Solutions for Homework #5

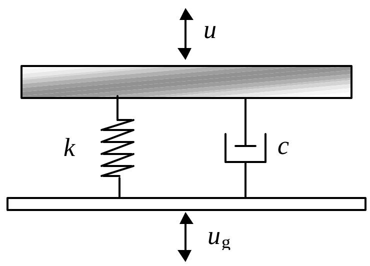
PROBLEM 1(<u>P. 18</u> on page 374**)** An optics experiment is set up on a vibration isolation table, which may be modeled as a single degree of freedom system for vertical oscillations. The system has very little damping.

- The floor has considerable vertical vibration at 30Hz. What must be the natural frequency of the table be in order to reduce the motion of the table and optics experiment by a factor of four, when compared to the motion of the floor?
- Each time you work on the experiment, you bump the table. This results in motion that takes a long time to die out. You decide to add 10% damping to the system. Will this increase or reduce the vibration of the table in response to floor motions at 30 Hz?

SOLUTION

The ground motion is harmonic, and can be described by the following function: $u_g(t) = \tilde{u}_g \; e^{iwt}$.

The equation of motion in terms of absolute displacements, is $m\ddot{u} + c\dot{u} + ku = c\dot{u}_g + k u_g$, where $u(t) = \tilde{u}\; e^{iwt}$.



The transfer function for absolute displacement of the system, due to harmonic ground motion is:

$$H(r) = \frac{1 + 2ixr}{(1-r^2) + 2ixr}$$, where $r = w / w_n$ = tuning ratio.

Therefore, the response of the system is:

$$u(t) = \tilde{u}_g \, A(r) \, e^{i(wt - f)}$$

$$A(r) = \sqrt{\frac{1+4x^2 r^2}{(1-r^2)^2 + 4x^2 r^2}} \qquad f = a\tan\frac{2x\,r^3}{1-(1-4x^2)r^2}$$

For $w = 2\pi \cdot 30 = 60\,\pi$ [rad/sec] and for damping $x = 0$, we want to reduce the motion of the table by a factor of four, i.e.:

$$A(r) = \sqrt{\frac{1}{(1-r^2)^2}} = \frac{1}{\left|1-r^2\right|} = \frac{1}{4}$$

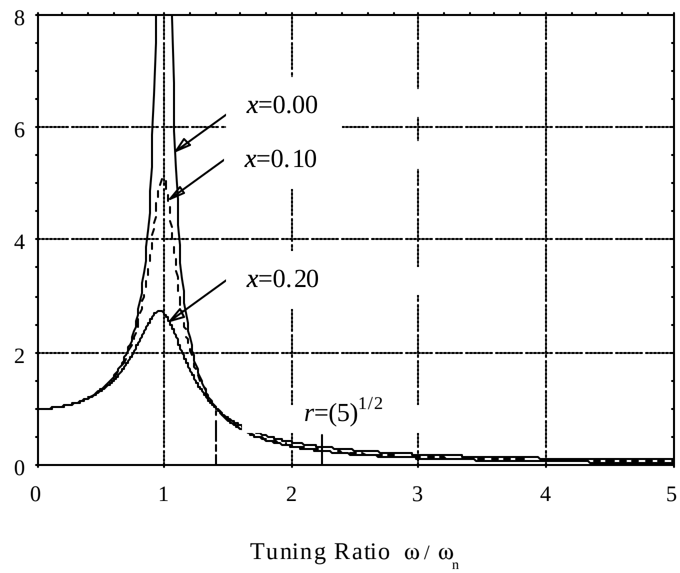
$$\therefore \; r = \frac{w}{w_n} = \sqrt{5} \;\Rightarrow\; w_n = \frac{w}{\sqrt{5}} = 84.3 \text{ [rad/sec]}$$

If we add damping to the system, $x = 0.10$, the amplification function will be:

$$A(r)=\sqrt{\frac{1+4x^2r^2}{(1-r^2)^2+4x^2r^2}}$$

$$A\left(\sqrt{5}\right)=\sqrt{\frac{1+4\cdot 0.1^2\cdot 5}{(1-5)^2+4\cdot 0.1^2\cdot 5}}=0.272>0.25$$

Therefore, the value of the amplification function for the tuning ratio $r^2 = 5 > 2$, increased with the addition of damping to the system.



