# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td></td>
<td>iii</td>
</tr>
<tr>
<td>1.</td>
<td>Rings and Ideals</td>
<td>1</td>
</tr>
<tr>
<td>2.</td>
<td>Prime Ideals</td>
<td>7</td>
</tr>
<tr>
<td>3.</td>
<td>Radicals</td>
<td>11</td>
</tr>
<tr>
<td>4.</td>
<td>Modules</td>
<td>17</td>
</tr>
<tr>
<td>5.</td>
<td>Exact Sequences</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>Appendix: Fitting Ideals</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td>Direct Limits</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>Appendix: Jacobson Rings</td>
<td></td>
</tr>
<tr>
<td>7.</td>
<td>Filtered Direct Limits</td>
<td>35</td>
</tr>
<tr>
<td>8.</td>
<td>Tensor Products</td>
<td>42</td>
</tr>
<tr>
<td>9.</td>
<td>Flatness</td>
<td>48</td>
</tr>
<tr>
<td>10.</td>
<td>Cayley-Hamilton Theorem</td>
<td>54</td>
</tr>
<tr>
<td>11.</td>
<td>Localization of Rings</td>
<td>60</td>
</tr>
<tr>
<td>12.</td>
<td>Localization of Modules</td>
<td>66</td>
</tr>
<tr>
<td>13.</td>
<td>Support</td>
<td>72</td>
</tr>
<tr>
<td>14.</td>
<td>Krull-Cohen-Seidenberg Theory</td>
<td>77</td>
</tr>
<tr>
<td>15.</td>
<td>Noether Normalization</td>
<td>84</td>
</tr>
<tr>
<td></td>
<td>Appendix: Jacobson Rings</td>
<td>88</td>
</tr>
<tr>
<td>16.</td>
<td>Chain Conditions</td>
<td>93</td>
</tr>
<tr>
<td>17.</td>
<td>Associated Primes</td>
<td>96</td>
</tr>
<tr>
<td>18.</td>
<td>Primary Decomposition</td>
<td>101</td>
</tr>
<tr>
<td>19.</td>
<td>Length</td>
<td>106</td>
</tr>
<tr>
<td>20.</td>
<td>Hilbert Functions</td>
<td>112</td>
</tr>
<tr>
<td></td>
<td>Appendix: Homogeneity</td>
<td>116</td>
</tr>
<tr>
<td>21.</td>
<td>Dimension</td>
<td>122</td>
</tr>
<tr>
<td>22.</td>
<td>Completion</td>
<td>124</td>
</tr>
<tr>
<td>23.</td>
<td>Discrete Valuation Rings</td>
<td>130</td>
</tr>
<tr>
<td></td>
<td>Appendix: Cohen-Macaulayness</td>
<td>138</td>
</tr>
<tr>
<td>24.</td>
<td>Dedekind Domains</td>
<td>143</td>
</tr>
<tr>
<td>25.</td>
<td>Fractional Ideals</td>
<td>148</td>
</tr>
<tr>
<td>26.</td>
<td>Arbitrary Valuation Rings</td>
<td>152</td>
</tr>
<tr>
<td></td>
<td>Appendix: Cohen-Macaulayness</td>
<td>157</td>
</tr>
<tr>
<td>Solutions</td>
<td></td>
<td>162</td>
</tr>
<tr>
<td></td>
<td>Appendix: Fitting Ideals</td>
<td></td>
</tr>
<tr>
<td>1.</td>
<td>Rings and Ideals</td>
<td>162</td>
</tr>
<tr>
<td>2.</td>
<td>Prime Ideals</td>
<td>164</td>
</tr>
<tr>
<td>3.</td>
<td>Radicals</td>
<td>166</td>
</tr>
<tr>
<td>4.</td>
<td>Modules</td>
<td>173</td>
</tr>
<tr>
<td>5.</td>
<td>Exact Sequences</td>
<td>175</td>
</tr>
<tr>
<td></td>
<td>Appendix: Fitting Ideals</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td>Direct Limits</td>
<td>179</td>
</tr>
<tr>
<td>7.</td>
<td>Filtered Direct Limits</td>
<td>182</td>
</tr>
<tr>
<td>8.</td>
<td>Tensor Products</td>
<td>185</td>
</tr>
<tr>
<td>9.</td>
<td>Flatness</td>
<td>188</td>
</tr>
<tr>
<td>10.</td>
<td>Cayley-Hamilton Theorem</td>
<td>191</td>
</tr>
<tr>
<td>11.</td>
<td>Localization of Rings</td>
<td>194</td>
</tr>
<tr>
<td>Chapter</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>---------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>12</td>
<td>Localization of Modules</td>
<td>198</td>
</tr>
<tr>
<td>13</td>
<td>Support</td>
<td>201</td>
</tr>
<tr>
<td>14</td>
<td>Krull-Cohen-Seidenberg Theory</td>
<td>211</td>
</tr>
<tr>
<td>15</td>
<td>Noether Normalization</td>
<td>214</td>
</tr>
<tr>
<td>16</td>
<td>Chain Conditions</td>
<td>218</td>
</tr>
<tr>
<td>17</td>
<td>Associated Primes</td>
<td>220</td>
</tr>
<tr>
<td>18</td>
<td>Primary Decomposition</td>
<td>221</td>
</tr>
<tr>
<td>19</td>
<td>Length</td>
<td>224</td>
</tr>
<tr>
<td>20</td>
<td>Hilbert Functions</td>
<td>226</td>
</tr>
<tr>
<td>21</td>
<td>Dimension</td>
<td>229</td>
</tr>
<tr>
<td>22</td>
<td>Completion</td>
<td>232</td>
</tr>
<tr>
<td>23</td>
<td>Discrete Valuation Rings</td>
<td>236</td>
</tr>
<tr>
<td>24</td>
<td>Dedekind Domains</td>
<td>241</td>
</tr>
<tr>
<td>25</td>
<td>Fractional Ideals</td>
<td>243</td>
</tr>
<tr>
<td>26</td>
<td>Arbitrary Valuation Rings</td>
<td>245</td>
</tr>
<tr>
<td></td>
<td><strong>Bibliography</strong></td>
<td>249</td>
</tr>
<tr>
<td></td>
<td>Disposition of the Exercises in [3]</td>
<td>250</td>
</tr>
<tr>
<td></td>
<td><strong>Index</strong></td>
<td>253</td>
</tr>
</tbody>
</table>
Preface

There is no shortage of books on Commutative Algebra, but the present book is different. Most books are monographs, with extensive coverage. But there is one notable exception: Atiyah and Macdonald’s 1969 classic [3]. It is a clear, concise, and efficient textbook, aimed at beginners, with a good selection of topics. So it has remained popular. However, its age and flaws do show. So there is need for an updated and improved version, which the present book aims to be.

Atiyah and Macdonald explain their philosophy in their introduction. They say their book “has the modest aim of providing a rapid introduction to the subject. It is designed to be read by students who have had a first elementary course in general algebra. On the other hand, it is not intended as a substitute for the more voluminous tracts on Commutative Algebra. . . . The lecture-note origin of this book accounts for the rather terse style, with little general padding, and for the condensed account of many proofs.” They “resisted the temptation to expand it in the hope that the brevity of [the] presentation will make clearer the mathematical structure of what is by now an elegant and attractive theory.” They endeavor “to build up to the main theorems in a succession of simple steps and to omit routine verifications.”

Their successful philosophy is wholeheartedly embraced below (it is a feature, not a flaw!), and also refined a bit. The present book also “grew out of a course of lectures.” That course was based primarily on their book, but has been offered a number of times, and has evolved over the years, influenced by other publications and the reactions of the students. Their book comprises eleven chapters, split into forty-two sections. The present book comprises twenty-six sections; each represents a single lecture, and is self-contained.

Atiyah and Macdonald “provided . . . exercises at the end of each chapter.” They “provided hints, and sometimes complete solutions, to the hard” exercises. Moreover, they developed a significant amount of the main content in the exercises. By contrast, in the present book, the exercises are integrated into the development, and complete solutions are given at the end of the book. Doing so lengthened the book considerably. In particular, it led to the addition of appendices on Fitting Ideals and on Cohen–Macaulayness. (All four appendices elaborate on important issues arising in the main text.)

There are 324 exercises below. They include about half the exercises in Atiyah and Macdonald’s book; eventually, all will be handled. The disposition of those exercises is indicated in a special index preceding the main index. The 324 also include many exercises that come from other publications and many that originate here. Here the exercises are tailored to provide a means for students to check, to solidify, and to expand their understanding of the material. The exercises are intentionally not difficult, tricky, or involved. Rarely do they introduce new techniques, although some introduce new concepts and many statements are used later.

Students are encouraged to try to solve each and every exercise, and to do so before looking up its solution. If they become stuck, then they should review the relevant material; if they remain stuck, then they should change tack by studying the given solution, possibly discussing it with others, but always making sure they can eventually solve the whole exercise entirely on their own. In any event, students
should read the given solution, even if they think they already know it, just to make sure; also, some exercises provide enlightening alternative solutions.

Instructors are encouraged to examine their students, possibly orally at a blackboard, possibly via written tests, on a small, randomly chosen subset of all the exercises that have been assigned over the course of the term for the students to write up in their own words. For use during each exam, instructors should provide students with a special copy of the book that does include the solutions.

Atiyah and Macdonald explain that “a proper treatment of Homological Algebra is impossible within the confines of a small book; on the other hand, it is hardly sensible to ignore it completely.” So they “use elementary homological methods — exact sequence, diagrams, etc. — but . . . stop short of any results requiring a deep study of homology.” Again, their philosophy is embraced and refined in the present book. Notably, below, elementary methods are used, not Tor’s as they do, to prove the Ideal Criterion for flatness, and to relate flat modules and free modules over local rings. Also, projective modules are treated below, but not in their book.

In the present book, Category Theory is a basic tool; in Atiyah and Macdonald’s, it seems like a foreign language. Thus they discuss the universal (mapping) property (UMP) of localization of a ring, but provide an ad hoc characterization. They also prove the UMP of tensor product of modules, but do not name it this time. Below, the UMP is fundamental: there are many standard constructions; each has a UMP, which serves to characterize the resulting object up to unique isomorphism owing to one general observation of Category Theory. For example, the Left Exactness of Hom is viewed simply as expressing in other words that the kernel and the cokernel of a map are characterized by their UMPs; by contrast, Atiyah and Macdonald prove the Left Exactness via a tedious elementary argument.

Atiyah and Macdonald prove the Adjoint-Associativity Formula. They note it says that Tensor Product is the left adjoint of Hom. From it and the Left Exactness of Hom, they deduce the Right Exactness of Tensor Product. They note that this derivation shows that any “left adjoint is right exact.” More generally, as explained below, this derivation shows that any left adjoint preserves arbitrary direct limits, ones indexed by any small category. Atiyah and Macdonald consider only direct limits indexed by a directed set, and sketch an ad hoc argument showing that tensor product preserves direct limit. Also, arbitrary direct sums are direct limits indexed by a discrete category (it is not a directed set); hence, the general result yields that Tensor Product and other left adjoints preserve arbitrary Direct Sum.

Below, left adjoints are proved unique up to unique isomorphism. Therefore, the functor of localization of a module is canonically isomorphic to the functor of tensor product with the localized base ring, as both are left adjoints of the same functor, Restriction of Scalars from the localized ring to the base ring. There is an alternative argument. Since Localization is a left adjoint, it preserves Direct Sum and Cokernel; whence, it is isomorphic to that tensor-product functor by Watts Theorem, which characterizes all tensor-product functors as those linear functors that preserve Direct Sum and Cokernel. Atiyah and Macdonald’s treatment is ad hoc. However, they do use the proof of Watts Theorem directly to show that, under the appropriate conditions, Completion of a module is Tensor Product with the completed base ring.

Below, Direct Limit is also considered as a functor, defined on the appropriate category of functors. As such, Direct Limit is a left adjoint. Hence, direct limits
preserve other direct limits. Here the theory briefly climbs to a higher level of abstraction. The discussion is completely elementary, but by far the most abstract in the book. The extra abstraction can be difficult, especially for beginners.

Below, filtered direct limits are treated too. They are closer to the kind of limits treated by Atiyah and Macdonald. In particular, filtered direct limits preserve exactness and flatness. Further, they appear in the following lovely form of Lazard’s Theorem: in a canonical way, every module is the direct limit of free modules of finite rank; moreover, the module is flat if and only if that direct limit is filtered.

Atiyah and Macdonald treat primary decomposition in a somewhat dated fashion. First, they study primary decompositions of ideals in rings. Then, in the exercises, they indicate how to translate the theory to modules. The decompositions need not exist, as the rings and modules need not be Noetherian. Associated primes play a secondary role: they are defined as the radicals of the primary components, and then characterized as the primes that are the radicals of annihilators of elements. Finally, they prove that, when the rings and modules are Noetherian, decompositions exist and the associated primes are annihilators. To prove existence, they use irreducible modules. Nowadays, associated primes are normally defined as prime annihilators of elements, and studied on their own at first; sometimes, as below, irreducible modules are not considered at all in the main development.

There are several other significant differences between Atiyah and Macdonald’s treatment and the one below. First, the Noether Normalization Lemma is proved below in a stronger form for nested sequences of ideals; consequently, for algebras that are finitely generated over a field, dimension theory can be developed directly without treating Noetherian local rings first. Second, in a number of results below, the modules are assumed to be finitely presented over an arbitrary ring, rather than finitely generated over a Noetherian ring. Third, there is an elementary treatment of regular sequences below and a proof of Serre’s Criterion for Normality. Fourth, below, the Adjoint-Associativity Formula is proved over a pair of base rings; hence, it yields both a left and a right adjoint to the functor of restriction of scalars.

The present book is a second beta edition. Please do the community a service by sending the authors comments and corrections. Thanks!

Allen B. Altman and Steven L. Kleiman
31 August 2013
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1. Rings and Ideals

We begin by reviewing basic notions and conventions to set the stage. Throughout this book, we emphasize universal mapping properties (UMPs); they are used to characterize notions and to make constructions. So, although polynomial rings and residue rings should already be familiar in other ways, we present their UMPs immediately, and use them extensively. We close this section with a brief treatment of idempotents and the Chinese Remainder Theorem.

(1.1) (Rings). — Recall that a ring $R$ is an abelian group, written additively, with an associative multiplication that is distributive over the addition.

Throughout this book, every ring has a multiplicative identity, denoted by 1.

Further, every ring is commutative (that is, $xy = yx$ in it), with an occasional exception, which is always marked (normally, it’s a ring of matrices).

As usual, the additive identity is denoted by 0. Note that, for any $x$ in $R$,

$$x \cdot 0 = 0;$$

indeed, $x \cdot 0 = x(0 + 0) = x \cdot 0 + x \cdot 0$, and $x \cdot 0$ can be canceled by adding $-(x \cdot 0)$.

We allow 1 = 0. If 1 = 0, then $R = 0$; indeed, $x = x \cdot 1 = x \cdot 0 = 0$ for any $x$.

A unit is an element $u$ with a reciprocal $1/u$ such that $u \cdot 1/u = 1$. Alternatively, $1/u$ is denoted $u^{-1}$ and is called the multiplicative inverse of $u$. The units form a multiplicative group, denoted $R^\times$.

For example, the ordinary integers form a ring $\mathbb{Z}$, and its units are 1 and $-1$.

A ring homomorphism, or simply a ring map, $\varphi: R \to R'$ is a map preserving sums, products, and 1. Clearly, $\varphi(R^\times) \subset R'^\times$. We call $\varphi$ an isomorphism if it is bijective, and then we write $\varphi: R \cong R'$. We call $\varphi$ an endomorphism if $R' = R$. We call $\varphi$ an automorphism if it is bijective and if $R' = R$.

If there is an unnamed isomorphism between rings $R$ and $R'$, then we write $R = R'$ when it is canonical; that is, it does not depend on any artificial choices, so that for all practical purposes, $R$ and $R'$ are the same — they are just copies of each other. For example, the polynomial rings $R[X]$ and $R[Y]$ in variables $X$ and $Y$ are canonically isomorphic when $X$ and $Y$ are identified. (Recognizing that an isomorphism is canonical can provide insight and obviate verifications. The notion is psychological, and depends on the context.) Otherwise, we write $R \cong R'$.

A subset $R'' \subset R$ is a subring if $R''$ is a ring and the inclusion $R'' \hookrightarrow R$ a ring map. For example, given a ring map $\varphi: R \to R'$, its image $\text{Im}(\varphi) := \varphi(R)$ is a subring of $R'$.

An $R$-algebra is a ring $R'$ that comes equipped with a ring map $\varphi: R \to R'$, called the structure map. An $R$-algebra homomorphism, or $R$-algebra map, $R' \to R''$ is a ring map between $R$-algebras compatible with their structure maps.

(1.2) (Boolean rings). — The simplest nonzero ring has two elements, 0 and 1. It is unique, and denoted $\mathbb{F}_2$.

Given any ring $R$ and any set $X$, let $R^X$ denote the set of functions $f: X \to R$. Then $R^X$ is, clearly, a ring under valuewise addition and multiplication.

For example, take $R := \mathbb{F}_2$. Given $f: X \to R$, put $S := f^{-1}\{1\}$. Then $f(x) = 1$ if $x \in S$, and $f(x) = 0$ if $x \notin S$; in other words, $f$ is the characteristic function $\chi_S$. Thus the characteristic functions form a ring, namely, $\mathbb{F}_2^X$. 

1
2 Rings and Ideals (1.4)

Given $T \subset X$, clearly $\chi_S \cdot \chi_T = \chi_{S \cap T}$. Further, $\chi_S + \chi_T = \chi_{S \Delta T}$, where $S \Delta T$ is the symmetric difference:

$$S \Delta T := (S \cup T) - (S \cap T) = (S - T) \cup (T - S);$$

here $S - T$ denotes, as usual, the set of elements of $S$ not in $T$. Thus the subsets of $X$ form a ring: sum is symmetric difference, and product is intersection. This ring is canonically isomorphic to $\mathbb{F}_2^X$.

A ring $B$ is said to be Boolean if $f^2 = f$ for all $f \in B$. Clearly, $\mathbb{F}_2^X$ is Boolean.

Suppose $X$ is a topological space, and give $\mathbb{F}_2$ the discrete topology; that is, every subset is both open and closed. Consider the continuous functions $f : X \to \mathbb{F}_2$. Clearly, they are just the $\chi_X$ where $S$ is both open and closed. Clearly, they form a Boolean subring of $\mathbb{F}_2^X$. Conversely, Stone’s Theorem (13.25) asserts that every Boolean ring is canonically isomorphic to the ring of continuous functions from a compact Hausdorff topological space $X$ to $\mathbb{F}_2$, or equivalently, isomorphic to the ring of open and closed subsets of $X$.

(1.3) (Polynomial rings). — Let $R$ be a ring, $P := R[X_1, \ldots, X_n]$ the polynomial ring in $n$ variables (see [24, pp. 352–3] or [3, p. 268]). Recall that $P$ has this Universal Mapping Property (UMP): given a ring map $\varphi : R \to R'$ and given an element $x_i$ of $R'$ for each $i$, there is a unique ring map $\pi : P \to R'$ with $\pi|R = \varphi$ and $\pi(X_i) = x_i$. In fact, since $\pi$ is a ring map, necessarily $\pi$ is given by the formula:

$$\pi \left( \sum a_{(i_1, \ldots, i_n)} X_1^{i_1} \cdots X_n^{i_n} \right) = \sum \varphi(a_{(i_1, \ldots, i_n)}) X_1^{i_1} \cdots X_n^{i_n}.$$ 

In other words, $P$ is universal among $R$-algebras equipped with a list of $n$ elements: $P$ is one, and it maps uniquely to any other.

Similarly, let $P' := R[\{X_\lambda\}_{\lambda \in \Lambda}]$ be the polynomial ring in an arbitrary list of variables: its elements are the polynomials in any finitely many of the $X_\lambda$; sum and product are defined as in $P$. Thus $P'$ contains as a subring the polynomial ring in any finitely many $X_\lambda$, and $P'$ is the union of these subrings. Clearly, $P'$ has essentially the same UMP as $P$: given $\varphi : R \to R'$ and given $x_\lambda \in R'$ for each $\lambda$, there is a unique $\pi : P' \to R'$ with $\pi|R = \varphi$ and $\pi(X_\lambda) = x_\lambda$.

(1.4) (Ideals). — Let $R$ be a ring. Recall that a subset $a$ is called an ideal if

1. $0 \in a$,
2. whenever $a, b \in a$, also $a + b \in a$, and
3. whenever $x \in R$ and $a \in a$, also $xa \in a$.

Given elements $a_\lambda \in R$ for $\lambda \in \Lambda$, by the ideal $\langle a_\lambda \rangle_{\lambda \in \Lambda}$ they generate, we mean the smallest ideal containing them all. If $\Lambda = \emptyset$, then this ideal consists just of $0$.

Any ideal containing all the $a_\lambda$ contains any (finite) linear combination $\sum x_\lambda a_\lambda$ with $x_\lambda \in R$ and almost all 0. Form the set $a$, or $\sum Ra_\lambda$, of all such linear combinations; clearly, $a$ is an ideal containing all $a_\lambda$. Thus $a$ is the ideal generated by the $a_\lambda$.

Given a single element $a$, we say that the ideal $\langle a \rangle$ is principal. By the preceding observation, $\langle a \rangle$ is equal to the set of all multiples $xa$ with $x \in R$.

Similarly, given ideals $a_\lambda$ of $R$, by the ideal they generate, we mean the smallest ideal $\sum a_\lambda$ that contains them all. Clearly, $\sum a_\lambda$ is equal to the set of all finite linear combinations $\sum x_\lambda a_\lambda$ with $x_\lambda \in R$ and $a_\lambda \in a_\lambda$. 
Given two ideals a and b, consider these three nested sets:

\[
\begin{align*}
    a + b & := \{ a + b \mid a \in a \text{ and } b \in b \}, \\
    a \cap b & := \{ a \mid a \in a \text{ and } a \in b \}, \\
    ab & := \{ \sum a_i b_i \mid a_i \in a \text{ and } b_i \in b \}.
\end{align*}
\]

They are clearly ideals. They are known as the **sum**, **intersection**, and **product** of a and b. Further, for any ideal c, the distributive law holds: \(a(b + c) = ab + ac\).

Let a be an ideal. Then a = R if and only if 1 ∈ a. Indeed, if 1 ∈ a, then \(x = x \cdot 1 \in a\) for every \(x \in R\). It follows that a = R if and only if a contains a unit. Further, if \(\langle x \rangle = R\), then x is a unit, since then there is an element y such that \(xy = 1\). If a ≠ R, then a is said to be proper.

Let \(\varphi : R \to R'\) be a ring map. Let \(aR'\) denote the ideal of \(R'\) generated by \(\varphi(a)\); we call \(aR'\) the **extension** of a. Let \(a'\) be an ideal of \(R'\). Clearly, the preimage \(\varphi^{-1}(a')\) is an ideal of R; we call \(\varphi^{-1}(a')\) the **contraction** of \(a'\).

**Exercise (1.5).** — Let \(\varphi : R \to R'\) be a map of rings, a an ideal of R, and b an ideal of \(R'\). Set \(a^e := \varphi(a)R'\) and \(b^e := \varphi^{-1}(b)\). Prove these statements:

1. Then \(a^{e e} \supset a\) and \(b^{e e} \subset b\).
2. Then \(a^{e e} = a^e\) and \(b^{e e} = b^e\).
3. If b is an extension, then \(b^e\) is the largest ideal of R with extension b.
4. If two extensions have the same contraction, then they are equal.

**Residue rings.** — Let \(\varphi : R \to R'\) be a ring map. Recall its **kernel** \(\text{Ker}(\varphi)\) is defined to be the ideal \(\varphi^{-1}(0)\) of R. Recall \(\text{Ker}(\varphi) = 0\) if and only if \(\varphi\) is injective. Conversely, let \(a\) be an ideal of \(R\). Form the set of cosets of \(a\):

\[
R/a := \{ x + a \mid x \in R \}.
\]

Recall that \(R/a\) inherits a ring structure, and is called the **residue ring** (or **quotient ring** or **factor ring**) of \(R\) modulo \(a\). Form the **quotient map**

\[
\kappa : R \to R/a \quad \text{by} \quad \kappa x := x + a.
\]

The element \(\kappa x \in R/a\) is called the **residue** of \(x\). Clearly, \(\kappa\) is surjective, \(\kappa\) is a ring map, and \(\kappa\) has kernel \(a\). Thus every ideal is a kernel!

Note that \(\text{Ker}(\varphi) \supset a\) if and only if \(\varphi a = 0\).

