CHAPTER 1

Combinatorial Analysis

1.1 INTRODUCTION

Here is a typical problem of interest involving probability. A communication system is to consist of n seemingly identical antennas that are to be lined up in a linear order. The resulting system will then be able to receive all incoming signals—and will be called *functional*—as long as no two consecutive antennas are defective. If it turns out that exactly m of the n antennas are defective, what is the probability that the resulting system will be functional? For instance, in the special case where $n = 4$ and $m = 2$ there are 6 possible system configurations—namely,

$$\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{array}$$

where 1 means that the antenna is working and 0 that it is defective. As the resulting system will be functional in the first 3 arrangements and not functional in the remaining 3, it seems reasonable to take $\frac{3}{6} = \frac{1}{2}$ as the desired probability. In the case of general n and m, we could compute the probability that the system is functional in a similar fashion. That is, we could count the number of configurations that result in the system being functional and then divide by the total number of all possible configurations.

From the above we see that it would be useful to have an effective method for counting the number of ways that things can occur. In fact, many problems in probability theory can be solved simply by counting the number of different ways that a certain event can occur. The mathematical theory of counting is formally known as *combinatorial analysis*.

1.2 THE BASIC PRINCIPLE OF COUNTING

The following principle of counting will be basic to all our work. Loosely put, it states that if one experiment can result in any of m possible outcomes and if another experiment can result in any of n possible outcomes, then there are mn possible outcomes of the two experiments.

The basic principle of counting

Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if for each outcome of experiment 1 there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.

Proof of the Basic Principle: The basic principle may be proved by enumerating all the possible outcomes of the two experiments as follows:

$$\begin{array}{c}(1, 1), (1, 2), \ldots, (1, n)\\ (2, 1), (2, 2), \ldots, (2, n)\\ \vdots \\ (m, 1), (m, 2), \ldots, (m, n)\end{array}$$

where we say that the outcome is (i, j) if experiment 1 results in its ith possible outcome and experiment 2 then results in the jth of its possible outcomes. Hence the set of possible outcomes consists of m rows, each row containing n elements, which proves the result.

Example 2a. A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

Solution By regarding the choice of the woman as the outcome of the first experiment and the subsequent choice of one of her children as the outcome of the second experiment, we see from the basic principle that there are $10 \times 3 = 30$ possible choices. ∎

When there are more than two experiments to be performed, the basic principle can be generalized as follows.

The generalized basic principle of counting

If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes, and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment, and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment, and if . . . , then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

Example 2b. A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?

Solution We may regard the choice of a subcommittee as the combined outcome of the four separate experiments of choosing a single representative from each of the classes. Hence it follows from the generalized version of the basic principle that there are $3 \times 4 \times 5 \times 2 = 120$ possible subcommittees. ∎

Example 2c. How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

Solution By the generalized version of the basic principle the answer is $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175{,}760{,}000$. ∎

Example 2d. How many functions defined on n points are possible if each functional value is either 0 or 1?

Solution Let the points be $1, 2, \ldots, n$. Since $f(i)$ must be either 0 or 1 for each $i = 1, 2, \ldots, n$, it follows that there are 2^n possible functions. ∎

Example 2e. In Example 2c, how many license plates would be possible if repetition among letters or numbers were prohibited?

Solution In this case there would be $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 78{,}624{,}000$ possible license plates. ∎

1.3 PERMUTATIONS

How many different ordered arrangements of the letters a, b, and c are possible? By direct enumeration we see that there are 6: namely, *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*. Each arrangement is known as a *permutation*. Thus there are 6 possible permutations of a set of 3 objects. This result could also have been obtained from the basic principle, since the first object in the permutation can be any of the 3, the second object in the permutation can then be chosen from any of the remaining

2, and the third object in the permutation is then chosen from the remaining 1. Thus there are $3 \cdot 2 \cdot 1 = 6$ possible permutations.

Suppose now that we have n objects. Reasoning similar to that we have just used for the 3 letters shows that there are

$$n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1 = n!$$

different permutations of the n objects.

Example 3a. How many different batting orders are possible for a baseball team consisting of 9 players?

Solution There are $9! = 362{,}880$ possible batting orders. ∎

Example 3b. A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.

(a) How many different rankings are possible?

(b) If the men are ranked just among themselves and the women among themselves, how many different rankings are possible?

Solution (a) As each ranking corresponds to a particular ordered arrangement of the 10 people, we see that the answer to this part is $10! = 3{,}628{,}800$.

(b) As there are $6!$ possible rankings of the men among themselves and $4!$ possible rankings of the women among themselves, it follows from the basic principle that there are $(6!)(4!) = (720)(24) = 17{,}280$ possible rankings in this case. ∎

Example 3c. Mr. Jones has 10 books that he is going to put on his bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Jones wants to arrange his books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

Solution There are $4!\,3!\,2!\,1!$ arrangements such that the mathematics books are first in line, then the chemistry books, then the history books, and then the language book. Similarly, for each possible ordering of the subjects, there are $4!\,3!\,2!\,1!$ possible arrangements. Hence, as there are $4!$ possible orderings of the subjects, the desired answer is $4!\,4!\,3!\,2!\,1! = 6912$. ∎

We shall now determine the number of permutations of a set of n objects when certain of the objects are indistinguishable from each other. To set this straight in our minds, consider the following example.

Example 3d. How many different letter arrangements can be formed using the letters $P\,E\,P\,P\,E\,R$?

