More on Complex Eigenvalues

The most general form used for solutions of linear systems that involve complex eigenvalues is mentioned in the text, but without a detailed explanation. A simpler form, when the eigenvalues are purely imaginary, is addressed in the notes Complex Eigenvectors. The example considered here began as a question by a student, and was supposed to be somewhere in between the extremes, and even then, as may be seen, the algebra becomes unwieldy. I’d like to use these notes to show how the algebra may be tamed somewhat, and more importantly, that the several different methods yield the same solution.

The system under consideration was

$$x' = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} A \\ B \end{bmatrix},$$

where $A$ and $B$ are arbitrary (but real; that will be important). Here, boldface $x$ is a column vector, and the prime denotes differentiation with respect to $t$.

So, the two complex eigenvalues

$$\lambda_\alpha = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \lambda_\beta = -\frac{1}{2} - \frac{i\sqrt{3}}{2},$$

are easily found. (Note that I switched the subscripts to $\alpha, \beta$. I’ve been wanting to do this all along, and now’s my chance.) For the above eigenvalues, we have

$$\lambda_\alpha \lambda_\beta = 1, \quad \lambda_\alpha + \lambda_\beta = -1, \quad \lambda_\alpha - \lambda_\beta = i\sqrt{3}, \quad \lambda_\alpha = \overline{\lambda_\beta},$$

where in the last relation, the text’s notation for the complex conjugate has been used. The alternate notation,

$$\lambda_\alpha = e^{i2\pi/3}, \quad \lambda_\beta = e^{-i2\pi/3},$$

while still valid, is not going to be used in these notes (but you are welcome to rederive everything using this notation).

A minor (that means do it yourself) amount of algebra leads to the corresponding eigenvectors

$$x_\alpha = \begin{bmatrix} 1 \\ \lambda_\beta \end{bmatrix}, \quad x_\beta = \begin{bmatrix} 1 \\ \lambda_\alpha \end{bmatrix},$$
where those subscripts are not typos; in finding this expression for the eigenvectors, the relations $-1 - \lambda_\alpha = \lambda_\beta$, $-1 - \lambda_\beta = \lambda_\alpha$ have been used. So, we have the general solution

$$x(t) = C_1 x_\alpha e^{\lambda_\alpha t} + C_2 x_\beta e^{\lambda_\beta t} = C_1 \begin{bmatrix} 1 \\ \lambda_\beta \end{bmatrix} e^{\lambda_\alpha t} + C_2 \begin{bmatrix} 1 \\ \lambda_\alpha \end{bmatrix} e^{\lambda_\beta t}. \quad (*)$$

It’s important to realize that in the above, the elements of the eigenvectors are complex, and so the constants $C_1$ and $C_2$ must be complex if $A$ and $B$ are to be real, and this property is what can cause discomfort in doing the algebra. In what follows, the most difficult way will be done first, in the hopes of convincing you that the less elegant way is the easier. It happens.

From the initial conditions, we have that

$$C_1 \begin{bmatrix} 1 \\ \lambda_\beta \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix},$$

from which (again, do it yourself to make sure)

$$C_1 = \frac{A \lambda_\alpha - B}{\lambda_\alpha - \lambda_\beta}, \quad C_2 = \frac{B - A \lambda_\beta}{\lambda_\alpha - \lambda_\beta}.$$

Inserting these into Equation $(*)$ above, and collecting terms with $A$ and $B$,

$$x(t) = \frac{e^{-t/2}}{\lambda_\alpha - \lambda_\beta} \left( A \begin{bmatrix} \lambda_\alpha e^{i \sqrt{3} t} - \lambda_\beta e^{-i \sqrt{3} t} \\ e^{i \sqrt{3} t} - e^{-i \sqrt{3} t} \end{bmatrix} + B \begin{bmatrix} -e^{i \sqrt{3} t} + e^{-i \sqrt{3} t} \\ -\lambda_\beta e^{i \sqrt{3} t} + \lambda_\alpha e^{-i \sqrt{3} t} \end{bmatrix} \right). \quad (***)$$

Phew. What a mess. Before giving up, though, note that two of the terms reduce quite nicely to $\pm 2i \sin \left( \left(\sqrt{3}/2 \right) t \right)$. So emboldened, the others can be put in terms of sines and cosines to arrive at

$$x(t) = e^{-t/2} \left( A \begin{bmatrix} \cos \frac{\sqrt{3}}{2} t - \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \\ \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \end{bmatrix} + B \begin{bmatrix} -\frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \\ \cos \frac{\sqrt{3}}{2} t + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \end{bmatrix} \right). \quad (***)$$