Recall that, if \(\text{Ker}(\varphi) \supset a\), then there is a ring map \(\psi : R/a \to R'\) with \(\psi \kappa = \varphi\); that is, the following diagram is commutative:

\[
\begin{array}{ccc}
R & \xrightarrow{\kappa} & R/a \\
\downarrow{\varphi} & & \downarrow{\psi} \\
R' & \xrightarrow{\kappa} & R/a
\end{array}
\]

Conversely, if \(\psi\) exists, then \(\text{Ker}(\varphi) \supset a\), or \(\varphi a = 0\), or \(aR' = 0\), since \(\kappa a = 0\).

Further, if \(\psi\) exists, then \(\psi\) is unique as \(\kappa\) is surjective.

Finally, as \(\kappa\) is surjective, if \(\psi\) exists, then \(\psi\) is surjective if and only if \(\varphi\) is so. In addition, then \(\psi\) is injective if and only if \(a = \text{Ker}(\varphi)\). Hence then \(\psi\) is an isomorphism if and only if \(\varphi\) is surjective and \(a = \text{Ker}(\varphi)\). Therefore, always

\[
R/\text{Ker}(\varphi) \xrightarrow{\sim} \text{Im}(\varphi).
\]

In practice, it is usually more productive to view \(R/a\) not as a set of cosets, but simply as another ring \(R'\) that comes equipped with a surjective ring map \(\varphi : R \to R'\) whose kernel is the given ideal \(a\).

Finally, \(R/a\) has, as we saw, this UMP: \(\kappa(a) = 0\), and given \(\varphi : R \to R'\) such that...
Let \( \varphi(a) = 0 \), there is a unique ring map \( \psi: \frac{R}{a} \to R' \) such that \( \psi_k = \varphi \). In other words, \( \frac{R}{a} \) is universal among \( R \)-algebras \( R' \) such that \( aR' = 0 \).

Above, if \( a \) is the ideal generated by elements \( a_\lambda \), then the UMP can be usefully rephrased as follows: \( \kappa(a_\lambda) = 0 \) for all \( \lambda \), and given \( \varphi: R \to R' \) such that \( \varphi(a_\lambda) = 0 \) for all \( \lambda \), there is a unique ring map \( \psi: \frac{R}{a} \to R' \) such that \( \psi_k = \varphi \).

The UMP serves to determine \( R/a \) up to unique isomorphism. Indeed, say \( R' \), equipped with \( \varphi: R \to R' \), has the UMP too. Then \( \varphi(a) = 0 \); so there is a unique \( \psi: \frac{R}{a} \to R' \) with \( \psi_k = \varphi \). And \( \kappa(a) = 0 \); so there is a unique \( \psi': R' \to \frac{R}{a} \) with \( \psi'\varphi = \kappa \). Then, as shown, \( (\psi'\psi)\kappa = \kappa \), but \( 1 \circ \kappa = \kappa \) where 1 is the identity map of \( \frac{R}{a} \): hence, \( \psi'\psi = 1 \) by uniqueness. Similarly, \( \psi\psi' = 1 \) where 1 now stands for the identity map of \( R' \). Thus \( \psi \) and \( \psi' \) are inverse isomorphisms.

The preceding proof is completely formal, and so works widely. There are many more constructions to come, and each one has an associated UMP, which therefore serves to determine the construction up to unique isomorphism.

**Exercise (1.7).** — Let \( R \) be a ring, \( a \) an ideal, and \( P := \frac{R[X_1, \ldots, X_n]}{} \) the polynomial ring. Prove \( P/aP = \frac{(R/a)[X_1, \ldots, X_n]}{} \).

**Proposition (1.8).** — Let \( R \) be a ring, \( P := \frac{R[X]}{} \) the polynomial ring in one variable, \( a \in R \), and \( \pi: P \to R \) the \( R \)-algebra map defined by \( \pi(X) := a \). Then \( \text{Ker}(\pi) = (X-a) \), and \( \frac{R[X]}{(X-a)} \xrightarrow{\pi} R \).

**Proof:** Given \( F(X) \in P \), the Division Algorithm yields \( F(X) = G(X)(X-a) + b \) with \( G(X) \in P \) and \( b \in R \). Then \( \pi(F(X)) = b \). Hence \( \text{Ker}(\pi) = (X-a) \). Finally, \( \xrightarrow{\pi} \) yields \( \frac{R[X]}{(X-a)} \xrightarrow{\pi} R \).

**Exercise (1.9) (Nested ideals).** — Let \( R \) be a ring, \( a \) an ideal, and \( \kappa: R \to \frac{R}{a} \) the quotient map. Given an ideal \( b \supseteq a \), form the corresponding set of cosets of \( a \):

\[
 b/a := \{b + a \mid b \in b\} = \kappa(b).
\]

Clearly, \( b/a \) is an ideal of \( R/a \). Also \( b/a = b(R/a) \).

Clearly, the operations \( b \mapsto b/a \) and \( b' \mapsto \kappa^{-1}(b') \) are inverse to each other, and establish a bijective correspondence between the set of ideals \( b \) of \( R \) containing \( a \) and the set of all ideals \( b' \) of \( R/a \). Moreover, this correspondence preserves inclusions.

Given an ideal \( b \supseteq a \), form the composition of the quotient maps

\[
 \varphi: R \to \frac{R}{a} \to \frac{(R/a)}{(b/a)}.
\]

Clearly, \( \varphi \) is surjective, and \( \text{Ker}(\varphi) = b \). Hence, owing to (1.8), \( \varphi \) factors through the canonical isomorphism \( \psi \) in this commutative diagram:

\[
 \begin{array}{ccc}
 R & \xrightarrow{\psi} & R/b \\
 \downarrow & & \downarrow \simeq \\
 R/a & \xrightarrow{(R/a)/(b/a)} & 
\end{array}
\]
**Exercise (1.10).** — Let $R$ be a ring, and $P := R[X_1, \ldots, X_n]$ the polynomial ring. Let $m \leq n$ and $a_1, \ldots, a_m \in R$. Set $p := (X_1 - a_1, \ldots, X_m - a_m)$. Prove that $P/p = R[X_{m+1}, \ldots, X_n]$.

**Remark (1.11) (Idempotents).** — Let $R$ be a ring. Let $e \in R$ be an idempotent; that is, $e^2 = e$. Then $Re$ is a ring with $e$ as 1, because $(xe)e = xe$. But $Re$ is not a subring of $R$ unless $e = 1$, although $Re$ is an ideal.

Set $e' := 1 - e$. Then $e'$ is idempotent and $e \cdot e' = 0$. We call $e$ and $e'$ complementary idempotents. Conversely, if two elements $e_1, e_2 \in R$ satisfy $e_1 + e_2 = 1$ and $e_1e_2 = 0$, then they are complementary idempotents, as for each $i$,

$$e_i = e_i \cdot 1 = e_i(e_1 + e_2) = e_i^2.$$

We denote the set of all idempotents by $\text{Idem}(R)$. Let $\varphi : R \to R'$ be a ring map. Then $\varphi(e)$ is idempotent. So the restriction of $\varphi$ to $\text{Idem}(R)$ is a map $\text{Idem}(\varphi): \text{Idem}(R) \to \text{Idem}(R')$.

**Example (1.12).** — Let $R := R' \times R''$ be a product of two rings: its operations are performed componentwise. The additive identity is $(0,0)$; the multiplicative identity is $(1,1)$. Set $e := (1,0)$ and $e' := (0,1)$. Then $e$ and $e'$ are complementary idempotents. The next proposition shows this example is the only one possible.

**Proposition (1.13).** — Let $R$ be a ring with complementary idempotents $e$ and $e'$. Set $R' := Re$ and $R'' := Re'$, and form the map $\varphi : R \to R' \times R''$ defined by $\varphi(x) := (xe, xe')$. Then $\varphi$ is a ring isomorphism.

**Proof:** Define a map $\varphi' : R \to R'$ by $\varphi'(x) := xe$. Then $\varphi'$ is a ring map since $xye = xye^2 = (xe)(ye)$. Similarly, define $\varphi'' : R \to R''$ by $\varphi''(x) := xe'$; then $\varphi''$ is a ring map. So $\varphi$ is a ring map. Further, $\varphi$ is surjective, since $(xe, xe') = \varphi(xe + xe')$. Also, $\varphi$ is injective, since if $xe = 0$ and $xe' = 0$, then $x = xe + xe' = 0$. Thus $\varphi$ is an isomorphism. \[\square\]

**Exercise (1.14) (Chinese Remainder Theorem).** — Let $R$ be a ring.

1. Let $a$ and $b$ be comaximal ideals; that is, $a + b = R$. Prove

   (a) $ab = a \cap b$ and (b) $R/ab = (R/a) \times (R/b)$.

2. Let $a$ be comaximal to both $b$ and $b'$. Prove $a$ is also comaximal to $bb'$.

3. Let $a, b$ be comaximal, and $m, n \geq 1$. Prove $a^m$ and $b^n$ are comaximal.

4. Let $a_1, \ldots, a_n$ be pairwise comaximal. Prove

   (a) $a_1$ and $a_2 \cdots a_n$ are comaximal;

   (b) $a_1 \cap \cdots \cap a_n = a_1 \cdots a_n$;

   (c) $R/(a_1 \cdots a_n) \rightarrow \bigoplus (R/a_i)$.

**Exercise (1.15).** — First, given a prime number $p$ and a $k \geq 1$, find the idempotents in $\mathbb{Z}/(p^k)$. Second, find the idempotents in $\mathbb{Z}/(12)$. Third, find the number of idempotents in $\mathbb{Z}/(n)$ where $n = \prod_{i=1}^{N} p_i^{n_i}$ with $p_i$ distinct prime numbers.

**Exercise (1.16).** — Let $R := R' \times R''$ be a product of rings, $a \subset R$ an ideal. Show $a = a' \times a''$ with $a' \subset R'$ and $a'' \subset R''$ ideals. Show $R/a = (R'/a') \times (R''/a'')$. 

Rings and Ideals (1.17) 5
Exercise (1.17). — Let $R$ be a ring, and $e, e'$ idempotents. (See (10.7) also.)

1. Set $a := \langle e \rangle$. Show $a$ is idempotent; that is, $a^2 = a$.
2. Let $a$ be a principal idempotent ideal. Show $a(f)$ with $f$ idempotent.
3. Set $e'' := e + e' - ee'$. Show $\langle e, e' \rangle = \langle e'' \rangle$ and $e''$ is idempotent.
4. Let $e_1, \ldots, e_r$ be idempotents. Show $\langle e_1, \ldots, e_r \rangle = \langle f \rangle$ with $f$ idempotent.
5. Assume $R$ is Boolean. Show every finitely generated ideal is principal.
2. Prime Ideals

Prime ideals are the key to the structure of commutative rings. So we review the basic theory. Specifically, we define prime ideals, and show their residue rings are domains. We show maximal ideals are prime, and discuss examples. Finally, we use Zorn’s Lemma to prove the existence of maximal ideals in every nonzero ring.

**Definition (2.1).** — Let \( R \) be a ring. An element \( x \) is called a zero divisor if there is a nonzero \( y \) with \( xy = 0 \); otherwise, \( x \) is called a nonzerodivisor. Denote the set of zero divisors by \( \text{z.div}(R) \).

A subset \( S \) is called multiplicative if \( 1 \in S \) and if \( x, y \in S \) implies \( xy \in S \).

An ideal \( \mathfrak{p} \) is called prime if its complement \( R - \mathfrak{p} \) is multiplicative, or equivalently, if \( 1 \notin \mathfrak{p} \) and if \( xy \in \mathfrak{p} \) implies \( x \in \mathfrak{p} \) or \( y \in \mathfrak{p} \).

**Exercise (2.2).** — Let \( a \) and \( b \) be ideals, and \( \mathfrak{p} \) a prime ideal. Prove that these conditions are equivalent: (1) \( a \subseteq \mathfrak{p} \) or \( b \subseteq \mathfrak{p} \); and (2) \( \mathfrak{p} \cap \mathfrak{b} \subseteq \mathfrak{p} \); and (3) \( \mathfrak{a} \mathfrak{b} \subseteq \mathfrak{p} \).

**Exercise (2.3).** (Fields, Domains). — A ring is called a field if \( 1 \neq 0 \) and if every nonzero element is a unit. Standard examples include the rational numbers \( \mathbb{Q} \), the real numbers \( \mathbb{R} \), and the complex numbers \( \mathbb{C} \).

A ring is called an integral domain, or simply a domain, if \( 0 \) is prime, or equivalently, if \( R \) is nonzero and has no nonzero zero divisors.

Every domain \( R \) is a subring of its fraction field \( \text{Frac}(R) \), which consists of the fractions \( x/y \) with \( x, y \in R \) and \( y \neq 0 \). Conversely, any subring \( R \) of a field \( K \), including \( K \) itself, is a domain; indeed, any nonzero \( x \in R \) cannot be a zero divisor, because, if \( xy = 0 \), then \((1/x)(xy) = 0 \), so \( y = 0 \). Further, \( \text{Frac}(R) \) has this UMP: the inclusion of \( R \) into any field \( L \) extends uniquely to an inclusion of \( \text{Frac}(R) \) into \( L \). For example, the ring of integers \( \mathbb{Z} \) is a domain, and \( \text{Frac}(\mathbb{Z}) = \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \).

Let \( R \) be a domain, and \( R[X] \) the polynomial ring in one variable. Then \( R[X] \) is a domain too. In fact, given two nonzero polynomials \( f \) and \( g \), not only is their product \( fg \) nonzero, but its leading term is the product of those of \( f \) and \( g \); so

\[
\text{deg}(fg) = \text{deg}(f) \text{deg}(g).
\]

By induction, the polynomial ring in \( n \) variables \( R[X_1, \ldots, X_n] \) is a domain, since

\[
R[X_1, \ldots, X_n] = R[X_1, \ldots, X_{n-1}][X_n].
\]

Hence the polynomial ring in an arbitrary set of variables \( R[\{X_\lambda\}_{\lambda \in \Lambda}] \) is a domain, since any two elements lie in a polynomial subring in finitely many of the \( X_\lambda \).

Similarly, if \( f, g \in R[X] \) with \( fg = 1 \), then \( f, g \in R \), because the product of the leading terms of \( f \) and \( g \) is constant. So by induction, if \( f, g \in R[X_1, \ldots, X_n] \) with \( fg = 1 \), then \( f, g \in R \). This reasoning can fail if \( R \) is not a domain. For example, if \( a^2 = 0 \) in \( R \), then \((1 + aX)(1 - aX) = 1 \) in \( R[X] \).

The fraction field \( \text{Frac}(R[\{X_\lambda\}_{\lambda \in \Lambda}]) \) is called the field of rational functions, and is also denoted by \( K(\{X_\lambda\}_{\lambda \in \Lambda}) \) where \( K := \text{Frac}(R) \).

**Exercise (2.4).** — Given a prime number \( p \) and an integer \( n \geq 2 \), prove that the residue ring \( \mathbb{Z}/(p^n) \) does not contain a domain as a subring.

**Exercise (2.5).** — Let \( R := R' \times R'' \) be a product of two rings. Show that \( R \) is a domain if and only if either \( R' \) or \( R'' \) is a domain and the other is 0.
(2.6) (Unique factorization). — Let \( R \) be a domain, \( p \) a nonzero nonunit. We call \( p \) prime if, whenever \( p \mid xy \) (that is, there exists \( z \in R \) such that \( pz = xy \)), either \( p \mid x \) or \( p \mid y \). Clearly, \( p \) is prime if and only if the ideal \( (p) \) is prime.

We call \( p \) irreducible if, whenever \( p = yz \), either \( y \) or \( z \) is a unit. We call \( R \) a Unique Factorization Domain (UFD) if every nonzero element is a product of irreducible elements in a unique way up to order and units.

In general, prime elements are irreducible; in a UFD, irreducible elements are prime. Standard examples of UFDs include any field, the integers \( \mathbb{Z} \), and a polynomial ring in \( n \) variables over a UFD; see [2], p. 398, p. 401, [3] Cor. 18.23, p. 297.

**Lemma (2.7).** — Let \( \varphi : R \to R' \) be a ring map, and \( T \subset R' \) a subset. If \( T \) is multiplicative, then \( \varphi^{-1}T \) is multiplicative; the converse holds if \( \varphi \) is surjective.

**Proof:** Set \( S := \varphi^{-1}T \). If \( T \) is multiplicative, then \( 1 \in S \) as \( \varphi(1) = 1 \in T \), and \( x, y \in S \) implies \( xy \in S \) as \( \varphi(xy) = \varphi(x)\varphi(y) \in T \); thus \( S \) is multiplicative.

If \( S \) is multiplicative, then \( 1 \in T \) as \( 1 \in S \) and \( \varphi(1) = 1 \); further, \( x, y \in S \) implies \( \varphi(x), \varphi(y), \varphi(xy) \in T \). If \( \varphi \) is surjective, then every \( x' \in T \) is of the form \( x' = \varphi(x) \) for some \( x \in S \). Thus if \( \varphi \) is surjective, then \( T \) is multiplicative if \( \varphi^{-1}T \) is.

**Proposition (2.8).** — Let \( \varphi : R \to R' \) be a ring map, and \( q \subset R' \) an ideal. If \( q \) is prime, then \( \varphi^{-1}q \) is prime; the converse holds if \( \varphi \) is surjective.

**Proof:** By (2.7), \( R - p \) is multiplicative if and only if \( R' - q \) is. So the assertion results from Definitions (2.1).

**Corollary (2.9).** — Let \( R \) be a ring, \( p \) an ideal. Then \( p \) is prime if and only if \( R/p \) is a domain.

**Proof:** By (2.8), \( p \) is prime if and only if \( \langle 0 \rangle \subset R/p \) is. So the assertion results from the definition of domain in (2.3).

**Exercise (2.10).** — Let \( R \) be a domain, and \( R[X_1, \ldots, X_n] \) the polynomial ring in \( n \) variables. Let \( m \leq n \), and set \( p := \langle X_1, \ldots, X_m \rangle \). Prove \( p \) is a prime ideal.

**Exercise (2.11).** — Let \( R := R' \times R'' \) be a product of rings, \( p \subset R \) an ideal. Show \( p \) is prime if and only if either \( p = p' \times R'' \) with \( p' \subset R' \) prime or \( p = R' \times p'' \) with \( p'' \subset R'' \) prime.

**Exercise (2.12).** — Let \( R \) be a domain, and \( x, y \in R \). Assume \( \langle x \rangle = \langle y \rangle \). Show \( x = uy \) for some unit \( u \).

**Definition (2.13).** — An ideal \( m \) is said to be maximal if \( m \) is proper and if there is no proper ideal \( a \) with \( m \subsetneq a \).

**Example (2.14).** — Let \( R \) be a domain. In the polynomial ring \( R[X, Y] \) in two variables, \( \langle X \rangle \) is prime by (2.11). However, \( \langle X \rangle \) is not maximal since \( \langle X \rangle \subsetneq \langle X, Y \rangle \). Moreover, \( \langle X, Y \rangle \) is maximal if and only if \( R \) is a field by (1.10) and by (2.17) below.

**Proposition (2.15).** — A ring \( R \) is a field if and only if \( \langle 0 \rangle \) is a maximal ideal.

**Proof:** Suppose \( R \) is a field. Let \( a \) be a nonzero ideal, and \( a \) a nonzero element of \( a \). Since \( R \) is a field, \( a \in R^\times \). So (1.2) yields \( a = R \).

Conversely, suppose \( \langle 0 \rangle \) is maximal. Take \( x \neq 0 \). Then \( \langle x \rangle \neq \langle 0 \rangle \). So \( \langle x \rangle = R \). So \( x \) is a unit by (1.2). Thus \( R \) is a field.
Let

Clearly, given two prime ideals, their sum is prime.

The complement of a multiplicative subset is a prime ideal.

A field is a domain by (2.25). Hence the assertion results from (2.15).

Exercise (2.16). — Let \( R \) be a ring, \( p \) a prime ideal, \( R[X] \) the polynomial ring. Show that \( pR[X] \) and \( pR[X] + \langle X \rangle \) are prime ideals of \( R[X] \), and that if \( p \) is maximal, then so is \( pR[X] + \langle X \rangle \).

Exercise (4.15). — Let \( B \) be a Boolean ring. Show that every prime \( p \) is maximal, and \( B/p = \mathbb{F}_2 \).

Exercise (2.20). — Let \( R \) be a ring. Assume that, given \( x \in R \), there is \( n \geq 2 \) with \( x^n = x \). Show that every prime \( p \) is maximal.

Example (2.21). — Let \( k \) be a field, \( a_1, \ldots, a_n \in k \), and \( P := k[X_1, \ldots, X_n] \) the polynomial ring in \( n \) variables. Set \( m := \langle X_1 - a_1, \ldots, X_n - a_n \rangle \). Then \( P/m = k \) by (1.10); so \( m \) is maximal by (2.17).

Exercise (4.16). — Prove the following statements or give a counterexample.

1. The complement of a multiplicative subset is a prime ideal.
2. Given two prime ideals, their intersection is prime.
3. Given two prime ideals, their sum is prime.
4. Given a ring map \( \varphi : R \to R' \), the operation \( \varphi^{-1} \) carries maximal ideals of \( R' \) to maximal ideals of \( R \).
5. In (1.9), an ideal \( n' \subset R/a \) is maximal if and only if \( \kappa^{-1} n' \subset R \) is maximal.

Exercise (2.22). — Let \( k \) be a field, \( P := k[X_1, \ldots, X_n] \) the polynomial ring, \( f \in P \) nonzero. Let \( d \) be the highest power of any variable appearing in \( f \).

1. Let \( S \subset k \) have at least \( d + 1 \) elements. Proceeding by induction on \( n \), find \( a_1, \ldots, a_n \in S \) with \( f(a_1, \ldots, a_n) \neq 0 \).
2. Using the algebraic closure \( K \) of \( k \), find a maximal ideal \( m \) of \( P \) with \( f \notin m \).

Corollary (2.24). — In a ring, every maximal ideal is prime.

Proof: A field is a domain by (2.3). So (2.8) and (2.17) yield the result. □

(2.25) (PIDs). — A domain \( R \) is called a Principal Ideal Domain (PID) if every ideal is principal. Examples include a field \( k \), the polynomial ring \( k[X] \) in one variable, and the ring \( \mathbb{Z} \) of integers. Every PID is a UFD by (2.12), p. 396, [3, Thm. 18.11, p. 291].

Let \( R \) be a PID, and \( p \in R \) irreducible. Then \( \langle p \rangle \) is maximal; indeed, if \( \langle p \rangle \subsetneq \langle x \rangle \), then \( p = xy \) for some nonunit \( y \), and so \( x \) must be a unit since \( p \) is irreducible. So (2.17) implies that \( R/\langle p \rangle \) is a field.

Exercise (4.26). — Prove that, in a PID, elements \( x \) and \( y \) are relatively prime (share no prime factor) if and only if the ideals \( \langle x \rangle \) and \( \langle y \rangle \) are comaximal.

Example (2.27). — Let \( R \) be a PID, and \( p \in R \) a prime. Set \( k := R/\langle p \rangle \). Let \( P := R[X] \) be the polynomial ring in one variable. Take \( g \in P \), let \( g' \) be its image in \( k[X] \), and assume \( g' \) is irreducible. Set \( m := \langle p, g \rangle \). Then \( P/m \cong k[X]/\langle g' \rangle \) by (4.20) and (4.14), and \( k[X]/\langle g' \rangle \) is a field by (4.25); hence, \( m \) is maximal by (4.11).
THEOREM (2.28). — Let $R$ be a PID. Let $P := R[X]$ be the polynomial ring in one variable, and $p$ a prime ideal of $P$.

1. Then $p = (0)$, or $p = (f)$ with $f$ prime, or $p$ is maximal.

2. Assume $p$ is maximal. Then either $p = (f)$ with $f$ prime, or $p = (p, g)$ with $p \in R$ prime and $g \in P$ with image $g' \in (R/(p))[X]$ prime.

Proof: Assume $p \neq (0)$. Take a nonzero $f_1 \in p$. Since $p$ is prime, $p$ contains a prime factor $f'_1$ of $f_1$. Replace $f_1$ by $f'_1$. Assume $p \neq (f_1)$. Then there is a prime $f_2 \in p - (f_1)$. Set $K := \text{Frac}(R)$. Gauss’s Lemma [2, p. 401], [8, Thm. 18.15, p. 295] implies that $f_1$ and $f_2$ are also prime in $K[X]$. So $f_1$ and $f_2$ are relatively prime in $K[X]$. So (2.25) and (2.26) yield $g_1, g_2 \in P$ and $c \in R$ with $(g_1/c)f_1 + (g_2/c)f_2 = 1$. So $c = g_1f_1 + g_2f_2 \in R \cap p$. Hence $R \cap p \neq 0$. But $R \cap p$ is prime, and $R$ is a PID; so $R \cap p = (p)$ where $p$ is prime by (2.6).

