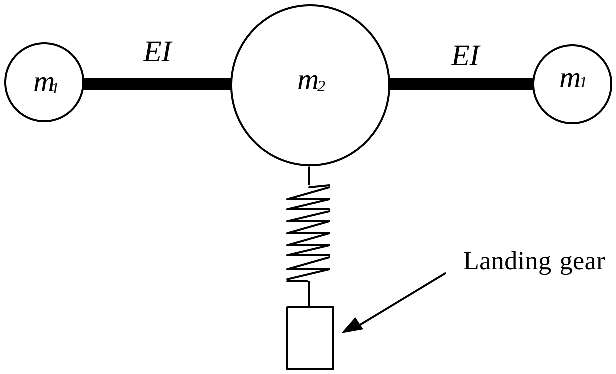
Solutions for Homework #8

<u>PROBLEM 1.</u> (**<u>P. 29</u>** on page 378 in the note) An airplane is modeled as a beam with 3 masses as shown below:



$$\sqrt{\frac{EI}{m_1 L^3}} = 2p \ \text{[rad/sec]}$$

$$\frac{m_1}{m_2} = 0.25$$

- Find the stiffness and mass matrices.
- Find all oscillator modes and frequencies.
- If the landing gear has stiffness k such as it deflects 0.1m under it's own weight (when parked), determine the frequencies and mode shapes of the plane after landing. Explain how you would determine the maximum force in the landing gear at the time of landing, assuming that the vertical descent rate at time of landing is 10 m/sec.

SOLUTION

The system analyzed has 2 rigid body motions (namely one translation and one rotation).

By inspection, the modal shape for the translational rigid body motion is:

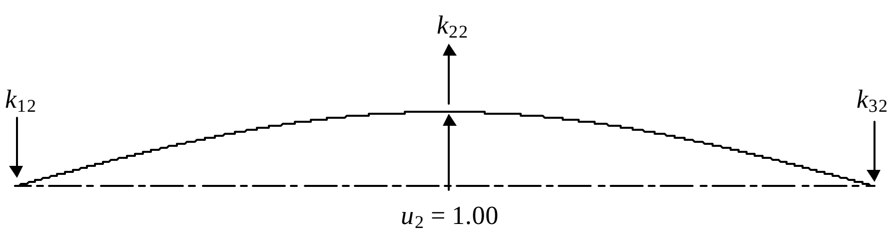
$$f_1^T = \{1 \quad 1 \quad 1\}$$

Similarly, the modal shape for the rotational rigid body motion is:

$$f_2^T = \{-1 \quad 0 \quad 1\}$$

In order to determine the stiffness matrix of the structure, we proceed as follows:

We calculate the stiffness coefficients k_{12}, k_{22} and k_{32} by calculating the force needed to prevent translation at the 1st and 3rd degree of freedom, when the 2nd degree of freedom has a unit deformation, namely:



Therefore, the stiffness coefficients are calculated as follows:

$$
\begin{aligned}
k_{22} &= 2\,\frac{3EI}{(L/2)^3} = \frac{48EI}{L^3} = k \\
k_{12} &= -\frac{3EI}{(L/2)^3} = -\frac{24EI}{L^3} = -0.5k \\
k_{32} &= -\frac{3EI}{(L/2)^3} = -\frac{24EI}{L^3} = -0.5k
\end{aligned}
\qquad \{1\}
$$

Due to symmetry of the stiffness matrix, we also have:

$$k_{21} = k_{12} \text{ and } k_{23} = k_{32} \qquad \{2\}$$

The stiffness matrix is now:

$$\boldsymbol{K} = \begin{bmatrix} k_{11} & -0.5k & k_{13} \\ -0.5k & k & -0.5k \\ k_{31} & -0.5k & k_{33} \end{bmatrix} \qquad \text{where: } k_{13} = k_{31}$$

