Matched filter

\[ u(t) \rightarrow h(t) \rightarrow y(t) \]

\[ t_0 = 0 \]

BPSK

\[ u(t) = \pm g(t) + w(t) \quad : \text{one bit only transmitted} \]

\[ d: \text{information bit} \]

\[ w(t) = w_c(t) \quad : \text{zero-mean Gaussian, p.s.d. No} \]

\[ \text{while} \Rightarrow h(t) = g(-t) \quad (h(t) = G^*(f)) \]

\[ \Rightarrow y(t) = u(t) \ast h(t) = d \cdot g(t) \ast g(-t) + w(t) \ast g(-t) = \int_{-\infty}^{\infty} u(\xi) g(t-\xi) d\xi \]

Zero-mean, Gaussian

\[ S_y(f) = S_w(f) |H^*(f)|^2 = No \cdot |G^2(f)| : \text{"colored"} \]

\[ R_y(t) = No \cdot x(t) \]
Sequence of symbols

$u(t) = \sum_{n} d(n) g(t-nT)$

$y(t) = u(t) * h(t) + w(t) * h(t)$

$\bar{y}(t) = u(t) * g(-t) = \sum_{n} d(n) g(t-nT) * g(-t)$

$x(t-nT)$

Waveform does not look much like $u(t)$, but samples do look like $\{d(n)\}$ (they are equal)

See: importance of having correct sampling times

If optimal sampling time is missed, there will be interference between adjacent symbols (ISI: InterSymbol Interf.)
Optimal sampling time => no ISI.

\[
\begin{align*}
\bar{y}(0) &= d(0) \cdot x(0) \\
\bar{y}(t) &= d(t) \cdot x(0) \\
\bar{y}(2T) &= d(2T) \cdot x(0) \\
\vdots & \\
\bar{y}(kT) &= d(kT) \cdot x(0) + \int_{-\infty}^{+\infty} |q^2(t)| \, dt = Eq
\end{align*}
\]

\[
\bar{y}(t) = u(t) * h(t) = \sum_{n} d(n) x(t-nT) \quad \text{simplified:}
\]

\[
\bar{y}(kT) = \sum_{n} d(n) x(kT-nT) = d(k) x(0) + 0
\]

So, after matched filtering and optimal sampling, we can recover the entire sequence by looking at samples \( \bar{y}(kT) \) one at a time.

\[
\text{no ISI:} \quad x(kT-nT) = 0 \quad \text{for} \quad k \neq 0
\]
Noise

\[ y(t) = \left[ \sum_n \xi d(n) g(t-nT) + w(t) \right] \ast g(-t) = \]

\[ = \sum_n \xi d(n) z(t-nT) + z(t) \]

\[ \bar{y}(t) \]

\[ y(kT) = \bar{y}(kT) + x(kT) \quad \text{in fact, we observe the symbol sequence in noise} \]

\[ d(k) \circ x(0) \quad \text{Eg} > 0 \]

Q: Are the noise samples independent?
If yes, we can still recover the entire sequence by looking at samples \( y(kT) \) one at a time.
(There is nothing to learn about \( z(kT) \) from other samples \( \ldots z(kT-t), z(kT+t), \ldots \) )
Know: $x(t)$ is Gaussian, zero-mean

$$S_x(f) = Sw(f) \cdot |H^2(f)| = Sw \cdot |G^2(f)| = Sw \cdot X(f)$$

$$\Rightarrow R_x(t) = Sw \cdot Z(t)$$

$$R_x(t) = E \{ x(t+T) x^*(t) \}$$

$$\Rightarrow R_x(mT) = E \{ x(t+mT) x^*(t) \}$$

$$= 0 \text{ for } m = \pm 1, \pm 2, \ldots$$

$$R_x(0) = \sigma_x^2 = P_x = Sw \cdot E_g$$

$\Rightarrow$ Noise samples are uncorrelated.

Because they are Gaussian random variables, they are also independent.

Look at joint p.d.f. of two zero-mean Gaussian r.v.s $x_1 = x(T), x_2 = x(2T)$

$$P_{x_1x_2}(x_1, x_2) = \frac{1}{\sqrt{2\pi} \cdot \sqrt{\sigma_x^4(1-\varphi^2)}} \cdot e^{-\frac{x_1^2 - 2\varphi x_1 x_2 + x_2^2}{2\sigma_x^2(1-\varphi^2)}}$$

$$\Rightarrow P_{x_1}(x_1) \cdot P_{x_2}(x_2)$$

for $\varphi = 0$

$E \{ x_1 x_2 \} = \varphi \cdot \sigma_x^2$
So, after matched filtering \& sampling, we have a sequence of (discrete-time) observations:

\[ y(kT) = d(k) x(0) + \mathbf{z}(kT) \]

\[ k = 0, 1, 2, \ldots \]

\[ \mathbf{z}(kT) \]

Independent zero-mean Gaussian random variables with identical variance \( \mathbf{z} = \sqrt{\mathbf{z}} \cdot \mathbf{Eg} \).