Set $k := R/(p)$. Then $k$ is a field by (2.24). Set $q := p/(p) \subset k[X]$. Then $k[X]/q = P/p$ by (1.7) and (1.8). But $P/p$ is a domain as $p$ is prime. Hence $q = (g')$ where $g'$ is prime in $k[X]$ by (2.4). Then $q$ is maximal by (2.4.6). So $p$ is maximal by (1.3). Take $g \in p$ with image $g'$. Then $p = (p, g)$ as $p/(p) = (g')$. □

EXERCISE (2.27). — Preserve the setup of (2.28). Let $f := a_0X^n + \ldots + a_n$ be a polynomial of positive degree $n$. Assume that $R$ has infinitely many prime elements $p$, or simply that there is a $p$ such that $p \nmid a_0$. Show that $(f)$ is not maximal.

THEOREM (2.30). — Every proper ideal $a$ is contained in some maximal ideal.

Proof: Set $S := \{ \text{ideals } b \mid b \supset a \text{ and } b \neq 1 \}$. Then $a \in S$, and $S$ is partially ordered by inclusion. Given a totally ordered subset $\{b_\lambda\}$ of $S$, set $b := \bigcup b_\lambda$. Then $b$ is clearly an ideal, and $1 \notin b$; so $b$ is an upper bound of $\{b_\lambda\}$ in $S$. Hence by Zorn’s Lemma [11, pp. 25, 26], [11, p. 880, p. 884], $S$ has a maximal element, and it is the desired maximal ideal. □

COROLLARY (2.31). — Let $R$ be a ring, $x \in R$. Then $x$ is a unit if and only if $x$ belongs to no maximal ideal.

Proof: By (1.3), $x$ is a unit if and only if $(x)$ is not proper. Apply (2.30). □
3. Radicals

Two radicals of a ring are commonly used in Commutative Algebra: the Jacobson radical, which is the intersection of all maximal ideals, and the nilradical, which is the set of all nilpotent elements. Closely related to the nilradical is the radical of a subset. We define these three radicals, and discuss examples. In particular, we study local rings; a local ring has only one maximal ideal, which is then its Jacobson radical. We prove two important general results: Prime Avoidance, which states that, if an ideal lies in a finite union of primes, then it lies in one of them, and the Scheinnullstellensatz, which states that the nilradical of an ideal is equal to the intersection of all the prime ideals containing it.

**Definition (3.1).** — Let $R$ be a ring. Its (Jacobson) radical $\text{rad}(R)$ is defined to be the intersection of all its maximal ideals.

**Proposition (3.2).** — Let $R$ be a ring, $x \in R$, and $u \in R^\times$. Then $x \in \text{rad}(R)$ if and only if $u - xy \in \text{rad}(R)$ is a unit for all $y \in R$. In particular, the sum of an element of $\text{rad}(R)$ and a unit is a unit.

**Proof:** Assume $x \in \text{rad}(R)$. Let $m$ be a maximal ideal. Suppose $u - xy \in m$. Since $x \in m$ too, also $u \in m$, a contradiction. Thus $u - xy$ is a unit by (3.1). In particular, taking $y := -1$ yields $u + x \in R^\times$.

Conversely, assume $x \notin \text{rad}(R)$. Then there is a maximal ideal $m$ with $x \notin m$. So $(x) + m = R$. Hence there exist $y \in R$ and $m \in m$ such that $xy + m = u$. Then $u - xy = m \in m$. So $u - xy$ is not a unit by (3.1), or directly by (1.1).

**Exercise (3.3).** — Let $R$ be a ring, $a \subseteq \text{rad}(R)$ an ideal, $w \in R$, and $w' \in R/a$ its residue. Prove that $w \in R^\times$ if and only if $w' \in (R/a)^\times$. What if $a \notin \text{rad}(R)$?

**Corollary (3.4).** — Let $R$ be a ring, $a$ an ideal, $\kappa: R \to R/a$ the quotient map. Assume $a \subseteq \text{rad}(R)$. Then $\text{Idem}(\kappa)$ is injective.

**Proof:** Given $e, e' \in \text{Idem}(R)$ with $\kappa(e) = \kappa(e')$, set $x := e - e'$. Then $x^3 = e^3 - 3e^2e' + 3ee'^2 - e'^3 = e - e' = x$.

Hence $x(1 - x^2) = 0$. But $\kappa(x) = 0$; so $x \in a$. But $a \subseteq \text{rad}(R)$. Hence $1 - x^2$ is a unit by (3.2). Thus $x = 0$. Thus $\text{Idem}(\kappa)$ is injective.

**Definition (3.5).** — A ring $A$ is called **local** if it has exactly one maximal ideal, and **semilocal** if it has at least one and at most finitely many.

**Lemma (3.6) (Nonunit Criterion).** — Let $A$ be a ring, $n$ the set of nonunits. Then $A$ is local if and only if $n$ is an ideal; if so, then $n$ is the maximal ideal.

**Proof:** Every proper ideal $a$ lies in $n$ as $a$ contains no unit. So, if $n$ is an ideal, then it is a maximal ideal, and the only one. Thus $A$ is local.

Conversely, assume $A$ is local with maximal ideal $m$. Then $A - n = A - m$ by (3.1). So $n = m$. Thus $n$ is an ideal.

**Example (3.7).** — The product ring $R' \times R''$ is not local by (3.1) if both $R'$ and $R''$ are nonzero. Indeed, $(1,0)$ and $(0,1)$ are nonunits, but their sum is a unit.
EXERCISE (3.8). — Let $A$ be a local ring. Find its idempotents $e$.

EXERCISE (3.9). — Let $A$ be a ring, $m$ a maximal ideal such that $1 + m$ is a unit for every $m \in m$. Prove $A$ is local. Is this assertion still true if $m$ is not maximal?

EXAMPLE (3.10). — Let $R$ be a ring. A formal power series in the $n$ variables $X_1, \ldots, X_n$ is a formal infinite sum of the form $\sum a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n}$ where $a_{i_1, \ldots, i_n} \in R$ and where $(i) := (i_1, \ldots, i_n)$ with each $i_j \geq 0$. The term $a_{0, \ldots, 0}$ where $(0) := (0, \ldots, 0)$ is called the constant term. Addition and multiplication are performed as for polynomials; with these operations, these series form a ring $R[[X_1, \ldots, X_n]]$.

Set $P := R[[X_1, \ldots, X_n]]$ and $a := (X_1, \ldots, X_n)$. Then $\sum a_{0, \ldots, 0} X_1^{i_1} \cdots X_n^{i_n} \to a_{0, \ldots, 0}$ is a canonical surjective ring map $P \to R$ with kernel $a$; hence, $P/a = R$.

Given an ideal $m \subset R$, set $n := a + m^p$. Then (3.8) yields $P/n = R/m$.

A power series $f$ is a unit if and only if its constant term $a_{0, \ldots, 0}$ is a unit. Indeed, if $ff' = 1$, then $a_{0, \ldots, 0}a_{0, \ldots, 0}' = 1$ where $a_{0, \ldots, 0}'$ is the constant term of $f'$. Conversely, if $a_{0, \ldots, 0}$ is a unit, then $f = a_{0, \ldots, 0}(1 - g)$ with $g \in a$. Set $f' := a_{0, \ldots, 0}^{-1}(1 + g + g^2 + \cdots)$; this sum makes sense as the component of degree $d$ involves only the first $d + 1$ summands. Clearly $f \cdot f' = 1$.

Suppose $R$ is a local ring with maximal ideal $m$. Given a power series $f \not\in n$, its initial term lies outside $m$, so is a unit by (3.6). So $f$ itself is a unit. Hence the nonunits constitute $n$. Thus (3.6) implies $P$ is local with maximal ideal $n$.

EXAMPLE (3.11). — Let $k$ be a ring, and $A := k[[X]]$ the formal power series ring in one variable. A formal Laurent series is a formal sum of the form $\sum_{i = -m}^{\infty} a_i X^i$ with $a_i \in k$ and $m \in \mathbb{Z}$. These series form a ring $k[[X]]$. Set $K := k[[X]]$.

Set $f := \sum_{i = -m}^{\infty} a_i X^i$. If $a_{-m} \in k^\times$, then $f \in K^\times$; indeed, $f = a_{-m}X^{-m}(1 + g + g^2 + \cdots) = 1$.

Assume $k$ is a field. If $f \neq 0$, then $f = X^{-m} u$ where $u \in A^\times$. Let $a \subset A$ be a nonzero ideal. Suppose $f \in a$. Then $X^{-m} \in a$. Let $n$ be the smallest integer such that $X^n \in a$. Then $-m \geq n$. Set $b := X^{-m-n} u$. Then $b \in A$ and $f = bX^n$. Hence $a = (X^n)$. Thus $A$ is a PID.

Further, $K$ is a field. In fact, $K = \text{Frac}(A)$ because any nonzero $f \in K$ is of the form $f = u/X^m$ where $u, X^m \in A$.

Let $A[Y]$ be the polynomial ring in one variable, and $\iota: A \hookrightarrow K$ the inclusion. Define $\varphi: A[Y] \to K$ by $\varphi(A) = \iota$ and $\varphi(Y) = X^{-1}$. Then $\varphi$ is surjective. Set $m := \ker(\varphi)$. Then $m$ is maximal by (2.47) and (1.6). So by (2.28), $m$ has the form $\langle f \rangle$ with $f$ irreducible, or the form $\langle p, g \rangle$ with $p \in A$ irreducible and $g \in A[Y]$. But $m \cap A = 0$ as $\iota$ is injective. So $m = \langle f \rangle$. But $XY - 1$ belongs to $m$, and is clearly irreducible; hence, $XY - 1 = fu$ with $u$ a unit. Thus $\langle XY - 1 \rangle$ is maximal.

In addition, $\langle X, Y \rangle$ is maximal. Indeed, $A[Y]/\langle Y \rangle = A$ by (1.6), and so (3.10) yields $A[Y]/\langle X, Y \rangle = A/\langle X \rangle = k$. However, $\langle X, Y \rangle$ is not principal, as no nonunit of $A[Y]$ divides both $X$ and $Y$. Thus $A[Y]$ has both principal and nonprincipal maximal ideals, the two types allowed by (2.25).

PROPOSITION (3.12). — Let $R$ be a ring, $S$ a multiplicative subset, and $a$ an ideal with $a \cap S = \emptyset$. Set $\mathcal{S} := \{\text{ideals } b \mid b \supset a \text{ and } b \cap S = \emptyset\}$. Then $\mathcal{S}$ has a maximal element $p$, and every such $p$ is prime.

PROOF: Clearly, $a \in \mathcal{S}$, and $\mathcal{S}$ is partially ordered by inclusion. Given a totally ordered subset $\{b_\lambda\}$ of $\mathcal{S}$, set $b := \bigcup b_\lambda$. Then $b$ is an upper bound for $\{b_\lambda\}$ in $\mathcal{S}$. The two types allowed by (2.25).
So by Zorn’s Lemma, $S$ has a maximal element $p$. Let’s show $p$ is prime.

Take $x, y \in R - p$. Then $p + (x)$ and $p + (y)$ are strictly larger than $p$. So there are $p, q \in p$ and $a, b \in R$ with $p + ax \in S$ and $q + by \in S$. Since $S$ is multiplicative, $pq + pby + qax + abxy \in S$. But $pq + pby + qax \in p$, so $xy \notin p$. Thus $p$ is prime. \hfill \square

**Exercise (3.13).** — Let $\varphi : R \to R'$ be a ring map, $p$ an ideal of $R$. Prove

1. there is an ideal $q$ of $R'$ with $\varphi^{-1}(q) = p$ if and only if $\varphi^{-1}(pR') = p$;
2. if $p$ is prime with $\varphi^{-1}(pR') = p$, then there’s a prime $q$ of $R'$ with $\varphi^{-1}(q) = p$.

**Exercise (3.14).** — Use Zorn’s lemma to prove that any prime ideal $p$ contains a prime ideal $q$ that is minimal containing any given subset $s \subset p$.

(3.15) (**Saturated multiplicative subsets**). — Let $R$ be a ring, and $S$ a multiplicative subset. We say $S$ is saturated if, given $x, y \in R$ with $xy \in S$, necessarily $x, y \in S$.

For example, the following statements are easy to check. The group of units $R^\times$ and the subset of nonzerodivisors $S_0 := R - z \cdot \text{div}(R)$ are saturated multiplicative subsets. Further, let $\varphi : R \to R'$ be a ring map, $T \subset R'$ a subset. If $T$ is saturated multiplicative, then so is $\varphi^{-1}T$. The converse holds if $\varphi$ is surjective.

**Exercise (3.16).** — Let $R$ be a ring, $S$ a subset. Show that $S$ is saturated multiplicative if and only if $R - S$ is a union of primes.

**Exercise (3.17).** — Let $R$ be a ring, and $S$ a multiplicative subset. Define its saturation to be the subset

$$\overline{S} := \{ x \in R \mid \text{there is } y \in R \text{ with } xy \in S \}.$$

1. Show (a) that $\overline{S} \supset S$, and (b) that $\overline{S}$ is saturated multiplicative, and (c) that any saturated multiplicative subset $T$ containing $S$ also contains $\overline{S}$.

2. Show that $R - \overline{S}$ is the union $U$ of all the primes $p$ with $p \cap S = \emptyset$.

3. Let $a$ be an ideal; assume $S = 1 + a$; set $W := \bigcup_{p \in V(a)} p$. Show $R - \overline{S} = W$.

4. Given $f \in R$, let $\overline{S}_f$ denote the saturation of the multiplicative subset of all powers of $f$. Given $f, g \in R$, show $\overline{S}_f \subset \overline{S}_g$ if and only if $\sqrt{(f)} \supset \sqrt{(g)}$.

**Exercise (3.18).** — Let $R$ be a nonzero ring, $S$ a subset. Show $S$ is maximal in the set $\mathcal{S}$ of multiplicative subsets $T$ of $R$ with $0 \notin T$ if and only if $R - S$ is a minimal prime — that is, it is a prime containing no smaller prime.

**Lemma (3.19) (Prime Avoidance).** — Let $R$ be a ring, $a$ a subset of $R$ that is stable under addition and multiplication, and $p_1, \ldots, p_n$ ideals such that $p_3, \ldots, p_n$ are prime. If $a \not\subseteq p_j$ for all $j$, then there is an $x \in a$ such that $x \notin p_j$ for all $j$; or equivalently, if $a \subseteq \bigcup_{i=1}^n p_i$, then $a \subseteq p_i$ for some $i$.

**Proof:** Proceed by induction on $n$. If $n = 1$, the assertion is trivial. Assume that $n \geq 2$ and by induction that, for every $i$, there is an $x_i \in a$ such that $x_i \notin p_j$ for all $j \neq i$. We may assume $x_i \in p_i$ for every $i$, else we’re done. If $n = 2$, then clearly $x_1 + x_2 \notin p_j$ for $j = 1, 2$. If $n \geq 3$, then $(x_1 \cdots x_{n-1}) + x_n \notin p_j$ for all $j$ as, if $j = n$, then $x_n \in p_n$ and $p_n$ is prime, and if $j < n$, then $x_n \notin p_j$ and $x_j \in p_j$. \hfill \square

**Exercise (3.20).** — Let $k$ be a field, $S \subset k$ a subset of cardinality $d$ at least 2.

1. Let $P := k[x_1, \ldots, x_n]$ be the polynomial ring, $f \in P$ nonzero. Assume the highest power of any $x_i$ in $f$ is less than $d$. Proceeding by induction on $n$, show there are $a_1, \ldots, a_n \in S$ with $f(a_1, \ldots, a_n) \neq 0$.

2. Let $V$ be a $k$-vector space, and $W_1, \ldots, W_r$ proper subspaces. Assume $r < d$. 

Radicals (3.20) 13
Let \( R = \bigcup_i W_i \). Show \( \bigcup_i W_i \neq V \).

(3) In (2), let \( W \subseteq \bigcup_i W_i \) be a subspace. Show \( W \subseteq W_i \) for some \( i \).

(4) Let \( R \) a \( k \)-algebra, \( a, a_1, \ldots, a_r \) ideals with \( a \subseteq \bigcup_i a_i \). Show \( a \subseteq a_i \) for some \( i \).

**EXERCISE (3.21).** — Let \( k \) be a field, \( R := k[X, Y] \) the polynomial ring in two variables, \( m := (X, Y) \). Show \( m \) is a union of strictly smaller primes.

**Exercise (3.22) (Nilradical).** — Let \( R \) be a ring, \( a \) a subset. Then the **radical** of \( a \) is the set \( \sqrt{a} \) defined by the formula \( \sqrt{a} := \{ x \in R \mid x^n \in a \text{ for some } n = n(x) \geq 1 \} \).

Notice \( \sqrt{\sqrt{a}} = \sqrt{a} \). Also, if \( a \) is an intersection of prime ideals, then \( \sqrt{a} = a \).

We call \( \sqrt{0} \) the **nilradical**, and sometimes denote it by \( \text{nil}(R) \). We call an element \( x \in R \) **nilpotent** if \( x \) belongs to \( \sqrt{0} \), that is, if \( x^n = 0 \) for some \( n \geq 1 \).

Note that, if \( x^n = 0 \) with \( n \geq 1 \) and if \( m \) is any maximal ideal, then \( x^n \in m \) and so \( x \in m \) as \( m \) is prime by (3.21). Thus

\[
\text{nil}(R) \subset \text{rad}(R)
\]

We call \( R \) **reduced** if \( \text{nil}(R) = (0) \), that is, if \( R \) has no nonzero nilpotents.

**EXERCISE (3.22).** — Find the nilpotents in \( \mathbb{Z}/(n) \). In particular, take \( n = 12 \).

**EXERCISE (3.23).** — Let \( R \) be a ring. (1) Assume every ideal not contained in \( \text{nil}(R) \) contains a nonzero idempotent. Prove that \( \text{nil}(R) = \text{rad}(R) \). (2) Assume \( R \) is Boolean. Prove that \( \text{nil}(R) = \text{rad}(R) = (0) \).

**EXERCISE (3.24).** — Let \( \varphi : R \to R' \) be a ring map, \( b \subseteq R' \) a subset. Prove

\[
\varphi^{-1} \sqrt{b} = \sqrt{\varphi^{-1}b}
\]

**EXERCISE (3.25).** — Let \( e, e' \in \text{Idem}(R) \). Assume \( \sqrt{\langle e \rangle} = \sqrt{\langle e' \rangle} \). Show \( e = e' \).

**EXERCISE (3.26).** — Let \( R \) be a ring, \( a_1, a_2 \) comaximal ideals with \( a_1 a_2 \subseteq \text{nil}(R) \). Show there are complementary idempotents \( e_1, e_2 \) with \( e_i \in a_i \).

**EXERCISE (3.27).** — Let \( R \) be a ring, \( a \) an ideal, \( \kappa : R \to R/a \) the quotient map. Assume \( a \subseteq \text{nil}(R) \). Show \( \text{Idem}(\kappa) \) is bijective.

**Theorem (3.29) (Scheinnullstellensatz).** — Let \( R \) be a ring, \( a \) an ideal. Then

\[
\sqrt{a} = \bigcap_{p \supseteq a} p
\]

where \( p \) runs through all the prime ideals containing \( a \). (By convention, the empty intersection is equal to \( R \).)

**Proof:** Take \( x \notin \sqrt{a} \). Set \( S := \{1, x, x^2, \ldots\} \). Then \( S \) is multiplicative, and \( a \cap S = \emptyset \). By (3.12), there is a \( p \supseteq a \), but \( x \notin p \). So \( x \notin \bigcap_{p \supseteq a} p \). Thus \( \sqrt{a} \cap \bigcap_{p \supseteq a} p \).

Conversely, take \( x \in \sqrt{a} \). Say \( x^n \in a \subset p \). Then \( x \in p \). Thus \( \sqrt{a} = \bigcap_{p \supseteq a} p \). \( \square \)

**EXERCISE (3.28).** — Let \( R \) be a ring. Prove the following statements equivalent:

1. \( R \) has exactly one prime \( p \);
2. every element of \( R \) is either nilpotent or a unit;
3. \( R/\text{nil}(R) \) is a field.

**Proposition (3.31).** — Let \( R \) be a ring, \( a \) an ideal. Then \( \sqrt{a} \) is an ideal.
Proof: Take \(x, y \in \sqrt{a}\); say \(x^n \in a\) and \(y^m \in a\). Then
\[
(x + y)^{n+m-1} = \sum_{i+j=m+n-1} \binom{n+m-1}{j} x^i y^j.
\]
This sum belongs to \(a\) as, in each summand, either \(x^i\) or \(y^j\) does, since, if \(i \leq n - 1\) and \(j \leq m - 1\), then \(i + j \leq m + n - 2\). Thus \(x + y \in \sqrt{a}\). So clearly \(\sqrt{a}\) is an ideal.

Alternatively, given any collection of ideals \(a_\lambda\), note that \(\bigcap a_\lambda\) is also an ideal. So \(\sqrt{a}\) is an ideal owing to (5.24).

Exercise (5.32). — Let \(R\) be a ring, and \(a\) an ideal. Assume \(\sqrt{a}\) is finitely generated. Show \((\sqrt{a})^n \subset a\) for all large \(n\).

Exercise (5.33). — Let \(R\) be a ring, \(q\) an ideal, \(p\) a finitely generated prime. Prove that \(p = \sqrt{q}\) if and only if there is \(n \geq 1\) such that \(p \supset q \supset p^n\).

Proposition (3.34). — A ring \(R\) is reduced and has only one minimal prime \(q\) if and only if \(R\) is a domain.

Proof: Suppose \(R\) is reduced, or \((0) = \sqrt{(0)}\). Then \((0)\) is equal to the intersection of all the prime ideals \(p\) by (4.14). By (4.14), every \(p\) contains \(q\). So \((0) = q\). Thus \(R\) is a domain. The converse is obvious.

Exercise (5.35). — Let \(R\) be a ring. Assume \(R\) is reduced and has finitely many minimal prime ideals \(p_1, \ldots, p_n\). Prove \(\varphi: R \to \prod(R/p_i)\) is injective, and for each \(i\), there is some \((x_1, \ldots, x_n) \in \text{Im}(\varphi)\) with \(x_i \neq 0\) but \(x_j = 0\) for \(j \neq i\).

Exercise (5.36). — Let \(R\) be a ring, \(X\) a variable, \(f := a_0 + a_1 X + \cdots + a_n X^n\) and \(g := b_0 + b_1 X + \cdots + b_m X^m\) polynomials with \(a_n \neq 0\) and \(b_m \neq 0\). Call \(f\) primitive if \((a_0, \ldots, a_n) = R\). Prove the following statements:
1. Then \(f\) is nilpotent if and only if \(a_0, \ldots, a_n\) are nilpotent.
2. Then \(f\) is a unit if and only if \(a_0\) is a unit and \(a_1, \ldots, a_n\) are nilpotent.
3. If \(f\) is a zerodivisor, then there is a nonzero \(b \in R\) with \(bf = 0\); in fact, if \(fg = 0\) with \(m\) minimal, then \(fb_m = 0\) (or \(m = 0\)).
4. Then \(fg\) is primitive if and only if \(f\) and \(g\) are primitive.

Exercise (5.37). — Generalize (5.36) to the polynomial ring \(P := R[X_1, \ldots, X_r]\). For (3), reduce to the case of one variable \(Y\) via this standard device: take \(d\) suitably large, and define \(\varphi: P \to R[Y]\) by \(\varphi(X_i) := Y^d\).

Exercise (5.38). — Let \(R\) be a ring, \(X\) a variable. Show that
\[
\text{rad}(R[X]) = \text{nil}(R[X]) = \text{nil}(R)R[X].
\]

Exercise (5.39). — Let \(R\) be a ring, \(a\) an ideal, \(X\) a variable, \(R[[X]]\) the formal power series ring, \(M \subset R[[X]]\) be a maximal ideal, and \(f := \sum a_n X^n \in R[[X]]\). Set \(m := M \cap R\) and \(A := \{ \sum b_n X^n \mid b_n \in a \}\). Prove the following statements:
1. If \(f\) is nilpotent, then \(a_n\) is nilpotent for all \(n\). The converse is false.
2. Then \(f \in \text{rad}(R[[X]])\) if and only if \(a_0 \in \text{rad}(R)\).
3. Assume \(X \in M\). Then \(X\) and \(m\) generate \(M\).
4. Assume \(M\) is maximal. Then \(X \in M\) and \(m\) is maximal.
5. If \(a\) is finitely generated, then \(aR[[X]] = A\). The converse may fail.
Example (3.40). — Let $R$ be a ring, $R[[X]]$ the formal power series ring. Then every prime $p$ of $R$ is the contraction of a prime of $R[[X]]$. Indeed, $pR[[X]] \cap R = p$. So by (3.13), there is a prime $q$ of $R[[X]]$ with $q \cap R = p$. In fact, a specific choice for $q$ is the set of series $\sum a_n X^n$ with $a_n \in p$. Indeed, the canonical map $R \rightarrow R/p$ induces a surjection $R[[X]] \rightarrow R/p$ with kernel $q$; hence, $R[[X]]/q = (R/p)[[X]]$. Plainly $(R/p)[[X]]$ is a domain. But (3.19) (5) shows $q$ may not be equal to $pR[[X]]$. 
4. Modules

In Commutative Algebra, it has proven advantageous to expand the study of rings to include modules. Thus we obtain a richer theory, which is more flexible and more useful. We begin the expansion here by discussing residue modules, kernels, and images. In particular, we identify the UMP of the residue module, and use it to construct the Noether isomorphisms. We also construct free modules, direct sums, and direct products, and we describe their UMPs.

