Lectures 8&9  Signal Detection in Noise

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Noise in communication systems

- Noise is additional “unwanted” signal that interferes with the transmitted signal
  - Generated by electronic devices

- The noise is a random process
  - Each “sample” of $n(t)$ is a random variable

- Typically, the noise process is modeled as “Additive White Gaussian Noise” (AWGN)
  - White: Flat frequency spectrum
  - Gaussian: noise distribution

$S(t)$  Channel  $r(t)$

\[ r(t) = S(t) + n(t) \]
Random Processes

- The auto-correlation of a random process $x(t)$ is defined as
  - $R_{xx}(t_1,t_2) = E[x(t_1)x(t_2)]$

- A random process is Wide-sense-stationary (WSS) if its mean and auto-correlation are not a function of time. That is
  - $m_x(t) = E[x(t)] = m$
  - $R_{xx}(t_1,t_2) = R_x(\tau)$, where $\tau = t_1 - t_2$

- If $x(t)$ is WSS then:
  - $R_x(\tau) = R_x(-\tau)$
  - $|R_x(\tau)| \leq |R_x(0)|$ (max is achieved at $\tau = 0$)

- The power content of a WSS process is:

$$P_x = E[\lim_{t \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt] = \lim_{t \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} R_x(0) dt = R_x(0)$$
Power Spectrum of a random process

- If $x(t)$ is WSS then the power spectral density function is given by:
  \[ S_x(f) = F[R_x(\tau)] \]
- The total power in the process is also given by:

\[
P_x = \int_{-\infty}^{\infty} S_x(f) df = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} R_x(t) e^{-j2\pi ft} dt \right] df
\]

\[
= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} R_x(t) e^{-j2\pi ft} df \right] dt
\]

\[
= \int_{-\infty}^{\infty} R_x(t) \left[ \int_{-\infty}^{\infty} e^{-j2\pi ft} df \right] dt = \int_{-\infty}^{\infty} R_x(t) \delta(t) dt = R_x(0)
\]
White noise

- The noise spectrum is flat over all relevant frequencies
  - White light contains all frequencies

\[ S_n(f) \]

\[ \frac{N_o}{2} \]

- Notice that the total power over the entire frequency range is infinite
  - But in practice we only care about the noise content within the signal bandwidth, as the rest can be filtered out

- After filtering the only remaining noise power is that contained within the filter bandwidth (B)

\[ S_{BP}(f) \]

\[ \frac{N_o}{2} \]
• The effective noise content of bandpass noise is $B_{N_0}$
  
  - Experimental measurements show that the pdf of the noise samples can be modeled as zero mean gaussian random variable

  \[ f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/2\sigma^2} \]

  - AKA Normal r.v., $N(0,\sigma^2)$
  
  - $\sigma^2 = P_X = B_{N_0}$

• The CDF of a Gaussian R.V.,

  \[ F_X(\alpha) = P[X \leq \alpha] = \int_{-\infty}^{\alpha} f_X(x)dx = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/2\sigma^2} dx \]

• This integral requires numerical evaluation
  
  - Available in tables
AWGN, continued

- $X(t) \sim N(0, \sigma^2)$

- $X(t_1)$, $X(t_2)$ are independent unless $t_1 = t_2$

- $R_x(\tau) = E[X(t+\tau)X(t)] = \begin{cases} E[X(t+\tau)]E[X(t)] & \tau \neq 0 \\ E[X^2(t)] & \tau = 0 \end{cases}$

  $= \begin{cases} 0 & \tau \neq 0 \\ \sigma^2 & \tau = 0 \end{cases}$

- $R_x(0) = \sigma^2 = P_x = BN_0$
Detection of signals in AWGN

Observe: \( r(t) = S(t) + n(t), \ t \in [0,T] \)

Decide which of \( S_1, \ldots, S_m \) was sent

- **Receiver filter**
  - Designed to maximize signal-to-noise power ratio (SNR)

\[
\begin{align*}
\text{r(t)} & \xrightarrow{\text{filter}} h(t) \xrightarrow{\text{y(t)}} \text{"sample at t}=T\text{"} \\
& \xrightarrow{\text{decide}}
\end{align*}
\]