However, increasing the damping ratio, the transient response due to the bumping of the table, mainly controlled by the table frequency (ω_d) will decay as follows:

$$N_{50\%}\simeq\frac{1}{0.10\cdot 10}=1\ \text{cycle}$$

Therefore, the amplitude will decrease 50% every cycle, and for the frequency of the table being 13Hz, i.e. 13 cycles per second, after 1 sec, the transient response due to the table bumping will be $(1/2)^{13}=1.22\times10^{-4}$.

PROBLEM 2(<u>P. 19</u> on page 374) A one-story structure that can be modeled as a single degree of freedom system has a natural frequency of 10Hz. This structure is subjected to an earthquake whose response spectrum can be idealized by the function

$$S_d(w) = \frac{u_{g0}}{\sqrt{\left[1-\left(\frac{w}{w_0}\right)^2\right]^2 + 4b^2\left(\frac{w}{w_0}\right)^2}}$$

in which:

u_{g0} = 0.03 [m] = peak ground displacement
w_0 = 4π [rad/s] = dominant frequency of the earthquake (=2 Hz)
b = 0.25 = an earthquake parameter

Determine:

- the maximum ground acceleration
- the maximum absolute acceleration of the structure
- the maximum relative displacement
- sketch the response spectrum on a tripartite logarithmic paper (Hint: $w/w_0 = f/f_0 = T_0/T$)

SOLUTION

The maximum absolute acceleration is evaluated as follows:

$$S_a(w) = w^2\, S_d(w) = \frac{w^2\, u_{g0}}{\sqrt{\left[1-\left(\frac{w}{w_0}\right)^2\right]^2 + 4b^2\left(\frac{w}{w_0}\right)^2}} \qquad \{1\}$$

The ground acceleration will coincide with the absolute acceleration of the structure for an infinitely rigid structure, i.e.

$$\tilde{\ddot{u}}_g = \lim_{w\to\infty} \tilde{\ddot{u}}(w) = \lim_{w\to\infty} \frac{w^2\, u_{g0}}{\sqrt{\left[1-\left(\frac{w}{w_0}\right)^2\right]^2 + 4b^2\left(\frac{w}{w_0}\right)^2}} = u_{g0}\; w_0^2 \lim_{w\to\infty} \frac{\left(\frac{w}{w_0}\right)^2}{\sqrt{\left[1-\left(\frac{w}{w_0}\right)^2\right]^2 + 4b^2\left(\frac{w}{w_0}\right)^2}}$$

$$\tilde{\ddot{u}}_g^{\max} = u_{g0}\; w_0^2 = 0.47\, g$$

The maximum absolute acceleration of the structure is given by:

$$S_a(20p) = (20p)^2\, S_d(20p) = \frac{(20p)^2\, 0.03}{\sqrt{\left[1-\left(\frac{20p}{4p}\right)^2\right]^2 + 4\cdot 0.25^2\left(\frac{20p}{4p}\right)^2}} = 4.908\; m/\sec^2 \simeq 0.5\, g$$

The maximum relative displacement is given by:

