1.0 Johnston, 4-10

We use Cramer’s Rule for a three by three matrix to calculate the determinant here. This is just a short cut for 3X3 matrices that lets us avoid working with co-factors, but ends up with the same thing (you can do this by assigning letters to the elements of the matrix, and working through a 3X3 matrix using the cofactor expansion method).

\[ |A| = 6 + 189 + 24 - 84 - 54 - 6 = 75 \]

\[ |E| = 0 + 0 + 0 - 0 - 1 - 0 = -1 \]

Note that the matrix E is a permutation matrix. It switches row 1 and row 2 when it pre-multiplies any 3X3 matrix. We know from switching rows, that this will cause the determinant to flip sign. This is one way to prove it (taking for now as given the theorem that the determinant of a multiple of two matrices is equal to the multiple of the determinants of those matrices). Here we show it using an example:

\[ |B| = \begin{vmatrix} 2 & 6 & 9 \\ 1 & 3 & 2 \\ 7 & 6 & 1 \end{vmatrix} = 6 + 84 + 54 - 189 - 24 - 6 = -75 \]

2.0 Johnston, 4-13

I am going to simply outline a proof here: When you add or subtract one line (or some multiple of one line) from another, you do not change the determinant of a matrix. Thus, we can subtract a multiple of the bottom row in an upper triangular matrix from the second to bottom row, and thereby eliminate the off diagonal element. Specifically, we add

\[-\frac{a_{(n-1),n}}{a_{nn}} \text{ times the bottom row, to the second to bottom row. This in effect will add zero to all of the elements in the second to bottom row except the last one, and will add} \]

\[-\frac{a_{(n-1),n}}{a_{nn}} \cdot a_{n,n} = -a_{(n-1),n} \text{ to the last element in the second to last row (which equals } a_{(n-1),n}) \]

This will cancel the last term in that row out, leaving only the on-diagonal element equal to non-zero. We can recursively apply this rule all the way up the matrix, working from the bottom to the top, and eliminate all of the off diagonal elements without changing any of the on diagonal elements. We further know that the determinant of any
diagonal matrix is just the product of the diagonal elements. Thus, we can now say that the determinant of any triangular matrix is the product of the diagonal elements.

Note: what happens if one of the diagonal elements in a triangular matrix is zero? Suddenly, we can’t eliminate one or more of the off diagonal pieces. It turns out that we are OK, because if one of the on-diagonal pieces is zero, then we can show that that row is collinear with another row, and hence we know the matrix is singular, and the determinant is zero (thus, no inverse).

Recursive matrices with no zero elements on the diagonal are very important, because systems of equations involving them are easily solved, and easily manipulated. The inverse is quite easy to calculate, but often this isn’t even necessary when solving a system of equations represented by a triangular matrix. We can simply begin with the shortest row, which solves out easily, and then substitute recursively into upper rows.

3.0 Johnston, 4-16

This is an important question - it stands at the root of forming confidence intervals for the basic OLS model.

First, consider the quadratic form $u' Au$. We know this collapses to a single number, because we can write the dimensions:

$$(1 \times n)(n \times n)(n \times 1) \Rightarrow 1 \times 1$$

This is always a useful exercise when looking at matrix multiplication. Now we wonder what the single scalar value output looks like? Consider a two by two case:

$$\begin{bmatrix} u_1 & u_2 \\ a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1a_1 + u_2a_3 \\ u_1a_2 + u_2a_4 \end{bmatrix}$$

$$u_1a_1u_1 + u_2a_3u_1 + u_1a_2u_2 + u_2a_4u_2 = u_1^2a_1 + (a_2 + a_3)u_2u_1 + u_2^2a_4$$

This is obviously a quadratic form. Next, we take the expectation for the two by two case:

$$E[u_1^2a_1 + (a_2 + a_3)u_2u_1 + u_2^2a_4] = \sigma^2a_1 + 0 + \sigma^2a_4 = (a_1 + a_2)\sigma^2$$

We can generalize this to n by n matrices using summation notation. The quadratic expression using summation notation is:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} u_i u_j a_{i, j} = \sum_{i=1}^{n} u_i^2 a_{i, i} + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} u_i u_j a_{i, j} + \sum_{i=1}^{n} \sum_{j=1}^{n} u_i^2 a_{j, j}$$
Taking the expectation, we notice that the terms with cross products between \( u_i \) and \( u_j \) are all multiplied by an off-diagonal element of the matrix \( A \). The terms that involve the square of \( u_i \) are all multiplied by an on-diagonal element of the matrix \( A \) (that is, the \( i \)'th row of \( A \) and the \( i \)'th column of \( A \)). The elements of \( A \) are constants, so when we take the expectation all the non-diagonal pieces fall out (equal zero), while we add up the diagonal pieces. The expectation of the squared \( u_i \) terms are equal to \( \sigma^2 \) across all the \( i \)'s, so we gather the diagonal elements of \( A \) into the sum, and end up with:

\[
E\left[ \sum_{i=1}^{n} u_i^2 a_{i,i} \right] = \sigma^2 \sum_{i=1}^{n} a_{i,i}
\]

This is our outline of the proof. Note that we can summarize the conditions given in the problem with the one matrix condition:

\[
E[uu'] = \sigma^2 I
\]

As you can see, this is a very loaded condition.