To calculate the remaining coefficients, we recall that the rigid body motions store no energy in the system, i.e.:

$$
\begin{aligned}
&\boldsymbol{K}\,\boldsymbol{f}_1 = \begin{bmatrix} k_{11} & -0.5k & k_{13} \\ -0.5k & k & -0.5k \\ k_{31} & -0.5k & k_{33} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow k_{11} - 0.5\,k + k_{13} = 0 \\
&\boldsymbol{K}\,\boldsymbol{f}_2 = \begin{bmatrix} k_{11} & -0.5k & k_{13} \\ -0.5k & k & -0.5k \\ k_{31} & -0.5k & k_{33} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{cases} k_{11} - k_{13} = 0 \\ k_{31} - k_{33} = 0 \end{cases}
\end{aligned}
\qquad \{3\}
$$

Therefore, from the system of equations {3}, we have:

$$k_{11} = k_{13} = k_{31} = k_{33} = 0.25\,k$$

The stiffness and mass matrices of the structure are:

$$\boldsymbol{K} = k \begin{bmatrix} 0.25 & -0.5 & 0.25 \\ -0.5 & 1 & -0.5 \\ 0.25 & -0.5 & 0.25 \end{bmatrix} \qquad \boldsymbol{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 \end{bmatrix}$$

The natural frequencies associated with the rigid body motions are:

$$w_1 = w_2 = 0$$

To calculate the third modal shape, we recall that the modal shapes are orthogonal with respect to the mass and the stiffness matrix, namely:

$$\boldsymbol{f}_1^{\,T}\,\boldsymbol{M}\,\boldsymbol{f}_3 = \{1 \quad 1 \quad 1\}\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 \end{bmatrix}\begin{Bmatrix} 1 \\ f_{23} \\ f_{33} \end{Bmatrix} = 0 \quad \Rightarrow \quad m_1 + f_{23}\, m_2 + f_{33}\, m_1 = 0$$

$$\boldsymbol{f}_2^{\,T}\,\boldsymbol{M}\,\boldsymbol{f}_3 = \{1 \quad 0 \quad -1\}\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 \end{bmatrix}\begin{Bmatrix} 1 \\ f_{23} \\ f_{33} \end{Bmatrix} = 0 \quad \Rightarrow \quad m_1 - f_{33}\, m_1 = 0 \tag{4}$$

Therefore, the coefficients f_{23}, f_{33} are evaluated by solving the system of equations in {4} as follows:

$$\boldsymbol{f}_3 = \begin{Bmatrix} 1 \\ f_{23} \\ f_{33} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -\dfrac{2\,m_1}{m_2} \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.5 \\ 1 \end{Bmatrix} \tag{5}$$

To calculate the frequency w_3, we use the Rayleigh quotient, which leads to the exact value of the frequency, when the exact modal shape is used:

$$w_3^2 = \frac{\boldsymbol{f}_3^{\,T}\,\boldsymbol{K}\,\boldsymbol{f}_3}{\boldsymbol{f}_3^{\,T}\,\boldsymbol{M}\,\boldsymbol{f}_3} = \frac{\{1 \quad -0.5 \quad 1\}\, k \begin{bmatrix} 0.25 & -0.5 & 0.25 \\ -0.5 & 1 & -0.5 \\ 0.25 & -0.5 & 0.25 \end{bmatrix}\begin{Bmatrix} 1 \\ -0.5 \\ 1 \end{Bmatrix}}{\{1 \quad -0.5 \quad 1\}\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 \end{bmatrix}\begin{Bmatrix} 1 \\ -0.5 \\ 1 \end{Bmatrix}} = \frac{2.25k}{3\,m_1} = 0.75\,\frac{48\,EI}{L^3\,m_1} = 36\,\frac{EI}{L^3\,m_1}$$

$$w_3 = \sqrt{36\,\frac{EI}{L^3\,m_1}} = 6\,\sqrt{\frac{EI}{L^3\,m_1}} = 12p \;\; rad\,/\mathrm{sec}$$