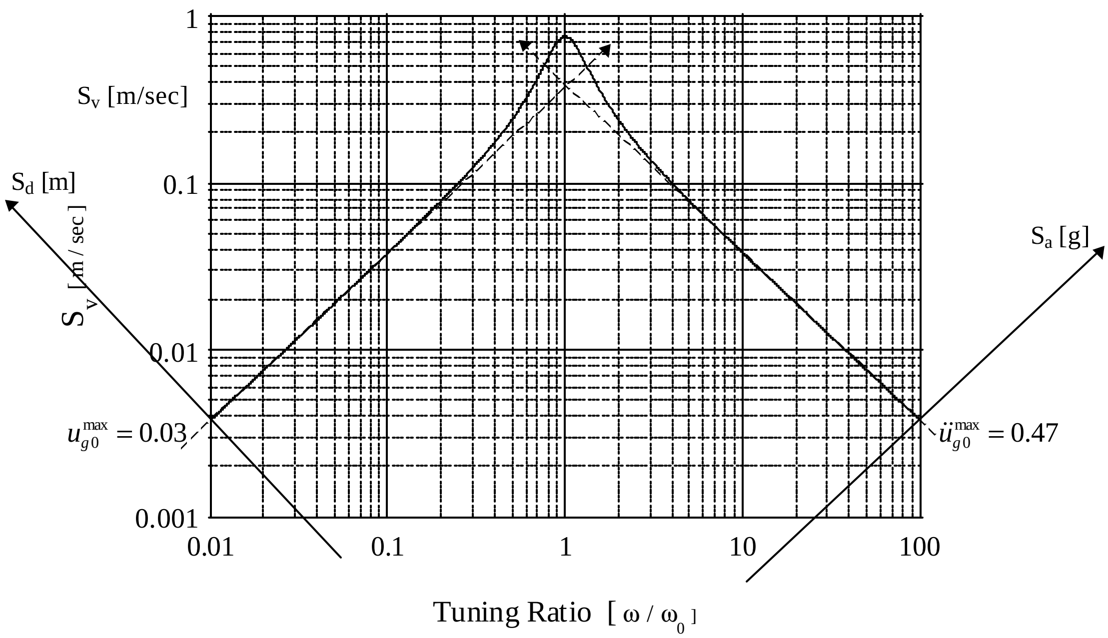
$$S_d(20p) = \frac{S_a(20p)}{(20p)^2} = 1.24 \cdot 10^{-3}\, m$$

The response spectrum is shown in the figure below.

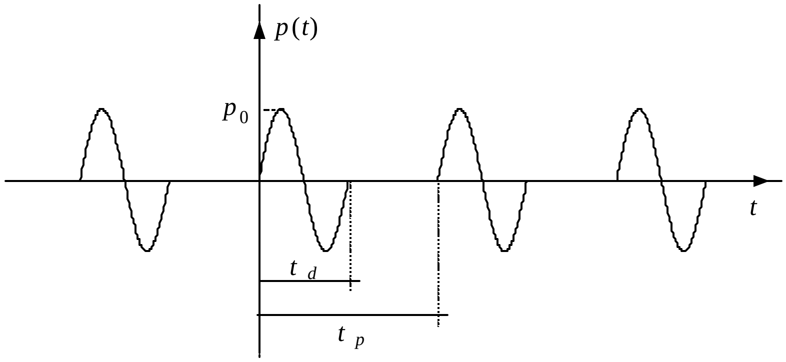
Note.- $\log S_v = \log \omega + \log S_d$

$\log S_v = \log S_v$

$\log S_v = -\log \omega + \log S_a$



PROBLEM 3(<u>P. 20</u> on page 374**)** Consider the following periodic forcing function:



where: p_0 = 1 N, t_d = 5 sec and t_p = 10 sec.

- Compute the coefficients of the Fourier series.
- Sketch the amplitude spectrum $\left|\tilde{p}_j\right|$
- Sketch the amplitude of the product $\left|H\ \tilde{p}_j\right|$, assuming f_n = 0.4 Hz, x = 0.05 and m = 1kg.
- What is the response at t=0 ? (Consider a finite number N of terms in the Fourier series. How many?)
- Using MATLAB, sketck the response for N = 16, 32, 64 terms.