(4.1) (Modules). — Let $R$ be a ring. Recall that an $R$-module $M$ is an abelian group, written additively, with a scalar multiplication, $R \times M \to M$, written $(x; m) \mapsto xm$, which is

1. **distributive**, $x(m + n) = xm + xn$ and $(x + y)m = xm + xm$,
2. **associative**, $x(ym) = (xy)m$, and
3. **unitary**, $1 \cdot m = m$.

For example, if $R$ is a field, then an $R$-module is a vector space. Moreover, a $\mathbb{Z}$-module is just an abelian group; multiplication is repeated addition.

As in (1.1), for any $x \in R$ and $m \in M$, we have $x \cdot 0 = 0$ and $0 \cdot m = 0$.

A **submodule** $N$ of $M$ is a subgroup that is closed under multiplication; that is, $xn \in N$ for all $x \in R$ and $n \in N$. For example, the ring $R$ is itself an $R$-module, and the submodules are just the ideals. Given an ideal $a$, let $aN$ denote the smallest submodule containing all products $an$ with $a \in a$ and $n \in N$. Similar to (1.4), clearly $aN$ is equal to the set of finite sums $\sum a_i n_i$ with $a_i \in a$ and $n_i \in N$.

Given $m \in M$, we call the set of $x \in R$ with $xm = 0$ the **annihilator** of $m$, and denote it $\text{Ann}(m)$. We call the set of $x \in R$ with $xm = 0$ for all $m \in M$ the **annihilator** of $M$, and denote it $\text{Ann}(M)$. Clearly, $\text{Ann}(m)$ and $\text{Ann}(M)$ are ideals.

(4.2) (Homomorphisms). — Let $R$ be a ring, $M$ and $N$ modules. Recall that a **homomorphism**, or $R$-linear map, is a map $\alpha : M \to N$ such that:

$$\alpha(xm + ym) = x(\alpha m) + y(\alpha m).$$

Associated to a homomorphism $\alpha : M \to N$ are its **kernel** and its **image**

$$\text{Ker}(\alpha) := \alpha^{-1}(0) \subset M \quad \text{and} \quad \text{Im}(\alpha) := \alpha(M) \subset N.$$  

They are defined as subsets, but are obviously submodules.

A homomorphism $\alpha$ is called an **isomorphism** if it is bijective. If so, then we write $\alpha : M \xrightarrow{\sim} N$. Then the set-theoretic inverse $\alpha^{-1} : N \to M$ is a homomorphism too. So $\alpha$ is an isomorphism if and only if there is a set map $\beta : N \to M$ such that $\beta \alpha = 1_M$ and $\alpha \beta = 1_N$, where $1_M$ and $1_N$ are the identity maps, and then $\beta = \alpha^{-1}$. If there is an unnamed isomorphism between $M$ and $N$, then we write $M \simeq N$ when it is **canonical** (that is, it does not depend on any artificial choices), and we write $M \cong N$ otherwise.

The set of homomorphisms $\alpha$ is denoted by $\text{Hom}_R(M, N)$ or simply $\text{Hom}(M, N)$. It is an $R$-module with addition and scalar multiplication defined by

$$(\alpha + \beta)m := \alpha m + \beta m \quad \text{and} \quad (x\alpha)m := x(\alpha m) = \alpha(xm).$$
Homomorphisms $\alpha: L \to M$ and $\beta: N \to P$ induce, via composition, a map
$$\text{Hom}(\alpha, \beta): \text{Hom}(M, N) \to \text{Hom}(L, P),$$
which is obviously a homomorphism. When $\alpha$ is the identity map $1_M$, we write $\text{Hom}(M, \beta)$ for $\text{Hom}(1_M, \beta)$; similarly, we write $\text{Hom}(\alpha, N)$ for $\text{Hom}(\alpha, 1_N)$.

**Exercise (1.3).** — Let $R$ be a ring, $M$ a module. Consider the set map
$$\rho: \text{Hom}(R, M) \to M \text{ defined by } \rho(\theta) := \theta(1).$$
Show that $\rho$ is an isomorphism, and describe its inverse.

(4.4) *Endomorphisms*. — Let $R$ be a ring, $M$ a module. An endomorphism of $M$ is a homomorphism $\alpha: M \to M$. The module of endomorphisms $\text{Hom}(M, M)$ is also denoted $\text{End}_R(M)$. It is a ring, usually noncommutative, with multiplication given by composition. Further, $\text{End}_R(M)$ is a subring of $\text{End}_\mathbb{Z}(M)$.

Given $x \in R$, let $\mu_x: M \to M$ denote the map of multiplication by $x$, defined by $\mu_x(m) := xm$. It is an endomorphism. Further, $x \mapsto \mu_x$ is a ring map
$$\mu_R: R \to \text{End}_R(M) \subset \text{End}_\mathbb{Z}(M).$$
(Thus we may view $\mu_R$ as representing $R$ as a ring of operators on the abelian group $M$.) Note that $\text{Ker}(\mu_R) = \text{Ann}(M)$.

Conversely, given an abelian group $N$ and a ring map
$$\nu: R \to \text{End}_\mathbb{Z}(N),$$
we obtain a module structure on $N$ by setting $xn := (\nu x)(n)$. Then $\mu_R = \nu$.

We call $M$ faithful if $\mu_R: R \to \text{End}_R(M)$ is injective, or $\text{Ann}(M) = 0$. For example, $R$ is a faithful $R$-module, as $x \cdot 1 = 0$ implies $x = 0$.

(4.5) *Algebras*. — Fix two rings $R$ and $R'$. Suppose $R'$ is an $R$-algebra with structure map $\varphi$. Let $M'$ be an $R'$-module. Then $M'$ is also an $R$-module by restriction of scalars: $x m := \varphi(x)m$. In other words, the $R$-module structure on $M'$ corresponds to the composition
$$R \xrightarrow{\varphi} R' \xrightarrow{\mu_{R'}} \text{End}_\mathbb{Z}(M').$$

In particular, $R'$ is an $R$-module; further, for all $x \in R$ and $y, z \in R'$,
$$(xy)z = x(yz).$$
Indeed, $R'$ is an $R'$-module, so an $R$-module by restriction of scalars: further, $(xy)z = x(yz)$ since $(\varphi(x)y)z = \varphi(x)(yz)$ by associativity in $R'$.

Conversely, suppose $R'$ is an $R$-module such that $(xy)z = x(yz)$. Then $R'$ has an $R$-algebra structure that is compatible with the given $R$-module structure. Indeed, define $\varphi: R \to R'$ by $\varphi(x) := x \cdot 1$. Then $\varphi(x)z = xz$ as $(x \cdot 1)z = x(1 \cdot z)$. So the composition $\mu_{R'} \varphi: R \to R' \to \text{End}_\mathbb{Z}(R')$ is equal to $\mu_R$. Hence $\varphi$ is a ring map, because $\mu_R$ is one and $\mu_{R'}$ is injective by (1.3). Thus $R'$ is an $R$-algebra, and restriction of scalars recovers its given $R$-module structure.

Suppose that $R' = R/\mathfrak{a}$ for some ideal $\mathfrak{a}$. Then an $R$-module $M$ has a compatible $R'$-module structure if and only if $\mathfrak{a}M = 0$; if so, then the $R'$-structure is unique. Indeed, the ring map $\mu_R: R \to \text{End}_\mathbb{Z}(M)$ factors through $R'$ if and only if $\mu_R(\mathfrak{a}) = 0$ by (1.10), so if and only if $\mathfrak{a}M = 0$; as $\text{End}_\mathbb{Z}(M)$ may be noncommutative, we must apply (1.16) to $\mu_R(R)$, which is commutative.

Again suppose $R'$ is an arbitrary $R$-algebra with structure map $\varphi$. A subalgebra $R''$ of $R'$ is a subring such that $\varphi$ maps into $R''$. The subalgebra generated by
\[ x_1, \ldots, x_n \in R' \] is the smallest \( R \)-subalgebra that contains them. We denote it by \( R[x_1, \ldots, x_n] \). It clearly contains all polynomial combinations \( f(x_1, \ldots, x_n) \) with coefficients in \( R \). In fact, the set \( R'' \) of these polynomial combinations is itself clearly an \( R \)-subalgebra; hence, \( R'' = R[x_1, \ldots, x_n] \).

We say \( R' \) is a \textbf{finitely generated} \( R \)-\textbf{algebra} or is \textbf{algebra finite over} \( R \) if there exist \( x_1, \ldots, x_n \in R' \) such that \( R' = R[x_1, \ldots, x_n] \).

\textbf{(4.6) (Residue modules).} — Let \( R \) be a ring, \( M \) a module, \( M' \subset M \) a submodule. Form the set of cosets, or set of residues,
\[
M/M' := \{ m + M' \mid m \in M \}.
\]
Recall that \( M/M' \) inherits a module structure, and is called the \textbf{residue module}, or \textbf{quotient}, of \( M \) \textbf{modulo} \( M' \). Form the \textbf{quotient map}
\[
\kappa: M \to M/M' \quad \text{by} \quad \kappa(m) := m + M'.
\]
Clearly \( \kappa \) is surjective, \( \kappa \) is linear, and \( \kappa \) has kernel \( M' \).

Let \( \alpha: M \to N \) be linear. Note that \( \text{Ker}(\alpha) \supset M' \) if and only if \( \alpha(M') = 0 \).

Recall that, if \( \text{Ker}(\alpha) \supset M' \), then there exists a homomorphism \( \beta: M/M' \to N \) such that \( \beta \kappa = \alpha \); that is, the following diagram is commutative:
\[
\begin{array}{ccc}
M & \xrightarrow{\kappa} & M/M' \\
\alpha \downarrow & & \beta \downarrow \\
N & & \text{Ker}(\alpha)
\end{array}
\]
Conversely, if \( \beta \) exists, then \( \text{Ker}(\alpha) \supset M' \), or \( \alpha(M') = 0 \), as \( \kappa(M') = 0 \).

Further, if \( \beta \) exists, then \( \beta \) is unique as \( \kappa \) is surjective.

Finally, since \( \kappa \) is surjective, if \( \beta \) exists, then \( \beta \) is surjective if and only if \( \alpha \) is so. In addition, then \( \beta \) is injective if and only if \( M' = \text{Ker}(\alpha) \). Hence \( \beta \) is an \textbf{isomorphism} if and only if \( \alpha \) is surjective and \( M' = \text{Ker}(\alpha) \). In particular, always
\[
M/\text{Ker}(\alpha) \xrightarrow{\sim} \text{Im}(\alpha). \tag{4.6.1}
\]

In practice, it is usually more productive to view \( M/M' \) not as a set of cosets, but simply another module \( M'' \) that comes equipped with a surjective homomorphism \( \alpha: M \to M'' \) whose kernel is the given submodule \( M' \).

Finally, as we have seen, \( M/M' \) has the following \textbf{UMP}: \( \kappa(M') = 0 \), and given \( \alpha: M \to N \) such that \( \alpha(M') = 0 \), there is a unique homomorphism \( \beta: M/M' \to N \) such that \( \beta \kappa = \alpha \). Formally, the UMP determines \( M/M' \) up to unique isomorphism.

\textbf{(4.7) (Cyclic modules).} — Let \( R \) be a ring. A module \( M \) is said to be \textbf{cyclic} if there exists \( m \in M \) such that \( M = Rm \). If so, form \( \alpha: R \to M \) by \( x \mapsto xm \); then \( \alpha \) induces an isomorphism \( R/\text{Ann}(m) \to M \) as \( \text{Ker}(\alpha) = \text{Ann}(m) \); see \((3.1.11)\). Note that \( \text{Ann}(m) = \text{Ann}(M) \). Conversely, given any ideal \( a \), the \( R \)-module \( R/a \) is cyclic, generated by the coset of 1, and \( \text{Ann}(R/a) = a \).

\textbf{(4.8) (Noether Isomorphisms).} — Let \( R \) be a ring, \( N \) a module, and \( L \) and \( M \) submodules.

First, assume \( L \subset M \subset N \). Form the following composition of quotient maps:
\[
\alpha: N \to N/L \to (N/L)/(M/L).
\]
Clearly \( \alpha \) is surjective, and \( \text{Ker}(\alpha) = M \). Hence owing to \((3.1.11)\), \( \alpha \) \textbf{factors through}
the isomorphism $\beta$ in this commutative diagram:

$$
\begin{array}{ccc}
N & \longrightarrow & N/M \\
\downarrow & & \downarrow \cong \\
N/L & \longrightarrow & (N/L)/(M/L)
\end{array}
$$

(4.8.1)

Second, let $L + M$ denote the set of all sums $\ell + m$ with $\ell \in L$ and $m \in M$. Clearly $L + M$ is a submodule of $N$. It is called the sum of $L$ and $M$.

Form the composition $\alpha'$ of the inclusion map $L \hookrightarrow L + M$ and the quotient map $(L + M) \twoheadrightarrow (L + M)/M$. Clearly $\alpha'$ is surjective and $\operatorname{Ker}(\alpha') = L \cap M$. Hence owing to (4.6.1), $\alpha'$ factors through the isomorphism $\beta'$ in this commutative diagram:

$$
\begin{array}{ccc}
L & \longrightarrow & L/(L \cap M) \\
\downarrow & & \downarrow \cong \\
L + M & \longrightarrow & (L + M)/M
\end{array}
$$

(4.8.2)

The isomorphisms of (4.6.1) and (4.8.1) and (4.8.2) are called Noether’s First, Second, and Third Isomorphisms.

(4.9) (Cokernels, coimages). — Let $R$ be a ring, $\alpha: M \rightarrow N$ a linear map. Associated to $\alpha$ are its cokernel and its coimage,

$$
\operatorname{Coker}(\alpha) := N/\operatorname{Im}(\alpha) \quad \text{and} \quad \operatorname{Coim}(\alpha) := M/\operatorname{Ker}(\alpha);
$$

they are quotient modules, and their quotient maps are both denoted by $\kappa$.

Note (4.7) yields the UMP of the cokernel: $\kappa \alpha = 0$, and given a map $\beta: N \rightarrow P$ with $\beta \alpha = 0$, there is a unique map $\gamma$: $\operatorname{Coker}(\alpha) \rightarrow P$ with $\gamma \kappa = \beta$ as shown below

$$
\begin{array}{ccc}
M & \overset{\alpha}{\longrightarrow} & N \overset{\alpha}{\longrightarrow} \operatorname{Coker}(\alpha) \\
\downarrow {\beta} & & \downarrow {\gamma} \\
& P & 
\end{array}
$$

Further, (4.6.1) becomes $\operatorname{Coim}(\alpha) \cong \operatorname{Im}(\alpha)$.

(4.10) (Free modules). — Let $R$ be a ring, $\Lambda$ a set, $M$ a module. Given elements $m_{\lambda} \in M$ for $\lambda \in \Lambda$, by the submodule they generate, we mean the smallest submodule that contains them all. Clearly, any submodule that contains them all contains any (finite) linear combination $\sum x_{\lambda} m_{\lambda}$ with $x_{\lambda} \in R$. On the other hand, consider the set $N$ of all such linear combinations; clearly, $N$ is a submodule containing the $m_{\lambda}$. Thus $N$ is the submodule generated by the $m_{\lambda}$.

The $m_{\lambda}$ are said to be free or linearly independent if, whenever $\sum x_{\lambda} m_{\lambda} = 0$, also $x_{\lambda} = 0$ for all $\lambda$. Finally, the $m_{\lambda}$ are said to form a (free) basis of $M$ if they are free and generate $M$; if so, then we say $M$ is free on the $m_{\lambda}$.

We say $M$ is finitely generated if it has a finite set of generators. We say $M$ is free if it has a free basis. If so, then by either (5.32) (2) or (10.5) below, any two free bases have the same number $\ell$ of elements, and we say $M$ is free of rank $\ell$, and we set $\operatorname{rank}(M) := \ell$.

For example, form the set of restricted vectors

$$
R^{\oplus \Lambda} := \{(x_{\lambda}) \mid x_{\lambda} \in R \text{ with } x_{\lambda} = 0 \text{ for almost all } \lambda\}.
$$

It is a module under componentwise addition and scalar multiplication. It has a standard basis, which consists of the vectors $e_{\mu}$ whose $\mu$th component is the value.
of the Kronecker delta function; that is,

\[ e_\mu := (\delta_{\mu \lambda}) \quad \text{where} \quad \delta_{\mu \lambda} := \begin{cases} 1, & \text{if } \lambda = \mu; \\ 0, & \text{if } \lambda \neq \mu. \end{cases} \]

Clearly the standard basis is free. If \( \Lambda \) has a finite number \( \ell \) of elements, then \( R^{\oplus \Lambda} \) is often written \( R^\ell \) and called the \textbf{direct sum of \( \ell \) copies of} \( R \).

The free module \( R^{\oplus \Lambda} \) has the following UMP: \textit{given a module \( M \) and elements \( m_\lambda \in M \) for \( \lambda \in \Lambda \), there is a unique homomorphism}

\[ \alpha : R^{\oplus \Lambda} \to M \text{ with } \alpha(e_\lambda) = m_\lambda \text{ for each } \lambda \in \Lambda; \]

\textit{namely,} \( \alpha((x_\lambda)) = \sum x_\lambda e_\lambda \). Note the following obvious statements:

1. \( \alpha \) is surjective if and only if the \( \lambda \) generate \( M \).
2. \( \alpha \) is injective if and only if the \( \lambda \) are linearly independent.
3. \( \alpha \) is an isomorphism if and only if the \( \lambda \) form a free basis.

Thus \( M \) is \textit{free of rank} \( \ell \) if and only if \( M \cong R^\ell \).

**Example (4.11).** — Take \( R := \mathbb{Z} \) and \( M := \mathbb{Q} \). Then any two \( x, y \) in \( M \) are not free; indeed, if \( x = a/b \) and \( y = -c/d \), then \( bcx + ady = 0 \). So \( M \) is not free.

Also \( M \) is not finitely generated. Indeed, given any \( m_1/n_1, \ldots, m_r/n_r \in M \), let \( d \) be a common multiple of \( n_1, \ldots, n_r \). Then \( (1/d)\mathbb{Z} \) contains every linear combination \( x_1(m_1/n_1) + \cdots + x_\ell(m_\ell/n_\ell) \), but \( (1/d)\mathbb{Z} \neq \mathbb{M} \).

Moreover, \( \mathbb{Q} \) is not algebra finite over \( \mathbb{Z} \). Indeed, let \( p \in \mathbb{Z} \) be any prime not dividing \( n_1 \cdots n_r \). Then \( 1/p \notin \mathbb{Z}[m_1/n_1, \ldots, m_r/n_r] \).

**Exercise (4.12).** — Let \( R \) be a domain, and \( x \in R \) nonzero. Let \( M \) be the submodule of \( \text{Frac}(R) \) generated by \( 1, x^{-1}, x^{-2}, \ldots \). Suppose that \( M \) is finitely generated. Prove that \( x^{-1} \in R \), and conclude that \( M = R \).

**Exercise (4.13).** — A finitely generated free module \( F \) has finite rank.

**Theorem (4.14).** — Let \( R \) be a PID, \( E \) a free module, \{\( e_\lambda \)\}_{\lambda \in \Lambda} \text{ a (free) basis, and} \ F \text{ a submodule. Then} F \text{ is free, and has a basis indexed by a subset of} \Lambda.

**Proof:** Well order \( \Lambda \). For all \( \lambda \), let \( \pi_\lambda : E \to R \) be the \( \lambda \)-th projection. For all \( \mu \), set \( E_\mu := \bigoplus_{\lambda \leq \mu} R e_\lambda \) and \( F_\mu := F \cap E_\mu \). Then \( \pi_\mu(F_\mu) = \{a_\mu\} \) for some \( a_\mu \in R \) as \( R \) is a PID. Choose \( f_\mu \in F_\mu \) with \( \pi_\mu(f_\mu) = a_\mu \). Let \( \Lambda_0 := \{ \mu \in \Lambda \mid a_\mu \neq 0 \} \).

Say \( \sum_{\mu \in \Lambda_0} c_\mu f_\mu = 0 \) for some \( c_\mu \in R \). Set \( \Lambda_1 := \{ \mu \in \Lambda_0 \mid c_\mu \neq 0 \} \). Suppose \( \Lambda_1 \neq 0 \). Note \( \Lambda_1 \) is finite. Let \( \mu_1 \) be the greatest element of \( \Lambda_1 \). Then \( \pi_{\mu_1}(f_{\mu_1}) = 0 \) for \( \mu < \mu_1 \) as \( f_{\mu_1} \in E_{\mu_1} \). So \( \pi_{\mu_1}(\sum_{\mu \leq \mu_1} c_\mu f_\mu) = c_{\mu_1} a_{\mu_1} \). So \( c_{\mu_1} a_{\mu_1} = 0 \). But \( c_{\mu_1} \neq 0 \) and \( a_{\mu_1} \neq 0 \), a contradiction. Thus \( \{f_\mu\}_{\mu \in \Lambda_0} \) is linearly independent.

Note \( F = \bigcup_{\lambda \in \Lambda_0} F_\lambda \). Given \( \lambda \in \Lambda_0 \), set \( \Lambda_\lambda := \{ \mu \in \Lambda_0 \mid \mu \leq \lambda \} \). Suppose \( \lambda \) is least such that \( \{f_\mu\}_{\mu \in \Lambda_\lambda} \) does not generate \( F_\lambda \). Given \( f \in F_\lambda \), say \( f = \sum_{\mu \leq \lambda} c_\mu e_\mu \) with \( c_\mu \in R \). Then \( \pi_\lambda(f) = c_\lambda \). But \( \pi_\lambda(F_\lambda) = \{a_\lambda\} \). So \( c_\lambda = b_\lambda a_\lambda \) for some \( b_\lambda \in R \). Set \( g := f - b_\lambda f_\lambda \). Then \( g \in F_\lambda \), and \( \pi_\lambda(g) = 0 \). So \( g \in F_{\nu} \) for some \( \nu \in \Lambda_0 \) with \( \nu < \lambda \). Hence \( g = \sum_{\mu \in \Lambda_\nu} b_{\mu} f_{\mu} \) for some \( b_{\mu} \in R \). So \( f = \sum_{\mu \in \Lambda_\lambda} b_{\mu} f_{\mu} \), a contradiction. Hence \( \{f_\mu\}_{\mu \in \Lambda_0} \) generates \( F_\lambda \). Thus \( \{f_\mu\}_{\mu \in \Lambda_0} \) is a basis of \( F \). \( \square \)

**4.15 (Direct Products, Direct Sums).** — Let \( R \) be a ring, \( \Lambda \) a set, \( M_\lambda \) a module for \( \lambda \in \Lambda \). The \textbf{direct product} of the \( M_\lambda \) is the set of arbitrary vectors:

\[ \prod M_\lambda := \{ (m_\lambda) \mid m_\lambda \in M_\lambda \}. \]
Clearly, $\prod M_\lambda$ is a module under componentwise addition and scalar multiplication.

The **direct sum** of the $M_\lambda$ is the subset of **restricted vectors**: 

$$\bigoplus M_\lambda := \{ (m_\lambda) \mid m_\lambda = 0 \text{ for almost all } \lambda \} \subset \prod M_\lambda.$$ 

Clearly, $\bigoplus M_\lambda$ is a submodule of $\prod M_\lambda$. Clearly, $\bigoplus M_\lambda = \prod M_\lambda$ if $\Lambda$ is finite. If $\Lambda = \{ \lambda_1, \ldots, \lambda_n \}$, then $\bigoplus M_\lambda$ is also denoted by $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_n}$. Further, if $M_\lambda = M$ for all $\lambda$, then $\bigoplus M_\lambda$ is also denoted by $M^A$, or by $M^n$ if $\Lambda$ has just $n$ elements.

The direct product comes equipped with projections

$$\pi_\kappa: \prod M_\lambda \to M_\kappa \quad \text{given by} \quad \pi_\kappa((m_\lambda)) := m_\kappa.$$ 

It is easy to see that $\prod M_\lambda$ has this UMP: **given homomorphisms** $\alpha_\kappa: L \to M_\kappa$, there is a unique homomorphism $\alpha: L \to \prod M_\lambda$ satisfying $\pi_\kappa \alpha = \alpha_\kappa$ for all $\kappa \in \Lambda$; namely, $\alpha(n) = (\alpha_\kappa(n))$. Often, $\alpha$ is denoted $\langle \alpha_\lambda \rangle$. In other words, the $\pi_\lambda$ induce a bijection of sets,

$$\text{Hom}(L, \prod M_\lambda) \to \prod \text{Hom}(L, M_\lambda). \quad (4.15.1)$$

Clearly, this bijection is an isomorphism of modules.