- **Goal:** find \( h(t) \) that maximized SNR
Receiver filter

\[ y(t) = r(t) \ast h(t) = \int_0^t r(\tau)h(t - \tau)d\tau \]

Sampling at \( t = T \) \( \Rightarrow \) \[ y(T) = \int_0^T r(\tau)h(T - \tau)d\tau \]

\[ r(\tau) = s(\tau) + n(\tau) \Rightarrow \]

\[ y(T) = \int_0^T s(\tau)h(T - \tau)d\tau + \int_0^T n(\tau)h(T - \tau)d\tau = Y_s(T) + Y_n(T) \]

\[ SNR = \frac{Y_s^2(T)}{E[Y_n^2(T)]} = \frac{\left[ \int_0^T s(\tau)h(T - \tau)d\tau \right]^2}{\frac{N_0}{2} \int_0^T h^2(T - t)dt} = \frac{\left[ \int_0^T h(\tau)s(T - \tau)d\tau \right]^2}{\frac{N_0}{2} \int_0^T h^2(T - t)dt} \]
Matched filter: maximizes SNR

Cauchy-Schwartz Inequality:

\[
\left[ \int_{-\infty}^{\infty} g_1(t)g_2(t)dt \right] \leq \int_{-\infty}^{\infty} (g_1(t))^2 \int_{-\infty}^{\infty} (g_2(t))^2
\]

Above holds with equality iff: \( g_1(t) = cg_2(t) \) for arbitrary constant \( c \)

\[
SNR = \frac{\left[ \int_{0}^{T} s(\tau)h(T-\tau)d\tau \right]^2}{\frac{N_0}{2} \int_{0}^{T} h^2(T-t)dt} \leq \frac{\int_{0}^{T} (s(\tau))^2 d\tau \int_{0}^{T} h^2(T-\tau)d\tau}{\frac{N_0}{2} \int_{0}^{T} h^2(T-t)dt} = \frac{2}{N_0} \int_{0}^{T} (s(\tau))^2 d\tau = \frac{2E_s}{N_0}
\]

Above maximum is obtained iff: \( h(T-\tau) = cS(\tau) \)

\[
\Rightarrow h(t) = cS(T-t) = S(T-t)
\]

\( h(t) \) is said to be “matched” to the signal \( S(t) \)
Example: PAM

\[ S_m(t) = A_m g(t), \quad t \in [0,T] \]

\[ A_m \] is a constant: Binary PAM \( A_m \in \{0,1\} \)

Matched filter is matched to \( g(t) \)

\[ g(t) \]

\[ \text{“matched filter”} \]

\[ g(T-t) \]
Example, continued

\[ Y_s(t) = \int_0^t S(\tau) h(t - \tau) d\tau, \quad h(t) = g(T - t) \Rightarrow h(t - \tau) = g(T + \tau - t) \]

\[ Y_s(t) = \int_0^t g(\tau) g(T + \tau - t) d\tau = \int_0^t g(\tau) g(T - t + \tau) d\tau \]

\[ Y_s(T) = \int_0^T g^2(\tau) d\tau \]

- Sample at \( t = T \) to obtain maximum value
Matched filter receiver

\[ U(t) \rightarrow 2\cos(2\pi f_c t) \rightarrow \hat{r}_x(t) \rightarrow g(T-t) \rightarrow \hat{r}_x(kT) \]

Sample at \( t = kT \)

\[ U(t) \rightarrow 2\sin(2\pi f_c t) \rightarrow \hat{r}_y(t) \rightarrow g(T-t) \rightarrow \hat{r}_y(kT) \]

Sample at \( t = kT \)
Binary PAM example, continued

0 => \( S_1 = g(t) \)
1 => \( S_2 = -g(t) \)
Alternative implementation: correlator receiver

\[ r(t) = S(t) + n(t) \]

Sample at \( t = kT \)

\[ Y(T) = \int_0^T r(t)S(t)dt = \int_0^T S^2(t)dt + \int_0^T n(t)S(t)dt = Y_s(T) + Y_n(T) \]

Notice resemblance to matched filter
Signal Detection

• After matched filtering we receive $r = S_m + n$
  - $S_m \in \{S_1, \ldots, S_M\}$

• How do we determine from $r$ which of the $M$ possible symbols was sent?
  - Without the noise we would receive what sent, but the noise can transform one symbol into another