Solution We first note that there are $6!$ permutations of the letters $P_1\,E_1\,P_2\,P_3\,E_2\,R$ when the 3 P's and the 2 E's are distinguished from each other. However, consider any one of these permutations—for instance, $P_1\,P_2\,E_1\,P_3\,E_2\,R$. If we now permute the P's among themselves and the E's among themselves, then the resultant arrangement would still be of the form $P\,P\,E\,P\,E\,R$. That is, all 3! 2! permutations

$$\begin{array}{ll} P_1\,P_2\,E_1\,P_3\,E_2\,R & P_1\,P_2\,E_2\,P_3\,E_1\,R \\ P_1\,P_3\,E_1\,P_2\,E_2\,R & P_1\,P_3\,E_2\,P_2\,E_1\,R \\ P_2\,P_1\,E_1\,P_3\,E_2\,R & P_2\,P_1\,E_2\,P_3\,E_1\,R \\ P_2\,P_3\,E_1\,P_1\,E_2\,R & P_2\,P_3\,E_2\,P_1\,E_1\,R \\ P_3\,P_1\,E_1\,P_2\,E_2\,R & P_3\,P_1\,E_2\,P_2\,E_1\,R \\ P_3\,P_2\,E_1\,P_1\,E_2\,R & P_3\,P_2\,E_2\,P_1\,E_1\,R \end{array}$$

are of the form $P\,P\,E\,P\,E\,R$. Hence there are $6!/3!\,2! = 60$ possible letter arrangements of the letters $P\,E\,P\,P\,E\,R$. ∎

In general, the same reasoning as that used in Example 3d shows that there are

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, . . . , n_r are alike.

Example 3e. A chess tournament has 10 competitors of which 4 are Russian, 3 are from the United States, 2 from Great Britain, and 1 from Brazil. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?

Solution There are

$$\frac{10!}{4!\,3!\,2!\,1!} = 12{,}600$$

possible outcomes. ∎

Example 3f. How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same color are identical?

Solution There are

$$\frac{9!}{4!\,3!\,2!} = 1260$$

different signals. ∎

1.4 COMBINATIONS

We are often interested in determining the number of different groups of r objects that could be formed from a total of n objects. For instance, how many different groups of 3 could be selected from the 5 items A, B, C, D, and E? To answer this, reason as follows: Since there are 5 ways to select the initial item, 4 ways to then select the next item, and 3 ways to select the final item, there are thus $5 \cdot 4 \cdot 3$ ways of selecting the group of 3 when the order in which the items are selected is relevant. However, since every group of 3, say, the group consisting of items A, B, and C, will be counted 6 times (that is, all of the permutations ABC, ACB, BAC, BCA, CAB, and CBA will be counted when the order of selection is relevant), it follows that the total number of groups that can be formed is

$$\frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$$

In general, as $n(n-1)\cdots(n-r+1)$ represents the number of different ways that a group of r items could be selected from n items when the order of selection is relevant, and as each group of r items will be counted $r!$ times in this count, it follows that the number of different groups of r items that could be formed from a set of n items is

$$\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!\,r!}$$

Notation and terminology

We define $\binom{n}{r}$, for $r \leq n$, by

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

and say that $\binom{n}{r}$ represents the number of possible combinations of n objects taken r at a time.†

Thus $\binom{n}{r}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of selection is not considered relevant.

Example 4a. A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Solution There are $\binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140$ possible committees. ∎

† By convention, 0! is defined to be 1. Thus $\binom{n}{0} = \binom{n}{n} = 1$. We also take $\binom{n}{i}$ to be equal to 0 when either $i < 0$ or $i > n$.

Example 4b. From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed? What if 2 of the men are feuding and refuse to serve on the committee together?

Solution As there are $\binom{5}{2}$ possible groups of 2 women, and $\binom{7}{3}$ possible groups of 3 men, it follows from the basic principle that there are $\binom{5}{2}\binom{7}{3} = \left(\frac{5 \cdot 4}{2 \cdot 1}\right)\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 350$ possible committees consisting of 2 women and 3 men.

On the other hand, if 2 of the men refuse to serve on the committee together, then, as there are $\binom{2}{0}\binom{5}{3}$ possible groups of 3 men not containing either of the 2 feuding men and $\binom{2}{1}\binom{5}{2}$ groups of 3 men containing exactly 1 of the feuding men, it follows that there are $\binom{2}{0}\binom{5}{3} + \binom{2}{1}\binom{5}{2} = 30$ groups of 3 men not containing both of the feuding men. Since there are $\binom{5}{2}$ ways to choose the 2 women, it follows that in this case there are $30\binom{5}{2} = 300$ possible committees. ∎

Example 4c. Consider a set of n antennas of which m are defective and $n - m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

Solution Imagine that the $n - m$ functional antennas are lined up among themselves. Now, if no two defectives are to be consecutive, then the spaces between the functional antennas must each contain at most one defective antenna. That is, in the $n - m + 1$ possible positions—represented in Figure 1.1 by carets—between the $n - m$ functional antennas, we must select m of these in which to put the defective antennas. Hence there are

$$\wedge\, 1 \,\wedge\, 1 \,\wedge\, 1 \ldots \wedge\, 1 \,\wedge\, 1 \,\wedge$$

1 = functional

$\wedge$ = place for at most one defective

Figure 1.1

$\binom{n-m+1}{m}$ possible orderings in which there is at least one functional antenna between any two defective ones. ∎

A useful combinatorial identity is

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \qquad 1 \leq r \leq n \tag{4.1}$$

Equation (4.1) may be proved analytically or by the following combinatorial argument. Consider a group of n objects and fix attention on some particular one of these objects—call it object 1. Now, there are $\binom{n-1}{r-1}$ groups of size r that contain object 1 (since each such group is formed by selecting $r-1$ from the remaining $n-1$ objects). Also, there are $\binom{n-1}{r}$ groups of size r that do not contain object 1. As there is a total of $\binom{n}{r}$ groups of size r, Equation (4.1) follows.

The values $\binom{n}{r}$ are often referred to as *binomial coefficients*. This is so because of their prominence in the binomial theorem.

The binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{4.2}$$

We shall present two proofs of the binomial theorem. The first is a proof by mathematical induction, and the second is a proof based on combinatorial considerations.

