\begin{align*}
p(\hat{x}_7) &= p(\hat{x}_{10}) = Q(\frac{a_9}{\sqrt{10}}) - Q(\frac{a_{10}}{\sqrt{10}}) = 0.1175 \\
p(\hat{x}_8) &= p(\hat{x}_9) = Q(\frac{a_8}{\sqrt{10}}) - Q(\frac{a_9}{\sqrt{10}}) = 0.1311
\end{align*}

Hence, the entropy of the quantized source is

\[ H(\hat{X}) = -\sum_{i=1}^{1} 6p(\hat{x}_i) \log_2 p(\hat{x}_i) = 3.6025 \]

This is the minimum number of bits per source symbol required to represent the quantized source.

4) Substituting \( \sigma^2 = 10 \) and \( D = 0.1154 \) in the rate-distortion bound, we obtain

\[ R = \frac{1}{2} \log_2 \frac{\sigma^2}{D} = 3.2186 \]

5) The distortion of the 16-level optimal quantizer is \( D_{16} = \sigma^2 \cdot 0.01154 \) whereas that of the 8-level optimal quantizer is \( D_8 = \sigma^2 \cdot 0.03744 \). Hence, the amount of increase in SQNR (db) is

\[ 10 \log_{10} \frac{\text{SQNR}_{16}}{\text{SQNR}_{8}} = 10 \cdot \log_{10} \frac{0.03744}{0.01154} = 5.111 \text{ db} \]

**Problem 4.47**

With 8 quantization levels and \( \sigma^2 = 400 \) we obtain

\[ \Delta = \sigma \cdot 0.5860 = 20 \cdot 0.5860 = 11.72 \]

Hence, the quantization levels are

\[ \hat{x}_1 = -\hat{x}_8 = -3 \cdot 11.72 - \frac{1}{2} 11.72 = -41.020 \]
\[ \hat{x}_2 = -\hat{x}_7 = -2 \cdot 11.72 - \frac{1}{2} 11.72 = -29.300 \]
\[ \hat{x}_3 = -\hat{x}_6 = -1 \cdot 11.72 - \frac{1}{2} 11.72 = -17.580 \]
\[ \hat{x}_4 = -\hat{x}_5 = \frac{1}{2} 11.72 = -5.860 \]

The distortion of the optimum quantizer is

\[ D = \sigma^2 \cdot 0.03744 = 14.976 \]

As it is observed the distortion of the optimum quantizer is significantly less than that of Example 4.51. The informational entropy of the optimum quantizer is found from Table 4.2 to be 2.761.

**Problem 4.48**

Using Table 4.3 we find the quantization regions and the quantized values for \( N = 16 \). These values should be multiplied by \( \sigma = \sigma_{1/2} = \sqrt{10} \), since Table 4.3 provides the optimum values for a unit variance Gaussian source.

\[ a_1 = a_{15} = -\sqrt{10} \cdot 2.401 = -7.5926 \]
\[ a_2 = a_{14} = -\sqrt{10} \cdot 1.844 = -5.8312 \]
\[ a_3 = a_{13} = -\sqrt{10} \cdot 1.437 = -4.5442 \]
\[ a_4 = a_{12} = -\sqrt{10} \cdot 1.099 = -3.4753 \]
Problem 4.51
\[ \bar{X} = \frac{X}{x_{\text{max}}} = X/2. \] Hence,
\[ E[\bar{X}^2] = \frac{1}{4} \int_{-2}^{2} \frac{X^2}{4} \, dx = \frac{1}{16} \cdot x^3 \bigg|_{-2}^{2} = \frac{1}{3} \]
With \( \nu = 8 \) and \( \bar{X}^2 = \frac{1}{3} \), we obtain
\[ \text{SQNR} = 3 \cdot 4^8 \cdot \frac{1}{3} = 4^8 = 48.165(\text{db}) \]

Problem 4.52
1)
\[ \sigma^2 = E[X^2(t)] = R_X(\tau)_{|\tau=0} = \frac{A^2}{2} \]
Hence,
\[ \text{SQNR} = 3 \cdot 4^\nu \bar{X}^2 = 3 \cdot 4^\nu \frac{\bar{X}^2}{x_{\text{max}}} = 3 \cdot 4^\nu \frac{A^2}{2A^2} \]
With SQNR = 60 db, we obtain
\[ 10 \log_{10} \left( \frac{3 \cdot 4^9}{2} \right) = 60 \implies q = 9.6733 \]
The smallest integer larger that \( q \) is 10. Hence, the required number of quantization levels is \( \nu = 10 \).

2) The minimum bandwidth requirement for transmission of a binary PCM signal is \( BW = \nu W \). Since \( \nu = 10 \), we have \( BW = 10W \).

Problem 4.53
1)
\[ E[X^2(t)] = \int_{-2}^{0} x^2 \left( \frac{x + 2}{4} \right) \, dx + \int_{0}^{2} x^2 \left( \frac{-x + 2}{4} \right) \, dx \]
\[ = \frac{1}{4} \left( \frac{1}{4} x^4 + \frac{2}{3} x^3 \right) \bigg|_{-2}^{0} + \frac{1}{4} \left( -\frac{1}{4} x^4 + \frac{2}{3} x^3 \right) \bigg|_{0}^{2} \]
\[ = \frac{2}{3} \]
Hence,
\[ \text{SQNR} = \frac{3 \times 4^\nu \times \frac{2}{3}}{x_{\text{max}}} = \frac{3 \times 4^5 \times \frac{2}{3}}{2^2} = 512 = 27.093(\text{db}) \]