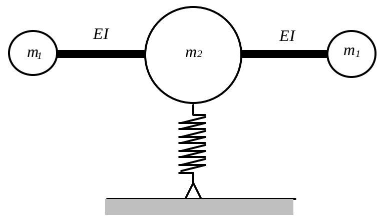
When parked, the force applied at the landing gear is the weight of the plane, namely:

$$W = m_1\,\mathrm{g} + m_2\,\mathrm{g} + m_1\,\mathrm{g} \tag{6}$$

Therefore, the stiffness of the landing gear, k_{g}, is calculated as follows:

$$k_g = \frac{W}{u_{st}} = \frac{2m_1 + m_2}{0.10}\,g = \frac{6\,m_1\,g}{0.10} = 600\,m_1 \tag{7}$$

The system is now modeled (when parked) as in the figure below, and there is only one rotational rigid body motion:



The mass matrix of the structure remains the same, and the stiffness matrix is now evaluated as follows:

$$k_{22} = 2\,\frac{3EI}{(L/2)^3} + k_g = \frac{48EI}{L^3} + k_g = k + 600m_1$$
$$k_{12} = -\frac{3EI}{(L/2)^3} = -\frac{24EI}{L^3} = -0.5k \qquad \{8\}$$
$$k_{32} = -\frac{3EI}{(L/2)^3} = -\frac{24EI}{L^3} = -0.5k$$

The rotational rigid body motion stores no energy in the system, and the translational motion, where all the masses have the same displacement, stores energy only due to the presence of the flexible support, namely:

$$\boldsymbol{K}\boldsymbol{f}_1 = \begin{bmatrix} k_{11} & -0.5k & k_{13} \\ -0.5k & k+kg & -0.5k \\ k_{31} & -0.5k & k_{33} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ kg \\ 0 \end{Bmatrix} \Rightarrow k_{11} - 0.5\,k + k_{13} = 0$$

$$\{9\}$$

$$\boldsymbol{K}\boldsymbol{f}_2 = \begin{bmatrix} k_{11} & -0.5k & k_{13} \\ -0.5k & k+k_g & -0.5k \\ k_{31} & -0.5k & k_{33} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{cases} k_{11} - k_{13} = 0 \\ k_{31} - k_{33} = 0 \end{cases}$$

Therefore, we have the stiffness matrix and the mass matrix for the parked case:

$$\boldsymbol{K} = k \begin{bmatrix} 0.25 & -0.5 & 0.25 \\ -0.5 & 1+\dfrac{k_g}{k} & -0.5 \\ 0.25 & -0.5 & 0.25 \end{bmatrix}, \qquad \boldsymbol{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 \end{bmatrix} \qquad \{10,11\}$$

There are three frequencies and mode shapes for this case. Again, by inspection, we can determine the first frequency and mode shape of the rotational rigid body motion as w_1=0 and $\mathbf{f}_1 = \{1 \quad 0 \quad -1\}^T$, respectively. Next, we determine the second and third frequencies w_2, w_3 and mode shapes $\mathbf{f}_2, \mathbf{f}_3$ by making use of the orthogonality conditions in connection with both the stiffness and mass matrices. To begin with, we assume the two mode shapes of the form.

$$\mathbf{f}_2 = \{1 \quad a \quad 1\}^T,\ \mathbf{f}_3 = \{1 \quad b \quad 1\}^T \qquad \{12,13\}$$

Then, consider the following two conditions.

$$\mathbf{f}_2^T \mathbf{M}\,\mathbf{f}_3^T = \{1 \quad a \quad 1\} \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 \end{bmatrix} \begin{Bmatrix} 1 \\ b \\ 1 \end{Bmatrix} = 2m_1 + abm_2 = 0 \qquad \{14\}$$

$$\mathbf{f}_2^T\mathbf{K}\,\mathbf{f}_3^T=\{1 \quad a \quad 1\}\,k\begin{bmatrix}0.25 & -0.5 & 0.25\\ -0.5 & 1+\dfrac{k_g}{k} & -0.5\\ 0.25 & -0.5 & 0.25\end{bmatrix}\begin{Bmatrix}1\\ b\\ 1\end{Bmatrix}=k[1-b+(1+k/k_g)ab-a]=0 \qquad \{15\}$$