Similarly, the direct sum comes equipped with injections

$$\iota_\kappa: M_\kappa \to \bigoplus M_\lambda \quad \text{given by} \quad \iota_\kappa(m) := (m_\lambda) \quad \text{where} \quad m_\lambda := \begin{cases} m, & \text{if } \lambda = \kappa; \\ 0, & \text{if } \lambda \neq \kappa. \end{cases}$$

It is easy to see that it has this UMP: **given homomorphisms** $\beta_\kappa: M_\kappa \to N$, there is a unique homomorphism $\beta: \bigoplus M_\lambda \to N$ satisfying $\beta \iota_\kappa = \beta_\kappa$ for all $\kappa \in \Lambda$; namely, $\beta((m_\lambda)) = \sum \beta_\lambda(m_\lambda)$. Often, $\beta$ is denoted $\langle \beta_\lambda \rangle$; often, $\langle \beta_\lambda \rangle$. In other words, the $\iota_\kappa$ induce this bijection of sets:

$$\text{Hom}(\bigoplus M_\lambda, N) \to \prod \text{Hom}(M_\lambda, N). \quad (4.15.2)$$

Clearly, this bijection is an isomorphism of modules.

For example, if $M_\lambda = R$ for all $\lambda$, then $\bigoplus M_\lambda = R^{\oplus \Lambda}$ by construction. Further, if $N_\lambda := N$ for all $\lambda$, then $\text{Hom}(R^{\oplus \Lambda}, N) = \prod N_\lambda$ by (1.16) and (1.3).

**Exercise 4.17.** — Let $\Lambda$ be an infinite set, $R_\Lambda$ a nonzero ring for $\lambda \in \Lambda$. Endow $\prod R_\Lambda$ and $\bigoplus R_\Lambda$ with componentwise addition and multiplication. Show that $\prod R_\Lambda$ has a multiplicative identity (so is a ring), but that $\bigoplus R_\Lambda$ does not (so is not a ring).

**Exercise 4.18.** — Let $R$ be a ring, $M$ a module, and $M'$, $M''$ submodules. Show that $M = M' \oplus M''$ if and only if $M = M' + M''$ and $M' \cap M'' = 0$.

**Exercise 4.19.** — Let $L$, $M$, and $N$ be modules. Consider a diagram

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N$$

where $\alpha$, $\beta$, $\rho$, and $\sigma$ are homomorphisms. Prove that

$$M = L \oplus N \quad \text{and} \quad \alpha = \iota_L, \beta = \pi_N, \sigma = \iota_N, \rho = \pi_L$$

if and only if the following relations hold:

$$\beta \alpha = 0, \beta \sigma = 1, \rho \sigma = 0, \rho \alpha = 1, \text{ and } \alpha \rho + \sigma \beta = 1.$$
Exercise (4.19). — Let $L$ be a module, $\Lambda$ a nonempty set, $M_\lambda$ a module for $\lambda \in \Lambda$. Prove that the injections $\iota_\lambda : M_\lambda \to \bigoplus M_\lambda$ induce an injection
\[
\bigoplus \text{Hom}(L, M_\lambda) \hookrightarrow \text{Hom}(L, \bigoplus M_\lambda),
\]
and that it is an isomorphism if $L$ is finitely generated.

Exercise (4.20). — Let $a$ be an ideal, $\Lambda$ a nonempty set, $M_\lambda$ a module for $\lambda \in \Lambda$. Prove $a(\bigoplus M_\lambda) = \bigoplus aM_\lambda$. Prove $a(\prod M_\lambda) = \prod aM_\lambda$ if $a$ is finitely generated.
5. Exact Sequences

In the study of modules, the exact sequence plays a central role. We relate it to the kernel and image, the direct sum and direct product. We introduce diagram chasing, and prove the Snake Lemma, which is a fundamental result in homological algebra. We define projective modules, and characterize them in four ways. Finally, we prove Schanuel’s Lemma, which relates two arbitrary presentations of a module.

In an appendix, we use determinants to study free modules.

**Definition (5.1)**. A (finite or infinite) sequence of module homomorphisms

\[ \cdots \to M_{i-1} \overset{\alpha_{i-1}}\to M_i \overset{\alpha_i}\to M_{i+1} \to \cdots \]

is said to be exact at \( M_i \) if \( \text{Ker}(\alpha_i) = \text{Im}(\alpha_{i-1}) \). The sequence is said to be exact if it is exact at every \( M_i \), except an initial source or final target.

**Example (5.2)**. (1) A sequence \( 0 \to L \overset{\alpha}\to M \) is exact if and only if \( \alpha \) is injective. If so, then we often identify \( L \) with its image \( \langle L \rangle \).

Dually—that is, in the analogous situation with all arrows reversed—a sequence \( M \overset{\beta}\to N \to 0 \) is exact if and only if \( \beta \) is surjective.

(2) A sequence \( 0 \to L \overset{\alpha}\to M \overset{\beta}\to N \) is exact if and only if \( L = \text{Ker}(\beta) \), where ‘\( = \)’ means “canonically isomorphic.” Dually, a sequence \( L \overset{\alpha}\to M \overset{\beta}\to N \to 0 \) is exact if and only if \( N = \text{Coker}(\alpha) \) owing to (1) and (4.6.1).

**Example (5.3)** (Short exact sequences). A sequence \( 0 \to L \overset{\alpha}\to M \overset{\beta}\to N \to 0 \) is exact if and only if \( \alpha \) is injective and \( N = \text{Coker}(\alpha) \), or dually, if and only if \( \beta \) is surjective and \( L = \text{Ker}(\beta) \). If so, then the sequence is called short exact, and often we regard \( L \) as a submodule of \( M \), and \( N \) as the quotient \( M/L \).

For example, the following sequence is clearly short exact:

\[ 0 \to L \overset{\iota_L}\to L \oplus N \overset{\pi_N}\to N \to 0 \quad \text{where} \quad \iota_L(l) := (l, 0) \quad \text{and} \quad \pi_N(l, n) := n. \]

Often, we identify \( L \) with \( \iota_L L \) and \( N \) with \( \iota_N N \).

**Proposition (5.4)**. For \( \lambda \in \Lambda \), let \( M_\lambda' \to M_\lambda \to M_\lambda'' \) be a sequence of module homomorphisms. If every sequence is exact, then so are the two induced sequences

\[ \bigoplus M_\lambda' \to \bigoplus M_\lambda \to \bigoplus M_\lambda'' \quad \text{and} \quad \prod M_\lambda' \to \prod M_\lambda \to \prod M_\lambda''. \]

Conversely, if either induced sequence is exact then so is every original one.

**Proof**: The assertions are immediate from (5.1) and (4.4.8). \( \square \)

**Exercise (5.5)**. Let \( M' \) and \( M'' \) be modules, \( N \subset M' \) a submodule. Set \( M := M' \oplus M'' \). Using (4.7.1) and (3.3.1) and (4.7.1), prove \( M/N = M'/N \oplus M'' \).

**Exercise (5.6)**. Let \( 0 \to M' \to M \to M'' \to 0 \) be a short exact sequence. Prove that, if \( M' \) and \( M'' \) are finitely generated, then so is \( M \).

**Proposition (5.7)**. Let \( 0 \to M' \overset{\alpha}\to M \overset{\beta}\to M'' \to 0 \) be a short exact sequence, and \( N \subset M \) a submodule. Set \( N' := \alpha^{-1}(N) \) and \( N'' := \beta(N) \). Then the induced sequence \( 0 \to N' \to N \to N'' \to 0 \) is short exact.
PROOF: It is simple and straightforward to verify the asserted exactness. □

(5.8) (Retraction, section, splits). — We call a linear map ρ: M → M’ a retraction of another α: M’ → M if ρα = 1M’. Then α is injective and ρ is surjective.

Dually, we call a linear map σ: M” → M a section of another β: M → M” if βσ = 1M”. Then β is surjective and σ is injective.

We say that a 3-term exact sequence M’ α M β M” splits if there is an isomorphism φ: M → M’ ⊕ M” with φα = iM’ and β = πM”φ.

PROPOSITION (5.9). — Let M’ α M β M” be a 3-term exact sequence. Then the following conditions are equivalent:

(1) The sequence splits.
(2) There exists a retraction ρ: M → M’ of α, and β is surjective.
(3) There exists a section σ: M” → M of β, and α is injective.

PROOF: Assume (1). Then there exists φ: M → M’ ⊕ M” such that φα = iM’ and β = πM”φ. Set ρ := πM”φ and σ := φ⁻¹iM”. Then plainly (2) and (3) hold.

Assume (2). Set σ′ := 1M – αρ. Then σ’α = α – αρα. But ρα = 1M as ρ is a retraction. Hence there exists σ: M” → M with σβ = σ’ by (4.22)(2) and the UMP of (1.9). Thus 1M = αρ + σβ.

Hence β = βαρ + βσβ. But βαρ = 0 as the sequence is exact. So β = βσβ. But β is surjective. Thus 1M = βσ; that is, (3) holds.

Similarly, σ = ασα + σβσ. But βσ = 1M” as (3) holds. So 0 = ασα. But α is injective, as ρ is a retraction of it. Thus ρα = 0. Thus (5.1) yields (1).

Assume (3). Then similarly (1) and (2) hold. □

EXAMPLE (5.10). — Let R be a ring, R’ an R-algebra, and M an R’-module. Set H := HomR(R’, M). Define α: M → H by α(m)(x) := xm, and ρ: H → M by ρ(θ) := θ(1). Then ρ is a retraction of α, as ρ(α(m)) = 1 · m. Let β: M → Coker(α) be the quotient map. Then (1.4) implies that M is a direct summand of H with α = iM and ρ = πM.


EXERCISE (5.12). — Criticize the following misstatement of (1.4): given a 3-term exact sequence M’ α M β M” there is an isomorphism M ≃ M’ ⊕ M” if and only if there is a section σ: M” → M of β and α is injective.

LEMMA (5.13) (Snake). — Consider this commutative diagram with exact rows:

\[
\begin{array}{ccc}
M’ & \xrightarrow{α} & M & \xrightarrow{β} & M” \rightarrow 0 \\
\downarrow{γ'} & & \downarrow{γ} & & \\
0 & \xrightarrow{γ’} & N & \xrightarrow{β’} & N”
\end{array}
\]

It yields the following exact sequence:

\[\text{Ker}(γ’) \xrightarrow{ε’} \text{Ker}(γ) \xrightarrow{ε} \text{Ker}(γ’') \xrightarrow{δ} \text{Coker}(γ’’) \xrightarrow{ψ’} \text{Coker}(γ’’).\] (5.13.1)

Moreover, if α is injective, then so is φ; dually, if β’ is surjective, then so is ψ’.

PROOF: Clearly α restricts to a map φ, because α(Ker(γ’)) ⊂ Ker(γ) since α’(Ker(γ’)) = 0. By the UMP discussed in (1.4), α factors through a unique map φ’ because M’ goes to 0 in Coker(γ). Similarly, β and β’ induce corresponding maps ψ and ψ’. Thus all the maps in (1.4) are defined except for δ.
To define \( \partial \), chase an \( m'' \in \ker(\gamma'') \) through the diagram. Since \( \beta \) is surjective, there is \( m \in M \) such that \( \beta(m) = m'' \). By commutativity, \( \gamma'' \beta(m) = \beta' \gamma(m) \). So \( \beta' \gamma(m) = 0 \). By exactness of the bottom row, there is a unique \( n' \in N' \) such that \( \alpha'(n') = \gamma(m) \). Define \( \partial(m'') \) to be the image of \( n' \) in \( \coker(\gamma') \).

To see \( \partial \) is well defined, choose another \( m_1 \in M \) with \( \beta(m_1) = m'' \). Let \( n'_1 \in N' \) be the unique element with \( \alpha'(n'_1) = \gamma(m_1) \) as above. Since \( \beta(m - m_1) = 0 \), there is an \( m' \in M' \) with \( \alpha(m') = m - m_1 \). But \( \alpha' \gamma' = \gamma \alpha \). So \( \alpha' \gamma'(m') = \alpha'(n' - n'_1) \). Hence \( \gamma'(m') = n' - n'_1 \) since \( \alpha' \) is injective. So \( n' \) and \( n'_1 \) have the same image in \( \coker(\gamma') \). Thus \( \partial \) is well defined.

Let’s show that \( (\text{Exercise } 5.16.1) \) is exact at \( \ker(\gamma'') \). Take \( m'' \in \ker(\gamma'') \). As in the construction of \( \partial \), take \( m \in M \) such that \( \beta(m) = m'' \) and take \( n' \in N' \) such that \( \alpha'(n') = \gamma(m) \). Suppose \( m'' \in \ker(\partial) \). Then the image of \( n' \) in \( \coker(\gamma') \) is equal to \( 0 \); so there is \( m' \in M' \) such that \( \gamma'(m') = n' \). Clearly \( \gamma \alpha(m') = \alpha' \gamma'(m') \). So \( \gamma \alpha(m') = \alpha(n') = \gamma(m) \). Hence \( m - \alpha(m') \in \ker(\gamma) \). Since \( \beta(m - \alpha(m')) = m'' \), clearly \( m'' = \psi(m - \alpha(m')) \); so \( m'' \in \im(\psi) \). Hence \( \ker(\partial) \subset \im(\psi) \).

Conversely, suppose \( m'' \in \im(\psi) \). We may assume \( m \in \ker(\gamma) \). So \( \gamma(m) = 0 \) and \( \alpha'(n') = 0 \). Since \( \alpha' \) is injective, \( n' = 0 \). Thus \( \partial(m'') = 0 \), and so \( \im(\psi) \subset \ker(\partial) \). Thus \( \ker(\partial) \) is equal to \( \im(\psi) \); that is, \( (\text{Exercise } 5.16.1) \) is exact at \( \ker(\gamma'') \).

The other verifications of exactness are similar or easier.

The last two assertions are clearly true. \( \square \)

**Exercise (5.16).** — Referring to (1.8), give an alternative proof that \( \beta \) is an isomorphism by applying the Snake Lemma to the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & M & \rightarrow & N & \rightarrow & N/M & \rightarrow & 0 \\
 & \downarrow{\kappa} & \downarrow{\beta} & & & & & \downarrow{=} \\
0 & \rightarrow & M/L & \rightarrow & N/L & \rightarrow & (N/L)/(M/L) & \rightarrow & 0
\end{array}
\]

**Exercise (5.16) (Five Lemma).** — Consider this commutative diagram:

\[
\begin{array}{cccccc}
M_4 & \xrightarrow{\alpha_4} & M_3 & \xrightarrow{\alpha_3} & M_2 & \xrightarrow{\alpha_2} & M_1 & \xrightarrow{\alpha_1} & M_0 \\
\gamma_4 & \downarrow{\gamma_3} & \gamma_2 & \downarrow{\gamma_1} & \gamma_0 & \downarrow{=} \\
N_4 & \xrightarrow{\beta_4} & N_3 & \xrightarrow{\beta_3} & N_2 & \xrightarrow{\beta_2} & N_1 & \xrightarrow{\beta_1} & N_0
\end{array}
\]

Assume it has exact rows. Via a chase, prove these two statements:

1. If \( \gamma_3 \) and \( \gamma_1 \) are surjective and if \( \gamma_0 \) is injective, then \( \gamma_2 \) is surjective.
2. If \( \gamma_3 \) and \( \gamma_1 \) are injective and if \( \gamma_4 \) is surjective, then \( \gamma_2 \) is injective.

**Exercise (5.16) (Nine Lemma).** — Consider this commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & L' & \rightarrow & L & \rightarrow & L'' & \rightarrow & 0 \\
& & \downarrow{=} & & \downarrow{=} & & \downarrow{=} \\
0 & \rightarrow & M' & \rightarrow & M & \rightarrow & M'' & \rightarrow & 0 \\
& & \downarrow{=} & & \downarrow{=} & & \downarrow{=} \\
0 & \rightarrow & N' & \rightarrow & N & \rightarrow & N'' & \rightarrow & 0
\end{array}
\]

(5.16.1)
Every short exact sequence

There is a module 
By
The module
By

that following induced sequence is exact:

Assume \( \alpha' \) and \( \gamma \) are surjective. Given \( n \in N \) and \( m'' \in M'' \) with \( \alpha''(m'') = \gamma'(n) \), show that there is \( m \in M \) such that \( \alpha(m) = n \) and \( \gamma(m) = m'' \).

**Theorem (5.18)** (Left exactness of Hom). — (1) Let \( M' \to M \to M'' \to 0 \) be a sequence of linear maps. Then it is exact if and only if, for all modules \( N \), the following induced sequence is exact:

\[
0 \to \text{Hom}(M'', N) \to \text{Hom}(M, N) \to \text{Hom}(M', N).
\]

(2) Let \( 0 \to N' \to N \to N'' \) be a sequence of module homomorphisms. Then it is exact if and only if, for all modules \( M \), the following induced sequence is exact:

\[
0 \to \text{Hom}(M, N') \to \text{Hom}(M, N) \to \text{Hom}(M, N'').
\]

**Proof:** By (5.23)(2), the exactness of \( M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0 \) means simply that \( M'' = \text{Coker}(\alpha) \). On the other hand, the exactness of (5.18.1) means that a \( \varphi \in \text{Hom}(M, N) \) maps to 0, or equivalently \( \varphi \alpha = 0 \), if and only if there is a unique \( \gamma: M'' \to N \) such that \( \gamma \beta = \varphi \). So (5.18.1) is exact if and only if \( M'' \) has the UMP of \( \text{Coker}(\alpha) \), discussed in (5.18); that is, \( M'' = \text{Coker}(\alpha) \). Thus (1) holds.

The proof of (2) is similar. \( \square \)

**Definition (5.19).** — A (free) **presentation** of a module \( M \) is an exact sequence

\[
G \to F \to M \to 0
\]

with \( G \) and \( F \) free. If \( G \) and \( F \) are free of finite rank, then the presentation is called **finite**. If \( M \) has a finite presentation, then \( M \) is said to be **finitely presented**.

**Proposition (5.20).** — Let \( R \) be a ring, \( M \) a module, \( m_\lambda \) for \( \lambda \in \Lambda \) generators. Then there is an exact sequence \( 0 \to K \to R^{\oplus \Lambda} \xrightarrow{\sigma\Sigma} M \to 0 \) with \( \alpha(e_\lambda) = m_\lambda \), where \( \{e_\lambda\} \) is the standard basis, and there is a presentation \( R^{\oplus \Sigma} \to R^{\oplus \Lambda} \xrightarrow{\sigma\Sigma} M \to 0 \).

**Proof:** By (5.14)(1), there is a surjection \( \alpha: R^{\oplus \Lambda} \to M \) with \( \alpha(e_\lambda) = m_\lambda \). Set \( K := \text{Ker}(\alpha) \). Then \( 0 \to K \to R^{\oplus \Lambda} \xrightarrow{\sigma\Sigma} M \to 0 \) is exact by (5.14). Take a set of generators \( \{k_\sigma\} \sigma \in \Sigma \) of \( K \), and repeat the process to obtain a surjection \( R^{\oplus \Sigma} \to K \). Then \( R^{\oplus \Sigma} \to R^{\oplus \Lambda} \to M \to 0 \) is a presentation. \( \square \)

**Definition (5.21).** — A module \( P \) is called **projective** if, given any surjective linear map \( \beta: M \twoheadrightarrow N \), every linear map \( \alpha: P \to N \) lifts to one \( \gamma: P \to M \); namely, \( \alpha = \beta \gamma \).

**Exercise (5.22).** — Show that a free module \( R^{\oplus \Lambda} \) is projective.

**Theorem (5.23).** — The following conditions on an \( R \)-module \( P \) are equivalent:

1. The module \( P \) is projective.
2. Every short exact sequence \( 0 \to K \to M \to P \to 0 \) splits.
3. There is a module \( K \) such that \( K \oplus P \) is free.
(4) Every exact sequence \( N' \to N \to N'' \) induces an exact sequence
\[
\text{Hom}(P, N') \to \text{Hom}(P, N) \to \text{Hom}(P, N'') \tag{5.23.1}
\]
(5) Every surjective homomorphism \( \beta : M \to N \) induces a surjection
\[
\text{Hom}(P, \beta) : \text{Hom}(P, M) \to \text{Hom}(P, N)
\]

**Proof:** Assume (1). In (2), the surjection \( M \to P \) and the identity \( P \to P \) yield a section \( P \to M \). So the sequence splits by (5.14). Thus (2) holds.

Assume (2). By (5.21), there is an exact sequence \( 0 \to K \to R^{\oplus A} \to P \to 0 \). Then (2) implies \( K \oplus P \simeq R^{\oplus A} \). Thus (3) holds.

Assume (3); say \( K \oplus P \simeq R^{\oplus A} \). For each \( \lambda \in \Lambda \), take a copy \( N_\lambda' \to N_\lambda \to N_\lambda'' \) of the exact sequence \( N' \to N \to N'' \) of (4). Then the induced sequence
\[
\prod N_\lambda' \to \prod N_\lambda \to \prod N_\lambda''
\]
is exact by (5.2). But by the end of (5.22), that sequence is equal to this one:
\[
\text{Hom}(R^{\oplus A}, N') \to \text{Hom}(R^{\oplus A}, N) \to \text{Hom}(R^{\oplus A}, N'').
\]
But \( K \oplus P \simeq R^{\oplus A} \). So owing to (5.15), the latter sequence is also equal to
\[
\text{Hom}(K, N') \oplus \text{Hom}(P, N') \to \text{Hom}(K, N) \oplus \text{Hom}(P, N) \to \text{Hom}(K, N'') \oplus \text{Hom}(P, N'').
\]
Hence (5.24) is exact by (5.21). Thus (4) holds.

Assume (4). Then every exact sequence \( M \xrightarrow{\beta} N \to 0 \) induces an exact sequence
\[
\text{Hom}(P, M) \xrightarrow{\text{Hom}(P, \beta)} \text{Hom}(P, N) \to 0.
\]
In other words, (5) holds.

Assume (5). Then every \( \alpha \in \text{Hom}(P, N) \) is the image under \( \text{Hom}(P, \beta) \) of some \( \gamma \in \text{Hom}(P, M) \). But, by definition, \( \text{Hom}(P, \beta)(\gamma) = \beta \gamma \). Thus (1) holds. \( \square \)

**Exercise (5.23).** — Let \( R \) be a ring, \( P \) and \( N \) finitely generated modules with \( P \) projective. Prove \( \text{Hom}(P, N) \) is finitely generated, and is finitely presented if \( N \) is.

**Lemma (5.25) (Schanuel).** — Given two short exact sequences
\[
0 \to L \xrightarrow{i} P \xrightarrow{\alpha} M \to 0 \quad \text{and} \quad 0 \to L' \xrightarrow{i'} P' \xrightarrow{\alpha'} M' \to 0
\]
with \( P \) and \( P' \) projective, there is an isomorphism of exact sequences — namely, a commutative diagram with vertical isomorphisms:
\[
\begin{array}{ccc}
0 & \to & L \oplus P' \\
\cong & & \cong \\
\beta & \downarrow & \gamma \\
0 & \to & P \oplus L' \\
\end{array}
\]
\[
\begin{array}{ccc}
\cong & & \cong \\
\lambda & \downarrow & \theta \\
0 & \to & K \\
\end{array}
\]
\[
\begin{array}{ccc}
0 & \to & P \oplus P' \\
\cong & & \cong \\
0 & \to & M \\
\end{array}
\]
Proof: First, let’s construct an intermediate isomorphism of exact sequences:
\[
\begin{array}{ccc}
0 & \to & L \oplus P' \\
\cong & & \cong \\
0 & \to & K \\
\end{array}
\]
\[
\begin{array}{ccc}
0 & \to & P \oplus P' \\
\cong & & \cong \\
0 & \to & M \\
\end{array}
\]
Take \( K := \text{Ker}(\alpha \alpha') \). To form \( \theta \), recall that \( P' \) is projective and \( \alpha \) is surjective. So there is a map \( \pi: P' \to P \) such that \( \alpha' = \alpha \pi \). Take \( \theta := \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} \).
Let \( \theta \) has \( \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \) as inverse. Further, the right-hand square is commutative:

\[
(\alpha \ 0)\theta = (\alpha \ 0)\left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) = (\alpha \ \alpha') = (\alpha \ \alpha').
\]

So \( \theta \) induces the desired isomorphism \( \lambda: K \rightarrow L \oplus P' \).

Symmetrically, form an automorphism \( \theta' \) of \( P \oplus P' \), which induces an isomorphism \( \lambda': K \rightarrow P \oplus L' \). Finally, take \( \gamma := \theta'\theta^{-1} \) and \( \beta := \lambda'\lambda^{-1} \). \( \square \)

**Exercise (5.28)**. Let \( R \) be a ring, and \( 0 \rightarrow L \rightarrow R^n \rightarrow M \rightarrow 0 \) an exact sequence. Prove \( M \) is finitely presented if and only if \( L \) is finitely generated.

