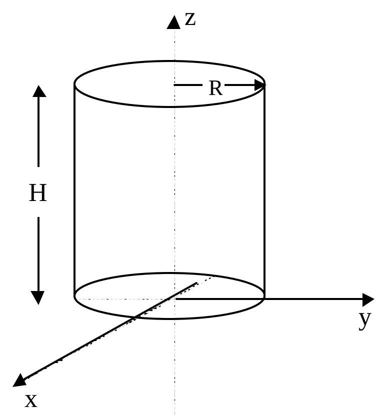
Solutions for Homework#7

PROBLEM 1 (**<u>P. 25</u>** on page 377) To a first approximation, a nuclear reactor building can be modeled as a rigid cylinder of radius R=20m and height H=60m, with an average mass density of r = 625 kg/m^3 (”=25% of solid concrete”). This building is supported by a soil with shear wave velocity C_s=300 m/s, mass density r_s=2000 kg/m^3, and Poisson’s ratio n=1/3. Using approximate expressions for the foundation stiffnesses, determine the six natural frequencies and modal shapes of this system.

SOLUTION

The nuclear reactor is modeled (to a first approximation!) as follows:



The mass of the cylinder is calculated as follows:

$$m_c = V_c \cdot r_c = \left(p \cdot R^2 \cdot H\right) \cdot r_c = p \cdot 20^2 \cdot 60 \cdot 625 \cong 47.12 \times 10^6 \; kg \qquad \{1\}$$

The shear modulus, G, of the soil is calculated as:

$$G_s = r_s \cdot C_s^{\,2} = 2000 \cdot 300^2 = 180 \times 10^6 \; N/m^2 \qquad \{2\}$$

The six natural frequencies and modal shapes of the system are:

<u>Vertical Oscillation</u>

Vertical Stiffness $$K_v = \frac{4\,G_s\,R}{1-v} = \frac{4 \cdot 180{\times}10^6 \cdot 20}{1-1/3} = 21.6 \times 10^9 \; N/m \qquad \{3\}$$

Therefore, the natural frequency for vertical oscillation is:

$$w_v = \sqrt{\frac{K_v}{m}} = \sqrt{\frac{21.6 \times 10^9}{47.12 \times 10^6}} = 21.41 \; rad/\mathrm{sec} \qquad \{4\}$$

Its mode shape is a any scalar.

<u>Torsional Oscillation</u>

Torsional Stiffness $$K_T = \frac{16\,G_s\,R^3}{3} = \frac{16\cdot 180\times10^6\cdot 20^3}{3} = 7.68\times10^{12}\ Nm \qquad \{5\}$$

The moment of inertia for the circular cross-section of the cylinder is defined as:

$$J_T = \frac{m_c\,R^2}{2} = \frac{47.12\times10^6\cdot 20^2}{2} = 9.42\times10^9\ kgm^2 \qquad \{6\}$$

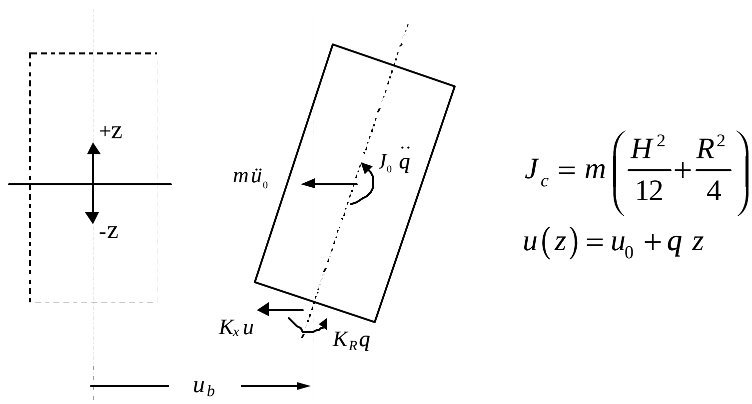
Therefore, the natural frequency for torsional oscillation is:

$$w_T = \sqrt{\frac{K_T}{J_c}} = \sqrt{\frac{7.68\times10^{12}}{9.42\times10^9}} = 28.55\ rad/\sec \qquad \{7\}$$

Its mode shape is a any scalar.