From equations {14} and {15}, we obtain two constants a and b as follows.

$$a=\frac{A+\sqrt{A^2-4B}}{2},\; b=\frac{A-\sqrt{A^2-4B}}{2} \qquad \{16,17\}$$

where $A=1-\dfrac{1}{2}\left(1+\dfrac{k}{k_g}\right)\dfrac{m_1}{m_2}$, and $B=-2\dfrac{m_1}{m_2}$. Then, we can calculate the two frequencies w_2, w_3 by using Rayleigh quotient as follows.

$$w_2=\sqrt{\frac{\mathbf{f}_2^T\mathbf{K}\mathbf{f}_2}{\mathbf{f}_2^T\mathbf{M}\mathbf{f}_2}},\; w_3=\sqrt{\frac{\mathbf{f}_3^T\mathbf{K}\mathbf{f}_3}{\mathbf{f}_3^T\mathbf{M}\mathbf{f}_3}} \qquad \{18,19\}$$

The force at the landing gear is given by: $F_g = k_g\, u_2$. Therefore, we need to determine the displacement time history of u_2, and calculate the moment when the displacement is maximized. For the response u_2 of the 2nd degree of freedom, we use modal superposition, namely:

$$u_2(t)=\sum_{j=1}^{3} f_{2j}q_j(t)=0\times q_1(t)+a\times q_2(t)+b\times q_3(t) \qquad \{20\}$$

For initial velocity of the system, $\dot{\boldsymbol{u}}_0^{\,T}=\{10 \quad 10 \quad 10\}$, we evaluate the modal components b_2 and b_3 as follows:

$$b_2=\frac{\boldsymbol{f}_2^T\boldsymbol{M}\,\dot{\boldsymbol{u}}_0}{\boldsymbol{f}_2^T\boldsymbol{M}\,\boldsymbol{f}_2} \qquad b_3=\frac{\boldsymbol{f}_3^T\boldsymbol{M}\,\dot{\boldsymbol{u}}_0}{\boldsymbol{f}_3^T\boldsymbol{M}\,\boldsymbol{f}_3} \qquad \{21,22\}$$

and the response of the system would be:

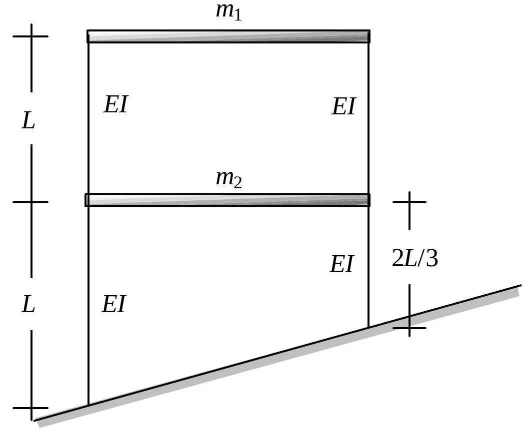
$$u_2(t)=a\frac{b_2}{w_2}\sin(w_2t)+b\frac{b_3}{w_3}\sin(w_3t) \qquad \{23\}$$

PROBLEM 2. (**P. 30** on page 378 in the note) Consider a 2-story frame with rigid girders (beams) and mass lumped at the level of each floor, which is supported on a sloping ground. It is known that