Hypothesis testing

• Objective: minimize the probability of a decision error

• Decision rule:
  - Choose $S_m$ such that $P(S_m \text{ sent} \mid r \text{ received})$ is maximized

• This is known as Maximum a posteriori probability (MAP) rule

• MAP Rule: Maximize the conditional probability that $S_m$ was sent given that $r$ was received
Notes:

- MAP rule requires prior probabilities
- MAP minimizes the probability of a decision error
- ML rule assumes equally likely symbols
- With equally likely symbols MAP and ML are the same

MAP detector: \( \max_{S_1 \ldots S_M} P(S_m \mid r) \)

\[
P(S_m \mid r) = \frac{P(S_m, r)}{P(r)} = \frac{P(r \mid S_m)P(S_m)}{P(r)}
\]

\[
P(S_m \mid r) = \frac{f_{r|s}(r \mid S_m)P(S_m)}{f_r(r)}
\]

\[
f_r(r) = \sum_{m=1}^{M} f_{r|s}(r \mid S_m)P(S_m)
\]

When \( P(S_m) = \frac{1}{M} \), Map rule becomes:

\[
\max_{S_1 \ldots S_M} f(r \mid S_m) \quad (\text{AKA Maximum Likelihood (ML) decision rule})
\]
Detection in AWGN
(Single dimensional constellations)

\[ f(r \mid S_m) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r - S_m)^2}{N_0}} \]

\[ \ln(f(r \mid S_m)) = - \ln(\sqrt{\pi N_0}) - \frac{(r - S_m)^2}{N_0} \]

\[ d_{rS_m} = (r - S_m)^2 \]

Maximum Likelihood decoding amounts to minimizing \[ d_{rS_m} = (r - S_m)^2 \]

- Also known as minimum distance decoding
  - Similar expression for multidimensional constellations
Detection of binary PAM

- $S_1(t) = g(t)$, $S_2(t) = -g(t)$
  - $S_1 = - S_2$ => “antipodal” signaling

- Antipodal signals with energy $E_b$ can be represented geometrically as

  \[
  S_2 \\
  \quad -\sqrt{E_b} \\
  \quad S_1 \\
  \quad \sqrt{E_b}
  \]

- If $S_1$ was sent then the received signal $r = S_1 + n$
- If $S_2$ was sent then the received signal $r = S_2 + n$

\[
\begin{align*}
  f_{r|s}(r \mid s_1) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-\sqrt{E_b})^2}{N_0}} \\
  f_{r|s}(r \mid s_2) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r+\sqrt{E_b})^2}{N_0}}
\end{align*}
\]
Detection of Binary PAM

• Decision rule: MLE => minimum distance decoding
  - => r > 0 decide S1
  - => r < 0 decide S2

• Probability of error
  - When S2 was sent the probability of error is the probability that noise exceeds $(Eb)^{1/2}$ similarly when S1 was sent the probability of error is the probability that noise exceeds $-(Eb)^{1/2}$

  - $P(e|S1) = P(e|S2) = P[r<0|S1]$
Probability of error for binary PAM

\[ P_e = \int_{-\infty}^{0} f_{r|s}(r | s_1) dr = \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} e^{- \left( \frac{r - \sqrt{E_b}}{\sqrt{N_0}} \right)^2} dr \]

\[ = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{-\sqrt{E_b} / \sqrt{N_0}} e^{-r^2 / N_0} dr \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{2E_b} / N_0} e^{-r^2 / 2} dr \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2E_b} / N_0}^{\infty} e^{-r^2 / 2} dr \]

\[ = Q(\sqrt{2E_b} / N_0) \text{ where}, \]

\[ Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-r^2 / 2} dr \]

- \( Q(x) = P(X>x) \) for X Gaussian with zero mean and \( \sigma^2 = 1 \)
- \( Q(x) \) requires numerical evaluation and is tabulated in many math books (Table 4.1 of text)
More on Q function

• Notes on Q(x)
  – Q(0) = 1/2
  – Q(-x) = 1 - Q(x)
  – Q(∞) = 0, Q(-∞) = 1

  – If X is N(m, σ²) Then P(X>x) = Q((x-m)/σ)