**Exercise (5.29)**. Let \( R \) be a ring, \( X_1, X_2, \ldots \) infinitely many variables. Set \( P := R[X_1, X_2, \ldots] \) and \( M := P/(X_1, X_2, \ldots) \). Is \( M \) finitely presented? Explain.

**Proposition (5.28)**. Let \( 0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0 \) be a short exact sequence with \( L \) finitely generated and \( M \) finitely presented. Then \( N \) is finitely presented.

**Proof**: Let \( R \) be the ground ring, \( \mu: R^n \rightarrow M \) any surjection. Set \( \nu := \beta \mu \), set \( K := \ker \nu \), and set \( \lambda := \mu|K \). Then the following diagram is commutative:

\[
\begin{array}{c}
0 \rightarrow K \xrightarrow{\lambda} R^n \xrightarrow{\nu} N \rightarrow 0 \\
\downarrow \mu \quad \downarrow 1_N \\
0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0
\end{array}
\]

The Snake Lemma (5.13) yields an isomorphism \( \ker \lambda \cong \ker \mu \). But \( \ker \mu \) is finitely generated by (5.26). So \( \ker \lambda \) is finitely generated. Also, the Snake Lemma implies \( \ker \lambda = 0 \) as \( \ker \mu = 0 \); so \( 0 \rightarrow \ker \lambda \rightarrow K \xrightarrow{\lambda} L \rightarrow 0 \) is exact. Hence \( K \) is finitely generated by (5.26). Thus \( K \) is finitely presented by (5.26). \( \square \)

**Exercise (5.29)**. Let \( 0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0 \) be a short exact sequence with \( M \) finitely generated and \( N \) finitely presented. Prove \( L \) is finitely generated.

**Proposition (5.30)**. Let \( 0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0 \) be a short exact sequence with \( L \) and \( N \) finitely presented. Then \( M \) is finitely presented too.

**Proof**: Let \( R \) be the ground ring, \( \lambda: R^\ell \rightarrow L \) and \( \nu: R^n \rightarrow N \) any surjections. Define \( \gamma: R^\ell \rightarrow M \) by \( \gamma := \alpha \lambda \). Note \( R^n \) is projective by (5.22), and define \( \delta: R^n \rightarrow M \) by lifting \( \nu \) along \( \beta \). Define \( \mu: R^\ell \oplus R^n \rightarrow M \) by \( \mu := \gamma + \delta \). Then the following diagram is, plainly, commutative, where \( \iota := \iota_R^\ell \) and \( \pi := \pi_{R^n} \):

\[
\begin{array}{c}
0 \rightarrow R^\ell \xrightarrow{\lambda} R^\ell \oplus R^n \xrightarrow{\mu} R^n \rightarrow 0 \\
\downarrow \lambda \quad \downarrow \mu \\
0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0
\end{array}
\]

Since \( \lambda \) and \( \nu \) are surjective, the Snake Lemma (5.13) yields an exact sequence

\[
0 \rightarrow \ker \lambda \rightarrow \ker \mu \rightarrow \ker \nu \rightarrow 0,
\]

and implies \( \ker \mu = 0 \). Also, \( \ker \lambda \) and \( \ker \nu \) are finitely generated by (5.26). So \( \ker \mu \) is finitely generated by (5.6). Thus \( M \) is finitely presented by (5.26). \( \square \)
5. Appendix: Fitting Ideals

(5.31) (The Ideals of Minors). — Let $R$ be a ring, $A := (a_{ij})$ an $m \times n$ matrix with $a_{ij} \in R$. Given $r \in \mathbb{Z}$, let $I_r(A)$ denote the ideal generated by the $r \times r$ minors of $A$; by convention, we have

$$I_r(A) = \begin{cases} (0), & \text{if } r > \min\{m, n\}; \\ R, & \text{if } r \leq 0. \end{cases} \quad (5.31.1)$$

Let $B := (b_{ij})$ be an $r \times r$ submatrix of $A$. Let $B_{ij}$ be the $(r-1) \times (r-1)$ submatrix obtained from $B$ by deleting the $i$th row and the $j$th column. For any $i$, expansion yields $\det(B) = \sum_{j=1}^{r} (-1)^{i+j} b_{ij} \det(B_{ij})$. So $I_r(A) \subset I_{r-1}(A)$. Thus

$$R = I_0(A) \supset I_1(A) \supset \cdots. \quad (5.31.2)$$

Let $U$ be an invertible $m \times m$ matrix. Then $\det(U)$ is a unit, as $UV = I$ yields $\det(U) \det(V) = 1$. So $I_m(U) = R$. Thus $I_r(U) = R$ for all $r \leq m$.

**Proposition (5.32).** — Let $R$ be a nonzero ring, $\alpha: R^n \to R^m$ a map. (1) If $\alpha$ is injective, then $n \leq m$. (2) If $\alpha$ is bijective, then $n = m$.

**Proof:** For (1), assume $n > m$, and let’s show $\alpha$ is not injective.

Let $A$ be the matrix of $\alpha$. Note (5.31.1) yields $I_n(A) = (0)$ as $n > m$ and $I_0(A) = R$. Let $r$ be the largest integer with $\operatorname{Ann}(I_r(A)) = (0)$. Then $0 \leq r < n$.

Take any nonzero $x \in I_{r+1}(A)$. If $r = 0$, set $z := (x, 0, \ldots, 0)$. Then $z \neq 0$ and $\alpha(z) = 0$; so $\alpha$ is not injective. So assume $r > 0$.

As $x \neq 0$, also $x \notin \operatorname{Ann}(I_r(A))$. So there’s an $r \times r$ submatrix $B$ of $A$ with $x \det(B) \neq 0$. By renumbering, we may assume that $B$ is the upper left $r \times r$ submatrix of $A$. Let $C$ be the upper left $(r+1) \times (r+1)$ submatrix.

Let $c_i$ be the cofactor of $a_{r+1,i}$ in $\det(C)$; so $\det(C) = \sum_{i=1}^{r+1} (-1)^{r+1} a_{r+1,i} c_i$. Then $c_{r+1} = \det(B)$. So $x_{r+1} \neq 0$. Set $z := (c_1, \ldots, c_{r+1}, 0, \ldots, 0)$. Then $z \neq 0$.

Let’s show $\alpha(z) = 0$. Denote by $A_k$ the $k$th row of $A$, by $D$ the matrix obtained by replacing the $(r+1)$st row of $C$ with the first $(r+1)$ entries of $A_k$, and by $z \cdot A_k$ the dot product. Then $z \cdot A_k = x \det(D)$. If $k \leq r$, then $D$ has two equal rows; so $z \cdot A_k = 0$. If $k \geq r+1$, then $D$ is an $(r+1) \times (r+1)$ submatrix of $A$; so $z \cdot A_k = 0$ as $xI_{r+1}(A) = 0$. Thus $\alpha(z) = 0$. Thus $\alpha$ is not injective. Thus (1) holds.

For (2), apply (1) to $\alpha^{-1}$ too; thus also $m \leq n$. Thus (2) holds.

**Lemma (5.33).** — Let $R$ be a ring, $A$ an $m \times n$ matrix, $B$ an $n \times p$ matrix, $U$ an invertible $m \times m$ matrix, and $V$ an invertible $n \times n$ matrix. Then for all $r$

(1) $I_r(AB) \subset I_r(A)I_r(B)$ and (2) $I_r(UAV) = I_r(A)$.

**Proof:** As a matter of notation, given a $p \times q$ matrix $X := (x_{ij})$, denote its $j$th column by $X^j$. Given sequences $I := (i_1, \ldots, i_r)$ with $1 \leq i_1 < \cdots < i_r \leq p$ and $J := (j_1, \ldots, j_r)$ with $1 \leq j_1 < \cdots < j_r \leq q$, set

$$X_{IJ} := \begin{pmatrix} x_{i_1j_1} & \cdots & x_{i_1j_r} \\ \vdots & \ddots & \vdots \\ x_{i_rj_1} & \cdots & x_{i_rj_r} \end{pmatrix} \quad \text{and} \quad X_I := \begin{pmatrix} x_{i_11} & \cdots & x_{i_1n} \\ \vdots & \ddots & \vdots \\ x_{i_r1} & \cdots & x_{i_rn} \end{pmatrix}. $$

For (1), say $A := (a_{ij})$ and $B := (b_{ij})$. Set $C := AB$. Given $I := (i_1, \ldots, i_r)$ with

(30)
First, assume $\begin{align*}
1 \leq i_1 < \cdots < i_r \leq m \text{ and } K := (k_1, \ldots, k_r) \text{ with } 1 \leq k_1 < \cdots < k_r \leq p,
\end{align*}$ note
\[
\det(C_{IK}) = \det(C_{IK}^1, \ldots, C_{IK}^r)
\]
\[
= \det\left(\sum_{j_1=1}^n A_{i_1j_1}^1 b_{j_1k_1}, \ldots, \sum_{j_r=1}^n A_{i_rj_r}^r b_{j_rk_r}\right)
\]
\[
= \sum_{j_1, \ldots, j_r=1}^n \det(A_{i_1j_1}^1, \ldots, A_{i_rj_r}^r) \cdot b_{j_1k_1} \cdots b_{j_rk_r}.
\]

In the last sum, each term corresponds to a sequence $J := (j_1, \ldots, j_r)$ with $1 \leq j_i \leq n$. If two $j_i$ are equal, then $\det(A_{i_1j_1}^1, \ldots, A_{i_rj_r}^r) = 0$ as two columns are equal. Suppose no two $j_i$ are equal. Then $J$ is a permutation $\sigma$ of $H := (h_1, \ldots, h_r)$ with $1 \leq h_1 < \cdots < h_r \leq q$; so $j_i = \sigma(h_i)$. Denote the sign of $\sigma$ by $(-1)^{\sigma}$. Then
\[
\det(A_{i_1j_1}^1, \ldots, A_{i_rj_r}^r) = (-1)^{\sigma} \det(A_{iH}).
\]

But $\det(B_{HK}) = \sum_{\sigma} (-1)^{\sigma} b_{\sigma(h_1)k_1} \cdots b_{\sigma(h_r)k_r}$. Hence
\[
\det(C_{IK}) = \sum_H \det(A_{iH}) \det(B_{HK}).
\]

Thus (1) holds.

For (2), note that $I_r(W) = R$ for $W = U, U^{-1}, V, V^{-1}$ by (5.31). So (1) yields
\[
I_r(A) = I_r(U^{-1} U A V V^{-1}) \subset I_r(U A V) \subset I_r(A).
\]
Thus (2) holds. \hfill \Box

**Lemma (5.34) (Fitting).** — Let $R$ be a ring, $M$ a module, $r$ an integer, and
\[
R^q \xrightarrow{\alpha} R^m \xrightarrow{\beta} M \rightarrow 0 \text{ and } R^q \xrightarrow{\beta} R^p \xrightarrow{\pi} M \rightarrow 0
\]
presentations. Represent $\alpha, \beta$ by matrices $A, B$. Then $I_{m-r}(A) = I_{p-r}(B)$.

**Proof:** First, assume $m = p$ and $\mu = \pi$. Set $K := \text{Ker}(\mu)$. Then $\text{Im}(\alpha) = K$ and $\text{Im}(\beta) = K$ by exactness; so $\text{Im}(\alpha) = \text{Im}(\beta)$. But $\text{Im}(\alpha)$ is generated by the columns of $A$. Hence each column of $B$ is a linear combination of the columns of $A$. So there’s a matrix $C$ such that $AC = B$. Set $s := m - r$.

Given $k$, denote by $I_k$ the $k \times k$ identity matrix. Denote by $0_{mq}$ the $m \times q$ zero matrix, and by $(A|B)$ and $(A|0_{mq})$ the juxtapositions of $A$ with $B$ and $0_{mq}$. Then, therefore, there is a block triangular matrix $V := (I_q \ 0_{qa})$ such that $(A|B)V = (A|0_{mq})$. But $V$ is invertible. So $I_s(A|B) = I_s(A|0_{mq})$ by (5.32). But $I_s(A|0_{mq}) = I_s(A)$. Thus $I_s(A|B) = I_s(A)$. Similarly, $I_s(A|B) = I_s(B)$. Thus $I_s(A) = I_s(B)$, as desired.

Second, assume $m = p$ and that there’s an isomorphism $\gamma: R^m \xrightarrow{\gamma} R^p$ with $\pi \gamma = \mu$. Represent $\gamma$ by a matrix $G$. Then $R^q \xrightarrow{\gamma^*} R^p \xrightarrow{\gamma} M \rightarrow 0$ is a presentation, and $GA$ represents $\gamma \alpha$. So, by the first paragraph, $I_s(B) = I_s(GA)$. But $G$ is invertible. So $I_s(GA) = I_s(A)$ by (5.32). Thus $I_s(A) = I_s(B)$, as desired.

Third, assume that $q = n + t$ and $p = m + t$ for some $t \geq 1$ and that $\beta = \alpha \oplus 1_{R^t}$ and $\pi = \mu + 0$. Then $B = \left(\begin{array}{cc} A & 0_{ae} \\ 0_{t} & I_{r}\end{array}\right)$.

Given an $s \times s$ submatrix $C$ of $A$, set $D := \left(\begin{array}{cc} C & 0_{st} \\ 0_{st} & I_{s}\end{array}\right)$. Then $D$ is an $(s+t) \times (s + t)$ submatrix of $B$, and $\det(D) = \det(C)$. Thus $I_s(A) \subset I_{s+t}(B)

For the opposite inclusion, given an $(s + t) \times (s + t)$ submatrix $D$ of $B$, assume $\det(D) \neq 0$. If $D$ includes part of the $(m+i)$th row of $B$, then $D$ must also include part of the $(n+i)$th column, or $D$ would have an all zero row. Similarly, if $D$
includes part of the \((n+i)\)th column, then \(D\) must include part of the \((m+i)\)th row. So \(D = \begin{pmatrix} C & 0_k \end{pmatrix}\) where \(h := s + t - k\) for some \(k \leq t\) and for some \(h \times h\) submatrix \(C\) of \(A\). But \(\det(D) = \det(C)\). So \(\det(D) \in I_h(A)\). But \(I_h(A) \subset I_s(A)\) by (5.31.2). So \(\det(D) \in I_s(A)\). Thus \(I_{s+t}(B) \subset I_s(A)\). Thus \(I_{s+t}(B) = I_s(A)\), or \(I_{m-r}(A) = I_{p-r}(B)\), as desired.

Finally, in general, Schanuel's Lemma (5.22) yields the commutative diagram
\[
\begin{array}{c}
R^n \oplus R^p \xrightarrow{\alpha \oplus 1_R} R^m \oplus R^p \\
\gamma \downarrow \quad \downarrow 1_M \\
R^m \oplus R^p \xrightarrow{1_R \oplus \beta} R^m \oplus R^p \xrightarrow{0 + \pi} M \to 0
\end{array}
\]
Thus, by the last two paragraphs, \(I_{m-r}(A) = I_{p-r}(B)\), as desired. \(\square\)

**Theorem (5.35) (Fitting Ideals).** — Let \(R\) be a ring, \(M\) a finitely presented module, \(r\) an integer. Take any presentation \(R^n \xrightarrow{\alpha} R^m \to M \to 0\), let \(A\) be the matrix of \(\alpha\), and define the \(r\)th **Fitting ideal** of \(M\) by
\[
F_r(M) := I_{m-r}(A).
\]
It is independent of the choice of presentation by (5.31). By definition, \(F_r(M)\) is finitely generated. Moreover, (5.31.2) yields
\[
\emptyset = F_{-1}(M) \subset F_0(M) \subset \cdots \subset F_m(M) = R. \tag{5.35.1}
\]

**Exercise (5.36).** — Let \(R\) be a ring, and \(a_1, \ldots, a_m \in R\) with \(\langle a_1 \rangle \supset \cdots \supset \langle a_m \rangle\). Set \(M := (R/\langle a_1 \rangle) \oplus \cdots \oplus (R/\langle a_m \rangle)\). Show that \(F_r(M) = \langle a_1 \cdots a_{m-r} \rangle\).

**Exercise (5.37).** — In the setup of (5.36), assume \(a_1\) is a nonunit.

1. Show that \(m\) is the smallest integer such that \(F_m(M) = R\).
2. Let \(n\) be the largest integer such that \(F_n(M) = \emptyset\); set \(k := m - n\). Assume \(R\) is a domain. Show (a) that \(a_i \neq 0\) for \(i < k\) and \(a_i = 0\) for \(i \geq k\), and (b) that \(M\) determines each \(a_i\) up to unit multiple.

**Theorem (5.38) (Elementary Divisors).** — Let \(R\) be a PID, \(M\) a free module, \(N\) a submodule. Assume \(N\) is free of rank \(n < \infty\). Then there exists a decomposition \(M = M' \oplus M''\) and elements \(x_1, \ldots, x_n \in M'\) and \(a_1, \ldots, a_n \in R\) such that
\[
M' = Rx_1 \oplus \cdots \oplus Rx_n, \quad N = Ra_1x_1 \oplus \cdots \oplus Ra_nx_n, \quad \langle a_1 \rangle \supset \cdots \supset \langle a_n \rangle \neq 0.
\]
Moreover, \(M\) and \(N\) determine \(M'\) and each \(a_i\) up to unit multiple.

**Proof:** Let’s prove existence by induction on \(n\). For \(n = 0\), take \(M' = 0\); no \(a_i\) or \(x_i\) are needed. So \(M'' = M\), and the displayed conditions are trivially satisfied.

Let \(\{e_\lambda\}\) be a free basis of \(M\), and \(\pi_\lambda : M \to R\) the \(\lambda\)th projection.

Assume \(n > 0\). Given any nonzero \(z \in N\), write \(z = \sum c_\lambda e_\lambda\) for some \(c_\lambda \in R\). Then some \(c_{\lambda_0} \neq 0\). But \(c_{\lambda_0} = \pi_{\lambda_0}(z)\). Thus \(\pi_{\lambda_0}(N) \neq 0\).

Consider the set \(S\) of nonzero ideals of the form \(\alpha(N)\) where \(\alpha : M \to R\) is a linear map. Partially order \(S\) by inclusion. Given a totally ordered subset \(\{\alpha_\lambda(N)\}\), set \(b := \bigcup \alpha_\lambda(N)\). Then \(b\) is an ideal. So \(b = \langle b \rangle\) for some \(b \in R\) as \(R\) is a PID. Then \(b \in \alpha_\lambda(N)\) for some \(\lambda\). So \(\alpha_\lambda(N) = b\). By Zorn’s Lemma, \(S\) has a maximal element, say \(\alpha_1(N)\). Fix \(a_1 \in R\) with \(\alpha_1(N) = \langle a_1 \rangle\), and fix \(y_1 \in N\) with \(\alpha_1(y_1) = a_1\).

Given any linear map \(\beta : M \to R\), set \(b := \beta(y_1)\). Then \(\langle a_1 \rangle + \langle b \rangle = \langle c \rangle\) for some \(c \in R\), as \(R\) is a PID. Write \(c = da_1 + \varepsilon b\) for \(d, \varepsilon \in R\), and set \(\gamma := da_1 + \varepsilon\beta\). Then \(\gamma(N) \supset \langle \gamma(y_1) \rangle\). But \(\gamma(y_1) = c\). So \(\langle c \rangle \subset \gamma(N)\). But \(\langle a_1 \rangle \subset \langle c \rangle\). Hence, by
maximality, \( \langle a_1 \rangle = \gamma(N) \). But \( \langle b \rangle \subset \langle c \rangle \). Thus \( \beta(y_1) = b \in \langle a_1 \rangle \).

Write \( y_1 = \sum c_\lambda e_\lambda \) for some \( c_\lambda \in R \). Then \( \pi_\lambda(y_1) = c_\lambda \). But \( c_\lambda = a_1d_\lambda \) for some \( d_\lambda \in R \) by the above paragraph with \( \beta := \pi_\lambda \). Set \( x_1 := \sum d_\lambda e_\lambda \). Then \( y_1 = a_1x_1 \).

So \( \alpha_1(y_1) = a_1\alpha_1(x_1) \). But \( \alpha_1(y_1) = a_1 \). So \( a_1\alpha_1(x_1) = a_1 \). But \( R \) is a domain and \( a_1 \neq 0 \). Thus \( \alpha_1(x_1) = 1 \).

Set \( M_1 := \ker(\alpha_1) \). As \( \alpha_1(x_1) = 1 \), clearly \( Rx_1 \cap M_1 = 0 \). Also, given \( x \in M \), write \( x = \alpha_1(x)x_1 + (x - \alpha_1(x)x_1) \); thus \( x \in Rx_1 + M_1 \). Hence \( (4.1.7) \) implies \( M = Rx_1 + M_1 \). Further, \( M_1 \) is free of rank \( 1 \).

Recall \( a_1x_1 = y_1 \in N \). So \( N \supset Ra_1x_1 \supset N_1 \). Conversely, given \( y \in N \), write \( y = bx_1 + m_1 \) with \( b \in R \) and \( m_1 \in M_1 \). Then \( \alpha_1(y) = b \), so \( b \in \langle a_1 \rangle \). Hence \( y \in Ra_1x_1 + N_1 \). Thus \( N = Ra_1x_1 \supset N_1 \).

Define \( \varphi : R \to Ra_1x_1 \) by \( \varphi(a) = a_1x_1 \). If \( \varphi(a) = 0 \), then \( a\alpha_1 = 0 \) as \( \alpha_1(x_1) = 1 \), and so \( a = 0 \). Thus \( \varphi \) is injective, so a homomorphism.

Note \( N_1 \cong R^m \) with \( m \leq n \) owing to \( (4.1.3) \) with \( N \) for \( E \). Hence \( N \cong R^{m+1} \). But \( N \cong R^n \). So \( (5.3.2)(2) \) yields \( m + 1 = n \).

By induction on \( n \), there exists a decomposition \( M_1 = M'_1 \oplus M'' \) and elements \( x_2, \ldots, x_n \in M'_1 \) and \( a_2, \ldots, a_n \in R \) such that

\[
M'_1 = Rx_2 \oplus \cdots \oplus Rx_n, \quad N_1 = Ra_2x_2 \oplus \cdots \oplus Ra_nx_n, \quad \langle a_2 \rangle \supset \cdots \supset \langle a_n \rangle \neq 0.
\]

Then \( M = M' \oplus M'' \) and \( M' = Rx_1 \oplus \cdots \oplus Rx_n \) and \( N = Ra_1x_1 \oplus \cdots \oplus Ra_nx_n \). Also \( \langle a_1 \rangle \supset \cdots \supset \langle a_n \rangle \neq 0 \). Thus existence is proved.

Finally, consider the projection \( \pi : M_1 \to R \) with \( \pi(x_j) = \delta_{2j} \), for \( j \leq 2 \leq n \) and \( \pi|M'' = 0 \). Define \( \rho : M \to R \) by \( \rho(ax_1 + m_1) := a \pi(m_1) \). Then \( \rho(a_1x_1) = a_1 \). So \( \rho(N) \supset \langle a_1 \rangle = a_1(N) \). By maximality, \( \rho(N) = a_1(N) \). But \( a_2 = \rho(a_2x_2) \in \rho(N) \). Thus \( \langle a_2 \rangle \subset \langle a_1 \rangle \), as desired.

Moreover, \( M' = \{ m \in M \mid xm \in N \} \) for some \( x \in R \). Thus \( M' \) is determined.

Also, by \( (5.3.2)(2) \) with \( M'/N \) for \( M \), each \( a_i \) is determined up to unit multiple. \( \square \)

**Theorem (5.39).** — Let \( A \) be a local ring, \( M \) a finitely presented module.

(1) Then \( M \) can be generated by \( n \) elements if and only if \( F_n(M) = A \).

(2) Then \( M \) is free of rank \( n \) if and only if \( F_n(M) = A \) and \( F_{n-1}(M) = \{0\} \).

**Proof:** For (1), assume \( M \) can be generated by \( n \) elements. Then \( (4.1.1)(1) \) and \( (5.3.1)(1) \) yield a presentation \( A^n \twoheadrightarrow A^m \to A \). So \( F_n(M) = A \) by \( (5.3.2) \).

For the converse, assume also \( M \) cannot be generated by \( n \) elements. Suppose \( F_k(M) = A \) with \( k < m \). Then \( F_{m-1}(M) = A \) by \( (5.3.1)(1) \). Hence one entry of the matrix \( (a_{ij}) \) of \( \alpha \) does not belong to the maximal ideal, so is a unit by \( (1.1) \).

By \( (5.3.2)(2) \), we may assume \( a_{11} = 1 \) and the other entries in the first row and first column of \( A \) are 0. Thus \( A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \) where \( B \) is an \( (m-1) \times (s-1) \) matrix. Then \( B \) defines a presentation \( A^{s-1} \to A^{m-1} \to A \). So \( M \) can be generated by \( n-1 \) elements, a contradiction. Thus \( F_k(M) \neq A \) for \( k < m \). Thus (1) holds.