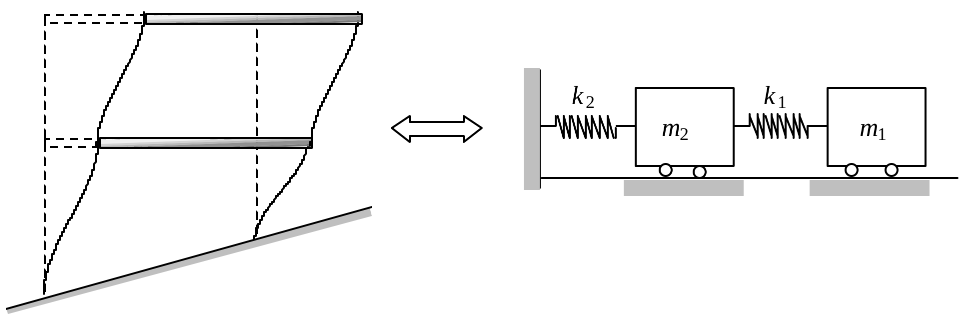
$m_1 = 5000$ kg

$m_2 / m_1 = 2,$

$$\frac{EI}{L^3} = 2500\ N/m$$



The deformed configuration and the equivalent 2-DOF model are



- Formulate the equations of motion for the system.
- Estimate the fundamental frequency, using Rayleigh's quotient.
- Find the exact modal shapes and frequencies
- What is the proportional damping matrix that gives uniform modal damping $x_1 = x_2 = 0.05$?

- This is a 2-degree of freedom system, with lateral displacements at the level of the floors, u_1 and u_2. The equation of motion for this system is formulated as follows:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} \dfrac{24EI}{L^3} & -\dfrac{24EI}{L^3} \\ -\dfrac{24EI}{L^3} & \dfrac{36\,EI}{L^3} + \dfrac{12\,EI}{(2L/3)^3} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$m_1 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{EI}{L^3} \begin{bmatrix} 24 & -24 \\ -24 & 76.5 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \{1\}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{EI}{m_1 L^3} \begin{bmatrix} 24 & -24 \\ -24 & 76.5 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- In order to estimate the fundamental frequency of the system, we use the Rayleigh quotient, estimating that the fundamental shape of the structure will be a straight line, i.e.:

$$v_0^{\,T} = \{2 \quad 1\}$$

$$w_1^2 \le \frac{v_0^{\,T}\,\boldsymbol{K}\,v_0}{v_0^{\,T}\,\boldsymbol{M}\,v_0} = \frac{\{2 \quad 1\}\,\dfrac{E\,I}{L^3}\begin{bmatrix}24 & -24\\ -24 & 76.5\end{bmatrix}\begin{Bmatrix}2\\1\end{Bmatrix}}{\{2 \quad 1\}\,m_1\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{Bmatrix}2\\1\end{Bmatrix}} = 0.5\,\frac{\{2 \quad 1\}\begin{bmatrix}24 & -24\\ -24 & 76.5\end{bmatrix}\begin{Bmatrix}2\\1\end{Bmatrix}}{\{2 \quad 1\}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{Bmatrix}2\\1\end{Bmatrix}} = 6.377\ [\,rad/\sec\,]^2$$

- The eigenvalue problem is formulated as follows:

$$\left(\boldsymbol{K} - w^2\,\boldsymbol{M}\right)\cdot \boldsymbol{f}_i = \boldsymbol{0} \qquad \{2\}$$

For the solution of {2} to be non-trivial, we demand:

$$\left|\boldsymbol{K} - w^2\,\boldsymbol{M}\right| = 0$$

$$\{3\}$$

$$\left|\frac{E\,I}{m_1\,L^3}\begin{bmatrix}24 & -24\\ -24 & 76.5\end{bmatrix} - w^2\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\right| = 0 \quad \Rightarrow \quad \left|\begin{bmatrix}24\dfrac{E\,I}{m_1\,L^3} - w^2 & -24\dfrac{E\,I}{m_1\,L^3}\\ -24\dfrac{E\,I}{m_1\,L^3} & 76.5\dfrac{E\,I}{m_1\,L^3} - 2w^2\end{bmatrix}\right| = 0$$

For $\dfrac{E\,I}{m_1\,L^3} = \dfrac{2500}{5000} = 0.5$, equation {3} is now evaluated as follows: $\left|\begin{bmatrix}12-l & -12\\ -12 & 38.25-2\,l\end{bmatrix}\right| = 0$ {4}
where $l = w^2$.