<u>Horizontal and Rocking Oscillation</u> [1](coupled)

Recall the solution of **<u>P. 1</u>** in the lecture note.



We formulate the equation of motion with respect to the center of mass:

Horizontal Equuilibrium: $$m\,\ddot{u}_0 + K_x\,u_b = 0 \qquad \{8\}$$

Moment Equilibrium: $$J_c\,\ddot{q} + K_R\,q - K_x u_b \frac{H}{2} = 0 \qquad \{9\}$$

We substitute in equations {8} and {9}, the displacement of the base u_b, as a function of the displacement of the center of mass u_0 and the angle of rotation, q, i.e.:

$$u_b = u\left(-\frac{H}{2}\right) = u_0 - q\,\frac{H}{2} \qquad \{10\}$$

[1] Note that the frequencies are the same for x and y direction, due to symmetry of the structure.

The equation of motion for the 2-dof system is now formulated as follows:

$$m\,\ddot{u}_0 + K_x\,u_0 - K_x\frac{H}{2}q = 0$$

$$J_c\,\ddot{q} + K_R\,q - K_x\,u_0\frac{H}{2} + K_x q\,\frac{H^2}{4} = 0 \qquad \{11\}$$

Equation {11} can be expressed in matrix form as follows:

$$\begin{bmatrix} m & 0 \\ 0 & J_c \end{bmatrix}\begin{Bmatrix} \ddot{u}_0 \\ \ddot{q} \end{Bmatrix} + \begin{bmatrix} K_x & -K_x\dfrac{H}{2} \\ -K_x\dfrac{H}{2} & K_R + K_x\dfrac{H^2}{4} \end{bmatrix}\begin{Bmatrix} u_0 \\ q \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \{12\}$$

We now calculate:

$$J_c = m\left(\frac{H^2}{12}+\frac{R^2}{4}\right) = 47.12\times10^6\left(\frac{60^2}{12}+\frac{20^2}{4}\right) = 18.85\times10^9\,kgm^2 \qquad \{13\}$$

$$K_x = K_y = \frac{8\cdot G_s\cdot R}{2-n} = \frac{8\cdot 180\times10^6\cdot 20}{2-1/3} = 17.28\times10^9\,N/m \qquad \{14\}$$

$$K_R^x = K_R^y = \frac{8\cdot G_s\cdot R^3}{3(1-n)} = \frac{8\cdot 180\times10^6\cdot 20^3}{3(1-1/3)} = 5.76\times10^{12}\;Nm \qquad \{15\}$$

The eigenvalue problem is formulated as follows:

$$\left(\boldsymbol{K} - w^2\,\boldsymbol{M}\right)\cdot\boldsymbol{f}_i = \boldsymbol{0} \qquad \{16\}$$

For the solution of {16} to be non-trivial, we demand:

$$\left|\boldsymbol{K} - w^2\,\boldsymbol{M}\right| = 0$$

$$\left|\begin{bmatrix} 17.28\times10^9 & -17.28\times10^9\cdot\dfrac{60}{2} \\ -17.28\times10^9\cdot\dfrac{60}{2} & 5.76\times10^{12} + 17.28\times10^9\cdot\dfrac{60^2}{4} \end{bmatrix} - w^2\begin{bmatrix} 47.12\times10^6 & 0 \\ 0 & 18.85\times10^9 \end{bmatrix}\right| = 0 \qquad \{17\}$$

$$\begin{vmatrix} 17.28\times10^9 - 47.12\times10^6\,w^2 & -17.28\times10^9\cdot\dfrac{60}{2} \\ -17.28\times10^9\cdot\dfrac{60}{2} & 5.76\times10^{12} + 17.28\times10^9\cdot\dfrac{60^2}{4} - 18.85\times10^9\,w^2 \end{vmatrix} = 0$$

The solution of the quadratic equation in {17}, has the following solutions:

$$w_1 = 8.89\ rad\ /\sec$$
$$w_2 = 37.66\ rad\ /\sec$$

To define the modal shapes, we set f_{1n} = 1.00, and solve for f_{2n}, using the eigenvalues calculated above. We therefore have:

$$\begin{bmatrix} 17.28\times10^9 - 47.12\times10^6 w_n^{\ 2} & -17.28\times10^9 \cdot \dfrac{60}{2} \\ -17.28\times10^9 \cdot \dfrac{60}{2} & 5.76\times10^{12} + 17.28\times10^9 \cdot \dfrac{60^2}{4} - 18.85\times10^9 w_n^{\ 2} \end{bmatrix} \begin{Bmatrix} 1.00 \\ f_{2n} \end{Bmatrix} = 0 \quad \{18\}$$