Proof of the Binomial Theorem by Induction: When $n = 1$, Equation (4.2) reduces to

$$x + y = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 = y + x$$

Assume Equation (4.2) for $n - 1$. Now,

$$\begin{aligned}(x+y)^n &= (x+y)(x+y)^{n-1} \\ &= (x+y) \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k}\end{aligned}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k}$$

Letting $i = k + 1$ in the first sum and $i = k$ in the second sum, we find that

$$\begin{aligned}
(x + y)^n &= \sum_{i=1}^{n} \binom{n-1}{i-1} x^i y^{n-i} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\
&= x^n + \sum_{i=1}^{n-1} \left[\binom{n-1}{i-1} + \binom{n-1}{i} \right] x^i y^{n-i} + y^n \\
&= x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i y^{n-i} + y^n \\
&= \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}
\end{aligned}$$

where the next-to-last equality follows by Equation (4.1). By induction the theorem is now proved.

Combinatorial Proof of the Binomial Theorem: Consider the product

$$(x_1 + y_1)(x_2 + y_2) \cdots (x_n + y_n)$$

Its expansion consists of the sum of 2^n terms, each term being the product of n factors. Furthermore, each of the 2^n terms in the sum will contain as a factor either x_i or y_i for each $i = 1, 2, \ldots, n$. For example,

$$(x_1 + y_1)(x_2 + y_2) = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2$$

Now, how many of the 2^n terms in the sum will have as factors k of the x_i's and $(n - k)$ of the y_i's? As each term consisting of k of the x_i's and $(n - k)$ of the y_i's corresponds to a choice of a group of k from the n values $x_1, x_2, \ldots, x_n$, there are $\binom{n}{k}$ such terms. Thus, letting $x_i = x$, $y_i = y$, $i = 1, \ldots, n$, we see that

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example 4d. Expand $(x + y)^3$.

Solution

$$\begin{aligned}
(x + y)^3 &= \binom{3}{0} x^0 y^3 + \binom{3}{1} x^1 y^2 + \binom{3}{2} x^2 y + \binom{3}{3} x^3 y^0 \\
&= y^3 + 3xy^2 + 3x^2y + x^3
\end{aligned}$$

∎

Example 4e. How many subsets are there of a set consisting of n elements?

Solution Since there are $\binom{n}{k}$ subsets of size k, the desired answer is

$$\sum_{k=0}^{n} \binom{n}{k} = (1 + 1)^n = 2^n$$

This result could also have been obtained by assigning to each element in the set either the number 0 or the number 1. To each assignment of numbers there corresponds, in a one-to-one fashion, a subset, namely, that subset consisting of all elements that were assigned the value 1. As there are 2^n possible assignments, the result follows.

Note that we have included as a subset the set consisting of 0 elements (that is, the null set). Hence the number of subsets that contain at least one element is $2^n - 1$. ∎

1.5 MULTINOMIAL COEFFICIENTS

In this section we consider the following problem: A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$. How many different divisions are possible? To answer this, we note that there are $\binom{n}{n_1}$ possible choices for the first group; for each choice of the first group there are $\binom{n - n_1}{n_2}$ possible choices for the second group; for each choice of the first two groups there are $\binom{n - n_1 - n_2}{n_3}$ possible choices for the third group; and so on. Hence it follows from the generalized version of the basic counting principle that there are

$$\binom{n}{n_1}\binom{n - n_1}{n_2} \cdots \binom{n - n_1 - n_2 - \cdots - n_{r-1}}{n_r}$$

$$= \frac{n!}{(n - n_1)!\, n_1!} \frac{(n - n_1)!}{(n - n_1 - n_2)!\, n_2!} \cdots \frac{(n - n_1 - n_2 - \cdots - n_{r-1})!}{0!\, n_r!}$$

$$= \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

possible divisions.

Notation

If $n_1 + n_2 + \cdots + n_r = n$, we define $\binom{n}{n_1, n_2, \ldots, n_r}$ by

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Thus $\binom{n}{n_1, n_2, \ldots, n_r}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

Example 5a. A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible?

Solution There are $\dfrac{10!}{5!\,2!\,3!} = 2520$ possible divisions. ∎

Example 5b. Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?

Solution There are $\dfrac{10!}{5!\,5!} = 252$ possible divisions. ∎

Example 5c. In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

Solution Note that this example is different from Example 5b because now the order of the two teams is irrelevant. That is, there is no A and B team but just a division consisting of 2 groups of 5 each. Hence the desired answer is

$$\frac{10!/5!\,5!}{2!} = 126$$ ∎

The proof of the following theorem, which generalizes the binomial theorem, is left as an exercise.

The multinomial theorem

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1, \ldots, n_r): \\ n_1 + \cdots + n_r = n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

That is, the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$.

The numbers $\binom{n}{n_1, n_2, \ldots, n_r}$ are known as *multinomial coefficients.*

Example 5d

$$\begin{aligned}(x_1 + x_2 + x_3)^2 &= \binom{2}{2, 0, 0} x_1^2 x_2^0 x_3^0 + \binom{2}{0, 2, 0} x_1^0 x_2^2 x_3^0 \\ &\quad + \binom{2}{0, 0, 2} x_1^0 x_2^0 x_3^2 + \binom{2}{1, 1, 0} x_1^1 x_2^1 x_3^0 \\ &\quad + \binom{2}{1, 0, 1} x_1^1 x_2^0 x_3^1 + \binom{2}{0, 1, 1} x_1^0 x_2^1 x_3^1 \\ &= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3\end{aligned}$$

*1.6 ON THE DISTRIBUTION OF BALLS IN URNS

There are r^n possible outcomes when n distinguishable balls are to be distributed into r distinguishable urns. This follows because each ball may be distributed into any of r possible urns. Let us now, however, suppose that the n balls are indistinguishable from each other. In this case, how many different outcomes are possible? As the balls are indistinguishable, it follows that the outcome of the experiment of distributing the n balls into r urns can be described by a vector $(x_1, x_2, \ldots x_r)$, where x_i denotes the number of balls that are distributed into the ith urn. Hence the problem reduces to finding the number of distinct nonnegative integer-valued vectors $(x_1, x_2, \ldots, x_r)$ such that

$$x_1 + x_2 + \cdots + x_r = n$$

To compute this, let us start by considering the number of positive integer-valued solutions. Toward this end, imagine that we have n indistinguishable objects lined

* Note that asterisks denote material that is optional.

up and that we want to divide them into r nonempty groups. To do so, we can select $r - 1$ of the $n - 1$ spaces between adjacent objects as our dividing points (see Figure 1.2). For instance, if we have $n = 8$ and $r = 3$ and choose the 2 divisors as shown

$$\text{ooo}|\text{ooo}|\text{oo}$$

then the vector obtained is $x_1 = 3$, $x_2 = 3$, $x_3 = 2$. As there are $\binom{n-1}{r-1}$ possible selections, we obtain the following proposition.