In (2), if \( M \) is free of rank \( m \), then there’s a presentation \( 0 \to A^m \to M \to 0 \); so \( F_m(M) = A \) and \( F_{m-1}(M) = \{0\} \) by \( (5.3.2) \). Conversely, if \( F_m(M) = A \), then \( (1) \) and \( (5.3.1)(1) \) yield a presentation \( A^n \twoheadrightarrow A^m \to M \). If also \( F_{m-1}(M) = \{0\} \), then \( \alpha = 0 \) by \( (5.3.1)(2) \). Thus \( M \) is free of rank \( m \); so (2) holds. \( \square \)

**Proposition (5.40).** — Let \( R \) be a ring, and \( M \) a finitely presented module. Say \( M \) can be generated by \( n \) elements. Set \( a := \text{Ann}(M) \). Then

(1) \( aF_r(M) \subset F_{r-1}(M) \) for all \( r > 0 \) and

(2) \( a^n \subset F_0(M) \subset a \).
Proof: As $M$ can be generated by $m$ elements, (5.11) and (5.20) yield a presentation $A^n \xrightarrow{\alpha} A^m \xrightarrow{\mu} M \to 0$. Say $\alpha$ has matrix $A$.

In (1), if $r > m$, then trivially $a_F(M) \subseteq F_{r-1}(M)$ owing to (5.25.1). So assume $r \leq m$ and set $s := m - r + 1$. Given $x \in a$, form the sequence

$$R^{n+m} \xrightarrow{\beta} R^m \xrightarrow{\mu} M \to 0$$

with $\beta := \alpha + x1_{R^n}$. Note that this sequence is a presentation. Also, the matrix of $\beta$ is $(A_j x I_m)$, obtained by juxtaposition, where $I_m$ is the $m \times m$ identity matrix.

Given an $(s-1) \times (s-1)$ submatrix $B$ of $A$, enlarge it to an $s \times s$ submatrix $B'$ of $(A_j x I_m)$ as follows: say the $i$th row of $A$ is not involved in $B$; form the $m \times s$ submatrix $B''$ of $(A_j x I_m)$ with the same columns as $B$ plus the $i$th column of $x I_m$ at the end; finally, form $B'$ as the $s \times s$ submatrix of $B''$ with the same rows as $B$ plus the $i$th row in the appropriate position.

Expanding along the last column yields $\det(B') = \pm x \det(B)$. By construction, $\det(B') \in I_s(A_j x I_m)$. But $I_s(A_j x I_m) = I_s(A)$ by (5.33). Furthermore, $x \in a$ is arbitrary, and $I_m(A)$ is generated by all possible $\det(B)$. Thus (1) holds.

For (2), apply (1) repeatedly to get $a^k F_r(M) \subseteq F_{r-k}(M)$ for all $r$ and $k$. But $F_m(M) = R$ by (5.35.1). So $a^m \subseteq F_0(M)$.

For the second inclusion, given any $m \times m$ submatrix $B$ of $A$, say $B = (b_{ij})$. Let $e_i$ be the $i$th standard basis vector of $R^n$. Set $m_i := \mu(e_i)$. Then $\sum b_{ij} m_j = 0$ for all $i$. Let $C$ be the matrix of cofactors of $B$: the $(i,j)$th entry of $C$ is $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting the $j$th row and the $i$th column of $B$. Then $C B = \det(B) I_m$. Hence $\det(B) m_i = 0$ for all $i$. So $\det(B) \in a$. But $I_m(A)$ is generated by all such $\det(B)$. Thus $F_0(M) \subseteq a$. Thus (2) holds. □
6. Direct Limits

Category theory provides the right abstract setting for certain common concepts, constructions, and proofs. Here we treat adjoints and direct limits. We elaborate on two key special cases of direct limits: coproducts (direct sums) and coequalizers (cokernels). Then we construct arbitrary direct limits of sets and of modules. Further, we prove direct limits are preserved by left adjoints; whence, direct limits commute with each other, and in particular, with coproducts and coequalizers.

Although this section is the most abstract of the entire book, all the material here is elementary, and none of it is very deep. In fact, many statements are just concise restatements in more expressive language; they can be understood through a simple translation of terms. Experience shows that it pays to learn this more abstract language, but that doing so requires determined, yet modest effort.

(6.1) (Categories). — A category \( \mathcal{C} \) is a collection of elements, called objects. Each pair of objects \( A, B \) is equipped with a set \( \text{Hom}_\mathcal{C}(A, B) \) of elements, called maps or morphisms. We write \( \alpha: A \to B \) or \( A \xrightarrow{\alpha} B \) to mean \( \alpha \in \text{Hom}_\mathcal{C}(A, B) \).

Further, given objects \( A, B, C \), there is a composition law

\[
\text{Hom}_\mathcal{C}(A, B) \times \text{Hom}_\mathcal{C}(B, C) \to \text{Hom}_\mathcal{C}(A, C), \quad (\alpha, \beta) \mapsto \beta \alpha,
\]

and there is a distinguished map \( 1_B \in \text{Hom}_\mathcal{C}(B, B) \), called the identity such that

1. composition is associative, or \( \gamma(\beta \alpha) = (\gamma \beta)\alpha \) for \( \gamma: C \to D \), and
2. \( 1_B \) is unitary, or \( 1_B \alpha = \alpha \) and \( \beta 1_B = \beta \).

We say \( \alpha \) is an isomorphism with inverse \( \beta: B \to A \) if \( \alpha \beta = 1_B \) and \( \beta \alpha = 1_A \).

For example, four common categories are those of sets ((Sets)), of rings ((Rings)), of \( R \)-modules ((R-mod)), and of \( R \)-algebras ((R-alg)); the corresponding maps are the set maps, and the ring, \( R \)-module, and \( R \)-algebra homomorphisms.

Given categories \( \mathcal{C} \) and \( \mathcal{C}' \), their product \( \mathcal{C} \times \mathcal{C}' \) is the category whose objects are the pairs \( (A, A') \) with \( A \) an object of \( \mathcal{C} \) and \( A' \) an object of \( \mathcal{C}' \) and whose maps are the pairs \( (\alpha, \alpha') \) of maps \( \alpha \) in \( \mathcal{C} \) and \( \alpha' \) in \( \mathcal{C}' \).

(6.2) (Functors). — A map of categories is known as a functor. Namely, given categories \( \mathcal{C} \) and \( \mathcal{C}' \), a (covariant) functor \( F: \mathcal{C} \to \mathcal{C}' \) is a rule that assigns to each object \( A \) of \( \mathcal{C} \) an object \( F(A) \) of \( \mathcal{C}' \) and to each map \( \alpha: A \to B \) of \( \mathcal{C} \) a map \( F(\alpha): F(A) \to F(B) \) of \( \mathcal{C}' \) preserving composition and identity; that is,

1. \( F(\beta \alpha) = F(\beta)F(\alpha) \) for maps \( \alpha: A \to B \) and \( \beta: B \to C \) of \( \mathcal{C} \), and
2. \( F(1_A) = 1_{F(A)} \) for any object \( A \) of \( \mathcal{C} \).

We also denote a functor \( F \) by \( F(\bullet) \), by \( A \mapsto F(A) \), or by \( A \mapsto F_A \).

Note that a functor \( F \) preserves isomorphisms. Indeed, if \( \alpha \beta = 1_B \) and \( \beta \alpha = 1_A \), then \( F(\alpha)F(\beta) = 1_{F(B)} \) and \( F(\beta)F(\alpha) = F(1_A) \).

For example, let \( R \) be a ring, \( M \) a module. Then clearly \( \text{Hom}_R(M, \bullet) \) is a functor from \( ((R\text{-mod})) \) to \( ((R\text{-mod})) \). A second example is the forgetful functor from \( ((R\text{-mod})) \) to \( ((\text{Sets})) \); it sends a module to its underlying set and a homomorphism to its underlying set map.

A map of functors is known as a natural transformation. Namely, given two functors \( F, F': \mathcal{C} \Rightarrow \mathcal{C}' \), a natural transformation \( \theta: F \to F' \) is a collection of maps \( \theta(A): F(A) \to F'(A) \), one for each object \( A \) of \( \mathcal{C} \), such that \( \theta(B)F(\alpha) = F'(\alpha)\theta(A) \)
for every map $\alpha: A \to B$ of $\mathcal{C}$; that is, the following diagram is commutative:

\[
\begin{array}{ccc}
F(A) & \xrightarrow{F(\alpha)} & F(B) \\
\theta(A) & \downarrow & \theta(B) \\
F'(A) & \xrightarrow{F'(\alpha)} & F'(B)
\end{array}
\]

For example, the identity maps $1_{F(A)}$ trivially form a natural transformation $1_F$ from any functor $F$ to itself. We call $F$ and $F'$ isomorphic if there are natural transformations $\theta: F \to F'$ and $\theta': F' \to F$ with $\theta'\theta = 1_F$ and $\theta\theta' = 1_{F'}$.

A contravariant functor $G$ from $\mathcal{C}$ to $\mathcal{C}'$ is a rule similar to $F$, but $G$ reverses the direction of maps; that is, $G(\alpha)$ carries $G(B)$ to $G(A)$, and $G$ satisfies the analogues of (1) and (2). For example, fix a module $N$; then $\text{Hom}(\bullet, N)$ is a contravariant functor from $((R\text{-mod}))$ to $((R\text{-mod}))$.

**Exercise (6.3).** — (1) Show that the condition (6.3.1) is equivalent to the commutativity of the corresponding diagram:

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{C}}(B, C) & \to & \text{Hom}_{\mathcal{C}'}(F(B), F(C)) \\
\downarrow & & \downarrow \\
\text{Hom}_{\mathcal{C}}(A, C) & \to & \text{Hom}_{\mathcal{C}'}(F(A), F(C))
\end{array}
\]

(6.3.1)\)

(2) Given $\gamma: C \to D$, show (6.3.1) yields the commutativity of this diagram:

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{C}}(B, C) & \to & \text{Hom}_{\mathcal{C}'}(F(B), F(C)) \\
\downarrow & & \downarrow \\
\text{Hom}_{\mathcal{C}}(A, D) & \to & \text{Hom}_{\mathcal{C}'}(F(A), F(D))
\end{array}
\]

(6.4) (Adjoints). — Let $F: \mathcal{C} \to \mathcal{C}'$ and $F': \mathcal{C}' \to \mathcal{C}$ be functors. We call $(F, F')$ an adjoint pair, $F$ the left adjoint of $F'$, and $F'$ the right adjoint of $F$ if, for every pair of objects $A \in \mathcal{C}$ and $A' \in \mathcal{C}'$, there is given a natural bijection

\[
\text{Hom}_{\mathcal{C}}(F(A), A') \simeq \text{Hom}_{\mathcal{C}'}(A, F'(A')).
\]

(6.4.1)

Here Natural means that maps $B \to A$ and $A' \to B'$ induce a commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{C}'}(F(A), A') & \simeq & \text{Hom}_{\mathcal{C}}(A, F'(A')) \\
\downarrow & & \downarrow \\
\text{Hom}_{\mathcal{C}'}(F(B), B') & \simeq & \text{Hom}_{\mathcal{C}}(B, F'(B'))
\end{array}
\]

Naturality serves to determine an adjoint up to canonical isomorphism. Indeed, let $F$ and $G$ be two left adjoints of $F'$. Given $A \in \mathcal{C}$, define $\theta(A): G(A) \to F(A)$ to be the image of $1_{F(A)}$ under the adjoint bijections

\[
\text{Hom}_{\mathcal{C}}(F(A), F(A)) \simeq \text{Hom}_{\mathcal{C}}(A, F'F(A)) \simeq \text{Hom}_{\mathcal{C}'}(G(A), F(A)).
\]

To see that $\theta(A)$ is natural in $A$, take a map $\alpha: A \to B$. It induces the following diagram, which is commutative owing to the naturality of the adjoint bijections:
Let $\phi$ be an adjoint pair. Let $F; F'$ be a pair of functors. $A$ is commutative. We call the natural transformation $\eta_A : A \to F(A)$ the unit of $(F, F')$. Conversely, instead of assuming $(F, F')$ is an adjoint pair, assume given a natural transformation $\eta: 1 \to F'$ satisfying (1) and (3). Prove the equation in (2) defines a natural bijection making $(F, F')$ an adjoint pair, whose unit is $\eta$. Dually, we can define a counit $\varepsilon : FF' \to 1_{F'}$. 

**Exercise**

1. Prove $\eta_A$ is natural in $A$; that is, given $g: A \to B$, the induced square

$$
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & F'FA \\
g \downarrow & & \downarrow F'Fg \\
B & \xrightarrow{\eta_B} & F'FB
\end{array}
$$

is commutative. We call the natural transformation $A \to \eta_A$ the **unit** of $(F, F')$.

2. Given $f': FA \to A'$, prove $\varphi_{A,A'}(f') = F'$.

3. Prove the natural map $\eta_A: A \to F'FA$ is **universal** from $A$ to $F'$; that is, given $f: A \to F'A'$, there is a unique map $f': FA \to A'$ with $F'f' \circ \eta_A = f$.

4. Conversely, instead of assuming $(F, F')$ is an adjoint pair, assume given a natural transformation $\eta: 1 \to F'$ satisfying (1) and (3). Prove the equation in (2) defines a natural bijection making $(F, F')$ an adjoint pair, whose unit is $\eta$. Dually, we can define a counit $\varepsilon : FF' \to 1_{F'}$, and prove analogous statements.

**Chase after 1**

Both map to $F(\alpha) \in \text{Hom}_C(G(A), F(B))$. So both map to the same image in $\text{Hom}_C(G(A), F(B))$. But, by naturality, the following diagram is commutative:

$$
\begin{array}{ccc}
\text{Hom}_C(F(A), F(A)) & \cong & \text{Hom}_C(A, F'F(A)) \\
\downarrow & & \downarrow \\
\text{Hom}_C(F(A), G(A)) & \cong & \text{Hom}_C(A, F'G(A))
\end{array}
$$

Similarly, there is a natural transformation $\theta: F \to G$. It remains to show $\theta \theta = 1_G$ and $\theta \theta = 1_F$. But, by naturality, the following diagram is commutative:

$$
\begin{array}{ccc}
\text{Hom}_C(F(A), F(A)) & \cong & \text{Hom}_C(A, F'F(A)) \\
\downarrow & & \downarrow \\
\text{Hom}_C(F(A), G(A)) & \cong & \text{Hom}_C(A, F'G(A))
\end{array}
$$

Similarly, the “free module” functor is the left adjoint of the forgetful functor from $((R\text{-mod}))$ to $((\text{Sets}))$, since by (6.10),

$$
\text{Hom}_{((R\text{-mod}))}(R^E, M) = \text{Hom}_{((\text{Sets}))}(A, M).
$$

Similarly, the “polynomial ring” functor is the left adjoint of the forgetful functor from $((R\text{-alg}))$ to $((\text{Sets}))$, since by (6.7),

$$
\text{Hom}_{((R\text{-alg}))}(R[X_1, \ldots, X_n], R') = \text{Hom}_{((\text{Sets}))}(\{X_1, \ldots, X_n\}, R').
$$

**Exercise**

Let $\mathcal{C}$ and $\mathcal{C}'$ be categories, $F: \mathcal{C} \to \mathcal{C}'$ and $F': \mathcal{C}' \to \mathcal{C}$ an adjoint pair. Let $\varphi_{A,A'} : \text{Hom}_C(FA, A') \cong \text{Hom}_C(A, F'A')$ denote the natural bijection, and set $\eta_A := \varphi_{A,F(A)}(1)$. Do the following:

1. Prove $\eta_A$ is natural in $A$; that is, given $g: A \to B$, the induced square

$$
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & F'FA \\
g \downarrow & & \downarrow F'Fg \\
B & \xrightarrow{\eta_B} & F'FB
\end{array}
$$

is commutative. We call the natural transformation $A \to \eta_A$ the **unit** of $(F, F')$.

2. Given $f': FA \to A'$, prove $\varphi_{A,A'}(f') = F'f' \circ \eta_A$.

3. Prove the natural map $\eta_A: A \to F'FA$ is **universal** from $A$ to $F'$; that is, given $f: A \to F'A'$, there is a unique map $f': FA \to A'$ with $F'f' \circ \eta_A = f$.

4. Conversely, instead of assuming $(F, F')$ is an adjoint pair, assume given a natural transformation $\eta: 1 \to F'$ satisfying (1) and (3). Prove the equation in (2) defines a natural bijection making $(F, F')$ an adjoint pair, whose unit is $\eta$.

5. Identify the units in the two examples in (6.3): the “free module” functor and the “polynomial ring” functor.
(6.6) (Direct limits). — Let \( \Lambda, \mathcal{C} \) be categories. Assume \( \Lambda \) is small; that is, its objects form a set. Given a functor \( \lambda \mapsto M_\lambda \) from \( \Lambda \) to \( \mathcal{C} \), its direct limit or colimit, denoted \( \operatorname{lim} M_\lambda \) or \( \operatorname{colim}_{\lambda \in \Lambda} M_\lambda \), is defined to be the object of \( \mathcal{C} \) universal among objects \( P \) equipped with maps \( \beta_\mu : M_\mu \to P \), called insertions, that are compatible with the transition maps \( \alpha^\mu_\nu : M_\nu \to M_\mu \), which are the images of the maps of \( \Lambda \). (Note: given \( \kappa \) and \( \mu \), there may be more than one map \( \kappa \to \mu \), and so more than one transition map \( \alpha^\mu_\nu \).) In other words, there is a unique map \( \beta \) such that all of the following diagrams commute:

\[
\begin{array}{ccc}
M_\nu & \xrightarrow{\alpha^\mu_\nu} & M_\mu \\
\downarrow{\beta_\nu} & & \downarrow{\beta_\mu} \\
\operatorname{lim} M_\lambda & \xrightarrow{\beta} & P
\end{array}
\]

To indicate this context, the functor \( \lambda \mapsto M_\lambda \) is often called a direct system.

As usual, universality implies that, once equipped with its insertions \( \alpha_\mu \), the limit \( \operatorname{lim} M_\lambda \) is determined up to unique isomorphism, assuming it exists. In practice, there is usually a canonical choice for \( \operatorname{lim} M_\lambda \), given by a construction. In any case, let us use \( \operatorname{lim} M_\lambda \) to denote a particular choice.

We say that \( \mathcal{C} \) has direct limits indexed by \( \Lambda \) if, for every functor \( \lambda \mapsto M_\lambda \) from \( \Lambda \) to \( \mathcal{C} \), the direct limit \( \operatorname{lim} M_\lambda \) exists. We say that \( \mathcal{C} \) has direct limits if it has direct limits indexed by every small category \( \Lambda \).

Given a functor \( F : \mathcal{C} \to \mathcal{C}' \), note that a functor \( \lambda \mapsto M_\lambda \) from \( \Lambda \) to \( \mathcal{C} \) yields a functor \( \lambda \mapsto F(M_\lambda) \) from \( \Lambda \) to \( \mathcal{C}' \). Furthermore, whenever the corresponding two direct limits exist, the maps \( F(\alpha^\mu_\nu) : F(M_\mu) \to F(\operatorname{lim} M_\lambda) \) induce a canonical map

\[
\phi : \operatorname{lim} F(M_\lambda) \to F(\operatorname{lim} M_\lambda).
\]

If \( \phi \) is always an isomorphism, we say \( F \) preserves direct limits. At times, given \( \operatorname{lim} M_\lambda \), we construct \( \operatorname{lim} F(M_\lambda) \) by showing \( F(\operatorname{lim} M_\lambda) \) has the requisite UMP.

Assume \( \mathcal{C} \) has direct limits indexed by \( \Lambda \). Then, given a natural transformation from \( \lambda \mapsto M_\lambda \) to \( \lambda \mapsto N_\lambda \), universality yields unique commutative diagrams

\[
\begin{array}{ccc}
M_\mu & \xrightarrow{\alpha^\mu_\nu} & M_\nu \\
\downarrow & & \downarrow \\
\operatorname{lim} M_\lambda & \xrightarrow{\beta} & N_\lambda
\end{array}
\]

To put it in another way, form the functor category \( \mathcal{C}^\Lambda \): its objects are the functors \( \lambda \mapsto M_\lambda \) from \( \Lambda \) to \( \mathcal{C} \); its maps are the natural transformations (they form a set as \( \Lambda \) is one). Then taking direct limits yields a functor \( \operatorname{lim} \) from \( \mathcal{C}^\Lambda \) to \( \mathcal{C} \).

In fact, it is just a restatement of the definitions that the “direct limit” functor \( \operatorname{lim} \) is the left adjoint of the diagonal functor

\[
\Delta : \mathcal{C} \to \mathcal{C}^\Lambda.
\]

By definition, \( \Delta \) sends each object \( M \) to the constant functor \( \Delta M \), which has the same value \( M \) at every \( \lambda \in \Lambda \) and has the same value \( 1_M \) at every map of \( \Lambda \); further, \( \Delta \) carries a map \( \gamma : M \to N \) to the natural transformation \( \Delta \gamma : \Delta M \to \Delta N \), which has the same value \( \gamma \) at every \( \lambda \in \Lambda \).

(6.7) (Coproducts). — Let \( \mathcal{C} \) be a category, \( \Lambda \) a set, and \( M_\lambda \) an object of \( \mathcal{C} \) for each \( \lambda \in \Lambda \). The coproduct \( \bigsqcup_{\lambda \in \Lambda} M_\lambda \), or simply \( \bigsqcup M_\lambda \), is defined as the object of \( \mathcal{C} \) universal among objects \( P \) equipped with a map \( \beta_\mu : M_\mu \to P \) for each \( \mu \in \Lambda \).
The maps \( \iota_\mu : M_\mu \to \coprod M_\lambda \) are called the **inclusions**. Thus, given such a \( P \), there exists a unique map \( \beta : \coprod M_\lambda \to P \) with \( \beta \iota_\mu = \beta_\mu \) for all \( \mu \in \Lambda \).

If \( \Lambda = \emptyset \), then the coproduct is an object \( B \) with a unique map \( \beta \) to every other object \( P \). There are no \( \mu \) in \( \Lambda \), so no inclusions \( \iota_\mu : M_\mu \to B \), so no equations \( \beta \iota_\mu = \beta_\mu \) to restrict \( \beta \). Such a \( B \) is called an **initial object**.

For instance, suppose \( \mathcal{C} = ((\text{R-mod})) \). Then the zero module is an initial object. For any \( \Lambda \), the coproduct \( \coprod M_\lambda \) is just the direct sum \( \bigoplus M_\lambda \) (a convention if \( \Lambda = \emptyset \)). Next, suppose \( \mathcal{C} = ((\text{Sets})) \). Then the empty set is an initial object. For any \( \Lambda \), the coproduct \( \coprod M_\lambda \) is the disjoint union \( \bigsqcup M_\lambda \) (a convention if \( \Lambda = \emptyset \)).

Note that the coproduct is a special case of the direct limit. Indeed, regard \( \Lambda \) as a **discrete** category: its objects are the \( \lambda \in \Lambda \), and it has just the required maps, namely, the \( 1_\lambda \). Then \( \coker \lim_{\longrightarrow} M_\lambda = \coprod M_\lambda \) with the insertions equal to the inclusions.

**Lemma (6.10).** — A category \( \mathcal{C} \) has direct limits if and only if \( \mathcal{C} \) has coproducts and coequalizers. If a category \( \mathcal{C} \) has direct limits, then a functor \( F : \mathcal{C} \to \mathcal{C}' \) preserves them if and only if \( F \) preserves coproducts and coequalizers.

**Proof:** If \( \mathcal{C} \) has direct limits, then \( \mathcal{C} \) has coproducts and coequalizers because they are special cases by (5.7) and (7.2). By the same token, if \( F : \mathcal{C} \to \mathcal{C}' \) preserves direct limits, then \( F \) preserves coproducts and coequalizers.

Conversely, assume that \( \mathcal{C} \) has coproducts and coequalizers. Let \( \Lambda \) be a small category, and \( \lambda \mapsto M_\lambda \) a functor from \( \Lambda \) to \( \mathcal{C} \). Let \( \Sigma \) be the set of all transition maps \( \alpha_\mu : M_\lambda \to M_\mu \). For each \( \sigma \in \Sigma \), set \( M_\sigma := M_\lambda \), set \( M := \coprod_{\sigma \in \Sigma} M_\sigma \) and \( N := \coprod_{\lambda \in \Lambda} M_\lambda \). For each \( \sigma \), there are two maps \( M_\sigma := M_\lambda \to N \) : the inclusion \( \iota_\lambda \) and the composition \( \iota_\sigma \alpha_\mu \). Correspondingly, there are two maps \( \alpha, \alpha' : M \to N \).

Let \( C \) be their coequalizer, and \( \eta : N \to C \) the insertion.

