period T increase with the wind speed and duration of the wind. To a first approximation, the relationship between wavelength, wave period and wave amplitude is as follows:

$$l \text{ [m]} = 1.56\, T^2, \quad T \text{ [sec]}$$

$$\frac{2A}{l} \cong \frac{1}{7} \qquad \text{(If } 2A > l \text{, the wave breaks)}$$

	T [sec]
Light Seas	5
Moderate Seas	10
Heavy Seas	15

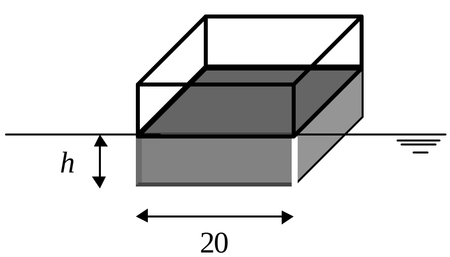




- What is the resonant frequency of the ship in vertical motions?
- For what ocean conditions will resonance occur?
- How does the wavelength causing resonance compare with the length of the ship?

SOLUTION

The displaced volume of the water is $V_d = h\ L\ b$. The mass of the displaced water is equal to the mass of the ship, namely:

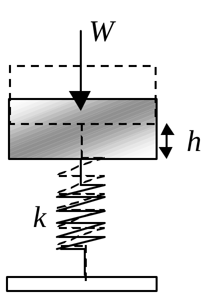


$$m_w = r_w h L b = m_s \qquad \{1\}$$

Therefore, the depth h is calculated as follows:

$$h = \frac{m_s}{r_w L b} = \frac{40 \cdot 10^6}{10^3 \cdot 20 \cdot 200} = 10\ m$$

In order to estimate the stiffness of the system, we proceed as follows. For W being the weight of the ship (= externally applied load), the deformation is h (the submerged part of the ship), i.e.:



$$W = g\ m_s = g\ m_w = \underbrace{g\ r_w L b}_{k}\ h = k\,h$$

$$\left.\begin{array}{l} k = g\ r_w L b \\ m_s = r_w\ L\ b\ h \end{array}\right\} \Rightarrow w_n = \sqrt{\frac{k}{m_s}} = \sqrt{\frac{g}{h}} = 1\ rad/\text{sec}$$

We want the resonant period of the waves to coincide with the resonant period of the ship, therefore:

$$T_w = T_n = 2p \text{ sec} = 6.28\text{sec}$$ Light – moderate sea conditions

For T_w = 6.28 sec, we estimate:

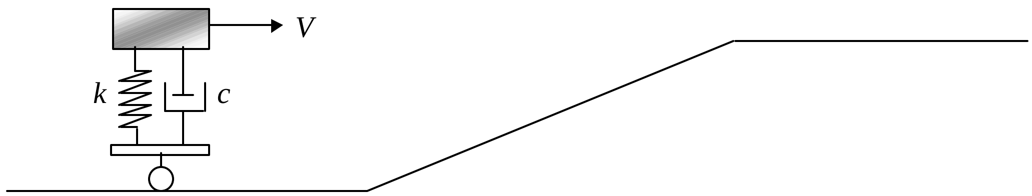
$$l = 1.56\, T_w^2 = 61.58\ m \qquad \text{Wavelength}$$

$$A \simeq \frac{1}{14}\, l = 4.40\ m \qquad \text{WaveAmplitude}$$

Comparing the wavelength at resonance with the length of the ship, we have:

$$\frac{L}{l} = \frac{200}{61.58} = 3.25 \qquad \frac{b}{l} = \frac{20}{61.58} = 0.325$$

PROBLEM 2 (P. 16 on page 373) A car that weighs 2000 kg travels on a bumpy road with constant velocity of 72 km/hr. To a first approximation, it can be modeled as a one degree-of-freedom system. The car is known to have a damped natural frequency of 1 Hz, and a fraction of critical damping of 50%. The road is flat, except for a ramp of length L=10 m and a height h=0.10 m.



a) Determine the equation of motion for this problem in terms of the absolute (inertial) vertical displacement of the car. Note that this motion is *not* the same as the distance to the road.
b) Neglecting damping, estimate the maximum absolute motion, and the time at which it occurs. The car reaches the ramp at t=0. (If you are familiar with MATLAB, you may also wish to evaluate and plot the damped response, and compare the two).
c) From the results of b), determine the deformation of the shock absorbers at the time of the maximum absolute motion. Notice that this is not necessarily the maximum deformation.
d) What problems would you encounter if you formulated this problem in terms of relative displacements? How would you overcome these?

