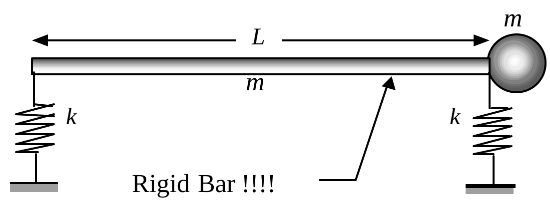
Solutions for Homework #9

PROBLEM 1. (**P. 32** on page 379 in the note) Consider a spring–mounted *rigid* bar of total mass m and length L, to which an additional mass m is lumped at the rightmost end. The system has *no* damping.

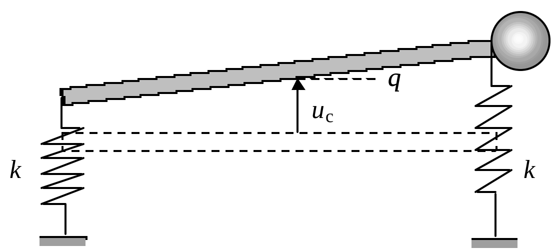
- Find the natural modes of vibration
- The left support is given an initial vertical displacement a and is then released. Find the response.
- Write down the impulse response functions for this system.

Careful: The rigid bar has rotational inertia (the mass of the bar is uniformly distributed).

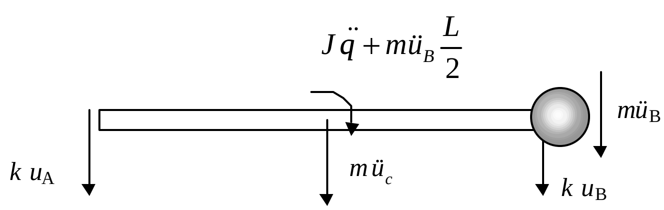


SOLUTION

The system has two degrees of freedom for the vertical translation u_C and rotation q. We first consider a general deformed shape for the system in terms of u_C and q, as shown below.



Then, we draw the free body diagramm and set up the equations of motion as follows.



where $J=mL^2/12$. Choosing as degrees of freedom the rotation of the rigid bar q and the translation of the center of mass, u_C, we have:

$$u_A = u_c - q\,\frac{L}{2} \qquad\qquad u_B = u_c + q\,\frac{L}{2} \qquad \{1\}$$

The equations of motion are formulated by defining equilibrium of vertical forces and moments around the center of mass, as follows:

$$\sum F_z = 0:\; ku_A + ku_B + m\ddot{u}_C + m\ddot{u}_B = 0$$

$$2m\ddot{u}_C + m\frac{L}{2}\ddot{q} + 2ku_C = 0 \qquad \{2\}$$

$$\sum M_C = 0:\; -ku_A\frac{L}{2} + ku_B\frac{L}{2} + J\ddot{q} + m\ddot{u}_B\frac{L}{2} = 0$$

$$\frac{mL}{2}\ddot{u}_C + \frac{mL^2}{3}\ddot{q} + \frac{kL^2}{2}q = 0 \qquad \{3\}$$

Express the above equations {2} and {3} in the following matrix form.

$$\begin{Bmatrix} 2m & \frac{mL}{2} \\ \frac{mL}{2} & \frac{mL^2}{3} \end{Bmatrix} \begin{Bmatrix} \ddot{u}_C \\ \ddot{q} \end{Bmatrix} + \begin{Bmatrix} 2k & 0 \\ 0 & \frac{kL^2}{2} \end{Bmatrix} \begin{Bmatrix} u_C \\ q \end{Bmatrix} = \begin{Bmatrix} P_z \\ M_q \end{Bmatrix} \qquad \{4\}$$

Manipulate equation {4} such that we shift the factor L from the second columns to the rotational component q, and then divide the second row by L to obtain the following form.