SOLUTION

The periodic function is of the form:

$$p(t) = p_0\ \sin\left(2p\ \frac{t}{t_d}\right) = p_0\ \sin(a\ t) \qquad \{1\}$$

where: P_0 = 1N, T_d = 5 sec and T_p = 10 sec

$$a = \frac{2p}{t_d} = \frac{2p}{5}$$
$$\Delta w = \frac{2p}{T_p} = \frac{2p}{2\ t_d} = \frac{a}{2} \qquad w_j = j\,\Delta w = \frac{a\ j}{2} \qquad \{2\}$$

The Fourier coefficients are evaluated by integration over 1 period, i.e. over T_p:

$$\tilde{p}_j = \int_0^{T_p} p(t)\, e^{-iw_j t}\, dt = \int_0^{t_d} p(t)\, e^{-iw_j t}\, dt + \int_{t_d}^{T_p} p(t)\, e^{-iw_j t}\, dt =$$

$$= \int_0^{t_d} \sin(a\,t)\, e^{-iw_j t}\, dt = \frac{1}{2i}\int_0^{t_d}\left[e^{iat} - e^{-iat}\right] e^{-iw_j t}\, dt =$$

$$= \frac{1}{2i}\int_0^{t_d}\left[e^{i(a-w_j)t} - e^{-i(a+w_j)t}\right]dt = \frac{1}{2i}\left[\frac{e^{i(a-w_j)t_d}-1}{i(a-w_j)} + \frac{e^{-i(a+w_j)t_d}-1}{i(a+w_j)}\right] =^{1} \qquad \{3\}$$

$$= \frac{1}{2i}\left[\frac{e^{i(a-a/2\cdot j)t_d}-1}{i(a-a/2\cdot j)} + \frac{e^{-i(a+a/2\cdot j)t_d}-1}{i(a+a/2\cdot j)}\right]$$

We now substitute in {3} for t_d: $\quad e^{\pm i(a\mp a/2\cdot j)t_d} = e^{\pm i(a\mp a/2\cdot j)\frac{2p}{a}} = e^{\pm i(2p\mp p\,j)}$ $\quad$ {4}

Therefore, from equation {4} we have: $\quad e^{\pm i(2p\mp p\,j)} = \begin{cases} e^{2pi} = 1 & \text{for } j = \text{even} \\ e^{pi} = -1 & \text{for } j = \text{odd} \end{cases}$ $\quad$ {5}

Equation {3} now becomes:

$$\tilde{p}_j\Big|_{\substack{j=\text{even}\\ j\neq 2}} = -\frac{1}{2}\left[\frac{1-1}{(a-a/2\cdot j)} + \frac{1-1}{(a+a/2\cdot j)}\right] = 0$$

$$\tilde{p}_j\Big|_{j=\text{odd}} = -\frac{1}{2}\left[\frac{-1-1}{(a-a/2\cdot j)} + \frac{-1-1}{(a+a/2\cdot j)}\right] = \frac{2a}{2}\left[\frac{(1+1/2\cdot j) + (1-1/2\cdot j)}{(a-a/2\cdot j)\cdot(a+a/2\cdot j)}\right] = \qquad \{6\}$$

$$= \frac{2}{a\,(1-j^2/4)} = \frac{2}{\frac{2p}{5}(1-j^2/4)} = \frac{5}{p\,(1-j^2/4)}$$

For the case of $j = 2$, i.e $w_j = a$, we have:

$$\tilde{p}_2 = \frac{1}{2i}\int_0^{t_d}\left[e^0 - e^{-i\,2w_j t}\right]dt = \frac{1}{2i}\int_0^{t_d}\left[e^0 - e^{-i\,2a\,t}\right]dt = \frac{1}{2i}\left[t_d + \frac{e^{-i\,2a\,t}}{i\,2a}\Bigg|_0^{t_d}\right] =$$

$$= \frac{1}{2i}\left[t_d + \frac{e^{-4p}-1}{i\,2a}\right] = \frac{1}{2i}\left[t_d + \frac{1-1}{i\,2a}\right] = \frac{t_d}{2i} = \frac{2.5}{i} \qquad \{7\}$$

In Figure 1, the Fourier coefficients are plotted as a function of circular frequency w_j, i.e. the Fourier amplitude spectrum:

[1] Note.- For $j = 2$, we encounter a zero denominator, therefore the case $j = 2$ will be solved separately.

