Spectrum of linearly modulated signals

\[ u(t) = \sum d(n) g(t-nT) \text{ band and linear mod.} \]

\[ \Downarrow \text{ random: } R_d(m) = E[d(n+m) d^*(n)] \]

\[ R_u(t+\tau, t) = E[u(t+\tau) u^*(t)] : \text{ autocorrelation} \]

\[ R_u(t+\tau, t) = E \left\{ \sum_{n} \sum_{k} d(n) g(t+\tau-nT) d^*(k) g^*(t-kT) \right\} = \]

\[ = \sum_{n} \sum_{k} E \left\{ d(n) d^*(k) \right\} g(t+\tau-nT) g^*(t-kT) = R_d(n-k) \]

\[ \Rightarrow \frac{R_d(n-k)}{m} \]

\[ = \sum_{n} \sum_{k} R_d(m) g(t+\tau-kT-mT) g^*(t-kT) : \text{ periodic in } t, \]

with period \( T \) (\( T = \text{symbol interval} \))

\[ \Rightarrow u(t) \text{ is not WSS.} \]

\[ u(t) \text{ is cyclostationary} : R_u(t_1, t_2) = R_u(t_1+T, t_2+T) \]
Def. \( \bar{R}_m(t) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} R_m(t+t, t) \, dt \)

\( \Rightarrow \bar{R}_m(t) = \frac{1}{T} \sum_m R_d(m) \int_{-\frac{T}{2}}^{\frac{T}{2}} g^*(t+\tau-kT-mT) g^*(t-kT) \, dt \)

\( \Rightarrow \bar{R}_m(t) = \frac{1}{T} \sum_m R_d(m) R_g(t-mT) \) ; \( R_g(t) = \int_{-\infty}^{+\infty} g(t+\tau) g^*(\tau) \, d\tau \)

Now it is possible to define spectrum:

\( S_m(f) = \int \bar{R}_m(t) e^{-j2\pi ft} \, dt \)

\( \Rightarrow S_m(f) = \frac{1}{T} S_d(f) S_g(f) \) ; \( S_d(f) = \sum_m R_d(m) e^{-j2\pi ft} \)

\( \Rightarrow S_m(f) = \frac{1}{T} S_d(f) |G^2(f)| \)
So, the spectrum of a linearly modulated signal is

\[ S_u(f) = \frac{1}{T} S_d(f) |G^2(f)| \]

- \( S_d(f) \): spectrum of data sequence
- \( g(t) \): basic pulse
- \( \tau \): symbol duration
- \( d(n) \): iid
- \( \delta(m) \): rectangular pulse

\[ G(f) = AT \frac{\sin \omega T}{\omega T} \cdot e^{-j\omega T} \]

\[ 1G^2(f) = (AT)^2 \left[ \frac{\sin \omega T/2}{\omega T/2} \right]^2 \]

\[ S_u(f) = \frac{1}{T} |G^2(f)| \]
Spectrum shaping

\[ S_u(f) = \begin{array}{c}
\text{line coding}
\end{array} \]
\[ \text{pulse shaping} \]

\[ R_d(n) = \sum d(n+m) \delta^n (n) \]

by introducing dependency among \( d(n) \), spectrum is shaped.

ex. mapping from logical bits "0", "1" into \( d(n) \):

1. NRZ
   "1" \( \rightarrow \) +1
   "0" \( \rightarrow \) -1

2. Manchester (biphase)  "1" \( \rightarrow \) + -
   "0" \( \rightarrow \) - +
   ensures transition in every \( T \)

3. Miller (delay mod)  "1" \( \rightarrow \)
   "0" \( \rightarrow \)
   or  or
   choose polarity to match previous bit (never 1/2 pulse)

- telephone lines
- magnetic recording
Pulse shaping

\[ d(u) \xrightarrow{g(u)} u(t) \]

\[ S_d(p) = 1 \quad S_u(p) = \frac{1}{T} |G^2(p)| \]

↓ choose \( g(t) \) other than rectangular to achieve better spectrum utilization → radio channels

Problem: narrower spectrum ⇒ longer pulse

↓ reduced intersymbol rate interference

Nyquist: it is possible to have both narrower spectrum and no ISI with a carefully designed pulse

First Nyquist criterion: Transmission with no ISI

\[ d(u) \xrightarrow{g_T(u)} c(t) \xrightarrow{\oplus} g_R(t) \xrightarrow{\downarrow w(t)} y(t+T) \]
\[ y(nT) = \sum_{k} d(k) x((nT - kT) + z(nT) \]

\[ d(n) z(0) + ISI \]

Want \( ISI = 0 \): \[ x(nT) = \begin{cases} 
  x(0), & m = 0 \\
  0, & m \neq 0 
\end{cases} \]

In spectrum:

\[ \sum_{m} x(nT)e^{-j2\pi nwT} \]

\[ = \frac{1}{T} \sum_{m} X(f + \frac{m}{T}) \]

\[ = x(0) = \text{const.} \quad \text{sum of shifted spectra} \quad X(f) = \mathcal{F}[x(t)] \]

So, sum of shifted spectra must be equal to a constant if we want no ISI at sampling times nt.

There are three cases, depending upon the width of \( X(f) \), that can be distinguished.
Case 1.

$X(f)$ is bandlimited

$X(f) + X(f - \frac{1}{T}) + X(f - \frac{2}{T})$

Sum of shifted spectra cannot be equal to a constant because there is a gap, i.e., $f_m < \frac{1}{2T}$

So, as long as $\frac{1}{T} > 2f_m$, it is not possible to avoid ISI.
Case 2. In which the shifted spectra touch.

\[ X(f + \frac{1}{T}) \quad X(f) \quad X(f - \frac{1}{T}) \quad X(f - \frac{2}{T}) \]

\[ \text{const.} \quad \rightarrow f \]

\[ \sum \text{can be equal to a constant} \]

\[ \text{If } \frac{f_m}{2T}, \text{ i.e. } R = B, \text{ transmission with no ISI is possible by having } X(f) \text{ as above.} \]
Case 3. In which the shifted spectra overlap.

If \( f_m > \frac{1}{2T} \), there are (infinitely) many shapes \( x(f) \) that can sum to a constant after shifting.

"Raised cosine" is a popular shape.
Spectral raised cosine is often used in practice. It provides:

- limited bandwidth
- no ISI
- well-behaved pulse

\[ R = \frac{1}{2T} \]

minimum bandwidth needed for no ISI transmission

(after, this pulse is not realizable)

\[ \Delta \in [0, 1] \]

by allowing transmission at

\[ R = \frac{B}{1+\Delta} \]

is possible with no ISI

Implementation on an ideal AWGN channel

\[ g(t) \rightarrow \mathbb{R} \rightarrow \sqrt{X(f)} \rightarrow \sqrt{X(f)} : \text{square root raised cosine} \]