2) If the available bandwidth of the channel is 40 KHz, then the maximum rate of transmission is \( \nu = 40/5 = 8 \). In this case the highest achievable SQNR is
\[ \text{SQNR} = \frac{3 \times 4^8 \times \frac{2}{3}}{2^2} = 32768 = 45.154(\text{db}) \]

3) In the case of a guard band of 2 KHz the sampling rate is \( f_s = 2W + 2000 = 12 \text{ KHz} \). The highest achievable rate is \( \nu = \frac{2BW}{f_s} = 6.6667 \) and since \( \nu \) should be an integer we set \( \nu = 6 \). Thus, the achievable SQNR is
\[ \text{SQNR} = \frac{3 \times 4^6 \times \frac{2}{3}}{2^2} = 2048 = 33.11(\text{db}) \]
Problem 4.54
1) The probabilities of the quantized source outputs are
\[
p(\hat{x}_1) = p(\hat{x}_4) = \int_{-2}^{-1} \frac{x + 2}{4} \, dx = \frac{1}{8} x^2 \bigg|_{-2}^{-1} + \frac{1}{2} x \bigg|_{-2}^{-1} = \frac{1}{8}
\]
\[
p(\hat{x}_2) = p(\hat{x}_3) = \int_{0}^{1} \frac{-x + 2}{4} \, dx = -\frac{1}{8} x^2 \bigg|_{0}^{1} + \frac{1}{2} x \bigg|_{0}^{1} = \frac{3}{8}
\]
Hence,
\[
H(\hat{X}) = - \sum_{\hat{x}_i} p(\hat{x}_i) \log_2 p(\hat{x}_i) = 1.8113 \text{ bits/output sample}
\]

2) Let $\tilde{X} = X - Q(X)$. Clearly if $|\tilde{X}| > 0.5$, then $p(\tilde{X}) = 0$. If $|\tilde{X}| \leq 0.5$, then there are four solutions to the equation $\tilde{X} = X - Q(X)$, which are denoted by $x_1, x_2, x_3$ and $x_4$. The solution $x_1$ corresponds to the case $-2 \leq X \leq -1$, $x_2$ is the solution for $-1 \leq X \leq 0$ and so on. Hence,
\[
f_X(x_1) = \frac{x_1 + 2}{4} = \frac{(\tilde{x} - 1.5) + 2}{4} \quad f_X(x_3) = \frac{-x_3 + 2}{4} = \frac{-(\tilde{x} + 0.5) + 2}{4}
\]
\[
f_X(x_2) = \frac{x_2 + 2}{4} = \frac{(\tilde{x} - 0.5) + 2}{4} \quad f_X(x_4) = \frac{-x_4 + 2}{4} = \frac{-(\tilde{x} + 1.5) + 2}{4}
\]
The absolute value of $(X - Q(X))'$ is one for $X = x_1, \ldots, x_4$. Thus, for $|\tilde{X}| \leq 0.5$
\[
f_X(\tilde{X}) = \sum_{i=1}^{4} \frac{f_X(x_i)}{|(x_i - Q(x_i))'|} = \frac{(\tilde{x} - 1.5) + 2}{4} + \frac{(\tilde{x} - 0.5) + 2}{4} + \frac{-(\tilde{x} + 0.5) + 2}{4} + \frac{-(\tilde{x} + 1.5) + 2}{4} = 1
\]

Problem 4.55
1)
\[
R_X(t + \tau, t) = E[X(t + \tau)X(t)] = E[Y^2 \cos(2\pi f_0(t + \tau) + \Theta) \cos(2\pi f_0 t + \Theta)] = \frac{1}{2} E[Y^2] E[\cos(2\pi f_0 \tau) + \cos(2\pi f_0 (2t + \tau) + 2\Theta)]
\]
and since
\[
E[\cos(2\pi f_0 (2t + \tau) + 2\Theta)] = \frac{1}{2\pi} \int_{0}^{2\pi} \cos(2\pi f_0 (2t + \tau) + 2\Theta) \, d\Theta = 0
\]
we conclude that
\[
R_X(t + \tau, t) = \frac{1}{2} E[Y^2] \cos(2\pi f_0 \tau) = \frac{3}{2} \cos(2\pi f_0 \tau)
\]

2)
\[
10 \log_{10} \text{SQNR} = 10 \log_{10} \left( \frac{3 \times 4^\nu \times R_X(0)}{x_{\text{max}}^2} \right) = 40
\]
Thus,
\[
\log_2 \left( \frac{4^\nu}{2} \right) = 4 \text{ or } \nu = 8
\]

134
Hence,

\[ g(x) = \begin{cases} 
  g(1) \left( x + 1 \right)^\frac{3}{2} - 1 & -1 \leq x < 0 \\
  g(1) \left[ 1 - (1 - x)^\frac{3}{2} \right] & 0 \leq x \leq 1 
\end{cases} \]

The next figure depicts \( g(x) \) for \( g(1) = 1 \). Since the resulting distortion is (see Equation 4.6.17)

\[
D = \frac{1}{12 \times 4^v} \left[ \int_{-\infty}^{\infty} \left( f_x(x) \right)^\frac{1}{2} \, dx \right]^3 = \frac{1}{12 \times 4^v} \left( \frac{3}{2} \right)^3
\]

we have

\[ SQNR = \frac{E[X^2]}{D} = \frac{32}{9} \times 4^v E[X^2] = \frac{32}{9} \times 4^v \cdot \frac{1}{6} = \frac{16}{27} 4^v. \]

**Problem 4.59**

The sampling rate is \( f_s = 44100 \) meaning that we take 44100 samples per second. Each sample is quantized using 16 bits so the total number of bits per second is \( 44100 \times 16 \). For a music piece of duration 50 min = 3000 sec the resulting number of bits is

\[ 44100 \times 16 \times 3000 = 2.1168 \times 10^9. \]