From the first equation in {18}, we evaluate f_{2n} as follows:

$$f_{2n} = \frac{17.28\times10^9 - 47.12\times10^6 w_n^{\ 2}}{17.28\times10^9 \cdot \dfrac{60}{2}} \qquad \{19\}$$

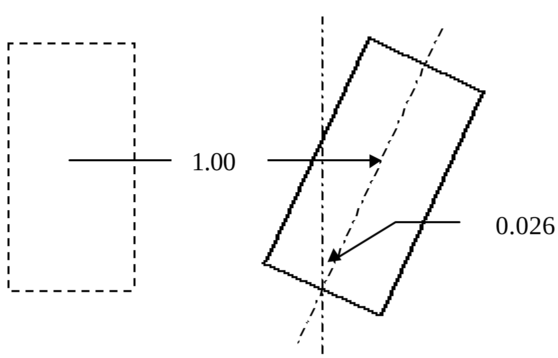
Substituting in {19} n=1,2, we evaluate:

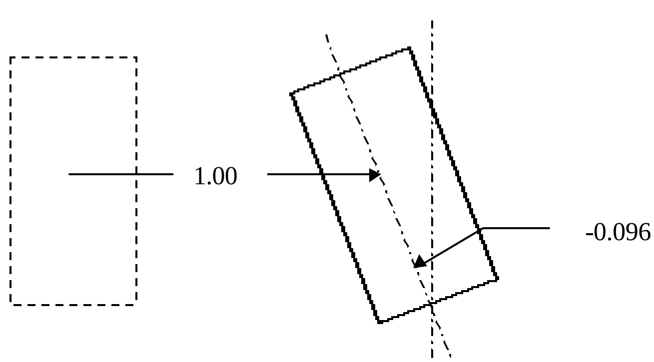
f_{21} = 0.026 *rad*

f_{22} = -0.096 *rad*

The modal shapes for the coupled horizontal – rocking motion are schematically shown below:

$$\boldsymbol{f}_1 = \begin{Bmatrix} 1.00 \\ 0.026 \end{Bmatrix}, \quad \boldsymbol{f}_2 = \begin{Bmatrix} 1.00 \\ -0.096 \end{Bmatrix}$$

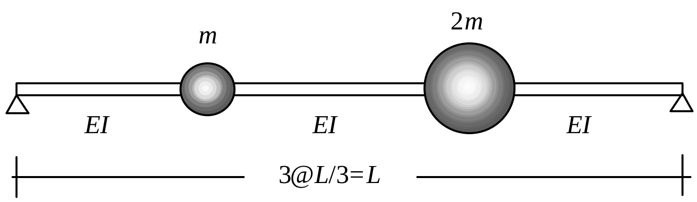




PROBLEM 2 (**P. 27** on page 377) A simply supported massless beam has two masses lumped, as shown.

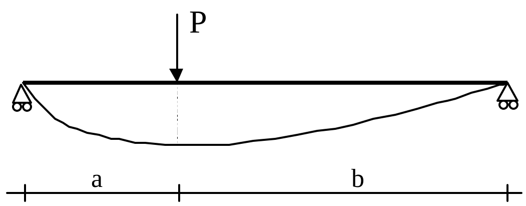
Determine

a) The frequencies and modal shapes
b) The dynamic response that the system would have if at t =0 gravity were to be suddenly "switched on" (physically, but not mentally impossible…). Basically, we are asking here to compute the modal contributions to the response, to write down the expressions, and perhaps sketch the response of one of the masses.