SOLUTION

The speed of the car is: V = 72 [km/hr] = 20 [m/sec]

The actual length that the car covers on the ramp is: $L = \sqrt{10^2 + 0.1^2} \cong 10\ m$

For t = 0, the car reaches the bottom of the ramp. We define t_d the time when the car reaches the top of the ramp, namely:

$$t_d = \frac{10\ m}{20\ m/\text{sec}} = 0.5\text{sec}$$

Therefore, the ground motion can be described as:

$$u_g(t) = \begin{cases} 0 & t < 0 \\ 0.1\dfrac{t}{t_d} = 0.2t & 0 \le t \le t_d = 0.5\text{sec} \\ 0.1 & t > t_d \end{cases}$$

The equation of motion, in terms of absolute displacements, u, is: $m\ddot{u} + c\dot{u} + ku = c\dot{u}_g + ku_g$
where: m = 2000 kg

$$w_D = 2p \;\; rad/\sec = w_n\sqrt{1-x^2}$$
$$w_n = 7.26 \; rad/\sec = \sqrt{k/m}$$
$$k = 105275.78 \;\; N/m$$
$$c = 2x\sqrt{km} = 14510.40 \;\; N\sec/m$$

To evaluate the response of the system, we interpret the ground motion as the superposition of the following functions:

$$u_g(t) = \begin{cases} 0.2t & t \ge 0 \\ -0.2\left(t-t_d\right) & t \ge t_d \end{cases}$$

Neglecting damping, the equation of motion is: $m\ddot{u} + ku = k \cdot u_g$ {1}

To evaluate the particular solution for the differential equation {1}, we proceed as follows. Try a linear function of time, t, namely:

$$\begin{aligned} u_p(t) &= A+Bt \\ \dot{u}_p(t) &= B \\ \ddot{u}_p(t) &= 0 \end{aligned} \qquad \{2\}$$

Substituting {2} in {1}, we have: $k\left(A+Bt\right) = k \cdot 0.2t$, from which: $A = 0$, $B = 0.2$.

Therefore, the solution of {1} for $t < t_d$ is:

$$u(t) = \left(u_0 - u_{p0}\right)\cos\left(w_n t\right) + \frac{\left(\dot{u}_0 - \dot{u}_{p0}\right)}{w_n}\sin\left(w_n t\right) + u_p(t)$$

$$\left.\begin{aligned} u_0 &= \dot{u}_0 = 0 \\ u_{p0} &= 0 \\ \dot{u}_{p0} &= 0.2 \end{aligned}\right\} \qquad \therefore \; u(t) = \frac{-0.2}{7.26}\sin\left(7.26t\right) + 0.2t \qquad \{3\}$$

To determine $u_{\max}$, we differentiate equation {3}, and set the derivative equal to zero, namely:

$$\begin{aligned} &\frac{d}{dt}u(t) = -0.2\cos\left(7.26t\right) + 0.2 = 0 \\ \Rightarrow \quad & \cos\left(7.26t\right) = 1 \\ \Rightarrow \quad & 7.26t = 2kp \\ \Rightarrow \quad & t = 0.865k \qquad k = 0,1\ldots \end{aligned}$$

Therefore, for $t < t_d$, the maximum displacement happens at: $u_{\max} = u\left(t_d\right) = 0.113 \; m$. {4}

The solution of {1} for $t \ge t_d$ is:

$$u(t) = \frac{-0.2}{7.26}\sin(7.26t) + 0.2t + \frac{0.2}{7.26}\sin(7.26(t-t_d)) - 0.2(t-t_d)$$

$$u(t) = 0.10 + \frac{0.2}{7.26}\left[\sin(7.26(t-t_d)) - \sin(7.26t)\right] \qquad \{5\}$$

For $t \geq t_d$, we determine the maximum displacement as follows:

$$\frac{d}{dt}u(t) = 0.2\left[\cos(7.26(t-t_d)) - \cos(7.26t)\right] = 0$$

$$\Rightarrow \qquad 7.26\,t = a\tan\left[\frac{1-\cos(7.26\,t_d)}{\sin(7.26\,t_d)}\right] + k\pi$$

$$\Rightarrow \qquad t = \frac{-1.32 + k\pi}{7.26} \qquad k = 0,1\ldots$$

For $k = 2$, $t = 0.683$ sec > 0.5 sec and $u_{max} = u(0.683) = 0.1535\ m$. {6}

From {5} and {6}, the overall maximum happens at $t = 0.683$ sec.[1]

