Problem Set 5

Multivariate Statistics, 3/10/2001

1.0 Johnston, 4-1

\((A + B)(A - B) = AA - AB + BA - BB\)

\((A - B)(A + B) = AA + AB - BA - BB\)

The two are only equivalent of the matrix \(AB\) is symmetric, in which case the center two terms cancel out.

2.0 Johnston, 4-2

\((AB)' = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 & 1 \\ 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 3 + 0 + 3 & 4 + 0 + 2 & 1 + 0 - 6 \\ 6 + 0 + 1 & 8 + 1 + 2 & 2 - 5 - 2 \end{pmatrix} = \begin{pmatrix} 6 & 6 & -5 \\ 7 & 11 & -5 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 6 & 11 \\ -5 & -5 \end{pmatrix}\)

\(B' A' = \begin{pmatrix} 3 & 0 & 1 \\ 4 & -1 & 2 \\ 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 + 0 + 3 & 6 + 0 + 1 \\ 4 + 0 + 2 & 8 + 1 + 2 \\ 1 + 0 - 6 & 2 - 5 - 2 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 6 & 11 \\ -5 & -5 \end{pmatrix}\)

\((AC)' = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 + 0 + 12 \\ 4 + 1 + 4 \end{pmatrix} = \begin{pmatrix} 14 \\ 9 \end{pmatrix} = \begin{pmatrix} 14 & 9 \end{pmatrix}\)

\(C' A' = \begin{pmatrix} 2 & -1 & 4 \\ 0 & -1 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 + 0 + 12 & 4 + 1 + 4 \end{pmatrix} = \begin{pmatrix} 14 & 9 \end{pmatrix}\)
3.0 Johnston, 4-3

\[
\begin{bmatrix}
0 & 1 \\
0 & 2
\end{bmatrix} \begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4 \\
B_5 \\
B_6
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 2
\end{bmatrix}
\]

Implies that B is a 2 by 3 matrix. If the 2X2 matrix premultiplying B were non-singular, we could simply invert it and premultiply the RHS by it. However, it is singular, indicating we will not obtain a unique solution for B. The easiest way to go about this is just to run through the scenarios:

No matter what, the first line of the matrix B can be anything... No part of it is incorporated into the RHS. How do we know this? Because the first column in the 2X2 matrix on the LHS consists of zeros. The second row is heavily constrained by the following equations:

First, defining \( B = \begin{bmatrix} B_1 & B_2 & B_3 \\ B_4 & B_5 & B_6 \end{bmatrix} \) we know that:

\[
\begin{align*}
1 \cdot B_4 &= 0 \\
1 \cdot B_5 &= 0 \\
1 \cdot B_6 &= 1 \\
2 \cdot B_4 &= 0 \\
2 \cdot B_5 &= 0 \\
2 \cdot B_6 &= 2
\end{align*}
\]

It is thus quite obvious that: \( B_4 = 0 \) \( B_5 = 0 \) and \( B_6 = 1 \)

Given the structure of the system, it was quite conceivable that we would get 2 conflicting answers for any of the parameters in the second row of B. This is because the 2X2 matrix only gives us one line worth of information - effectively it is a vector. Thus, we could drop either row of the 2X3 matrix on the RHS and still come up with the same solutions. We might ask ourselves, however, what would we do if the two sets of solutions from the first row and the second row of the RHS disagreed? Which would we use? [This is a trick question.]

4.0 Johnston, 4-7

J is a permutation matrix. In fact, any type of matrix with a single 1 in each row and all zeroes otherwise is a permutation matrix. (If you write some out, you will see that all matrices of this sort also have a single 1 in each column too...)

A permutation matrix rearranges the rows of some matrix A if you premultiply it with J (JA). It rearranges the columns of the matrix A if you postmultiply it with J (AJ). J-squared will be the identity matrix in this example, but that isn’t always the case. It is true that for any given permutation matrix of rank N, and take that matrix to the Nth power, you will end up with the identity matrix at least once during that cycling.
What’s the intuition behind permutation matrices? If you premultiply a permutation matrix by A (that is, take JA), look at the first row of J. The 1 will appear in the Tth column of the first row, and the rest of the numbers in the first row of J will be zeroes. When you do the matrix multiplication, the 1 will pick out each number in the Tth row of the matrix A, and will put that number in the first row of the new matrix JA. Nothing else from the A matrix will enter the first row of JA because there are zeroes everywhere else in the first row of J. We’ll work over this in section.