The solution of the quadratic equation in {4}, has the following solutions:

$$w_1^{\,2} = 6.36\ [\,rad/\sec\,]^2$$
$$w_2^{\,2} = 24.765\ [\,rad/\sec\,]^2$$

To define the modal shapes, we set f_{1n} = 1.00, and solve for f_{2n}, using the eigenvalues calculated above. We therefore have:

$$\begin{bmatrix}12-w_n^{\,2} & -12\\ -12 & 38.25-2\,w_n^{\,2}\end{bmatrix}\begin{Bmatrix}1.00\\ f_{2n}\end{Bmatrix} = 0 \qquad \{5\}$$

From the first equation in {5}, we evaluate f_{2n} as follows:

$$f_{2n} = \frac{12 - w_n^{\,2}}{12} = 1 - \frac{w_n^{\,2}}{12} \qquad \{6\}$$

Substituting in {6} n=1,2, we evaluate:

$f_{21} = 0.47\,m$, $\qquad f_{22} = -1.06\ m$

- The Rayleigh damping matrix for the 2-degree of freedom system, is formulated as follows: $\boldsymbol{C} = a_0\ \boldsymbol{M} + a_1\ \boldsymbol{K}$

We know from the class notes that the coefficients α_0 and α_1 are evaluated as follows:

$$\begin{Bmatrix} 2\ x_1\ w_1 \\ 2\ x_2\ w_2 \end{Bmatrix} = \begin{bmatrix} 1 & w_1^2 \\ 1 & w_2^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} \qquad \{7\}$$

Setting in equation {16} $x_1 = x_2 = 0.05$, we evaluate the coefficients as follows:

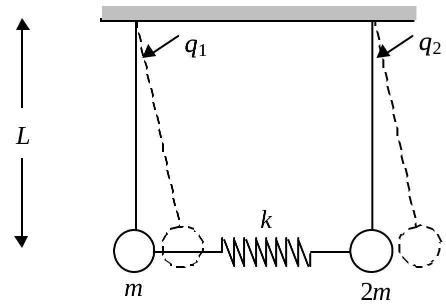
$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 6.36 \\ 1 & 24.765 \end{bmatrix}^{-1} \begin{Bmatrix} 2 \cdot 0.05 \cdot 6.36 \\ 2 \cdot 0.05 \cdot 24.765 \end{Bmatrix} = \begin{Bmatrix} 0.506 \\ 3.21310^{-3} \end{Bmatrix} \qquad \{8\}$$

Therefore, the damping matrix is now:

$$\boldsymbol{C} = a_0\ \boldsymbol{M} + a_1\ \boldsymbol{K} = 0.506\ m_1 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + 3.21310^{-3} \frac{E\,I}{L^3} \begin{bmatrix} 24 & -24 \\ -24 & 76.5 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 2.723 \cdot 10^3 & -192.78 \\ -192.78 & 5.764 \cdot 10^3 \end{bmatrix} \qquad \{9\}$$

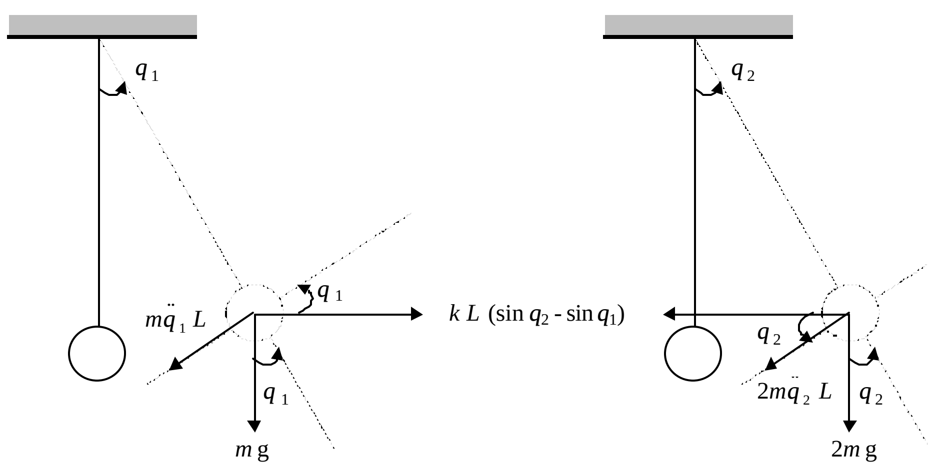
PROBLEM 3. (**P. 31** on page 379 in the note) Consider a set of coupled penduli, as shown in the figure below (gravity is “on”).



