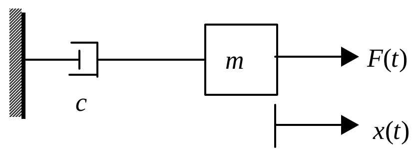
Solutions for Homework #6

PROBLEM 1 (**P. 22** on page 375) A simple 1-dof system consists of a mass and a damper only, as shown in the figure below. The excitation is the real part of $F(t) = F_0\, e^{iwt}$, where F_0 is real and positive.



where: m = 1kg, and c = 10 N sec / m.

- Draw a free body diagram of the mass showing all the forces.
- Find the equation of motion.
- Find the frequency response function $H_{x|F}(w)$, which is defined as the ratio of the response $X\, e^{iwt}$ to the input $F_0\, e^{iwt}$. Sketch the magnitude of $|H_{x|F}(w)|$.
- At very low frequency, damping is dominant. Find the response magnitude $|X|$ and the approximate phase angle between the force and the response when w = 0.5 rad /sec. Let F_0 = 5.0 N.
- At high frequency, the system is mass controlled. Find the response magnitude $|X|$ and the approximate phase angle between the force and the response when w = 200.00 rad /sec.

SOLUTION

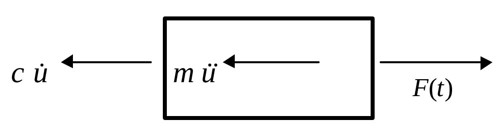
For the single degree of freedom system, the forcing function is of the following form:

$$F(t) = \mathrm{Re}\left(F_0 \cdot e^{iwt}\right) = \mathrm{Re}\left(F_0 \cos(wt) + i\, F_0 \sin(wt)\right) = F_0 \cos(wt) \qquad \{1\}$$

We will evaluate the response using complex notation, and the response will be the real part of the result, namely:

$$\begin{aligned} u(t) &= \mathrm{Re}\left(F_0\ H(w)\ e^{iwt}\right) = \mathrm{Re}\left(F_0\ A(w)\ e^{i(wt-f)}\right) \\ &= F_0\ A(w)\cos(wt-f) \end{aligned} \qquad \{2\}$$

- The free body diagram of the structure is simply as follows. Note also that $u(t)=x(t)$



- The equation of motion is then formulated as follows:

$$m\,\ddot{u} + c\,\dot{u} = F(t) = F_0\, e^{iwt} \qquad \{3\}$$

- For the response of the structure $u(t) = \tilde{u}(w)\, e^{iwt}$, we substitute in equation {3} to evaluate the frequency response function. Therefore we have (we cancel out the common non-zero exponential term):

$$-w^2 m\, \tilde{u}(w) + i\, w\, c\, \tilde{u}(w) = F_0$$

$$\tilde{u}(w) = \frac{F_0}{-w^2 m + i\, w\, c}$$

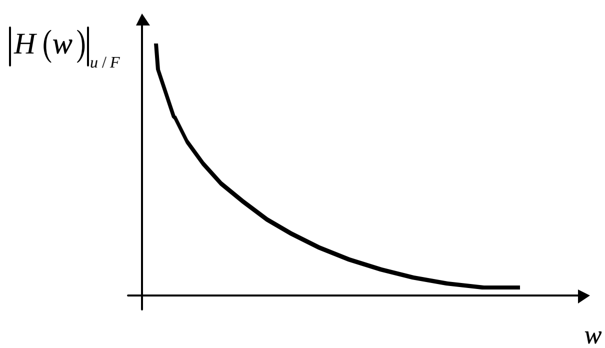
$$H(w) = \frac{\tilde{u}(w)}{F_0} = \frac{1}{-w^2 m + i w\, c} \qquad \{4\}$$

$$\left|H(w)\right| = \left|\frac{1}{-w^2 m + i w\, c}\right| = \frac{1}{\sqrt{\left(-w^2 m\right)^2 + \left(w\, c\right)^2}}$$

Therefore, the absolute value of the transfer function $H(w)$ is a hyperbolic function of the driving frequency w, where:

$$\left|H(w)\right| = \begin{cases} \infty & \text{for} \quad w \to 0 \\ 0 & \text{for} \quad w \to \infty \end{cases} \qquad \{5\}$$

We plot schematically the absolute value of the transfer function vs. the driving frequency w.