• Example: Pe = P[r<0|S1 was sent]

\[ f_{r|s}(r \mid s1) \sim N(\sqrt{E_b}, N_0 / 2) \Rightarrow m = \sqrt{E_b}, \sigma = \sqrt{N_0 / 2} \]

\[ P_e = 1 - P[r > 0 \mid s1] = 1 - Q\left(\frac{-\sqrt{E_b}}{\sqrt{N_0 / 2}}\right) = 1 - Q\left(-\sqrt{2E_b / N_0}\right) = Q\left(\sqrt{2E_b / N_0}\right) \]
Error analysis continued

• In general, the probability of error between two symbols separated by a distance $d$ is given by:

$$P_e(d) = Q\left(\sqrt[2]{\frac{d}{2N_0}}\right)$$

• For binary PAM $d = 2 \sqrt{E_b}$ Hence,

$$P_e = Q\left(\sqrt[2]{\frac{2E_b}{N_0}}\right)$$
Orthogonal signals

- Orthogonal signaling scheme (2 dimensional)

\[ P_e = Q\left(\sqrt[4]{\frac{d^2}{2N_0}} = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \right) \]
Orthogonal vs. Antipodal signals

- Notice from Q function that orthogonal signaling requires twice as much bit energy than antipodal for the same error rate
  - This is due to the distance between signal points
Probability of error for M-PAM

\[ S_M = A_M \sqrt{E_g}, \quad A_M = (2m - 1 - M) \tau_i \]

\[ d_{ij} = 2\sqrt{E_g} \text{ for } |i - j| = 1 \]

Decision rule: Choose \( s_i \) such that \( d(r, s_i) \) is minimized

\[ P[\text{error} | s_i] = P[\text{decode } s_{i-1} | s_i] + P[\text{decode } s_{i+1} | s_i] = 2P[\text{decode } s_{i+1} | s_i] \]

\[ Pe = 2Q \left[ \sqrt{\frac{d_{i,i+1}^2}{2N_0}} \right] = 2Q \left[ \sqrt{\frac{2E_g}{N_0}} \right], \quad P_{eb} = \frac{Pe}{\log_2(M)} \]

Notes:
1) the probability of error for \( s_1 \) and \( s_M \) is lower because error only occur in one direction

2) With Gray coding the bit error rate is \( P_e / \log_2(M) \)
Probability of error for M-PAM

\[ E_{av} = \frac{M^2 - 1}{3} E_g \Rightarrow E_{bav} = \frac{M^2 - 1}{3 \log_2(M)} E_g \]

\[ E_g = \frac{3 \log_2(M)}{M^2 - 1} E_{bav} \]

\[ P_e = 2Q \left[ \sqrt{\frac{6 \log_2(M)}{(M^2 - 1)N_0}} E_{bav} \right], \quad P_{eb} = \frac{P_e}{\log_2(M)} \]

accounting for effect of \( S_1 \) and \( S_M \) we get:

\[ P_e = 2 \left( \frac{M - 1}{M} \right) Q \left[ \sqrt{\frac{6 \log_2(M)}{(M^2 - 1)N_0}} E_{bav} \right] \]
Probability of error for PSK

- Binary PSK is exactly the same as binary PAM
- 4-PSK can be viewed as two sets of binary PAM signals
- For large $M$ (e.g., $M>8$) a good approximation assumes that errors occur between adjacent signal points

$$\theta = \frac{2\pi}{M}$$

$$d_{ij} = 2\sqrt{E_s \sin\left(\frac{\pi}{M}\right)}, \quad |i - j| = 1$$
Error Probability for PSK

\[ P[\text{error} \mid s_i] = P[\text{decode } s_{i-1} \mid s_i] + P[\text{decode } s_{i+1} \mid s_i] = 2P[\text{decode } s_{i+1} \mid s_i] \]

\[ P_{es} = 2Q \left[ \frac{d_{i,i+1}^2}{2N_0} \right] = 2Q \left[ \frac{2E_s}{N_0} \sin(\pi / M) \right] \]

\[ E_b = E_s / \log_2(M) \]

\[ P_{es} = 2Q \left[ \sqrt{\frac{2\log_2(M)E_b}{N_0}} \sin(\pi / M) \right], \quad P_{eb} = \frac{P_{es}}{\log_2(M)} \]