$$m\begin{Bmatrix} 2 & 1/2 \\ 1/2 & 1/3 \end{Bmatrix} \begin{Bmatrix} \ddot{u}_C \\ L\ddot{q} \end{Bmatrix} + k\begin{Bmatrix} 2 & 0 \\ 0 & 1/2 \end{Bmatrix} \begin{Bmatrix} u_C \\ Lq \end{Bmatrix} = \begin{Bmatrix} P_z \\ M_q / L \end{Bmatrix} \qquad \{5\}$$

or symbolically

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{P} \qquad \{6\}$$

Note that

$$\mathbf{u} = \begin{Bmatrix} u_C \\ Lq \end{Bmatrix}, \text{ and } \mathbf{P} = \begin{Bmatrix} P_z \\ M_q / L \end{Bmatrix} \qquad \{7,8\}$$

To find the natural modes of vibration, we establish the following eigenvalue problem from equation {5}.

$$\left[\begin{Bmatrix} 12 & 0 \\ 0 & 3 \end{Bmatrix} - l\begin{Bmatrix} 12 & 3 \\ 3 & 2 \end{Bmatrix}\right] \begin{Bmatrix} f_z \\ f_q \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \qquad \text{with } l = \frac{mw^2}{k} \qquad \{9\}$$

Then, the solutions for equation {9} are given as $l_1 = 0.735$, and $l_2 = 3.264$. Then, it follows that

$$w_1 = \sqrt{\frac{k}{m}l_1} \text{ and } w_2 = \sqrt{\frac{k}{m}l_2}$$

$$\mathbf{f}_1 = \begin{Bmatrix} 1 \\ \frac{4-4l_1}{l_1} \end{Bmatrix}, \text{ and } \mathbf{f}_2 = \begin{Bmatrix} 1 \\ \frac{4-4l_2}{l_2} \end{Bmatrix} \qquad \{10,11,12,13\}$$

Next, consider the response for the case that the left support at B is given an initial vertical displacement a and is then released. In connection with equation {7}, this initial condition can be expressed in the form.

$$\mathbf{u}_0 = \begin{Bmatrix} u_{C0} \\ Lq_0 \end{Bmatrix} = \begin{Bmatrix} a/2 \\ -a \end{Bmatrix}, \qquad \text{with } q_0 = -a/L \qquad \{14\}$$

Also, in terms of mode shapes, $\mathbf{u}_0$ is written in the form.

$$\mathbf{u}_0 = q_{01}\mathbf{f}_1 + q_{02}\mathbf{f}_2 \qquad \{15\}$$

From the orthogonality condition, we obtain

$$q_{01} = \frac{\mathbf{f}_1^T\mathbf{M}\mathbf{u}_0}{\mathbf{f}_1^T\mathbf{M}\mathbf{f}_1}, \text{ and } q_{02} = \frac{\mathbf{f}_2^T\mathbf{M}\mathbf{u}_0}{\mathbf{f}_2^T\mathbf{M}\mathbf{f}_2} \qquad \{16,17\}$$

Finally, we can express

$$\mathbf{u} = \sum_{j=1}^{2} q_j \mathbf{f}_j \qquad \{18\}$$

where

$$q_j(t) = q_{0j}\cos w_j t, \text{ and } \mathbf{f}_j = \begin{Bmatrix} 1 \\ \frac{4-4l_j}{l_j} \end{Bmatrix} \qquad \{19,20\}$$

However, remember that $\mathbf{u} = \{u_C \quad Lq\}^T$. Therefore, more precisely, we can express

$$\begin{Bmatrix} u_C \\ q \end{Bmatrix} = \sum_{j=1}^{2} q_{0j} \cos w_j t \begin{Bmatrix} 1 \\ \dfrac{4-4l_j}{l_j L} \end{Bmatrix} \qquad \{21\}$$

The impulse response functions for this system are expressed in the form.

$$\mathbf{u} = \sum_{j=1}^{2} q_j g_j \mathbf{f}_j \qquad \{22\}$$

where

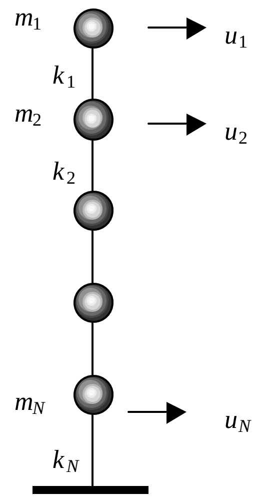
$$q_j(t) = \frac{1}{w_j} \sin w_j t \qquad \{23\}$$

$$g_j = \begin{cases} 1 & , \quad \text{for } P_z = 1 \text{ and } M_q = 0 \\ \dfrac{4-4l_j}{l_j L^2} & , \quad \text{for } P_z = 0 \text{ and } M_q = 1 \end{cases} \qquad \{24\}$$