- At very low frequency, damping is dominant. Find the response magnitude |X| and the approximate phase angle between the force and the response when $\omega = 0.5$ rad /sec. Let $F_0 = 5.0$ N.

We use equation {4} to evaluate the system response due to evaluate the response of the system as follows:

$$\left|\tilde{u}(w)\right| = F_0 \left|H(w)\right| = F_0 \frac{1}{\sqrt{\left(-w^2 m\right)^2 + \left(w\, c\right)^2}}$$

$$\left|\tilde{u}(0.5)\right| = 5 \frac{1}{\sqrt{\left(-0.5^2 \cdot 1\right)^2 + \left(0.5 \cdot 10\right)^2}} \cong 1.00 \qquad \{6\}$$

$$\tan(f) = \frac{w\, c}{-w^2 m} = -\frac{0.5 \cdot 10}{0.5^2 \cdot 1} = -20 \Rightarrow f = -87^o \approx -\frac{p}{2}\, rad$$

The system response is almost 90^o out of phase with the applied forcing function, therefore damping is the control factor of the response.

- At high frequency, the system is mass controlled. Find the response magnitude |X| and the approximate phase angle between the force and the response when ω = 200.00 rad /sec.

We use equation {4} to evaluate the system response due to evaluate the response of the system as follows:

$$\left|\tilde{u}(w)\right| = F_0 \left|H(w)\right| = F_0 \frac{1}{\sqrt{\left(-w^2 m\right)^2 + \left(w\ c\right)^2}}$$

$$\left|\tilde{u}(200)\right| = 5 \frac{1}{\sqrt{\left(-200^2 \cdot 1\right)^2 + \left(200 \cdot 10\right)^2}} = 1.2485 \times 10^{-4} \cong 0.00 \qquad \{7\}$$

$$\tan(f) = \frac{w\ c}{-w^2\ m} = -\frac{200 \cdot 10}{200^2 \cdot 1} = -0.05 \Rightarrow f = -2.86^o \approx 0^-\ rad$$

The system response is almost in phase with the applied forcing function, therefore inertia (mass) is the control factor of the response.

PROBLEM 2 (**<u>P. 23</u>** on page 375) If the force input to a single degree of freedom vibration system is $F(t) = F\, e^{\mathrm{i}wt}$ and the steady–state output is $X\, e^{\mathrm{i}wt}$, then the frequency response function is $H_{x|F}\ (w) = X\, e^{\mathrm{i}wt} / F\, e^{\mathrm{i}wt}$, which can be expressed as a magnitude and phase:

$$H_{x|F}\ (w) = |\, H_{x|F}\ (w)|\ e^{-\mathrm{i}f}$$
$$X\,(t) = |F\,|\ |\ H_{x|F}\ (w)| e^{-\mathrm{i}f}$$

- The Euler formula says $e^{\mathrm{i}q} = \cos(q) + \mathrm{i}\sin(q)$. Show that $\mathrm{i} = e^{\mathrm{i}p/2}$ and $-1 = e^{\mathrm{i}p}$.
- Find $H_{\dot{X}|F}(w)$ and $H_{\ddot{X}|F}(w)$ the frequency response functions when the response is the velocity or the acceleration.

Hint: $H_{\dot{X}|F}(w) = \dfrac{\dot{X}(t)}{F\, e^{\mathrm{i}wt}} = \left|H_{\dot{X}|F}(w)\right|\, e^{-\mathrm{i}f}$ $\qquad H_{\ddot{X}|F}(w) = \dfrac{\ddot{X}(t)}{F\, e^{\mathrm{i}wt}}$

Generalize your answer by explaining the effect of phase angle when you take the time derivative of the response.