- Find the equations of motion.
- Specialize the result of (a) for small angles q_1 and q_2.
- Find the frequencies and model shapes of the system (in terms of $\sqrt{k/m}$.

SOLUTION

To determine the equations of motion for the system above, we first draw the free body diagram (centripidal forces are not shown):



We formulate the equations of motion, by taking moment equilibrium around the pivot of each pendulum:

$$m\ddot{q}_1 L^2 + mg\sin q_1 L - kL^2\left(\sin q_2 - \sin q_1\right)\cos q_1 = 0$$

$$2m\ddot{q}_2 L^2 + 2mg\sin q_2 L + kL^2\left(\sin q_2 - \sin q_1\right)\cos q_2 = 0 \qquad \{1\}$$

We linearize equation {1}, assuming small displacements[1] as follows:

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[1] Small displacement assumption: $\sin q \cong q$, $\cos q \cong 1$.

$$m\ddot{q}_1 L^2 + mg\, q_1 L - k\, L^2 (q_2 - q_1) = 0$$

$$2m\ddot{q}_2 L^2 + 2mg q_2 L + k\, L^2 (q_2 - q_1) = 0 \qquad \{2\}$$

Equation {2} is written in matrix form as follows:

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} k + \frac{mg}{L} & -k \\ -k & k + \frac{2mg}{L} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \{3\}$$

The eigenvalue problem is formulated as follows:

$$\left(\boldsymbol{K} - w^2 \boldsymbol{M} \right) \cdot \boldsymbol{f}_i = \boldsymbol{0} \qquad \{4\}$$

For the solution of {4} to be non-trivial, we demand:

$$\left| \boldsymbol{K} - w^2 \boldsymbol{M} \right| = 0$$

$$\left| \begin{bmatrix} k + \frac{mg}{L} & -k \\ -k & k + \frac{2mg}{L} \end{bmatrix} - w^2 \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \right| = 0 \qquad \{5\}$$

We define $s = k / m$, and equation {5} is transformed as follows:

$$\left| \begin{bmatrix} 1 + \frac{g}{s\,L} & -1 \\ -1 & 1 + \frac{2g}{s\,L} \end{bmatrix} - \frac{w^2}{s} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right| = 0$$

$$2\,w^4 - \left(3s + 4\frac{g}{L} \right) w^2 + 2\left(\frac{g}{L} \right)^2 + 3\,\frac{g}{L}\,s = 0 \qquad \{6\}$$

The solutions of equation {6} (only positive solutions have physical meaning) are:

$$w_1^{\ 2} = \frac{g}{L} = w_p^{\ 2}$$

$$w_2^{\ 2} = \frac{g}{L} + 1.5\frac{k}{m} = w_p^{\ 2} + 1.5\,\frac{k}{m} \qquad \{7\}$$

where w_p is the natural frequency of the simple pendulum.

To define the modal shapes, we set $f_{1n} = 1.00$, and solve for f_{2n}, using the eigenvalues calculated above. We therefore have:

$$\begin{bmatrix} 1+\dfrac{g}{s\,L}-\dfrac{w^2}{s} & -1 \\ -1 & 1+\dfrac{2g}{s\,L}-\dfrac{2w^2}{s} \end{bmatrix} \begin{Bmatrix} 1.00 \\ f_{2n} \end{Bmatrix} = 0 \qquad \{8\}$$

From the first equation in {8}, we evaluate $f_{2\mathrm{n}}$ as follows:

$$f_{2n} = 1+\frac{g}{s\,L}-\frac{w^2}{s} \qquad \{9\}$$

Substituting in {9} n=1,2, we evaluate:

$$f_{21} = 1 \qquad\qquad f_{22} = -0.5$$