Since $\mathbf{u} = \{u_C \quad Lq\}^T$, we can express

$$\begin{Bmatrix} u_C \\ q \end{Bmatrix} = \sum_{j=1}^{2} \frac{\sin w_j t}{w_j} g_j \begin{Bmatrix} 1 \\ \dfrac{4-4l_j}{l_j L} \end{Bmatrix} \qquad \{25\}$$

<u>PROBLEM 2.</u> (**<u>P. 33</u>** on page 380 in the note) Consider a lumped mass structure which can be modeled as a shear beam (= a chain of *close-coupled* lateral springs and masses, i.e. the simple system that we often use to model high-rise buildings). The distance between floors (i.e. inter-story height) is constant. The masses are numbered from the top down, and while arbitrary, you should consider them as being *known*. The stiffness', however, are *not* known. However, from a vibration test, you have determined experimentally the fundamental frequency w_1 as well as the fundamental mode $\mathbf{f}_1$, and you have found that this mode is a straight line, i.e. $\mathbf{f}_1^T = \{N \quad N\text{-}1 \quad N\text{-}2 \quad \ldots \quad 3 \quad 2 \quad 1\}$

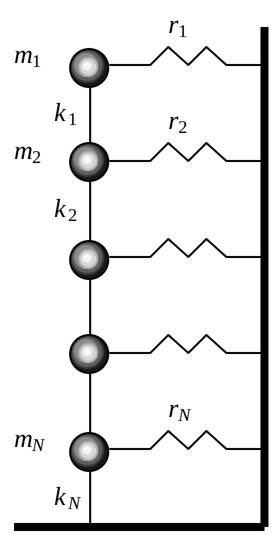


a) Determine the values of the springs in terms of the fundamental frequency and masses.

b) The overturning moment at the base is the moment exerted by the inertia forces, which is

$$M = \sum_{i=1}^{N} m_i \ddot{u}_i z_i$$

in which z_i is the height of the i^{th} mass above the ground. Show that because the first mode is a straight line, *only that mode* contributes to the overturning moment (notice, by the way, that the foundation must be able to resist this moment).



c) Suppose that there is a wall that runs parallel to the structure, and that there are springs of rigidity $r_i = a\, m_i$ connecting the masses and the wall, in which a is a *known* constant (i.e, the springs are proportional to the masses). If you knew *all* the frequencies and mode shapes of the original system *without* the added springs, could you find the frequencies and mode shapes for the structure with the added springs?

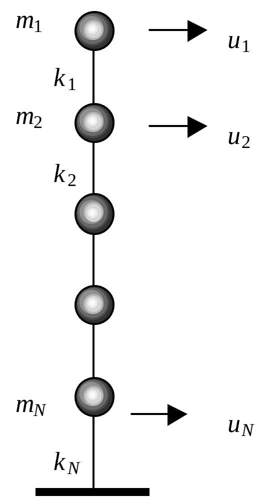
d) If N=2 (i.e. two floors only), find the second frequency and mode shape, again in terms of the known parameters (for the original system without added lateral springs!). Assume that both masses are equal.

e) Determine the proportional damping matrix for the 2-dof system in d) that would produce the same value of damping x in both modes. From this matrix, figure out the values and physical configuration of the dashpots.

f) Again for the 2-dof system considered in d), assume that it is subjected to a unit impulsive load acting on the bottom floor. Find the response in each floor, assuming no damping.