Proposition 6.1

There are $\binom{n-1}{r-1}$ distinct positive integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying

$$x_1 + x_2 + \cdots + x_r = n \qquad x_i > 0,\ i = 1, \ldots, r$$

To obtain the number of nonnegative (as opposed to positive) solutions, note that the number of nonnegative solutions of $x_1 + x_2 + \cdots + x_r = n$ is the same as the number of positive solutions of $y_1 + \cdots + y_r = n + r$ (seen by letting $y_i = x_i + 1$, $i = 1, \ldots, r$). Hence, from Proposition 6.1, we obtain the following proposition.

Proposition 6.2

There are $\binom{n+r-1}{r-1}$ distinct nonnegative integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying

$$x_1 + x_2 + \cdots + x_r = n \tag{6.1}$$

$0 \wedge 0 \wedge 0 \wedge \ldots \wedge 0 \wedge 0$

n objects 0

Choose $r - 1$ of the spaces $\wedge$.

Figure 1.2

Example 6a. How many distinct nonnegative integer-valued solutions of $x_1 + x_2 = 3$ are possible?

Solution There are $\binom{3 + 2 - 1}{2 - 1} = 4$ such solutions: (0, 3), (1, 2), (2, 1), (3, 0). ∎

Example 6b. An investor has 20 thousand dollars to invest among 4 possible investments. Each investment must be in units of a thousand dollars. If the total 20 thousand is to be invested, how many different investment strategies are possible? What if not all the money need be invested?

Solution If we let x_i, $i = 1, 2, 3, 4$, denote the number of thousands invested in investment number i, then, when all is to be invested, x_1, x_2, x_3, x_4 are integers satisfying

$$x_1 + x_2 + x_3 + x_4 = 20 \qquad x_i \geq 0$$

Hence, by Proposition 6.2, there are $\binom{23}{3} = 1771$ possible investment strategies. If not all of the money need be invested, then if we let x_5 denote the amount kept in reserve, a strategy is a nonnegative integer-valued vector $(x_1, x_2, x_3, x_4, x_5)$ satisfying

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

Hence, by Proposition 6.2, there are now $\binom{24}{4} = 10{,}626$ possible strategies. ∎

Example 6c. How many terms are there in the multinomial expansion of $(x_1 + x_2 + \cdots + x_r)^n$?

Solution

$$(x_1 + x_2 + \cdots + x_r)^n = \sum \binom{n}{n_1, \ldots, n_r} x_1^{n_1} \cdots x_r^{n_r}$$

where the sum is over all nonnegative integer-valued $(n_1, \ldots, n_r)$ such that $n_1 + \cdots + n_r = n$. Hence, by Proposition 6.2, there are $\binom{n + r - 1}{r - 1}$ such terms. ∎

Example 6d. Let us reconsider Example 4c, in which we have a set of n items, of which m are (indistinguishable and) defective and the remaining $n - m$ are (also indistinguishable and) functional. Our objective is to determine the number of linear orderings in which no two defectives are next to each other. To determine this quantity, let us imagine that the defective items are lined up among themselves and the functional ones are now to be put in position. Let us denote x_1 as the number of functional items to the left of the first defective, x_2 as the number of functional items between the first two defectives, and so on. That is, schematically we have

$$x_1\ 0\ x_2\ 0 \cdots x_m\ 0\ x_{m+1}$$

Now there will be at least one functional item between any pair of defectives as long as $x_i > 0$, $i = 2, \ldots, m$. Hence the number of outcomes satisfying the condition is the number of vectors $x_1, \cdots, x_{m+1}$ that satisfy

$$x_1 + \cdots + x_{m+1} = n - m \qquad x_1 \geq 0, x_{m+1} \geq 0, x_i > 0, i = 2, \ldots, m$$

But on letting $y_1 = x_1 + 1$, $y_i = x_i$, $i = 2, \ldots, m$, $y_{m+1} = x_{m+1} + 1$, we see that this is equal to the number of positive vectors $(y_1, \ldots, y_{m+1})$ that satisfy

$$y_1 + y_2 + \cdots + y_{m+1} = n - m + 2$$

Hence, by Proposition 6.1, there are $\binom{n - m + 1}{m}$ such outcomes, which is in agreement with the results of Example 4c.

Suppose now that we are interested in the number of outcomes in which each pair of defective items is separated by at least 2 functional ones. By the same reasoning as that applied above, this would equal the number of vectors satisfying

$$x_1 + \cdots + x_{m+1} = n - m \qquad x_1 \geq 0, x_{m+1} \geq 0, x_i \geq 2, i = 2, \ldots, m$$

Upon letting $y_1 = x_1 + 1$, $y_i = x_i - 1$, $i = 2, \ldots, m$, $y_{m+1} = x_{m+1} + 1$, we see that this is the same as the number of positive solutions of

$$y_1 + \cdots + y_{m+1} = n - 2m + 3$$

Hence, from Proposition 6.1, there are $\binom{n - 2m + 2}{m}$ such outcomes.

SUMMARY

The basic principle of counting states that if an experiment consisting of two phases is such that there are n possible outcomes of phase 1, and for each of these n outcomes there are m possible outcomes of phase 2, there are nm possible outcomes of the experiment.

There are $n! = n(n - 1) \cdots 3 \cdot 2 \cdot 1$ possible linear orderings of n items. The quantity $0!$ is defined to equal 1.