**4.0 Johnston, 4-22**

These are at first glance quadratic forms. Note that there are three \( x \)'s in each equation, which indicates that they are quadratic forms from a three by three matrix. We could write these as:

\[
x'Ax = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

Unfortunately, we don’t know the particular values of the \( a \)'s in the \( A \) matrix. We could try to find them, and then we could take the determinant and sub-determinants to see if its positive or not. Unfortunately, while we do know the values on the diagonal for certain, we only have three values to account for the six off diagonal pieces. Ugh. We can only get the sum of two off diagonal pieces, so we might write (for part b for instance):
Unfortunately, this is a tough nut to crack too. We’d need to take the determinant and show that it’s positive, and that every subdeterminant is positive.

An easier way, for a three by three, may be to look at the scalar version of the quadratic equation and see if we can show that it is always positive. For instance, if our scalar form in a two by two case was: $x_1^2 + y_1^2$ then we would know that this was from a positive definite matrix $A$. (In fact, this would be from the identity matrix - as you might have noticed, an identity matrix yields a perfect circle when used in a quadratic form.) It’s quite obvious in this two by two example because neither piece can be negative.

It’s usually easier to prove non-positive definiteness than positive definiteness. All I need to do is find one example of a negative result, which in these cases is pretty easy:

(a) We notice here that the coefficient on $x_1^2$ is rather small in comparison to the other coefficients, so we focus our attention on high negative values for $x_1$. We’d also like both $x_2$ and $x_3$ both to be negative or both positive - in this case we choose positive. Let’s try a value of negative 2 for $x_1$ and 1 for the other two $x$’s. This would yield:

$$24 + 49 + 51 - 82 - 40 + 8 = 10$$

Nope, didn’t work. Let’s try negative 2 for $x_1$ and 0.5 for the other two $x$’s:

$$1.5 + 12.25 + 12.75 - 20.5 - 20 + 4 = -10$$

So clearly this expression is not positive definite.

(b) Let’s try to see if we can find a combination of $x$’s that yields a negative result. Here’s our strategy - the coefficient on $x_3^2$ is small, but the coefficients on the cross products involving $x_3$ are large. What if we set $x_3 = -4$, and then set $x_1 = 1$ and $x_2 = 1$. When we multiply this out, we get:

$$4 + 9 + 32 - 32 - 24 + 6 = -5$$

Clearly the expression is not from a positive definite quadratic matrix form. Note that if we had chosen a value $x_3 = -10$ we would have gotten a positive result. Why? It has to do with the square term, and the coefficients on the cross-product. Consider the simple
positive definite quadratic form \( x^2 + 2xy + y^2 = (x + y)^2 \). The reason this is positive definite is because whenever we have a negative value for \( x \) and a positive value for \( y \), we can’t raise the \( 2xy \) piece faster than the squared pieces.

But what if we couldn’t find some combination to show non-positive definiteness. Then what? How would we positive definiteness without testing every feasible combination of \( x \)’s? Ideally, we could write this expression as a simple quadratic in scalar notation:

\[
(a_1x_1 + a_2x_2 + a_3x_3)^2.
\]

If we could, then we would know for certain that the expression is always positive. We can make a simple attempt with:

\[
(\sqrt{6}x_1 - 7x_2 + \sqrt{51}x_3)^2 = 6x_1^2 + 49x_2^2 + 51x_3^2 + 2(-7\sqrt{6}x_1x_2 + \sqrt{306}x_1x_3 - 7\sqrt{51}x_2x_3)
\]

(Note that we chose -7 on the middle \( x \) because both cross product terms with \( x_2 \) in our original expression had a negative sign.) We use our calculator to write out the square roots:

\[
(\sqrt{6}x_1 - 7x_2 + \sqrt{51}x_3)^2 = 6x_1^2 + 49x_2^2 + 51x_3^2 - 34.3x_1x_2 + 35x_1x_3 - 100x_2x_3
\]

We know that the term inside the quadratic can equal zero if all the \( x \)’s are zero. We know wonder what happens if we take out quadratic and subtract the one from the other. We might then subtract expression (a) from this new expression, and we’d end up with:

\[-30.3x_1x_2 + 15x_1x_3 - 18x_2x_3\]

Unfortunately, it’s not easy to see how this would behave - it could be negative or positive. Another method might be to try to wrap up all of the cross products in 2 by 2 quadratics, and show that the sum of these is less than the expression in part (a). Thus, we might try:

\[(2x_1 - x_2) = 4x_1^2 - 4x_1x_2 + x_2^2\]

This would exactly account for one of the cross product pieces. We might then see if we can construct a way to account for the other cross product pieces with similar expressions, add the expressions together, and see if we can get all positive values remaining on the quadratics.