Because the noise samples are independent it suffices to look at \( y(0) \) only for detecting \( d(0) \):

\[ y(1) \quad \text{\( d(1) \), etc.} \]

\[ \Rightarrow \text{Problem statement: Detection} \]

Observe \( y = d \cdot \mathbf{Eg} + \mathbf{z} \), \( \mathbf{z} \sim \mathcal{N}(0, \sigma^2) \)

determine \( d \), knowing that \( d \) is either +1 or -1.

or, in general, one of \( M \) possible symbol values \( \{d_1, d_2, \ldots, d_M\} \)

(PAM, M-QAM, M-PSK)
BPSK (Binary PAM, Binary Antipodal Signaling).

\[ y = d \cdot E_g + z, \quad z \sim N(0, \sigma_z^2) \]

\( d \in \{+1, -1\} \)

decision rule: \( y > y_T \) decide +1
\( y < y_T \) decide -1

Detection criterion = ?

Probability of error \( \rightarrow \text{minimal} \): \( y_T = ? \) optimal threshold

\[ P_e = P\{ \text{a bit decision is erroneous} \} = \]

\[ = P\{ \text{transmit +1 & decide -1 OR transmit -1 & decide +1} \} = \]

\[ = P\{ +x +1 \} \cdot P\{ \text{dec. -1 | +1 was +x} \} + P\{ +x -1 \} \cdot P\{ \text{dec. +1 | -1 was +x} \} \]

\[ \frac{1}{2} \quad \frac{1}{2} \]

\( y < y_T \quad y = +E_g + z \quad \frac{1}{2} \quad y > y_T \quad y = -E_g + z \)

\[ = \frac{1}{2} P\{ +E_g + z < y_T \} + \frac{1}{2} P\{ -E_g + z > y_T \} = \]

\[ = \frac{1}{2} P\{ z < -E_g + y_T \} + \frac{1}{2} P\{ z > +E_g + y_T \} \]
\[ z \sim N(0, \sigma_z^2) \quad \Rightarrow \quad p_z(z) = \frac{1}{\sqrt{2\pi}\sigma_z} e^{-\frac{z^2}{2\sigma_z^2}} \]

\[ P\{ z > E_g + y_T \} \implies p_z(z) \, dz = \frac{1}{2} \text{erfc} \left( \frac{E_g + y_T}{\sqrt{2} \sigma_z} \right) = Q \left( \frac{E_g + y_T}{\sigma_z} \right) \]

So,

\[ P_e = \frac{1}{2} \left[ Q \left( \frac{E_g + y_T}{\sigma_z} \right) + Q \left( \frac{E_g - y_T}{\sigma_z} \right) \right] \]

\[ \frac{dP_e}{dy_T} = 0 \quad \Rightarrow \quad y_T = 0 \quad : \quad \text{optimal threshold} \]

intuitively obvious for RPSK,

equally probable +1 and -1,

and equally important errors \( + \rightarrow - \)

and \( - \rightarrow + \)

(but in a system different than communications sys,

+1 and -1 may not be "equal" - medical decisions, for example)
\( P(+|-) = P(-|+) : \text{equal} \)

\( P_e = P(+) P(-|+) + P(-) P(+|-) = Q \)

\( S_x^2 = S_w \cdot E_g \)

\( \text{BPSK: } S_w = S_{wc} = N_0 \)

\( E_b = E_{\delta} = \frac{1}{2} E_g \)

\( e^{\text{erfc}}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} \, dt \)

\( Q(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-u^2/2} \, du \)

\( Q(x) = \frac{1}{2} e^{\text{erfc}} \frac{x}{\sqrt{2}} \)

\( \text{half distance between expected values} \)

\( \sqrt{\text{variance at imp. to decision making}} \)

\( E_g = \frac{E_g}{\sigma_x^2} = \sqrt{\frac{E_g}{S_w}} = \sqrt{\frac{S_w E_g}{N_0}} \)

\( P_e = Q \sqrt{\frac{2E_b}{N_0}} = \frac{1}{2} e^{\text{erfc}} \sqrt{\frac{E_b}{N_0}} \)

\[ \frac{E_b}{N_0} [\text{dB}] = 10 \log \frac{E_b}{N_0} \]

\[ \text{SNR per bit} \]