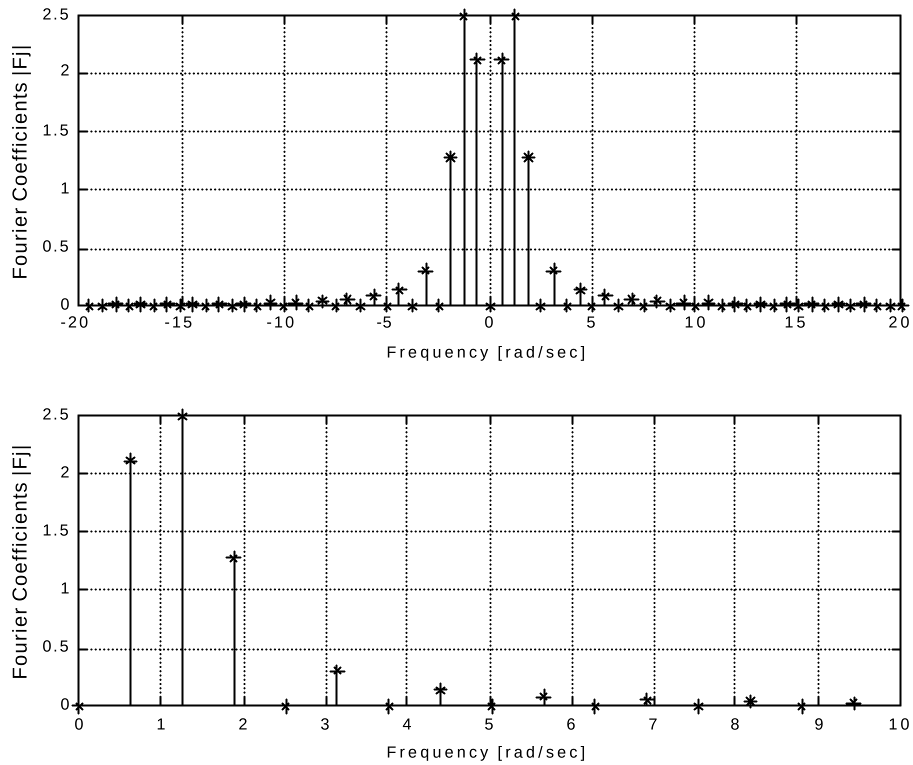


Figure 1. Fourier Amplitude Spectrum (in bottom Figure, the Fourier coefficients between 0-10 rad/sec are shown enlarged): Note that $F_J = \tilde{p}_j$ in the plots.

The single degree of freedom system has the following properties:

Natural frequency: $f_n = 0.4$ Hz $\quad \omega_n = 0.8\,\pi$ rad/sec

Damping, mass: $x = 0.05$ $\quad m = 1$kg

Stiffness: $k = \omega_n^2\, m = 0.64\,\pi^2$ N/m

The transfer funtion for the harmonic applied force is:

$$H\left(w_j\right) = \frac{1}{k\left[1 - \left(\frac{w_j}{w_n}\right)^2 + 2ix\left(\frac{w_j}{w_n}\right)\right]} \qquad \{8\}$$

The absolute value of $H\left(w_j\right)$ will be $1/k$ for $\omega_j = 0$ and $H\left(w_n\right) = \frac{1}{2xk} = \frac{1}{2 \cdot 0.05 \cdot 0.64 p^2} = 1.58$ at the resonant frequency of the structure.

In Figure 2, the absolute value of $H\left(w_j\right)$ as well as the product $\left|H\left(w_j\right) \cdot \tilde{p}_j\right|$ are plotted as a function of circular frequency w_j:

