A Continuation of the Example of May 12 2003

This problem is to be considered a practice problem for the Final Exam. It is longer and more computationally-intensive than what I would typically put on an exam, but it contains many of the elements that I would hope that you would be able to do. It is also typical of an exam problem in that you will be directed to use certain techniques at certain points, and when it comes to evaluating integrals, the needed substitutions (in this case the double-angle formulas) are given. It should be noted that these integrals are similar to those that were not quite done on May 13 due to time constraints.

Recall that we had found the eigenvalues of the matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\omega^2 & 0 & -(\omega^2 + \omega^2) & 0
\end{pmatrix}
\]

to be \(\pm i \omega^2, \pm i \omega^2\). So, if \(\omega^2 \neq \omega^2\), the eigenvalues are distinct and hence independent, and a similarity transform may be used to diagonalize the above matrix (see the notes Similarity Transformations with Complex Eigenvalues for an outline of how this could be done).

If \(\omega^2 = \omega^2\), it was stated without proof that with the repeated roots of \(\pm i \omega\), the eigenvalues were defective, and so the above matrix could not be diagonalized, although a similarity transformation would give a matrix that could be exponentiated with only a little difficulty. What is outlined here for you to do is to find the non-eigenvector solutions by using reduction of order.

To make life easy, set \(\omega_c = \omega = 1\), so that the above matrix becomes

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & -2 & 0
\end{bmatrix}.
\]

Start by showing that for the matrix given above, \(\det(A - \lambda I) = (\lambda^2 + 1)^2\).

We wish to solve \(x' = Ax\). Show that the two vectors

\[
x_{\alpha} = \begin{bmatrix}
cost \\
-sint \\
-cost \\
sint
\end{bmatrix} \quad x_{\beta} = \begin{bmatrix}
sint \\
cost \\
-sint \\
-cost
\end{bmatrix}
\]
are solutions to \( x' = A x \). Now, look for another solution of the form
\[
x_\gamma = \begin{bmatrix} x_\alpha & x_\beta \end{bmatrix}
\]
\[
z = \begin{bmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ -\cos t & -\sin t & 1 & 0 \\ \sin t & -\cos t & 0 & 1 \end{bmatrix} z = \begin{bmatrix} -R(-t) & 0_2 \\ R(-t) & I_2 \end{bmatrix} z,
\]
where \( R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \), \( 0_2 \) is the \( 2 \times 2 \) zero matrix and \( I_2 \) is the \( 2 \times 2 \) identity matrix.

Now, do the drill; assume \( x'_\gamma = A x_\gamma \), differentiate the given form for \( x_\gamma \), make the necessary substitutions and so arrive at an expression of the form \( z' = B z \), where \( B \) is a non-constant matrix whose first two columns are zero and whose third and fourth columns are quite simple.

There are many ways to do this, the easiest being to recognize that in the form of \( x_\gamma \), the first two columns were constructed especially so that great cancellation occurs. Also, you will need at some point to find the inverse of the matrix
\[
C = \begin{bmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ -\cos t & -\sin t & 1 & 0 \\ \sin t & -\cos t & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -R(-t) & 0_2 \\ R(-t) & I_2 \end{bmatrix}.
\]

To do this in a slick way, consider that you have a matrix of the form
\[
C = \begin{bmatrix} A_{11} & 0_2 \\ A_{21} & I_2 \end{bmatrix},
\]
where \( A_{11} \) and \( A_{21} \) are \( 2 \times 2 \) matrices. Show that
\[
C^{-1} = \begin{bmatrix} A_{11}^{-1} & 0_2 \\ -A_{21} A_{11}^{-1} & I_2 \end{bmatrix}
\]
by showing that \( C^{-1} C = C C^{-1} = I_4 \) (here, \( I_4 \) is the \( 4 \times 4 \) identity matrix). Then, use the fact that \( [R(\theta)]^{-1} = R(-\theta) \) to find a neat, clean expression for \( C^{-1} \).

Your expression for \( B \) should allow solving for \( z_3 \) and \( z_4 \) almost immediately; guessing, or recognizing that we’ve done this problem lots of times is allowed. Then, integrate your expressions for \( z'_1 \) and \( z'_2 \) to find \( z_1 \) and \( z_2 \).

**HINTS:** If you’ve followed the above procedure, you should get \( z'_1 \) and \( z'_2 \) as functions of \( z_3 \) only (not \( z_4 \), but with entries of \( B \) which are simple functions of time).

Further, in integrating your expressions for \( z'_1 \) and \( z'_2 \), you will want to use the double-angle formulas
\[
\sin t \cos t = \frac{1}{2} \sin 2t, \quad \sin^2 t = \frac{1}{2} (1 - \cos 2t), \quad \cos^2 t = \frac{1}{2} (1 + \cos 2t).
\]
So, now go back to $\mathbf{x}_\gamma = \mathbf{C} \mathbf{z}$ to find at least one more solution. If you’ve been clever, you should get at least two solutions, and if you get more than two, you should recognize that they duplicate $x_\alpha$ and $x_\beta$. If not, see if you can simplify your expressions, especially if you find yourself with products of \{sin $t$, cos $t$\} with \{sin $2t$, cos $2t$\}, perhaps using the above double-angle formulas in a constructive manner.