Let

$$\binom{n}{i} = \frac{n!}{(n - i)!\, i!}$$

when $0 \le i \le n$, and let it equal 0 otherwise. This quantity represents the number of different subgroups of size i that can be chosen from a set of size n. It is often called a *binomial coefficient* because of its prominence in the binomial theorem, which states that

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

For nonnegative integers $n_1, \ldots, n_r$ summing to n,

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

is the number of ways of dividing up n items into r distinct nonoverlapping subgroups of sizes $n_1, n_2, \ldots, n_r$.

PROBLEMS

1. (a) How many different 7-place license plates are possible if the first 2 places are for letters and the other 5 for numbers?

(b) Repeat part (a) under the assumption that no letter or number can be repeated in a single license plate.

2. How many outcome sequences are possible when a die is rolled four times, where we say, for instance, that the outcome is 3, 4, 3, 1 if the first roll landed on 3, the second on 4, the third on 3, and the fourth on 1?

3. Twenty workers are to be assigned to 20 different jobs, one to each job. How many different assignments are possible?

4. John, Jim, Jay, and Jack have formed a band consisting of 4 instruments. If each of the boys can play all 4 instruments, how many different arrangements are possible? What if John and Jim can play all 4 instruments, but Jay and Jack can each play only piano and drums?

5. For years, telephone area codes in the United States and Canada consisted of a sequence of three digits. The first digit was an integer between 2 and 9; the second digit was either 0 or 1; the third digit was any integer between 1 and 9. How many area codes were possible? How many area codes starting with a 4 were possible?

6. A well-known nursery rhyme starts as follows:

As I was going to St. Ives
I met a man with 7 wives.
Each wife had 7 sacks.
Each sack had 7 cats.
Each cat had 7 kittens.

How many kittens did the traveler meet?

7. **(a)** In how many ways can 3 boys and 3 girls sit in a row?
(b) In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are each to sit together?
(c) In how many ways if only the boys must sit together?
(d) In how many ways if no two people of the same sex are allowed to sit together?

8. How many different letter arrangements can be made from the letters
(a) FLUKE;
(b) PROPOSE;
(c) MISSISSIPPI;
(d) ARRANGE?

9. A child has 12 blocks, of which 6 are black, 4 are red, 1 is white, and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?

10. In how many ways can 8 people be seated in a row if
(a) there are no restrictions on the seating arrangement;
(b) persons A and B must sit next to each other;
(c) there are 4 men and 4 women and no 2 men or 2 women can sit next to each other;
(d) there are 5 men and they must sit next to each other;
(e) there are 4 married couples and each couple must sit together?

11. In how many ways can 3 novels, 2 mathematics books, and 1 chemistry book be arranged on a bookshelf if
(a) the books can be arranged in any order;
(b) the mathematics books must be together and the novels must be together;
(c) the novels must be together but the other books can be arranged in any order?

12. Five separate awards (best scholarship, best leadership qualities, and so on) are to be presented to selected students from a class of 30. How many different outcomes are possible if
(a) a student can receive any number of awards;
(b) each student can receive at most 1 award?

13. Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?

14. How many 5-card poker hands are there?

15. A dance class consists of 22 students, 10 women and 12 men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

16. A student has to sell 2 books from a collection of 6 math, 7 science, and 4 economics books. How many choices are possible if
(a) both books are to be on the same subject;
(b) the books are to be on different subjects?

17. A total of 7 different gifts are to be distributed among 10 children. How many distinct results are possible if no child is to receive more than one gift?

18. A committee of 7, consisting of 2 Republicans, 2 Democrats, and 3 Independents, is to be chosen from a group of 5 Republicans, 6 Democrats, and 4 Independents. How many committees are possible?

19. From a group of 8 women and 6 men a committee consisting of 3 men and 3 women is to be formed. How many different committees are possible if

(a) 2 of the men refuse to serve together;

(b) 2 of the women refuse to serve together;

(c) 1 man and 1 woman refuse to serve together?

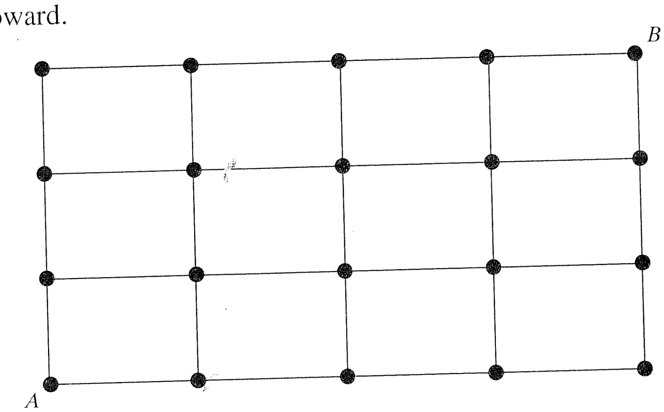
20. A person has 8 friends, of whom 5 will be invited to a party.

(a) How many choices are there if 2 of the friends are feuding and will not attend together?

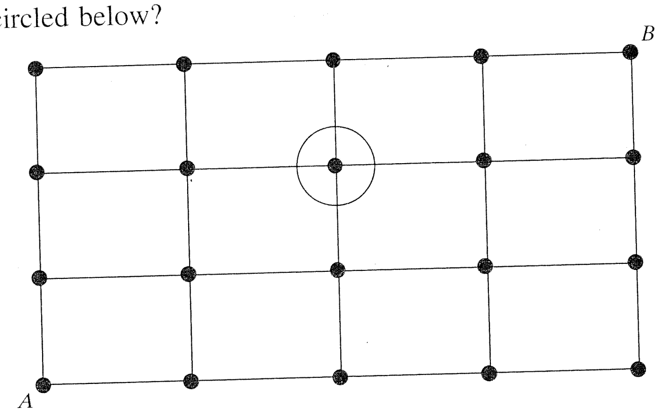
(b) How many choices if 2 of the friends will only attend together?

