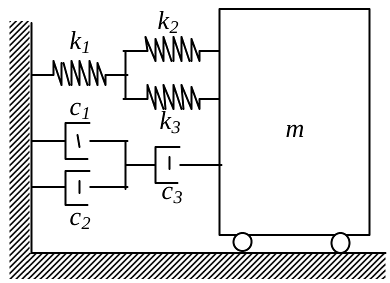
Solutions for Quiz#2 (Fall, 2001)

1. Find the equations of motion for the following systems.

a)



SOLUTION

We first find the equivalent stiffness and damping coefficient k_{eq} and c_{eq}.

$$\frac{1}{k_{\text{eq}}}=\frac{1}{k_1}+\frac{1}{k_2+k_3}, \qquad \therefore k_{\text{eq}}=\frac{k_1(k_2+k_3)}{k_1+k_2+k_3} \tag{1.1}$$

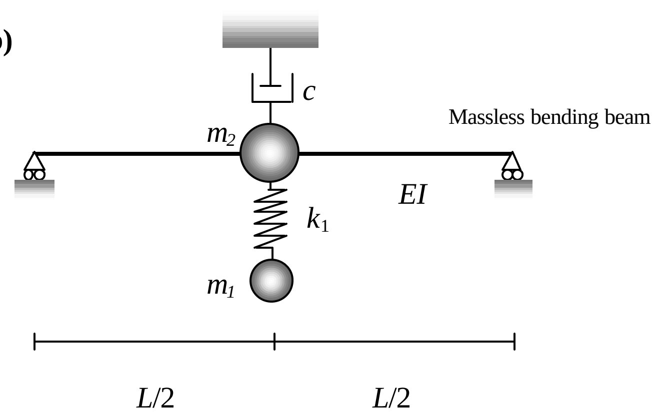
$$\frac{1}{c_{\text{eq}}}=\frac{1}{c_3}+\frac{1}{c_1+c_2}, \qquad \therefore c_{\text{eq}}=\frac{c_3(c_1+c_2)}{c_1+c_2+c_3} \tag{1.2}$$

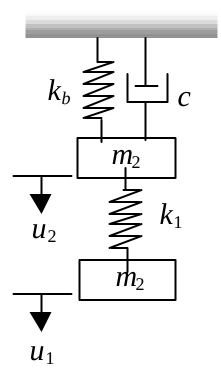
Then, we can simply construct the equation of motions in the following form.

$$m\ddot{u}+c_{\text{eq}}\dot{u}+k_{\text{eq}}u=0 \tag{1.3}$$

where u is the degree of freedom of the sytem above, and k_{eq} and c_{eq} are defined in equations (1.1 and 2).

b)





SOLUTION

To begin with, we model the given system as the right figure above, where k_b is the stiffness of the massless bending beam of the original system, which is given as

$$k_b=\frac{48EI}{L^3} \tag{1.4}$$

Finally, we can express the equation of motion in the following matrix form.

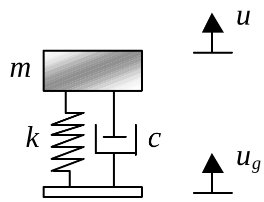
$$\begin{Bmatrix} m_1 & 0 \\ 0 & m_2 \end{Bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}+\begin{Bmatrix} 0 & 0 \\ 0 & c \end{Bmatrix}\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}+\begin{Bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_b \end{Bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1.5}$$

where k_b is given in equation (1.4).

2. A buoy that floats in water has a natural frequency w_n in vertical direction. This buoy oscillates up and down in response to harmonic water waves with frequency w. If the fraction of damping, which we can assume to be viscous, is x, then what is the transfer function for the *dynamic force* between the water and the buoy? (Note that *dynamic force* = total force – weight). Sketch the amplitude and phase of this transfer function.

SOLUTION

We can model this problem as the SDOF system subjected to a support motion of $u_g=u_0e^{iwt}$ as shown below.



The equation of motion is given as

$$m\ddot{u}+c\dot{v}+kv=0 \tag{2.1}$$

where $v=u-u_g$ is the relative displacement.

The transfer function H_{dyn} to be calculated, for the *dynamic force* between the water and the buoy, is given as

$$H_{dyn}=c\dot{v}+kv \tag{2.2}$$

Then, from equation (2.1), it follows that

$$H_{dyn}=c\dot{v}+kv=-m\ddot{u} \tag{2.3}$$

Also, we already know the transfer function $H_{u/ug}$ from class, which is of the form.

$$H_{u/u_g}=\frac{1+2ixr}{1-r^2+2ixr} \tag{2.4}$$

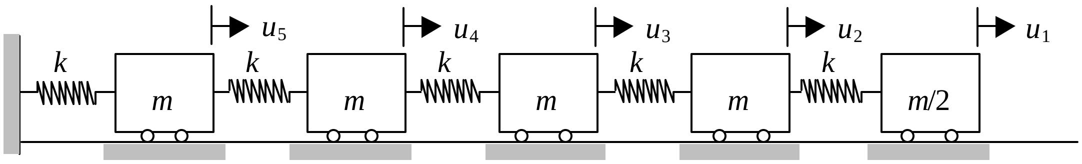
where $r=w/w_r$. Then, we can easily obtain $H_{\ddot{u}/ug}$ as follows.

$$H_{\ddot{u}/u_g}=\frac{d^2H_{u/u_g}}{dt^2}=(-w^2)\frac{1+2ixr}{1-r^2+2ixr} \tag{2.5}$$

Finally, as combining equations (2.3 and 5), we obtain the transfer function H_{dyn} as

$$H_{dyn}=-mH_{\ddot{u}/u_g}=-mw^2\frac{1+2ixr}{1-r^2+2ixr}=-mw_n^2\frac{w^2}{w_n^2}\frac{1+2ixr}{1-r^2+2ixr}=-kr^2\frac{1+2ixr}{1-r^2+2ixr} \tag{2.6}$$

3. Consider a 5-d.o.f. system, as shown below. Estimate the fundamental frequency by means of Rayleigh's quotient. Also, compare it with exact result obtained with the formula given in class.