SOLUTION

- We first formulate the mass and stiffness matrix of the structure:



$$\boldsymbol{M} = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \dots & \\ & & & m_N \end{bmatrix} \qquad \boldsymbol{K} = \begin{bmatrix} k_1 & -k_1 & & \\ -k_1 & k_1 + k_2 & -k_2 & \\ & -k_2 & \dots & -k_{N-1} \\ & & -k_{N-1} & k_{N-1} + k_N \end{bmatrix}$$

The fundamental eigenvalue and modal shape are an eigenpair, satisfying the eigenvalue problem, namely:

$$\boldsymbol{K}\ \boldsymbol{f}_1 = w_1^2\ \boldsymbol{M}\ \boldsymbol{f}_1 \qquad \{1\}$$

We substitute the mass and stiffness matrix in equation {1} and perform the algebraic operations in the left and right hand side of {1}.

$$w_1^2\ \boldsymbol{M}\ \boldsymbol{f}_1 = w_1^2 \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \dots & \\ & & & m_N \end{bmatrix} \begin{Bmatrix} N \\ N-1 \\ \dots \\ 1 \end{Bmatrix} = w_1^2 \begin{Bmatrix} m_1 N \\ m_2 (N-1) \\ \dots \\ m_N \end{Bmatrix} \qquad \{2\}$$

$$\boldsymbol{K}\ \boldsymbol{f}_1 = \begin{bmatrix} k_1 & -k_1 & & \\ -k_1 & k_1 + k_2 & -k_2 & \\ & -k_2 & \dots & -k_{N-1} \\ & & -k_{N-1} & k_{N-1} + k_N \end{bmatrix} \begin{Bmatrix} N \\ N-1 \\ \dots \\ 1 \end{Bmatrix} = \begin{Bmatrix} k_1 \\ k_2 - k_1 \\ \dots \\ k_N - k_{N-1} \end{Bmatrix} \qquad \{3\}$$

Substituting {2} and {3} in equation {1}, we get the following system of N equations with N unknown stiffness coefficients k_1 , k_2 ... k_N:

$$\begin{Bmatrix} k_1 \\ k_2 - k_1 \\ \dots \\ k_N - k_{N-1} \end{Bmatrix} = w_1^2 \begin{Bmatrix} m_1 N \\ m_2 (N-1) \\ \dots \\ m_N \end{Bmatrix} \Rightarrow \begin{cases} k_1 = w_1^2\ m_1 N \\ k_2 = w_1^2 \left[m_1 N + m_2 (N-1) \right] \\ \dots \\ k_N = w_1^2 \left[m_1 N + m_2 (N-1) + \dots + m_N \right] \end{cases}$$

$$k_i = w_1^2 \left[m_1 N + m_2 (N-1) + \dots + m_i \left(N - (i-1) \right) \right] \qquad \{4\}$$

- The overturning moment at the base is the moment exerted by the inertia forces is $M = \sum_{i=1}^{N} m_i \ddot{u}_i z_i$.

We want to prove that $\sum_{i=1}^{N} m_i \ddot{u}_i z_i = 0 \ \ \forall i \neq 1$.

We know that the response of the i th degree of freedom for the j th mode is given by: $u_i(t) = f_{ij}\ q_j(t)$.

Therefore the acceleration of the i th degree of freedom for the j th mode is given by: $\ddot{u}_i(t) = f_{ij}\ \ddot{q}_j(t)$.

So the overturning moment at the base, for the jth mode is evaluated as follows:

$$M_j = \ddot{q}_j \sum_{i=1}^{N} m_i f_{ij} z_i \qquad \{5\}$$

The inter-story height, *h*, i.e. the distance between floors, is constant. Therefore, the position of the masses can be written in vector form as follows:

$$\boldsymbol{z}^T = \{N h \quad (N-1)h \quad \ldots \quad h\} = h\ \{N \quad N-1 \quad \ldots \quad 1\} = h\ \boldsymbol{f}_1 \qquad \{6\}$$

We now write equation {5} in matrix form as follows:

$$M_j = \ddot{q}_j\, \boldsymbol{f}_j^T\, \boldsymbol{M}\, \boldsymbol{z} = \ddot{q}_j\, h\, \boldsymbol{f}_j^T\, \boldsymbol{M}\, \boldsymbol{f}_1 = 0 \quad \forall\ j \neq 1 \qquad \{7\}$$

due to the orthogonality of the modal shapes with respect to the mass matrix.

Therefore, from modal superposition, the total moment at the base is evaluated as follows:

$$M = \sum_{j=1}^{N} M_j = \sum_{j=1}^{N} \ddot{q}_j\, h\, \boldsymbol{f}_j^T\, \boldsymbol{M}\, \boldsymbol{f}_1 = \ddot{q}_1\, h\, \boldsymbol{f}_1^T\, \boldsymbol{M}\, \boldsymbol{f}_1 = \ddot{q}_1\, h\, m_1 \qquad \{8\}$$

- Suppose that there is a wall that runs parallel to the structure, and that there are springs of rigidity $r_i = \boldsymbol{a}\, m_i$ connecting the masses and the wall, in which $\boldsymbol{a}$ is a *known* constant (i.e, the springs are proportional to the masses).