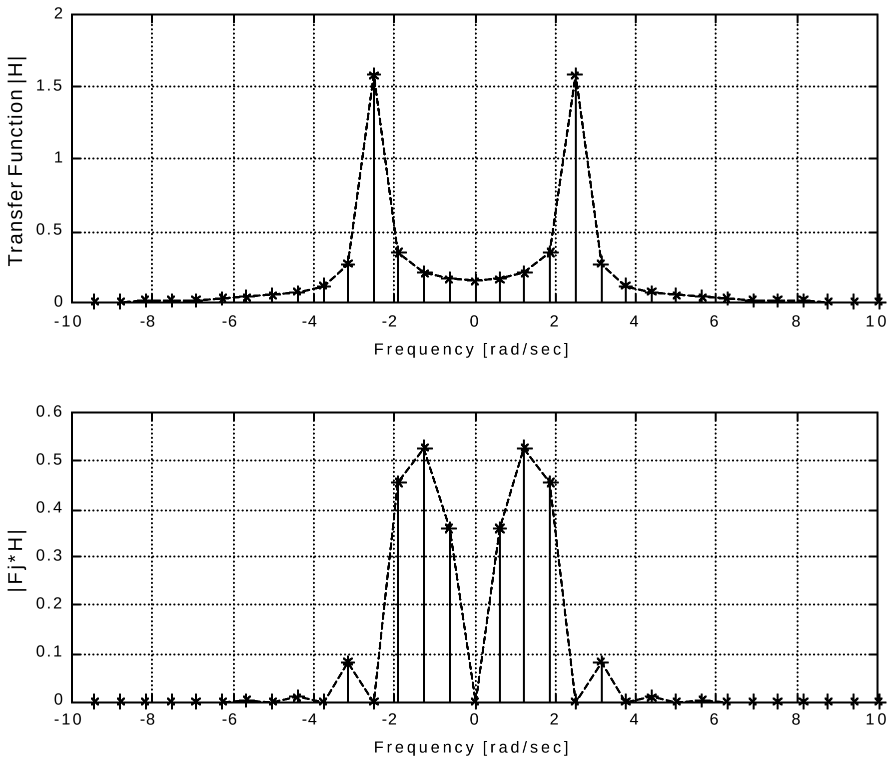


Figure 2. Transfer Function | H(ω_j) | and Response Spectrum $\left|H\left(w_j\right)\cdot \tilde{p}_j\right|$: Note that $F_J = \tilde{p}_j$ in the plots.

To evaluate the response at time t = 0, we proceed as follows:

The steady state response due to a periodic load is:

$$u(t) = \frac{1}{T_p}\sum_{-\infty}^{\infty} H\left(w_j\right)\,\tilde{p}_j\,e^{-iw_j t}$$
$$u(0) = \frac{1}{T_p}\sum_{-\infty}^{\infty} H\left(w_j\right)\,\tilde{p}_j \qquad \{9\}$$

Observing Figure 1, we can see that the coefficients 0-5 are enough to evaluate an acceptable result for the response at t = 0. Therefore we have:

$$u(0) \cong \frac{1}{T_p}\sum_{-5}^{5} H\left(w_j\right)\,\tilde{p}_j \qquad \{10\}$$

In Figure 3, the real and imaginary parts of the product $\left|H\left(w_j\right)\cdot \tilde{p}_j\right|$ are plotted. We can readily see that the imaginary parts of the symmetric coefficients cancel out, and the real parts have identical value, therefore:

$$u(0) \cong \frac{1}{T_p} \sum_{-5}^{5} \mathrm{Re}\left\{ H(w_j)\ \tilde{p}_j \right\} = \frac{1}{T_p} \sum_{0}^{5} 2\ \mathrm{Re}\left\{ H(w_j)\ \tilde{p}_j \right\} =$$

$$= \frac{1}{T_p}\ 2\left\{0.3581 - 0.035 - 0.4476 + 0.0813\right\} = -8.64 \times 10^{-3}\ m$$