The undamped and damped absolute responses are shown in the Figure below. Note that the vertical distance of the vehicle from the road, was arbitrarily set to 0.06m

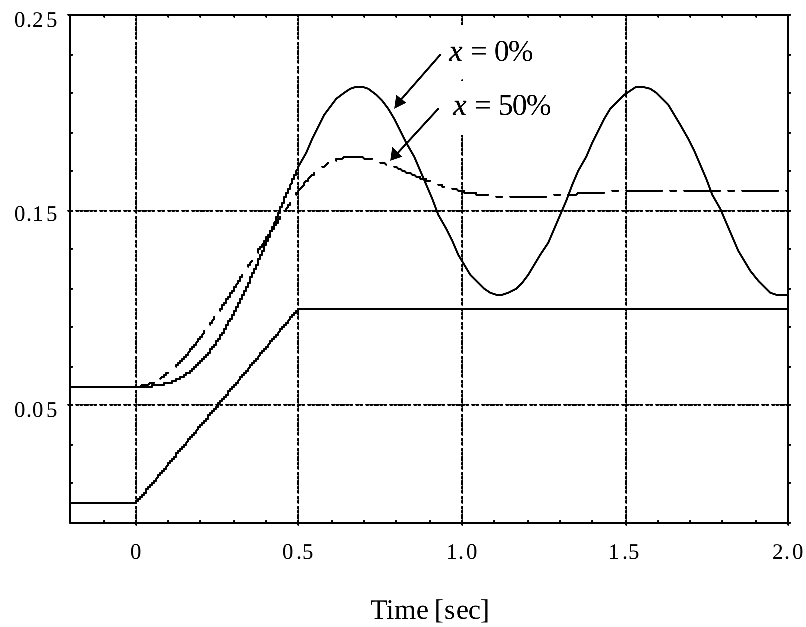
[1] If damping $\xi = 50\%$ is included, the system response is evaluated as follows:
$0 \leq t \leq 0.5\text{sec}$

$$u(t) = e^{-\xi\omega_n t}\left(\frac{-0.2}{2\pi}\sin(2\pi\,t)\right) + 0.2t$$

$t > 0.5\text{sec}$

$$u(t) = e^{-\xi\omega_n t}\left(\frac{-0.2}{2\pi}\sin(2\pi\,t)\right) + 0.2t + e^{-\xi\omega_n(t-0.5)}\left(\frac{0.2}{2\pi}\sin(2\pi(t-0.5))\right) - 0.2(t-0.5)$$

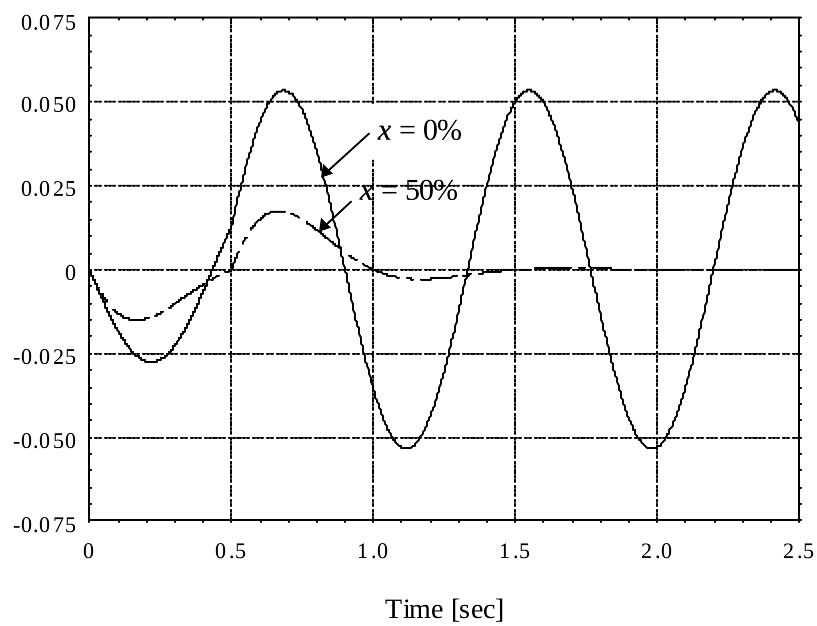
For this case, the overall maximum response occurs earlier, and the response decays very rapidly, as the system is very highly damped.