21. Consider the grid of points shown below. Suppose that starting at the point labeled *A* you can go one step up or one step to the right at each move. This is continued until the point labeled *B* is reached. How many different paths from *A* to *B* are possible?

HINT: Note that to reach *B* from *A* you must take 4 steps to the right and 3 steps upward.



22. In Problem 21, how many different paths are there from *A* to *B* that go through the point circled below?



23. A psychology laboratory conducting dream research contains 3 rooms, with 2 beds in each room. If 3 sets of identical twins are to be assigned to these 6 beds so that each set of twins sleeps in different beds in the same room, how many assignments are possible?

24. Expand $(3x^2 + y)^5$.
25. The game of bridge is played by 4 players, each of whom is dealt 13 cards. How many bridge deals are possible?
26. Expand $(x_1 + 2x_2 + 3x_3)^4$.
27. If 12 people are to be divided into 3 committees of respective sizes 3, 4, and 5, how many divisions are possible?
28. If 8 new teachers are to be divided among 4 schools, how many divisions are possible? What if each school must receive 2 teachers?
29. Ten weight lifters are competing in a team weight-lifting contest. Of the lifters, 3 are from the United States, 4 are from Russia, 2 are from China, and 1 is from Canada. If the scoring takes account of the countries that the lifters represent but not their individual identities, how many different outcomes are possible from the point of view of scores? How many different outcomes correspond to results in which the United States has 1 competitor in the top three and 2 in the bottom three?
30. Delegates from 10 countries, including Russia, France, England, and the United States, are to be seated in a row. How many different seating arrangements are possible if the French and English delegates are to be seated next to each other, and the Russian and U.S. delegates are not to be next to each other?

*31. If 8 identical blackboards are to be divided among 4 schools, how many divisions are possible? How many, if each school must receive at least 1 blackboard?

*32. An elevator starts at the basement with 8 people (not including the elevator operator) and discharges them all by the time it reaches the top floor, number 6. In how many ways could the operator have perceived the people leaving the elevator if all people look alike to him? What if the 8 people consisted of 5 men and 3 women and the operator could tell a man from a woman?

*33. We have 20 thousand dollars that must be invested among 4 possible opportunities. Each investment must be integral in units of 1 thousand dollars, and there are minimal investments that need to be made if one is to invest in these opportunities. The minimal investments are 2, 2, 3, and 4 thousand dollars. How many different investment strategies are available if
   (a) an investment must be made in each opportunity;
   (b) investments must be made in at least 3 of the 4 opportunities?

THEORETICAL EXERCISES

1. Prove the generalized version of the basic counting principle.
2. Two experiments are to be performed. The first can result in any one of m possible outcomes. If the first experiment results in outcome number i, then the second experiment can result in any of n_i possible outcomes, $i = 1, 2, \ldots, m$. What is the number of possible outcomes of the two experiments?

3. In how many ways can r objects be selected from a set of n if the order of selection is considered relevant?

4. There are $\binom{n}{r}$ different linear arrangements of n balls of which r are black and $n - r$ are white. Give a combinatorial explanation of this fact.

5. Determine the number of vectors $(x_1, \ldots, x_n)$, such that each x_i is either 0 or 1 and

$$\sum_{i=1}^{n} x_i \geq k$$

6. How many vectors $x_1, \ldots, x_k$ are there for which each x_i is a positive integer such that $1 \leq x_i \leq n$ and $x_1 < x_2 < \cdots < x_k$?

7. Give an analytic proof of Equation (4.1).

8. Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$$

HINT: Consider a group of n men and m women. How many groups of size r are possible?

9. Use Theoretical Exercise 8 to prove that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

10. From a group of n people, suppose that we want to choose a committee of k, $k \leq n$, one of whom is to be designated as chairperson.

(a) By focusing first on the choice of the committee and then on the choice of the chair, argue that there are $\binom{n}{k}k$ possible choices.

(b) By focusing first on the choice of the nonchair committee members and then on the choice of the chair, argue that there are $\binom{n}{k-1}(n - k + 1)$ possible choices.

(c) By focusing first on the choice of the chair and then on the choice of the other committee members, argue that there are $n\binom{n-1}{k-1}$ possible choices.

(d) Conclude from parts (a), (b), and (c) that

$$k\binom{n}{k} = (n - k + 1)\binom{n}{k-1} = n\binom{n-1}{k-1}$$

(e) Use the factorial definition of $\binom{m}{r}$ to verify the identity in part (d).

11. The following identity is known as Fermat's combinatorial identity.

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1} \qquad n \geq k$$

Give a combinatorial argument (no computations are needed) to establish this identity.

HINT: Consider the set of numbers 1 through n. How many subsets of size k have i as their highest-numbered member?

12. Consider the following combinatorial identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$$

(a) Present a combinatorial argument for the above by considering a set of n people and determining, in two ways, the number of possible selections of a committee of any size and a chairperson for the committee.

HINT: (i) How many possible selections are there of a committee of size k and its chairperson?
(ii) How many possible selections are there of a chairperson and the other committee members?

(b) Verify the following identity for $n = 1, 2, 3, 4, 5$:

$$\sum_{k=1}^{n} \binom{n}{k} k^2 = 2^{n-2}n(n+1)$$

For a combinatorial proof of the above, consider a set of n people, and argue that both sides of the identity above represent the number of different selections of a committee, its chairperson, and its secretary (possibly the same as the chairperson).

HINT: (i) How many different selections result in the committee containing exactly k people?
(ii) How many different selections are there in which the chairperson and the secretary are the same?
(ANSWER: $n2^{n-1}$.)
(iii) How many different selections result in the chairperson and the secretary being different?

(c) Now argue that

$$\sum_{k=1}^{n} \binom{n}{k} k^3 = 2^{n-3}n^2(n+3)$$

13. Show that for $n > 0$,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

HINT: Use the binomial theorem.

14. From a set of n people a committee of size j is to be chosen, and from this committee a subcommittee of size i, $i \le j$, is also to be chosen.