All the frequencies and mode shapes of the original system *without* the added springs are known.

The presence of the lateral springs, results in the following stiffness matrix for the structure:

$$\boldsymbol{K}^* = \begin{bmatrix} k_1 & -k_1 & & \\ -k_1 & k_1 + k_2 & -k_2 & \\ & -k_2 & \ldots & -k_{N-1} \\ & & -k_{N-1} & k_{N-1} + k_N \end{bmatrix} + a \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ldots & \\ & & & m_N \end{bmatrix} = \boldsymbol{K} + a\ \boldsymbol{M} \qquad \{9\}$$

The new eigenvalue problem is formulated as follows (assume that the modal shapes are identical to the previous structure and successively verify):

$$\begin{aligned} \boldsymbol{K}^* \boldsymbol{f}_i &= \left(w_i^*\right)^2\ \boldsymbol{M}\ \boldsymbol{f}_i \\ \left(\boldsymbol{K} + a\ \boldsymbol{M}\right) \boldsymbol{f}_i &= \left(w_i^*\right)^2\ \boldsymbol{M}\ \boldsymbol{f}_i \\ \boldsymbol{K}\, \boldsymbol{f}_i &= \left(\left(w_i^*\right)^2 - a\right)\ \boldsymbol{M}\ \boldsymbol{f}_i = w_i^2\ \boldsymbol{M}\ \boldsymbol{f}_i \end{aligned} \qquad \{10\}$$

Therefore, the new natural frequencies of the system are:

$$\left(w_i^*\right)^2 = w_i^2 + \boldsymbol{a} \qquad \{11\}$$

and the modal shapes of the modified structure remain the same.

- For the 2 degree of freedom system, the mass and stiffness matrices are:

$$\boldsymbol{K} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} \qquad \boldsymbol{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

From equation {4}, we calculate the stiffness coefficients, as a function of the masses and the fundamental frequency, as follows (N = 2):

$$k_1 = 2\ m\ \omega_1^2$$
$$k_2 = \omega_1^2\,[2\ m + m\,] = 3\ m\ \omega_1^2$$

Therefore, the stiffness matrix is now: $\boldsymbol{K} = m\ \omega_1^2 \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$ {12}

From the orthogonality properties of the modal shapes wrt. the mass matrix, we know that:

$$\boldsymbol{\phi}_1^{\,T} \boldsymbol{M}\ \boldsymbol{\phi}_2 = 0 \ \Rightarrow\ \begin{Bmatrix} 2 & 1 \end{Bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ \phi_{22} \end{Bmatrix} = 0 \Rightarrow \phi_{22} = -2 \qquad \{13\}$$

We also verify the orthogonality properties of the second mode wrt. the stiffness matrix:

$$\boldsymbol{\phi}_1^{\,T} \boldsymbol{K}\ \boldsymbol{\phi}_2 = 0 \ \Rightarrow\ \begin{Bmatrix} 2 & 1 \end{Bmatrix}\ m\ \omega_1^2 \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} = 0 \qquad \{14\}$$

The second frequency of the system is calculated using the Rayleigh quotient, which will give the exact frequency, when the second modal shape is used:

$$R = \omega_2^2 = \frac{\boldsymbol{\phi}_2^{\,T} \boldsymbol{K}\ \boldsymbol{\phi}_2}{\boldsymbol{\phi}_2^{\,T} \boldsymbol{M}\ \boldsymbol{\phi}_2} = \frac{\begin{Bmatrix} 1 & -2 \end{Bmatrix}\ m\ \omega_1^2 \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}}{\begin{Bmatrix} 1 & -2 \end{Bmatrix}\ m\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}} = \frac{30\ \omega_1^2}{5} = 6\ \omega_1^2 \qquad \{15\}$$

- The Rayleigh damping matrix for the 2 dof system, is formulated as follows: $\boldsymbol{C} = \alpha_0\ \boldsymbol{M} + \alpha_1\ \boldsymbol{K}$