The deformation of the shock absorbers is the *relative* displacement of the vehicle, $v = u - u_g$.

At the time of absolute displacement, we have: $u_{max} = 0.1535$ m
$u_g = 0.10$ m

Therefore, the maximum relative displacement, which for the particular problem occurs simultaneously with the maximum absolute displacement, is $v_{max} = 0.0535$ m. The relative displacement vs. time, both for the undamped and the damped case, are shown in the Figure below.

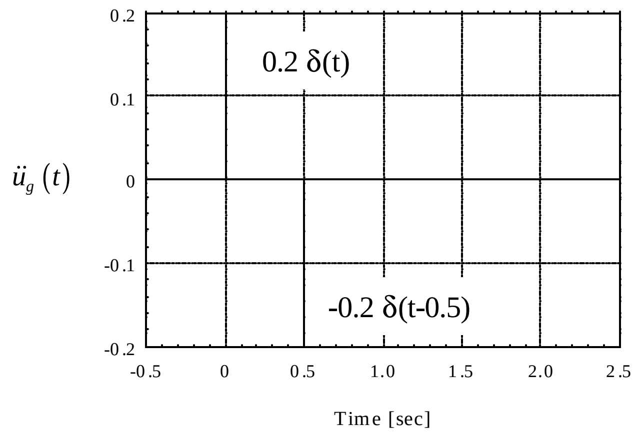


The equation of motion, in terms of relative displacements is formulated as follows:

$$m\ddot{v} + c\dot{v} + kv = -m \cdot \ddot{u}_g$$

The ground displacement is described by the following function: $u_g(t) = \begin{cases} 0.2t & t \geq 0 \\ -0.2(t-t_d) & t \geq t_d \end{cases}$

The ground acceleration is shown in the Figure below:



where δ(t) is the Dirac Delta function.
Therefore, the response of the system is evaluated as follows:

$$0 \leq t \leq 0.5\text{sec}$$

$$m\ddot{v} + c\dot{v} + kv = -m \cdot 0.2\, d(t)$$

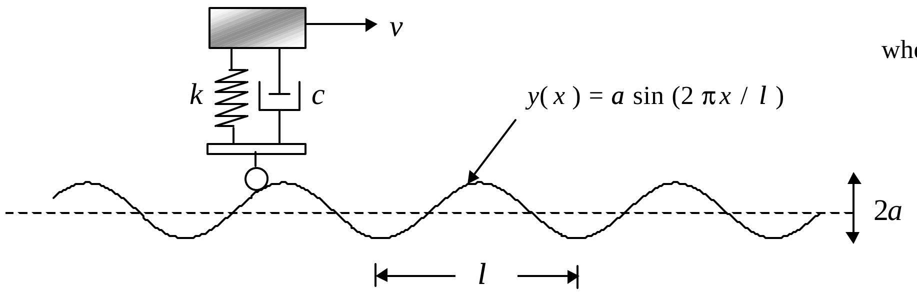
$$v(t) = \frac{-0.2}{7.26} \sin(7.26t)$$

$$t > 0.5\text{sec}$$

$$m\ddot{v} + c\dot{v} + kv = -m \cdot 0.2\, d(t) + m \cdot 0.2\, d(t-0.5)$$

$$v(t) = \frac{-0.2}{7.26} \sin(7.26\, t) + \frac{0.2}{7.26} \sin(7.26(t - 0.5))$$

PROBLEM 3 (P. 17 on page 373**)** An automobile with mass m=1000 kg is traveling with *constant* velocity v along a bumpy road, as shown below. To a first approximation, the car can be modeled as a single degree of freedom system that oscillates vertically with an *undamped* frequency f_n = 1 Hz, and a fraction of critical damping of 50%. On the other hand, the road roughness can be idealized as a sinusoid with a wavelength of 20 m and an amplitude of 5 cm (i.e. 0.05 m)



where: l = 20 m

a = 0.05 m

- Find the vertical absolute displacement as a function of the travel velocity, v.
- Define the travel velocity for which *undamped* resonance occurs.
- Find the power dissipated at resonance, i.e. at the *undamped* natural frequency.
- What is the maximum vertical acceleration at resonance?