SOLUTION

The Euler formula says: $e^{iq} = \cos(q) + i\,\sin(q)$ {1}

We use equation {1} as follows:

$$e^{ip/2} = \cos(\frac{p}{2}) + i\,\sin(\frac{p}{2}) = 0 + i = i$$
$$e^{ip} = \cos(p) + i\,\sin(p) = -1 + 0 = -1$$

The equation of motion for the simple harmonic oscillator subjected to harmonic force is:

$$m\ddot{u} + c\dot{u} + ku = F_0\, e^{iwt} \qquad \{2\}$$

Notice that $u(t)$=$X(t)$ in this present solution.

- For the velocity being of the form: $\dot{u} = \tilde{\dot{u}}\, e^{iwt}$

Displacement: $u(t) = \int \dot{u}\, dt = \int \tilde{\dot{u}}\, e^{iwt}\, dt = \dfrac{\tilde{\dot{u}}}{iw}\, e^{iwt}$ {3}

Acceleration: $\ddot{u}(t) = \dfrac{d}{dt}\dot{u}(t) = iw\, \tilde{\dot{u}} e^{iwt}$

We substitute {3} in {2} and cancel out the common non-zero exponential term.

$$miw\,\tilde{\dot{u}} + c\tilde{\dot{u}} + k\frac{\tilde{\dot{u}}}{iw} = F_0$$

$$\tilde{\dot{u}} = \frac{F_0}{miw + \dfrac{k}{iw} + c} = \frac{F_0}{\dfrac{k}{iw}\left[\left(1 - \left(\dfrac{w}{w_n}\right)^2\right) + 2ix\dfrac{w}{w_n}\right]} \qquad \{4\}$$

$$H_{\dot{u}/F(w)} = \frac{iw}{k\left[\left(1-\left(\frac{w}{w_n}\right)^2\right)+2\,ix\,\frac{w}{w_n}\right]} = i\,w\,H_{u/F(w)} \qquad \{5\}$$

- For the acceleration being of the form: $\ddot{u} = \tilde{\ddot{u}}\, e^{iwt}$

Velocity: $$\dot{u}(t) = \int \ddot{u}\; dt = \int \tilde{\ddot{u}}\; e^{iwt}\; dt = \frac{\tilde{\ddot{u}}}{iw}\, e^{iwt} \qquad \{6\}$$

Displacement: $$u(t) = \int \dot{u}\; dt = \int \frac{\tilde{\ddot{u}}}{iw}\, e^{iwt}\; dt = -\frac{\tilde{\ddot{u}}}{w^2}\, e^{iwt}$$

We substitute {3} in {2} and cancel out the common non-zero exponential term.

$$m\tilde{\ddot{u}} + \frac{c}{iw}\tilde{\ddot{u}} - \frac{k}{w^2}\tilde{\ddot{u}} = F_0$$

$$\tilde{\ddot{u}} = \frac{F_0}{m - \frac{k}{w^2} + \frac{c}{iw}} = \frac{F_0}{-\frac{k}{w^2}\left[\left(1-\left(\frac{w}{w_n}\right)^2\right)+2\,ix\,\frac{w}{w_n}\right]} \qquad \{7\}$$

$$H_{\ddot{u}/F(w)} = \frac{-w^2}{k\left[\left(1-\left(\frac{w}{w_n}\right)^2\right)+2\,ix\,\frac{w}{w_n}\right]} = -w^2\,H_{u/F(w)} \qquad \{8\}$$

We have proved:

- $H_{\dot{u}/F(w)} = i\,w\,H_{u/F(w)}$

  The velocity is 90^{o} out of phase with the displacement, i.e. when u = max , $\dot{u}$ = 0.

- $H_{\ddot{u}/F(w)} = -w^2\,H_{u/F(w)}$

  The acceleration is proportional to the displacement, but they have opposite direction i.e. when u = max , $\ddot{u}$ = min or $|\ddot{u}|$ = max.