SOLUTION

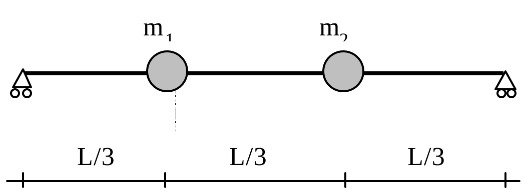
Recall the deflection shape of the beam, from Problem 1, Homework No. 1.



$$u(x) = \frac{PL^3}{6EI}\frac{b}{L}\frac{x}{L}\left[1-\left(\frac{b}{L}\right)^2-\left(\frac{x}{L}\right)^2\right] \qquad x \le a$$

$$= \frac{PL^3}{6EI}\frac{a}{L}\left(1-\frac{x}{L}\right)\left[1-\left(\frac{a}{L}\right)^2-\left(1-\frac{x}{L}\right)^2\right] \qquad x \ge a \qquad \{1\}$$

The structure under consideration is shown in the figure below. The system has 2 dynamic dof's, namely u_1 and u_2.



$m_1 = m$

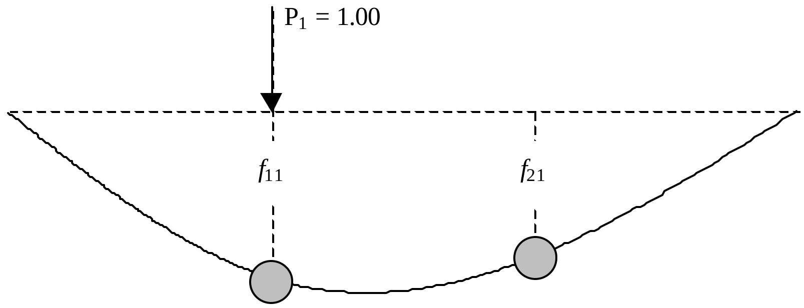
$m_2 = 2m$

The mass matrix of the structure is simply:

$$\boldsymbol{M} = \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \qquad \{2\}$$

To evaluate the *stiffness* matrix, we first construct the *flexibility* matrix as follows:

- We apply a unit load $P_1 = 1.00$ at the 1rst degree of freedom, and calculate the deflection at the 1rst degree of freedom, f_{11}, and at the 2nd degree of freedom, f_{21}.

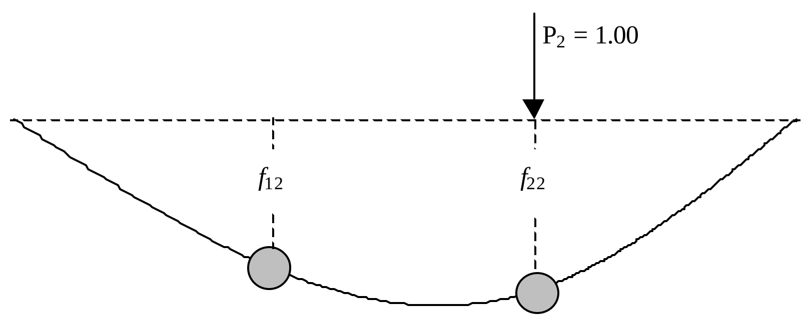


For a = L/3 and b = 2L/3, we evaluate the flexibility coefficients by substituting in equation {1}:

$$f_{11} = u\left(L/3\right) = \frac{L^3}{6EI}\,\frac{2L/3}{L}\,\frac{L/3}{L}\left[1-\left(\frac{2L/3}{L}\right)^2-\left(\frac{L/3}{L}\right)^2\right] = \frac{4}{243}\,\frac{L^3}{EI}$$

$$f_{21} = u\left(2L/3\right) = \frac{L^3}{6EI}\,\frac{L/3}{L}\left(1-\frac{2L/3}{L}\right)\left[1-\left(\frac{L/3}{L}\right)^2-\left(1-\frac{2L/3}{L}\right)^2\right] = \frac{7}{486}\,\frac{L^3}{EI}$$

- We apply a unit load $P_2 = 1.00$ at the 2nd degree of freedom, and calculate the deflection at the 1rst degree of freedom, f_{12}, and at the 2nd degree of freedom, f_{22}.