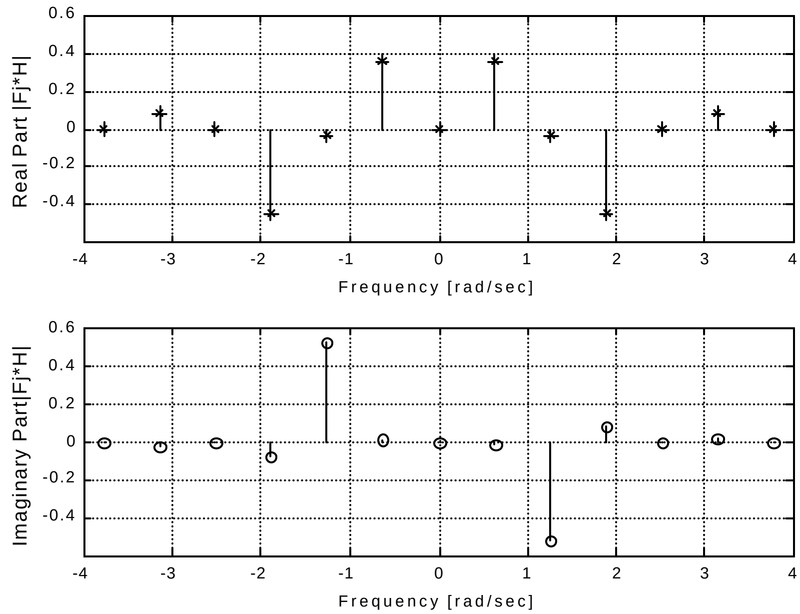


Figure 3. Real and Imaginary Part of the Response Spectrum $\left|H(w_j)\cdot \tilde{p}_j\right|$: Note that $F_J = \tilde{p}_j$ in the plots.

Figure 4 shows the displacement time history of the 1-degree of freedom system, when 16, 32 and 64 terms of the Fourier series are considered.

It can be readily seen that the contribution of the higher coefficients is minor to the response of the system, therefore, the result is essentially identical for the cases considered.

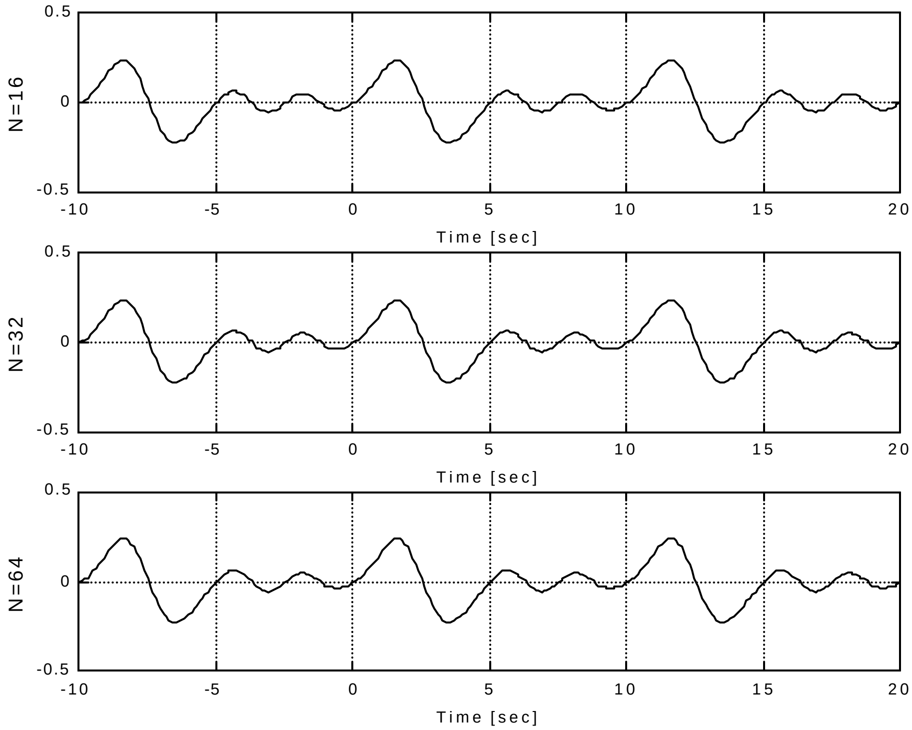
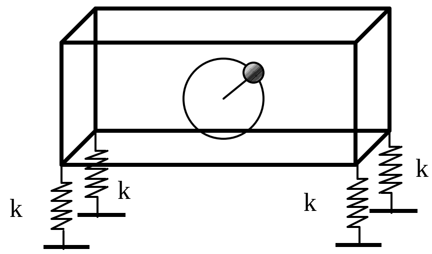


Figure 4. Response evaluated using N=16,32 and 64 terms of the Fourier Series

<u>PROBLEM 4(P. 21</u> on page 375**)** A reciprocating machine that weighs 100,000N is known to develop vertical harmonic forces having a maximum amplitude of 5000N at its operating speed of 30 Hz, when attached to a rigid support. In order to limit the vibrations in the building in which the machine is to be installed, it is to be supported by a spring at each corner of its base. What spring constant (stiffness) is required to reduce the total harmonic force trasmitted to the building to 1000N?