SOLUTION

Because the car's velocity is contant, its horizontal position increases linearly with time, that is, $x=V\,t$. Assuming also that the tires are infinitely stiff and that they always remain in contact with the road, their vertical displacement will be a harmonic function of the form $y = a\sin\omega t$ with effective frequency:

$$\omega = \frac{2\pi V}{l}$$

Hence, this problem is the same as that of a 1-dof system subjected to a *harmonic* support motion:

$$u_g(t) = y = a\,\mathrm{Im}(\mathrm{e}^{\mathrm{i}\omega t})$$

The equations of motion for *absolute* and *relative* vertical displacements of the car are then

$$m\ddot{u} + c\dot{u} + k\,u = k\,u_g + c\dot{u}_g \qquad \text{and} \qquad m\ddot{v} + c\dot{v} + k\,v = -m\ddot{u}_g$$

for which the frequency response functions are

$$\tilde{u} = a\,\frac{k + i\omega c}{k - \omega^2 m + i\omega c} = a\;A_u\,\mathrm{e}^{-\mathrm{i}\phi_u} \qquad \text{and} \qquad \tilde{v} = a\,\frac{\omega^2 m}{k - \omega^2 m + i\omega c} = a\;A_v\,\mathrm{e}^{-\mathrm{i}\phi_v}$$

with amplification functions

$$A_u = \sqrt{\frac{1+4x^2r^2}{\left(1-r^2\right)^2+4x^2r^2}} \qquad \text{and} \qquad A_v = \frac{r^2}{\sqrt{\left(1-r^2\right)^2+4x^2r^2}}$$

The tuning ratio r can be expressed in terms of the travel velocity

$$r = \frac{w}{w_n} = \frac{2pV}{w_n\, l} = \frac{V}{V_n} \qquad \text{with} \qquad V_n = \frac{w_n\, l}{2p} = \frac{l}{T_n} = \frac{20}{1} = 20 \text{ m/s} \;= \text{critical velocity}$$

The absolute and relative response functions for the actual support motion are then

$$u(t) = a\, A_u \sin\left(wt - f_u\right) \qquad \text{and} \qquad v(t) = a\, A_v \sin\left(w\,t - f_v\right)$$

It can be seen that when the car's velocity equals the critical velocity V_n = 20m/s = 72km/hr, the oscillations attain a frequency equal to the undamped resonant frequency. While for x=0.50 the most vigorous absolute response occurs at a tuning ratio somewhat less than unity, namely

$$\frac{w_{\max}}{w_n} = \frac{\sqrt{\sqrt{1+(8)(0.50)^2}-1}}{(2)(0.50)} = 0.86$$

the response at the undamped frequency will be nearly as strong as the theoretical maximum. Setting r=1 into the previous expressions for the amplification functions, we obtain the maximum absolute and relative responses at resonance

$$u_{\max} = a\,\frac{\sqrt{2}}{(2)(0.50)} = 0.05\sqrt{2} = 0.0701 \text{ m} \qquad \text{and} \qquad v_{\max} = a\,\frac{1}{(2)(0.5)} = 0.05 \text{ m}$$

The averge power dissipated per cycle of motion is the power dissipated by the dasahpot, that is

$$\left\langle \Pi \right\rangle = \frac{1}{T}\int_0^T c\dot{v}^2 dt = c\,a^2 w^2 A_v^2 \frac{1}{T}\int_0^T \cos^2(wt - f_u)\, dt = \tfrac{1}{2} c\,a^2 w^2 A_v^2 \equiv x\, m\, w_n\, a^2 w^2 A_v^2$$

which can be written as

$$\left\langle \Pi \right\rangle = x\, m\, a^2 w_n^3 \left[\frac{r^6}{\left(1-r^2\right)^2+4x^2r^2}\right]$$

The average power at resonanance is then

$$\left\langle \Pi \right\rangle = x\, m\, a^2 w_n^3 = (0.50)(1000)(0.05)^2(2p)^3 = 310 \text{ Watt}$$

On the other hand, the absolute acceleration is simply

$$\ddot{u}(t) = -w_n^2\, a\, A_u \sin\left(w\,t - f_u\right)$$

so that the maximum acceleration a resonace is

$$\ddot{u}_{\max} = \omega_n^2 \, \alpha \, A_u = (2\pi)^2 (0.05)(\sqrt{2}) = 2.79 \text{ m/s}^2 = 0.283 \text{ g}$$

As can be seen, the acceleration is large indeed.