For a = 2L/3 and b = L/3, we evaluate the flexibility coefficients by substituting in equation {1}:

$$f_{21} = u\left(L/3\right) = \frac{L^3}{6EI}\,\frac{L/3}{L}\,\frac{L/3}{L}\left[1-\left(\frac{L/3}{L}\right)^2-\left(\frac{L/3}{L}\right)^2\right] = \frac{7}{486}\,\frac{L^3}{EI}$$

$$f_{22} = u\left(2L/3\right) = \frac{L^3}{6EI}\,\frac{2L/3}{L}\left(1-\frac{2L/3}{L}\right)\left[1-\left(\frac{2L/3}{L}\right)^2-\left(1-\frac{2L/3}{L}\right)^2\right] = \frac{4}{243}\,\frac{L^3}{EI}$$

The flexibility matrix of the structure is formulated as follows:

$$\boldsymbol{F} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \frac{L^3}{EI} \begin{bmatrix} \frac{4}{243} & \frac{7}{486} \\ \frac{7}{486} & \frac{4}{243} \end{bmatrix} \quad \{2\}$$

Therefore, the stiffness matrix is:

$$\boldsymbol{F}^{-1} = \boldsymbol{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \frac{64.8\ EI}{L^3} \begin{bmatrix} 4 & -3.5 \\ -3.5 & 4 \end{bmatrix} \quad \{3\}$$

The eigenvalue problem is formulated as follows:

$$\left(\boldsymbol{K} - w^2\, \boldsymbol{M} \right) \cdot \boldsymbol{f}_i = \mathbf{0} \quad \{4\}$$

For the solution of {4} to be non-trivial, we demand:

$$\left| \boldsymbol{K} - w^2\, \boldsymbol{M} \right| = 0$$

$$\left| \frac{64.8\ EI}{L^3} \begin{bmatrix} 4 & -3.5 \\ -3.5 & 4 \end{bmatrix} - w^2 \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \right| = 0 \quad \{5\}$$

$$\begin{vmatrix} 4 - l & -3.5 \\ -3.5 & 4 - 2\,l \end{vmatrix} = 0 \qquad \text{where } \ l = \frac{w^2 L^3 m}{64.8\ EI}$$

The solution of the quadratic equation in {5}, has the following solutions:

$$\begin{aligned} l_1 &= 5.67 \Rightarrow w_1 = 19.17\sqrt{\frac{EI}{mL^3}} \\ l_2 &= 0.33 \Rightarrow w_2 = 4.62\sqrt{\frac{EI}{mL^3}} \end{aligned} \quad \{6\}$$

To define the modal shapes, we set f_{1n} = 1.00, and solve for f_{2n}, using the eigenvalues calculated above. We therefore have:

$$\begin{bmatrix} 4 - l & -3.5 \\ -3.5 & 4 - 2\,l \end{bmatrix} \begin{Bmatrix} 1.00 \\ f_{2n} \end{Bmatrix} = 0 \quad \{7\}$$

From the first equation in {7}, we evaluate f_{2n} as follows:

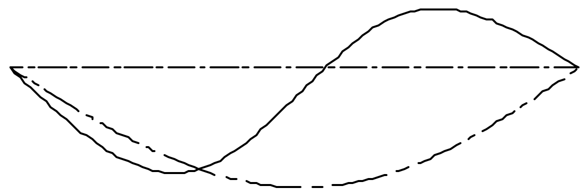
$$f_{2n} = \frac{4 - l_n}{3.5} \quad \{8\}$$

Substituting in {8} n=1,2, we evaluate:

f_{21} = -0.477 m, f_{22} = 1.049 m

The modal shapes are schematically shown in the figure below:

$$\boldsymbol{f}_1 = \begin{Bmatrix} 1.00 \\ -0.477 \end{Bmatrix} \qquad \boldsymbol{f}_2 = \begin{Bmatrix} 1.00 \\ 1.049 \end{Bmatrix}$$



We now calculate the response of the system, it at t=0 a step load of amplitude P_i = $-m_i$ g and infinite duration, is applied to the ith degree of freedom.

We first calculate the modal masses (μ_i) and modal stiffnesses (κ_i) as follows:

$$m_1 = \{1.00 \quad -0.477\} \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} 1.00 \\ -0.477 \end{Bmatrix} = 1.455m$$

$$m_2 = \{1.00 \quad 1.049\} \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} 1.00 \\ 1.049 \end{Bmatrix} = 3.20m$$