(a) Derive a combinatorial identity by computing, in two ways, the number of possible choices of the committee and subcommittee—first by supposing that the committee is chosen first and then the subcommittee, and second by supposing that the subcommittee is chosen first and then the remaining members of the committee are chosen.

(b) Use part (a) to prove the following combinatorial identity:

$$\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} = \binom{n}{i} 2^{n-i} \qquad i \le n$$

(c) Use part (a) and Theoretical Exercise 13 to show that

$$\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} (-1)^{n-j} = 0 \qquad i \le n$$

15. Let $H_k(n)$ be the number of vectors $x_1, \ldots, x_k$ for which each x_i is a positive integer satisfying $1 \le x_i \le n$ and $x_1 \le x_2 \le \cdots \le x_k$.

(a) Without any computations, argue that

$$H_1(n) = n$$

$$H_k(n) = \sum_{j=1}^{n} H_{k-1}(j) \qquad k > 1$$

HINT: How many vectors are there in which $x_k = j$?

(b) Use the preceding recursion to compute $H_3(5)$.

HINT: First compute $H_2(n)$ for $n = 1, 2, 3, 4, 5$.

16. Consider a tournament of n contestants in which the outcome is an ordering of these contestants, with ties allowed. That is, the outcome partitions the players into groups, with the first group consisting of the players that tied for first place, the next group being those that tied for the next best position, and so on. Let $N(n)$ denote the number of different possible outcomes. For instance, $N(2) = 3$ since in a tournament with 2 contestants, player 1 could be uniquely first, player 2 could be uniquely first, or they could tie for first.

(a) List all the possible outcomes when $n = 3$.

(b) With $N(0)$ defined to equal 1, argue, without any computations, that

$$N(n) = \sum_{i=1}^{n} \binom{n}{i} N(n-i)$$

HINT: How many outcomes are there in which i players tie for last place?

(c) Show that the formula of part (b) is equivalent to the following:

$$N(n) = \sum_{i=0}^{n-1} \binom{n}{i} N(i)$$

(d) Use the recursion to find $N(3)$ and $N(4)$.

17. Present a combinatorial explanation of why $\binom{n}{r} = \binom{n}{r, n-r}$.

18. Argue that

$$\binom{n}{n_1, n_2, \ldots, n_r} = \binom{n-1}{n_1 - 1, n_2, \ldots, n_r} + \binom{n-1}{n_1, n_2 - 1, \ldots, n_r} + \cdots + \binom{n-1}{n_1, n_2, \ldots, n_r - 1}$$

HINT: Use an argument similar to the one used to establish Equation (4.1).

19. Prove the multinomial theorem.

***20.** In how many ways can n identical balls be distributed into r urns so that the ith urn contains at least m_i balls, for each $i = 1, \ldots, r$? Assume that $n \geq \sum_{i=1}^{r} m_i$.

***21.** Argue that there are exactly $\binom{r}{k}\binom{n-1}{n-r+k}$ solutions of

$$x_1 + x_2 + \cdots + x_r = n$$

for which exactly k of the x_i are equal to 0.

***22.** Consider a function $f(x_1, \ldots, x_n)$ of n variables. How many different partial derivatives of order r does it possess?

***23.** Determine the number of vectors $(x_1, \ldots, x_n)$, such that each x_i is a nonnegative integer and

$$\sum_{i=1}^{n} x_i \leq k$$

SELF-TEST PROBLEMS AND EXERCISES

1. How many different linear arrangements are there of the letters A, B, C, D, E, F for which
(a) A and B are next to each other;
(b) A is before B;
(c) A is before B and B is before C;
(d) A is before B and C is before D;
(e) A and B are next to each other and C and D are also next to each other;
(f) E is not last in line?

2. If 4 Americans, 3 Frenchmen, and 3 Englishmen are to be seated in a row, how many seating arrangements are possible when people of the same nationality must sit next to each other?

3. A president, treasurer, and secretary, all different, are to be chosen from a club consisting of 10 people. How many different choices of officers are possible if
(a) there are no restrictions;
(b) A and B will not serve together;
(c) C and D will serve together or not at all;
(d) E must be an officer;
(e) F will serve only if he is president?

4. A student is to answer 7 out of 10 questions in an examination. How many choices has she? How many if she must answer at least 3 of the first 5 questions?

5. In how many ways can a man divide 7 gifts among his 3 children if the eldest is to receive 3 gifts and the others 2 each?

6. How many different 7-place license plates are possible when 3 of the entries are letters and 4 are digits? Assume that repetition of letters and numbers is allowed and that there is no restriction on where the letters or numbers can be placed.

7. Give a combinatorial explanation of the identity

$$\binom{n}{r} = \binom{n}{n-r}$$

8. Consider n-digit numbers where each digit is one of the 10 integers 0, 1, . . . , 9. How many such numbers are there for which
(a) no two consecutive digits are equal;
(b) 0 appears as a digit a total of i times, $i = 0, \ldots, n$?

9. Consider three classes, each consisting of n students. From this group of $3n$ students, a group of 3 students is to be chosen.
(a) How many choices are possible?
(b) How many choices are there in which all 3 students are in the same class?
(c) How many choices are there in which 2 of the 3 students are in the same class and the other student is in a different class?
(d) How many choices are there in which all 3 students are in different classes?
(e) Using the results of parts (a) through (d), write a combinatorial identity.

***10.** An art collection on auction consisted of 4 Dalis, 5 van Goghs, and 6 Picassos. At the auction were 5 art collectors. If a reporter noted only the number of Dalis, van Goghs, and Picassos acquired by each collector, how many different results could have been recorded if all works were sold?

***11.** Determine the number of vectors $(x_1, \ldots, x_n)$ such that each x_i is a positive integer and

$$\sum_{i=1}^{n} x_i \leq k$$

where $k \geq n$.