SOLUTION

The equation of motion for the system is:

$$m\ddot{u} + 4\,k\,u + m_e\left(\ddot{u} - r\,\Omega^2 e^{i\Omega t}\right) = 0 \;\Rightarrow$$
$$\left(m + m_e\right)\ddot{u} + 4\,k\,u = m_e r\,\Omega^2 e^{i\Omega t} \;\Rightarrow \qquad \{1\}$$
$$\left(m + m_e\right)\ddot{u} + 4\,k\,u = 5000\, e^{i\Omega t}$$

We assume that the eccentric mass, m_e, is very small compared to the total mass of the machine, therefore:

$$m + m_e \approx m = \frac{100000N}{10\ m/\sec^2} = 10000\ kg$$

For the response of the structure being $u\left(t\right) = \tilde{u}\left(\Omega\right) e^{i\Omega t}$, we substitute in equation {1}, and cancel out the common factor $e^{i\Omega t}$, to evaluate the transfer function:

$$-\Omega^2 m\tilde{u} + 4\,k\,\tilde{u} = m_e\,r\,\Omega^2 \qquad \Rightarrow$$

$$\tilde{u} = \frac{m_e\,r\,\Omega^2}{4\,k - \Omega^2 m} = \frac{\frac{m_e}{m} r\,\Omega^2}{w_n^2 - \Omega^2} = H\left(\Omega\right) \;\Rightarrow\; A\left(\Omega\right) = \sqrt{\frac{\left[\frac{m_e}{m} r\left(\frac{\Omega}{w_n}\right)^2\right]^2}{\left[1 - \left(\frac{\Omega}{w_n}\right)^2\right]^2}} \qquad \{2\}$$

We know that $m_e\,r\,\Omega^2 = 5000$ N and $\Omega = 60p\ \ rad/\sec$.

The force transmitted to the structure is:

$$F_s\ \left(t\right) = 4\ k\ u\left(t\right) \qquad = 4\ k\ \tilde{u}\left(\Omega\right) e^{i\Omega t}$$
$$\{3\}$$
$$F_s^{\max} = 4\,k\,\tilde{u}\left(\Omega\right) = 4\,k\ \tilde{u}\left(\Omega\right) = 1000\,N \ \text{(required)}$$

Therefore, introducing {2} in equation {3}, the required stiffness is calculated as follows:

$$F_s^{\max} = 4\,k\,\tilde{u}(\Omega) = 4\,k\sqrt{\frac{\left[\frac{m_e}{m}r\left(\frac{\Omega}{w_n}\right)^2\right]^2}{\left[1-\left(\frac{\Omega}{w_n}\right)^2\right]^2}} = 1000\,N$$

$$4\,k\,\frac{\frac{m_e}{m}r\left(\frac{\Omega}{w_n}\right)^2}{\left|1-\left(\frac{\Omega}{w_n}\right)^2\right|} = 4\,k\,\frac{\frac{m_e}{m}r\,\frac{\Omega^2}{4k/m}}{\left|1-\frac{\Omega^2}{4k/m}\right|} = \frac{m_e r\,\Omega^2}{\left|1-\frac{\Omega^2}{4k/m}\right|} = \frac{5000}{\left|1-\frac{(60p)^2}{4k/10000}\right|} = 1000\,N$$

$$\left|1-\frac{(60p)^2}{4k/10000}\right| = 5 \Rightarrow \frac{(60p)^2}{4k/10000} = 6 \Rightarrow k = 14804.41\ kN$$