We know from the class notes that the coefficients α_0 and α_1 are evaluated as follows:

$$\begin{Bmatrix} 2\ \xi_1\ \omega_1 \\ 2\ \xi_2\ \omega_2 \end{Bmatrix} = \begin{bmatrix} 1 & \omega_1^2 \\ 1 & \omega_2^2 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \end{Bmatrix} \qquad \{16\}$$

Setting in equation {16} $\xi_1 = \xi_2 = \xi$, $\omega_2^2 = 6\ \omega_1^2$, we evaluate the coefficients as follows:

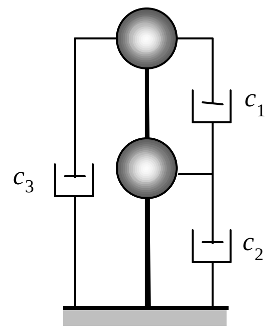
$$\begin{Bmatrix} \alpha_0 \\ \alpha_1 \end{Bmatrix} = \begin{bmatrix} 1 & \omega_1^2 \\ 1 & 6\,\omega_1^2 \end{bmatrix}^{-1} \begin{Bmatrix} 2\ \xi\ \omega_1 \\ 2\ \xi\ \sqrt{6}\,\omega_1 \end{Bmatrix} = \frac{2\ \xi}{5} \begin{Bmatrix} \left(6-\sqrt{6}\right)\omega_1 \\ \dfrac{\sqrt{6}-1}{\omega_1} \end{Bmatrix} \qquad \{17\}$$

Therefore, the damping matrix is now:

$$
\begin{aligned}
\boldsymbol{C} = a_0\, \boldsymbol{M} + a_1\, \boldsymbol{K} &= \frac{2x}{5}\left\{\left(6-\sqrt{6}\right)w_1 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} + \frac{\sqrt{6}-1}{w_1}\, m\, w_1^2 \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\right\} = \\
&= \frac{2x\, m\, w_1}{5}\left\{\left(6-\sqrt{6}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\sqrt{6}-1\right)\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\right\} = x\, m\, w_1 \begin{bmatrix} 2.58 & -1.16 \\ -1.16 & 4.32 \end{bmatrix}
\end{aligned}
\qquad \{18\}
$$

The damping matrix can be written as follows: $\boldsymbol{C} = x\, m\, w_1 \begin{bmatrix} 2.58 & -1.16 \\ -1.16 & 4.32 \end{bmatrix} = \begin{bmatrix} c_1 + c_3 & -c_1 \\ -c_1 & c_1 + c_2 \end{bmatrix}$

Due to the way the Rayleigh damping matrix was formulated (mass and stiffness proportional), the physical configuration of the dashpots is shown in the figure below:



- The force vector for a unit impulsive load acting on the bottom floor, is the following:

$$
\boldsymbol{p}^T = \left\{0 \quad d\left(t\right)\right\} \text{ where: } d\left(t\right) \text{ is the Dirac Delta function} \qquad \{19\}
$$

The modal properties of the structure, namely modal masses and forces are calculated below (damping is ignored):

$$
\begin{aligned}
m_1 &= \{2 \quad 1\}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = 5\, m \qquad & m_2 &= \{1 \quad -2\}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} 1 \\ -2 \end{Bmatrix} = 5\, m \\
p_1 &= \{2 \quad 1\}\begin{Bmatrix} 0 \\ d\left(t\right) \end{Bmatrix} = d\left(t\right) \qquad & p_2 &= \{1 \quad -2\}\begin{Bmatrix} 0 \\ d\left(t\right) \end{Bmatrix} = -2\, d\left(t\right)
\end{aligned}
\qquad \{20\}
$$

Therefore, the structure response is evaluated using modal superposition, as follows:

$$
\boldsymbol{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \boldsymbol{f}_1\, h_1 + \boldsymbol{f}_2\, h_2 = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}\frac{1}{5\, m\, w_1}\sin\left(w_1 t\right) + \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}\frac{-2}{5\, m\sqrt{6}\, w_1}\sin\left(\sqrt{6}\, w_1 t\right)
$$

where h_1 and h_2 are the impulse response functions corresponding to the 1st and 2nd mode.