{9}

$$k_1 = \{1.00 \quad -0.477\} \frac{64.8\,EI}{L^3} \begin{bmatrix} 4 & -3.5 \\ -3.5 & 4 \end{bmatrix} \begin{Bmatrix} 1.00 \\ -0.477 \end{Bmatrix} = 535.54\frac{EI}{L^3}$$

$$k_2 = \{1.00 \quad 1.049\} \frac{64.8\,EI}{L^3} \begin{bmatrix} 4 & -3.5 \\ -3.5 & 4 \end{bmatrix} \begin{Bmatrix} 1.00 \\ 1.049 \end{Bmatrix} = 68.62\frac{EI}{L^3}$$

The modal forces are calculated as follows:

$$p_1 = \{1.00 \quad -0.477\} \begin{Bmatrix} -mg \\ -2mg \end{Bmatrix} = -0.046\,mg$$

{10}

$$p_2 = \{1.00 \quad 1.049\} \begin{Bmatrix} -mg \\ -2mg \end{Bmatrix} = -3.098\,mg$$

The response of a 1-dof system with zero initial conditions, $u_0 = \dot{u}_0 = 0$, due to a step load of infinite duration is:

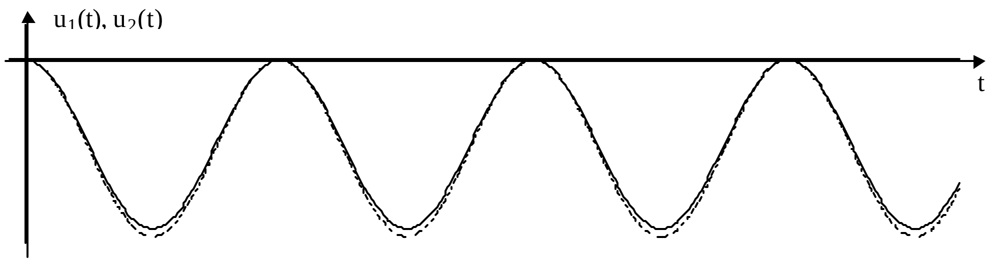
$$q_j(t) = \frac{p_j}{k_j}\left(1 - \cos\left(w_j t\right)\right) \qquad \{11\}$$

Therefore, the response of the system, is calculated using modal superposition, as follows:

$$u(t) = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \sum_{j=1}^{2} \boldsymbol{f}_j \; q_j(t)$$

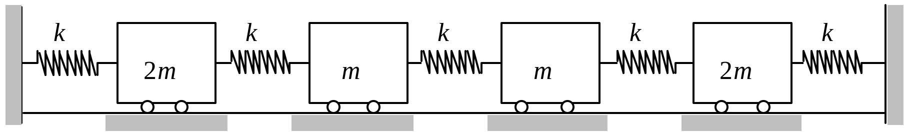
$$= -\begin{Bmatrix} 1.00 \\ -0.477 \end{Bmatrix} 8.59\times10^{-5} \frac{mgL^3}{EI} \left(1-\cos(w_1 t)\right) - \begin{Bmatrix} 1.00 \\ 1.049 \end{Bmatrix} 0.045 \frac{mgL^3}{EI} \left(1-\cos(w_2 t)\right)$$

In what follows, the displacement time history of the masses is plotted.



As it can be readily observed, the second mode mainly contributes to the response as the oscillation of the two masses is essentially in phase.

PROBLEM 3 (**P. 28** on page 378) In the structure shown, determine all frequencies and modal shapes (but think before you proceed…)



(You may verify your answers with MATLAB, but you should also work out this problem by hand.)

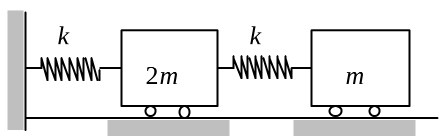
SOLUTION

Recall the solution of Problem 3, Homework No. 1. The system above can be analyzed using symmetry and antisymmetry concepts, as follows:

Antisymmetric mode

The two left masses are moving in phase with the two right masses, and due to the symmetry of the structure, the middle point is not moving.

Therefore, the system can be analyzed as follows:



For this system, the modal frequencies are calculated as follows:

$$\omega_n = 2\sqrt{\frac{k}{2m}}\sin\left(\frac{\pi}{4N}(2n-1)\right) \qquad \{1\}$$

which is derived on pages 333 to 340 in the lecture note for a discrete shear beam.