Well, good for us. However, the above is clearly far more complicated than we need. Before going on to the most direct method, take a minute to see what happens if we try to use a fundamental matrix. We would have

$$\Phi(t) = \begin{bmatrix} e^{\lambda_\alpha t} & e^{\lambda_\beta t} \\ \lambda_\beta e^{\lambda_\alpha t} & \lambda_\alpha e^{\lambda_\beta t} \end{bmatrix}, \quad \Phi(0) = \begin{bmatrix} 1 & 1 \\ \lambda_\beta & \lambda_\alpha \end{bmatrix}, \quad \Phi^{-1}(0) = \frac{1}{\lambda_\alpha - \lambda_\beta} \begin{bmatrix} \lambda_\alpha & -1 \\ -\lambda_\beta & 1 \end{bmatrix}.$$

Another matrix multiplication gives

$$\Phi(t) \Phi^{-1}(0) = \frac{1}{\lambda_\alpha - \lambda_\beta} \begin{bmatrix} \lambda_\alpha e^{\lambda_\alpha t} - \lambda_\beta e^{\lambda_\beta t} & -e^{\lambda_\alpha t} + e^{\lambda_\beta t} \\ \lambda_\beta e^{\lambda_\alpha t} - \lambda_\alpha e^{\lambda_\beta t} & e^{\lambda_\alpha t} - e^{\lambda_\beta t} \end{bmatrix}.$$
and another bout of algebra gives a result equivalent to (***)

Before going on, take another look at (*) above, and note that it could be rewritten as

\[ \mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ \lambda_{\beta} \end{bmatrix} e^{\lambda_{\alpha} t} + C_2 \begin{bmatrix} 1 \\ \lambda_{\beta} \end{bmatrix} \overline{e^{\lambda_{\alpha} t}} \]

and that for this to be real, \( C_2 = \overline{C_1} \); the truth is, we got this earlier, but didn’t say so explicitly. Applying the initial conditions,

\[ \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} C_1 + \overline{C_1} \\ \lambda_{\beta} C_1 + \overline{\lambda_{\beta} C_1} \end{bmatrix} \]

From this, we could solve for

\[ C_1 = \frac{A}{2} - i \frac{A + 2B}{2\sqrt{3}}, \]

which would still leave us with some algebra. This is completely equivalent to saying that two independent solutions are the real and imaginary parts of

\[ \begin{bmatrix} 1 \\ \lambda_{\beta} \end{bmatrix} e^{i\lambda_{\alpha} t}, \]

and this is the motivation for the text’s declaration of Equation (22) on Page 365.

Much along the same lines, we could take a deep breath after finding

\[ \lambda_{\alpha} = -\frac{1}{2} + i \frac{\sqrt{3}}{2}, \quad \lambda_{\beta} = -\frac{1}{2} - i \frac{\sqrt{3}}{2} \]

and acknowledge that any solution must be of the form

\[ \mathbf{x}(t) = e^{-t/2} \begin{bmatrix} a_1 \cos \omega t + a_2 \sin \omega t \\ b_1 \cos \omega t + b_2 \sin \omega t \end{bmatrix}, \]

where \( \omega = \frac{\sqrt{3}}{2} \). Then, we can cut to the chase, as it were, and recognize that we must have one solution with \( a_1 = 1, b_1 = 0 \), which corresponds to \( A = 1, B = 0 \). (NOTE: the coefficients \( a_1, a_2, b_1 \) and \( b_2 \) are not quite the same as the components of \( a \) and \( b \) in Equations (20), (21) and (22) on Page 365.)

So, seeking a solution of the form

\[ \mathbf{x}_{\gamma} = e^{-t/2} \begin{bmatrix} \cos \omega t + a_2 \sin \omega t \\ b_2 \sin \omega t \end{bmatrix}, \]

\[ \mathbf{x}_{\gamma} = e^{-t/2} \begin{bmatrix} (a_2 \omega - (1/2)) \cos \omega t - (a_2/2 + \omega) \sin \omega t \\ b_2 \omega \cos \omega t - (b_2/2) \sin \omega t \end{bmatrix} \]
and substituting into the original equation gives

\[
(a_2 \omega - (1/2)) \cos \omega t - (a_2/2 + \omega) \sin \omega t = -\cos \omega t - a_2 \sin \omega t - b_2 \sin \omega t
\]

\[
b_2 \omega \cos \omega t - (b_2/2) \sin \omega t = \cos \omega t + a_2 \sin \omega t.
\]

Setting the coefficients of the sines and cosines equal to zero in each equation and solving (the second is the easier to solve) gives

\[
b_2 = \frac{1}{\omega} = \frac{2}{\sqrt{3}}, \quad a_2 = -\frac{b_2}{2} = -\frac{1}{\sqrt{3}},
\]

and this is the first column in Equation (**). A similar calculation gives the second column (it can’t hurt to check), which we would call \(x_3\).

So, take your pick. You can do the algebra more or less any way you prefer, but it must be done to solve the problem completely.