APPENDIX A

Answers to Selected Problems

CHAPTER 1

1. 67,600,000; 19,656,000 **2.** 1296 **4.** 24; 4 **5.** 144; 18
6. 2401 **7.** 720; 72; 144; 72 **8.** 120; 1260; 34,650
9. 27,720 **10.** 40,320; 10,080; 1152; 2880; 384 **11.** 720; 72; 144
12. 24,300,000; 17,100,720 **13.** 190 **14.** 2,598,960
16. 42; 94 **17.** 604,800 **18.** 600 **19.** 896; 1000; 910
20. 36; 26 **21.** 35 **22.** 18 **23.** 48 **25.** $52!/(13!)^4$
27. 27,720 **28.** 65,536; 2520 **29.** 12,600; 945 **30.** 564,480
31. 165; 35 **32.** 1287; 14,112 **33.** 220; 572

CHAPTER 2

9. 74 **10.** .4; .1 **11.** 70; 2 **12.** .5; .32; 149/198
13. 20,000; 12,000; 11,000; 68,000; 10,000 **14.** 1.057
15. .0020; .4226; .0475; .0211; .00024 **17.** 9.10946×10^{-6}
18. .048 **19.** 5/18 **20.** .9017 **22.** $(n + 1)/2^n$ **23.** 5/12
25. .4 **26.** .492929 **27.** .58333 **28.** .2477; .2099
30. 1/18; 1/6; 1/2 **31.** 2/9; 1/9 **33.** 70/323 **35.** .8363
36. .0045; .0588 **37.** .0833; .5 **38.** 4 **39.** .48
40. 1/64; 21/64; 36/64; 6/64 **41.** .5177 **44.** .3; .2; .1 **46.** 5
48. 1.0604×10^{-3} **49.** .4329 **50.** 2.6084×10^{-6}
52. .09145; .4268 **53.** 12/35 **54.** .0511 **55.** .2198; .0342

APPENDIX B

Solutions to Self-Test Problems and Exercises

CHAPTER 1

1. (a) There are 4! different orderings of the letters C, D, E, F. For each of these orderings, we can obtain an ordering with A and B next to each other by inserting A and B, either in the order A, B or in the order B, A, in any of 5 places. Namely, either before the first letter of the permutation of C, D, E, F, or between the first and second, and so on. Hence, there are $2 \cdot 5 \cdot 4! = 240$ arrangements. Another way of solving this is to imagine that B is glued to the back of A. This yields that there are 5! orderings in which A is immediately before B. As there are also 5! orderings in which B is immediately before A, we again obtain a total of $2 \cdot 5! = 240$ different arrangements.

(b) There are a total of $6! = 720$ possible arrangements, and as there are as many with A before B as with B before A, there are 360 arrangements.

(c) Of the 720 possible arrangements, there are as many that have A before B before C, as have any of the 3! possible orderings of A, B, and C. Hence, there are $720/6 = 120$ possible orderings.

(d) Of the 360 arrangements that have A before B, half will have C before D and half D before C. Hence, there are 180 arrangements having A before B and C before D.

(e) Gluing B to the back of A, and D to the back of C, yields $4! = 24$ different orderings in which B immediately follows A and D immediately follows C. Since the order of A and B and of C and D can be reversed, there are thus $4 \cdot 24 = 96$ different arrangements.

(f) There are 5! orderings in which E is last. Hence, there are $6! - 5! = 600$ orderings in which E is not last.

2. 3!4!3!3! since there are 3! possible orderings of countries and then the countrymen must be ordered.

3. **(a)** $10 \cdot 9 \cdot 8 = 720$

(b) $8 \cdot 7 \cdot 6 + 2 \cdot 3 \cdot 8 \cdot 7 = 672$.
The preceding follows since there are $8 \cdot 7 \cdot 6$ choices not including A or B, and there are $3 \cdot 8 \cdot 7$ choices in which a specified one of A and B, but not the other, serves. The latter following since the serving member of the pair can be assigned to any of the 3 offices, the next position can then be filled by any of the other 8 people, and the final position by any of the remaining 7.

(c) $8 \cdot 7 \cdot 6 + 3 \cdot 2 \cdot 8 = 384$.

(d) $3 \cdot 9 \cdot 8 = 216$.

(e) $9 \cdot 8 \cdot 7 + 9 \cdot 8 = 576$.

4. **(a)** $\binom{10}{7}$ **(b)** $\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2}$

5. $\binom{7}{3,2,2} = 210$

6. There are $\binom{7}{3} = 35$ choices of the three places for the letters. For each choice, there are $(26)^3(10)^4$ different license plates. Hence, altogether there are a total of $35 \cdot (26)^3 \cdot (10)^4$ different plates.

7. Any choice of r of the n items is equivalent to a choice of $n - r$, namely, those items not selected.

8. **(a)** $10 \cdot 9 \cdot 9 \cdots 9 = 10 \cdot 9^{n-1}$

(b) $\binom{n}{i}9^{n-i}$, since there are $\binom{n}{i}$ choices of the i places to put the zeroes, and then each of the other $n - i$ positions can be any of the digits $1, \ldots, 9$.

9. **(a)** $\binom{3n}{3}$ **(b)** $3\binom{n}{3}$ **(c)** $\binom{3}{1}\binom{2}{1}\binom{n}{2}\binom{n}{1} = 3n^2(n-1)$ **(d)** n^3

(e) $\binom{3n}{3} = 3\binom{n}{3} + 3n^2(n-1) + n^3$

10. (number of solutions of $x_1 + \cdots + x_5 = 4$)(number of solutions of $x_1 + \cdots + x_5 = 5$)(number of solutions of $x_1 + \cdots + x_5 = 6$) $= \binom{8}{4}\binom{9}{4}\binom{10}{4}$

11. Since there are $\binom{j-1}{n-1}$ positive vectors whose sum is j, it follows that there are $\sum_{j=n}^{k}\binom{j-1}{n-1}$ such vectors.

CHAPTER 2

1. (a)
(d)
(e)
2. Let
purc
P(A
(a)
(b)
3. By s
and
num
Letti
4. Let
Subt
the f
or, P
5. (a)
6. Let
are t
7. (a)