Therefore, for N = 2, substituting in {1} n = 1,2, we calculate:

$$\omega_1 = 2\sqrt{\frac{k}{2m}}\sin\left(\frac{\pi}{8}(2-1)\right) = 0.54\sqrt{\frac{k}{m}}$$

$$\omega_2 = 2\sqrt{\frac{k}{2m}}\sin\left(\frac{\pi}{8}(4-1)\right) = 1.307\sqrt{\frac{k}{m}}$$

The modal shapes are now calculated as follows:

$$\phi_{in} = \cos\left(\frac{\pi}{2N}(i-1)(2n-1)\right) \qquad \{2\}$$

For n = 1, we have:

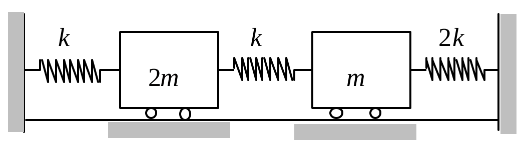
$$f_{11} = \cos\left(\frac{p}{4}(1-1)(2-1)\right) = 1.00$$
$$f_{21} = \cos\left(\frac{p}{4}(2-1)(2-1)\right) = 0.707$$

and for n = 2, we have:

$$f_{12} = \cos\left(\frac{p}{4}(1-1)(4-1)\right) = 1.00$$
$$f_{22} = \cos\left(\frac{p}{4}(2-1)(4-1)\right) = -0.707$$

<u>Symmetric mode</u>

The two left masses are moving to the opposite direction from the two right masses, and due to the symmetry of the structure, the system can be analyzed as follows:



The mass and stiffness matrix of the structure are shown below:

$$\boldsymbol{K} = k\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \qquad \boldsymbol{M} = m\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

The eigenvalue problem is formulated as follows:

$$\left(\boldsymbol{K} - w^2\,\boldsymbol{M}\right)\cdot \boldsymbol{f}_i = \boldsymbol{0} \qquad \{3\}$$

For the solution of {4} to be non-trivial, we demand:

$$\left|\boldsymbol{K} - w^2\,\boldsymbol{M}\right| = 0$$

$$\left| k\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} - w^2\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \right| = 0 \qquad \{4\}$$

$$\begin{vmatrix} 2-2l & -1 \\ -1 & 3-l \end{vmatrix} = 0 \qquad \text{where } l = \frac{w^2 m}{k}$$

The solution of the quadratic equation in {4}, has the following solutions:

$$\lambda_1 = 3.22 \Rightarrow \omega_1 = 1.80\sqrt{\frac{k}{m}}$$
$$\lambda_2 = 0.78 \Rightarrow \omega_2 = 0.88\sqrt{\frac{k}{m}} \qquad \{5\}$$

To define the modal shapes, we set $\phi_{1n} = 1.00$, and solve for ϕ_{2n}, using the eigenvalues calculated above. We therefore have:

$$\begin{bmatrix} 2-2\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} \begin{Bmatrix} 1.00 \\ \phi_{2n} \end{Bmatrix} = 0 \qquad \{6\}$$

From the first equation in {6}, we evaluate ϕ_{2n} as follows:

$$\phi_{2n} = 2 - 2\lambda_n \qquad \{7\}$$

Substituting in {7} n=1,2, we evaluate:

$\phi_{21} = -4.44$, $\phi_{22} = 0.44$

Therefore, the spectral and modal matrices of the structure are the following:[2]

$$\Omega = diag(\omega_i) = \sqrt{\frac{k}{m}} \begin{bmatrix} 0.54 & & & \\ & 1.307 & & \\ & & 1.80 & \\ & & & 0.88 \end{bmatrix}$$

$$\boldsymbol{\Phi} = \{\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \boldsymbol{\phi}_3 \quad \boldsymbol{\phi}_4\} = \begin{bmatrix} 0.707 & -0.707 & -4.44 & 0.44 \\ 1.00 & 1.00 & 1 & 1 \\ 1.00 & 1.00 & -1 & -1 \\ 0.707 & -0.707 & 4.44 & -0.44 \end{bmatrix}$$

[2] Note-.The eigenvalues and eigenvectors could have been obtained directly from the following EVP:

$$\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right)\cdot\boldsymbol{\phi}_i = \boldsymbol{0}$$

where: $\boldsymbol{K} = k\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ $\qquad \boldsymbol{M} = m\begin{bmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 2 \end{bmatrix}$

yet the problem demands much more computational effort.