SOLUTION

We construct first the stiffness and mass matrices **K** and **M** as follows.

$$\mathbf{K} = k\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \text{ and } \mathbf{M} = m\begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(3.1 and 2)}$$

We choose an arbitrary vector $\mathbf{v}_0$ to estimate the fundamental frequency such as

$$\mathbf{v}_0 = \{5 \quad 4 \quad 3 \quad 2 \quad 1\}^T \qquad (3.3)$$

Apply the Rayleigh quotient to obtain an approximation for the fundamental frequency as below.

$$R = \frac{\mathbf{v}_0^T \mathbf{K} \mathbf{v}_0}{\mathbf{v}_0^T \mathbf{M} \mathbf{v}_0} = 0.1176 \frac{k}{m} \qquad (3.4)$$

where we insert equations (3.1 to 3) for **K**, **M**, and $\mathbf{v}_0$. It follows that the estimated fundamental frequency is given as

$$w_R = \sqrt{R} = 0.343\sqrt{\frac{k}{m}} \qquad (3.5)$$

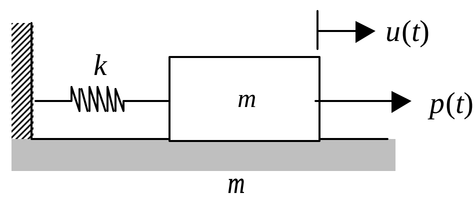
On the other hand, we have the exact solution w_1 as

$$w_1 = 2\sin(p/20)\sqrt{\frac{k}{m}} = 0.313\sqrt{\frac{k}{m}} \qquad (3.6)$$

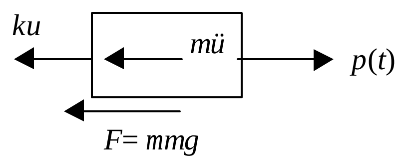
As seen in equations (3.5 and 6), the estimated fundamental frequency w_R by the Rayleigh quotient, as using the given vector $\mathbf{v}_0$ in equation (3.3), provides quite good agreement with the exact solution.

4. A 1-d.o.f. system slides on a rough surface, as shown in figure (a). The frictional force is $F=mmg$, where m is the coefficient of friction, and mg is the weight. This force on the oscillator always opposes the motion, that is, $N = -mmg\,\mathrm{sgn}(\dot{u})$. Figure (b) displays the free body diagram when the velocity is positive. The oscillator is driven by an external force $p(t)$ such that the system executes cyclic motions with amplitude u_m, as shown in the hysteresis diagram below, see Figure (c). Notice that when the motion reverses directions, the total change in forces N is $2F$.

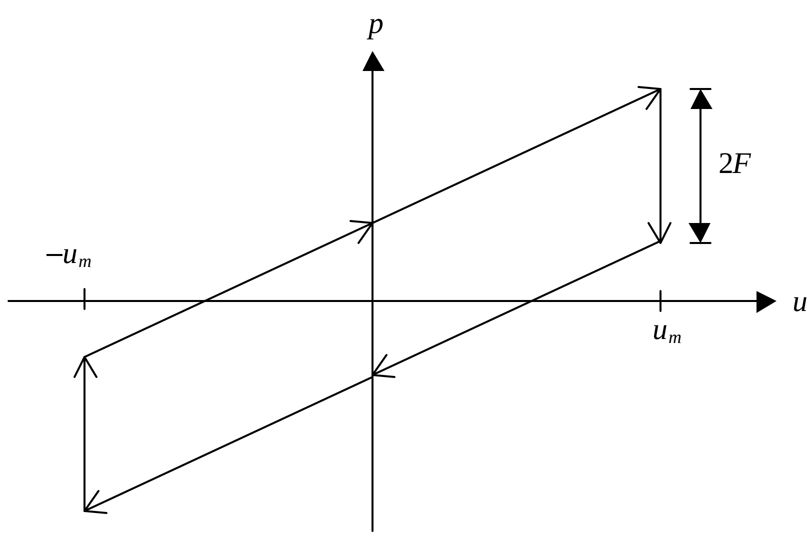
a) What is the average power dissipated?
b) What is the fraction of hysteretic damping?
c) How does this damping change with u_m?



(a) Oscillator sliding on a rough surface, subjected to $p(t)$



(b) Free body diagram for $\dot{u} > 0$



(c) Hysteresis diagram

SOLUTION

The dissipated energy, Π_{diss}, is just equal to the area of the hysteresis loop. So,

$$\Pi_{\text{diss}} = 2F \times 2u_m = 4Fu_m = 4mmgu_m \tag{4.1}$$

Then, the average power dissipated is

$$\langle \Pi_{\text{diss}} \rangle = \frac{4mmgu_m}{T} \tag{4.2}$$

where T is a period.

The fraction of hysteresis damping, x_{eq}, is given as

$$x_{\text{eq}} = \frac{1}{4ð}\frac{\Pi_{\text{diss}}}{\Pi_{\text{stored}}} = \frac{1}{4ð}\frac{\dfrac{4mmgu_m}{T}}{\dfrac{1}{2}ku_m^2} = \frac{2}{ð}\frac{mmg}{Tku_m} \tag{4.3}$$

As shown in equation (4.3), the damping x_{eq} is inversely proportional to u_m.