Given maps \( \beta_\lambda : M_\lambda \to P \) with \( \beta_\mu \alpha_\mu = \beta_\lambda \), there is a unique map \( \beta : M \to P \) with \( \beta \iota_\lambda = \beta_\lambda \) by the UMP of the coproduct. Clearly \( \beta \alpha = \beta \alpha' \); so \( \beta \) factors uniquely...
40 Direct Limits (6.14)

through $C$ by the UMP of the coequalizer. Thus $C = \lim M_\Lambda$, as desired.

Finally, if $F : \mathcal{C} \to \mathcal{C}'$ preserves coproducts and coequalizers, then $F$ preserves arbitrary direct limits as $F$ preserves the above construction.

\[ \text{Theorem (6.11).} \quad \text{The categories } ((\mathbb{R}-\text{mod})) \text{ and } ((\text{Sets})) \text{ have direct limits.} \]

\[ \text{Proof:} \quad \text{The assertion follows from (6.10) because } ((\mathbb{R}-\text{mod})) \text{ and } ((\text{Sets})) \text{ have coproducts by (6.7) and have coequalizers by (6.8).} \]

\[ \text{Theorem (6.12).} \quad \text{Every left adjoint } F : \mathcal{C} \to \mathcal{C}' \text{ preserves direct limits.} \]

\[ \text{Proof:} \quad \text{Let } \Lambda \text{ be a small category, } \lambda \mapsto M_\lambda \text{ a functor from } \Lambda \text{ to } \mathcal{C} \text{ such that } \lim M_\Lambda \text{ exists. Given an object } P' \text{ of } \mathcal{C}', \text{ consider all possible commutative diagrams} \]

\begin{equation}
\begin{array}{c}
F(M_\lambda) \xrightarrow{F(\alpha^\mu_\lambda)} F(M_\mu) \xrightarrow{F(\alpha^\mu_\lambda)} F(\lim M_\Lambda) \\
\downarrow{\beta'_\mu} \quad \downarrow{\beta'_\mu} \quad \downarrow{\beta'} \\
1 \quad 1 \quad 1
\end{array}
\end{equation}

(6.12.1)

where $\alpha^\mu_\lambda$ is any transition map and $\alpha_\mu$ is the corresponding insertion. Given the $\beta'_\mu$, we must show there is a unique $\beta'$.

Say $F$ is the left adjoint of $F' : \mathcal{C} \to \mathcal{C}'$. Then giving (6.12.1) is equivalent to giving this corresponding commutative diagram:

\[
\begin{array}{c}
M_\lambda \xrightarrow{\alpha^\mu_\lambda} M_\mu \xrightarrow{\alpha_\mu} \lim M_\Lambda \\
\downarrow{\beta_\lambda} \quad \downarrow{\beta_\mu} \quad \downarrow{\beta} \\
F'(P') \xrightarrow{1} F'(P') \xrightarrow{1} F'(P')
\end{array}
\]

(6.13.1)

However, given the $\beta_\lambda$, there is a unique $\beta$ by the UMP of $\lim M_\Lambda$. \[ \square \]

\[ \text{Proposition (6.13).} \quad \text{Let } \mathcal{C} \text{ be a category, } \Lambda \text{ and } \Sigma \text{ small categories. Assume } \mathcal{C} \text{ has direct limits indexed by } \Sigma. \text{ Then the functor category } \mathcal{C}^\Lambda \text{ does too.} \]

\[ \text{Proof:} \quad \text{Let } \sigma \mapsto (\lambda \mapsto M_{\sigma\lambda}) \text{ be a functor from } \Sigma \text{ to } \mathcal{C}^\Lambda. \text{ Then a map } \sigma \to \tau \text{ in } \Sigma \text{ yields a natural transformation from } \lambda \mapsto M_{\sigma\lambda} \text{ to } \lambda \mapsto M_{\tau\lambda}. \text{ So a map } \lambda \to \mu \text{ in } \Lambda \text{ yields a commutative square} \]

\begin{equation}
\begin{array}{c}
M_{\sigma\lambda} \to M_{\sigma\mu} \\
\downarrow \quad \downarrow \\
M_{\tau\lambda} \to M_{\tau\mu}
\end{array}
\end{equation}

(6.13.1)

in a manner compatible with composition in $\Sigma$. Hence, with $\lambda$ fixed, the rule $\sigma \mapsto M_{\sigma\lambda}$ is a functor from $\Sigma$ to $\mathcal{C}$.

By hypothesis, $\lim_{\sigma \in \Sigma} M_{\sigma\lambda}$ exists. So $\lambda \mapsto \lim_{\sigma \in \Sigma} M_{\sigma\lambda}$ is a functor from $\Lambda$ to $\mathcal{C}$. Further, as $\tau \in \Sigma$ varies, there are compatible natural transformations from the $\lambda \mapsto M_{\tau\lambda}$ to $\lambda \mapsto \lim_{\sigma \in \Sigma} M_{\sigma\lambda}$. Finally, the latter is the direct limit of the functor $\tau \mapsto (\lambda \mapsto M_{\tau\lambda})$ from $\Sigma$ to $\mathcal{C}^\Lambda$, because, given any functor $\lambda \mapsto P_\lambda$ from $\Lambda$ to $\mathcal{C}$ equipped with, for $\tau \in \Sigma$, compatible natural transformations from the $\lambda \mapsto M_{\tau\lambda}$ to $\lambda \mapsto P_\lambda$, there are, for $\lambda \in \Lambda$, compatible unique maps $\lim_{\sigma \in \Sigma} M_{\sigma\lambda} \to P_\lambda$. \[ \square \]
**Theorem (6.14)** (Direct limits commute). — Let $\mathcal{C}$ be a category with direct limits indexed by small categories $\Sigma$ and $\Lambda$. Let $\sigma \mapsto (\lambda \mapsto M_{\sigma, \lambda})$ be a functor from $\Sigma$ to $\mathcal{C}^\Lambda$. Then

$$
\lim_{\gamma \in \Sigma} \lim_{\lambda \in \Lambda} M_{\sigma, \lambda} = \lim_{\lambda \in \Lambda} \lim_{\sigma \in \Sigma} M_{\sigma, \lambda}.
$$

**Proof:** By (6.12), the functor $\lim_{\lambda \in \Lambda} : \mathcal{C}^\Lambda \to \mathcal{C}$ is a left adjoint. By (6.12), the category $\mathcal{C}^\Lambda$ has direct limits indexed by $\Sigma$. So (6.14) yields the assertion. □

**Corollary (6.15).** — Let $\Lambda$ be a small category, $R$ a ring, and $\mathcal{C}$ either $((\text{Sets}))$ or $((R\text{-mod}))$. Then functor $\lim_{\lambda} : \mathcal{C}^\Lambda \to \mathcal{C}$ preserves coproducts and coequalizers.

**Proof:** By (6.7) and (6.8), both coproducts and coequalizers are special cases of direct limits, and $\mathcal{C}$ has them. So (6.14) yields the assertion. □

**Exercise (6.16).** — Let $\mathcal{C}$ be a category, $\Sigma$ and $\Lambda$ small categories.

1. Prove $\mathcal{C}^{\Sigma \times \Lambda} = (\mathcal{C}^\Lambda)^\Sigma$ with $(\sigma, \lambda) \mapsto M_{\sigma, \lambda}$ corresponding to $\sigma \mapsto (\lambda \mapsto M_{\sigma, \lambda})$.

2. Assume $\mathcal{C}$ has direct limits indexed by $\Sigma$ and by $\Lambda$. Prove that $\mathcal{C}$ has direct limits indexed by $\Sigma \times \Lambda$ and that $\lim_{\lambda \in \Lambda} \lim_{\sigma \in \Sigma} = \lim_{(\sigma, \lambda) \in \Sigma \times \Lambda}.$

**Exercise (6.17).** — Let $\lambda \mapsto M_{\lambda}$ and $\lambda \mapsto N_{\lambda}$ be two functors from a small category $\Lambda$ to $((R\text{-mod}))$, and $\{\theta_{\lambda} : M_{\lambda} \to N_{\lambda}\}$ a natural transformation. Show

$$
\lim_{\Lambda} \text{Coker}(\theta_{\lambda}) = \text{Coker}(\lim_{\lambda} M_{\lambda} \to \lim_{\lambda} N_{\lambda}).
$$

Show that the analogous statement for kernels can be false by constructing a counterexample using the following commutative diagram with exact rows:

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\mu_2} & \mathbb{Z} \\
\downarrow{\mu_2} & & \downarrow{\mu_2} \\
\mathbb{Z} & \xrightarrow{\mu_2} & \mathbb{Z}/\langle 2 \rangle \\
\end{array}
$$

Show that the analogous statement for kernels can be false by constructing a counterexample using the following commutative diagram with exact rows:

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\mu_2} & \mathbb{Z} \\
\downarrow{\mu_2} & & \downarrow{\mu_2} \\
\mathbb{Z} & \xrightarrow{\mu_2} & \mathbb{Z}/\langle 2 \rangle \\
\end{array}
$$

0
7. Filtered Direct Limits

Filtered direct limits are direct limits indexed by a filtered category, which is a more traditional sort of index set. After making the definitions, we study an instructive example where the limit is $\mathbb{Q}$. Then we develop an alternative construction of filtered direct limits for modules. We conclude that forming them preserves exact sequences, and so commutes with forming the module of homomorphisms out of a fixed finitely presented source.

(7.1) (Filtered categories). — We call a small category $\Lambda$ filtered if

1. given objects $\kappa$ and $\lambda$, for some $\mu$ there are maps $\kappa \to \mu$ and $\lambda \to \mu$,
2. given two maps $\sigma, \tau: \eta \to \kappa$ with the same source and the same target, for some $\mu$ there is a map $\varphi: \kappa \to \mu$ such that $\varphi \sigma = \varphi \tau$.

Given a category $\mathcal{C}$, we say a functor $\lambda \mapsto M_\lambda$ from $\Lambda$ to $\mathcal{C}$ is filtered if it exists. If so, then we say the direct limit $\lim M_\lambda$ is filtered if it exists.

For example, let $\Lambda$ be a partially ordered set. Suppose $\Lambda$ is directed; that is, given $\lambda; \mu \in \Lambda$, there is a $\nu \in \Lambda$ with $\lambda \leq \mu$ and $\mu \leq \nu$. Regard $\Lambda$ as a category whose objects are its elements and whose sets Hom($\kappa; \lambda$) consist of a single element if $\kappa \leq \lambda$, and are empty if not; morphisms can be composed, because the ordering is transitive. Clearly, the category $\Lambda$ is filtered.

Exercise (7.2). — Let $R$ be a ring, $M$ a module, $\Lambda$ a set, $M_\lambda$ a submodule for each $\lambda \in \Lambda$. Assume $\bigcup M_\lambda = M$. Assume, given $\lambda; \mu \in \Lambda$, there is $\nu \in \Lambda$ such that $M_\lambda, M_\mu \subset M_\nu$. Order $\Lambda$ by inclusion: $\lambda \leq \mu$ if $M_\lambda \subset M_\mu$. Prove $M = \lim M_\lambda$.

Exercise (7.3). — Show that every module $M$ is the filtered direct limit of its finitely generated submodules.

Exercise (7.4). — Show that every direct sum of modules is the filtered direct limit of its finite direct subsums.

Example (7.5). — Let $\Lambda$ be the set of all positive integers, and for each $n \in \Lambda$, set $M_n := \{r/n \mid r \in \mathbb{Z}\} \subset \mathbb{Q}$. Then $\bigcup M_n = \mathbb{Q}$ and $M_m, M_n \subset M_{mn}$. Then (7.2) yields $\mathbb{Q} = \lim M_n$ where $\Lambda$ is ordered by divisibility of the $n \in \Lambda$.

However, $M_m \subset M_n$ if and only if $1/m = s/n$ for some $s$, if and only if $m \mid n$. Thus we may view $\Lambda$ as ordered by divisibility of the $n \in \Lambda$.

For each $n \in \Lambda$, set $R_n := \mathbb{Z}$, and define $\beta_n: R_n \to M_n$ by $\beta_n(r) := r/n$. Clearly, $\beta_n$ is a $\mathbb{Z}$-module isomorphism. And if $n = ms$, then this diagram is commutative:

$\begin{array}{ccc}
R_m & \xrightarrow{\mu} & R_n \\
\beta_m & \cong & \beta_n \\
M_m & \xrightarrow{i_m} & M_n
\end{array}$

where $\mu$ is the map of multiplication by $s$ and $i_m$ is the inclusion. Thus $\mathbb{Q} = \lim R_n$ where the transition maps are the $\mu_s$.

Exercise (7.6). — Keep the setup of (7.5). For each $n \in \Lambda$, set $N_n := \mathbb{Z}/(n)$; if $n = ms$, define $\alpha_n^m: N_m \to N_n$ by $\alpha_n^m(x) := xs \pmod{n}$. Show $\lim N_n = \mathbb{Q}/\mathbb{Z}$.
Theorem (7.7). — Let $\Lambda$ be a filtered category, $R$ a ring, and $\mathcal{C}$ either $((\text{Sets}))$ or $((\text{R-mod}))$ or $((\text{R-alg}))$. Let $\lambda \mapsto M_\lambda$ be a functor from $\Lambda$ to $\mathcal{C}$. Define a relation $\sim$ on the set-theoretic disjoint union $\bigsqcup M_\lambda$ as follows: $m_1 \sim m_2$ for $m_i \in M_{\lambda_i}$, if there are transition maps $\alpha^i_\mu^j : M_{\lambda_i} \to M_{\mu}$ such that $\alpha^i_\mu^j m_1 = \alpha^j_\mu^i m_2$. Then $\sim$ is an equivalence relation. Set $M := (\bigsqcup M_\lambda)/\sim$. Then $M = \lim_{\rightarrow} M_\lambda$, and for each $\mu$, the canonical map $\alpha_\mu : M_\lambda \to M$ is equal to the insertion map $M_\mu \to \lim_{\rightarrow} M_\lambda$.

Proof: Clearly $\sim$ is reflexive and symmetric. Let’s show it is transitive. Given $m_i \in M_{\lambda_i}$ for $i = 1, 2, 3$ with $m_1 \sim m_2$ and $m_2 \sim m_3$, there are $\alpha^i_\mu^j$ for $i = 1, 2$ and $\alpha^i_\nu^j$ for $i = 2, 3$ with $\alpha^i_\mu^j m_1 = \alpha^j_\mu^i m_2$ and $\alpha^i_\nu^j m_2 = \alpha^j_\nu^i m_3$. Then (7.1) yields $\alpha^j_\mu^i m_1 = \alpha^j_\nu^i m_3$. Thus $m_1 \sim m_3$.

If $\mathcal{C} = ((\text{R-mod}))$, define addition in $M$ as follows. Given $m_i \in M_{\lambda_i}$ for $i = 1, 2$, there are $\alpha^i_\mu$ by (7.1). Set

$$\alpha_{\lambda_i} m_1 + \alpha_{\lambda_2} m_2 := \alpha_\mu (\alpha^i_\mu m_1 + \alpha^j_\mu m_2).$$

We must check that this addition is well defined.

First, consider $\mu$. Suppose there are $\alpha^i_\nu$ too. Then (7.1) yields $\alpha^i_\mu$ and $\alpha^j_\nu$. Possibly, $\alpha^i_\mu \alpha^j_\nu \neq \alpha^j_\nu \alpha^i_\mu$, but (7.2) yields $\alpha^j_\nu$ with $\alpha^j_\nu (\alpha^i_\mu \alpha^j_\nu) = \alpha^j_\nu (\alpha^j_\nu \alpha^j_\nu)$, and then $\alpha^j_\nu$ with $\alpha^j_\nu (\alpha^j_\nu \alpha^j_\nu) = \alpha^j_\nu (\alpha^j_\nu \alpha^j_\nu)$. In sum, we have this diagram of indices:

$$\begin{array}{c}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{array}
\xymatrix{
\mu \\
\nu \\
\rho}
\xymatrix{
\sigma \\
\tau
}
$$

Hence, $(\alpha^j_\nu \alpha^j_\nu) = (\alpha^j_\nu \alpha^j_\nu)$. Thus $m_1 \sim m_3$.

If $\mathcal{C} = ((\text{R-alg}))$, then we can see similarly that $M$ is canonically an $R$-algebra.

Finally, let $\beta_\lambda : M_\lambda \to N$ be maps with $\beta_\lambda \alpha^i_\mu = \beta_\lambda$ for all $\alpha^i_\mu$. The $\beta_\lambda$ induce a map $\bigsqcup M_\lambda \to N$. Suppose $m_1 \sim m_2$ for $m_i \in M_{\lambda_i}$; that is, $\alpha_{\lambda_i} m_1 = \alpha_{\lambda_2} m_2$ for some $\alpha_{\lambda_i}$. Then $\beta_\lambda m_1 = \beta_\lambda m_2$ as $\beta_\mu \alpha_{\lambda_i} = \beta_\lambda$. So there is a unique map $\beta : M \to N$ with $\beta \alpha_{\lambda_i} = \beta_\lambda$ for all $\lambda$. Further, if $\mathcal{C} = ((\text{R-mod}))$ or $\mathcal{C} = ((\text{R-alg}))$, then clearly $\beta$ is a homomorphism. The proof is now complete.

Corollary (7.8). — Preserve the conditions of (7.7).

1. Given $m \in \lim_{\rightarrow} M_\lambda$, there are $\lambda$ and $m_\lambda \in M_\lambda$ such that $m = \alpha_{\lambda} m_\lambda$.

2. Given $m_i \in M_{\lambda_i}$ for $i = 1, 2$ such that $\alpha_{\lambda_1} m_1 = \alpha_{\lambda_2} m_2$, there are $\alpha^i_\mu$ such that $\alpha^i_\mu m_1 = \alpha^j_\mu m_2$.

3. Suppose $\mathcal{C} = ((\text{R-mod}))$ or $\mathcal{C} = ((\text{R-alg}))$. Then given $\lambda$ and $m_\lambda \in M_\lambda$ such that $\alpha_{\lambda} m_\lambda = 0$, there is $\alpha^i_\mu$ such that $\alpha^i_\mu m_\lambda = 0$.
PROOF: The assertions follow directly from (7.11). Specifically, (1) holds, since \( \lim M_\lambda \) is a quotient of the disjoint union \( \coprod M_\lambda \). Further, (2) holds owing to the definition of the equivalence relation involved. Finally, (3) is the special case of (2) where \( m_1 := m_\lambda \) and \( m_2 = 0 \). \( \square \)

EXERCISE (7.14). — Let \( R := \lim R_\lambda \) be a filtered direct limit of rings.

(1) Prove that \( R = 0 \) if and only if \( R_\lambda = 0 \) for some \( \lambda \).

(2) Assume that each \( R_\lambda \) is a domain. Prove that \( R \) is a domain.

(3) Assume that each \( R_\lambda \) is a field. Prove that \( R \) is a field.

EXERCISE (7.14). — Let \( M := \lim M_\lambda \) be a filtered direct limit of modules, with transition maps \( \alpha_\lambda^\mu : M_\lambda \to M_\mu \) and insertions \( \alpha_\lambda : M_\lambda \to M \). For each \( \lambda \), let \( N_\lambda \subset M_\lambda \) be a submodule, and let \( N \subset M \) be a submodule. Prove that \( N_\lambda = \alpha_\lambda^{-1}N \) for all \( \lambda \) if and only if (a) \( N_\lambda = (\alpha_\lambda^\mu)^{-1}N_\mu \) for all \( \alpha_\lambda^\mu \) and (b) \( \bigcup \alpha_\lambda N_\lambda = N \).

DEFINITION (7.11). — Let \( R \) be a ring. We say an algebra \( R' \) is \textit{finitely presented} if \( R' \cong R[X_1, \ldots, X_r]/\mathfrak{a} \) for some variables \( X_1 \) and finitely generated ideal \( \mathfrak{a} \).

PROPOSITION (7.12). — Let \( \Lambda \) be a filtered category, \( R \) a ring, \( \mathcal{C} \) either \((\text{-mod})\) or \((\text{-alg})\), \( \lambda \mapsto M_\lambda \) a functor from \( \Lambda \) to \( \mathcal{C} \). Given \( N \in \mathcal{C} \), form the map
\[
\theta : \lim \text{Hom}(N, M_\lambda) \to \text{Hom}(N, \lim M_\lambda).
\]

(1) If \( N \) is finitely generated, then \( \theta \) is injective.

(2) The following conditions are equivalent:

\begin{enumerate}
  \item \( N \) is finitely presented;
  \item \( \theta \) is bijective for all filtered categories \( \Lambda \) and all functors \( \lambda \mapsto M_\lambda \);
  \item \( \theta \) is surjective for all directed sets \( \Lambda \) and all \( \lambda \mapsto M_\lambda \).
\end{enumerate}

PROOF: Given a transition map \( \alpha_\lambda^\mu : M_\lambda \to M_\mu \), set \( \beta_\lambda^\mu := \text{Hom}(N, \alpha_\lambda^\mu) \). Then the \( \beta_\lambda^\mu \) are the transition maps of \( \lim \text{Hom}(N, M_\lambda) \). Denote by \( \alpha_\lambda \) and \( \beta_\lambda \) the limits of \( M_\lambda \) and \( \lim \text{Hom}(N, M_\lambda) \).

For (1), let \( n_1, \ldots, n_r \) generate \( N \). Given \( \varphi \) and \( \varphi' \in \lim \text{Hom}(N, M_\lambda) \) with \( \theta(\varphi) = \theta(\varphi') \), note that (7.13) (1) yields \( \lambda \) and \( \varphi_\lambda : N \to M_\lambda \) and \( \varphi'_\lambda : N \to M_\mu \) with \( \beta_\lambda(\varphi_\lambda) = \varphi \) and \( \beta_\mu(\varphi'_\lambda) = \varphi' \). Then \( \theta(\varphi) = \alpha_\lambda(\varphi_\lambda) \) and \( \theta(\varphi') = \alpha_\mu(\varphi'_\lambda) \) by construction of \( \theta \). Hence \( \alpha_\lambda(\varphi_\lambda) = \alpha_\mu(\varphi'_\lambda) \). So \( \alpha_\lambda(\varphi_\lambda(n_i)) = \alpha_\mu(\varphi'_\lambda(n_i)) \) for all \( i \).

For (2), let \( \nu_i \) and \( \alpha_\lambda^\nu_i \) and \( \alpha_\mu^\nu_i \) such that \( \alpha_\lambda^\nu_i(\varphi_\lambda(n_i)) = \alpha_\mu^\nu_i(n_j) \) for \( 1 \leq j \leq i \). Indeed, given \( \nu_i \) and \( \alpha_\lambda^\nu_i \) and \( \alpha_\mu^\nu_i \), by (7.13) (1), there are \( \rho_i \) and \( \alpha_\rho_i \) and \( \alpha_\lambda^\rho_i \). By (7.13) (2), there are \( \nu_i \) and \( \alpha_\lambda^\nu_i \) such that \( \alpha_\lambda^\nu_i \alpha_\rho^{-1} = \alpha_\rho^{-1} \alpha_\lambda^\rho \) and \( \alpha_\lambda^\nu_i \alpha_\rho^{-1} = \alpha_\rho^{-1} \alpha_\lambda^\rho \). Set \( \alpha_\lambda^\nu := \alpha_\mu^\nu_i \alpha_\rho_i \alpha_\lambda^\rho \) and \( \alpha_\mu^\nu_i := \alpha_\mu^\nu_i \alpha_\lambda^\rho \). Then \( \alpha_\lambda^\nu(\varphi_\lambda(n_j)) = \alpha_\mu^\nu_i(n_j) \) for \( 1 \leq j \leq i \), as desired.

Let \( \nu := \nu_i \). Then \( \alpha_\lambda^\nu(\varphi_\lambda(n_i)) = \alpha_\lambda^\nu_i(\varphi'_\lambda(n_i)) \) for all \( i \). Hence \( \alpha_\lambda^\nu(\varphi_\lambda) = \alpha_\mu^\nu_i(\varphi'_\lambda) \). But \( \varphi = \beta_\lambda(\varphi_\lambda) = \beta_\mu(\varphi'_\lambda) = \beta_\nu(\alpha_\lambda^\nu(\varphi_\lambda)) \). Hence \( \varphi = \varphi' \). Thus \( \theta \) is injective. Notice that this proof works equally well for \((\text{-mod})\) and \((\text{-alg})\). Thus (1) holds.

For (2), let’s treat the case \( \mathcal{C} = ((\text{-mod})\) first. Assume (a). Say \( N \cong \mathbb{F}/N' \) where \( F := \mathbb{F} \) and \( N' \) is finitely generated, say by \( n'_1, \ldots, n'_r \). Let \( n_i \) be the image of \( N \) in the \( i \th \) standard basis vector \( e_i \) of \( F \). Then there are homogeneous linear polynomials \( f_j \) with \( f_j(e_1, \ldots, e_r) = n'_j \) for all \( j \). So \( f_j(n_1, \ldots, n_r) = 0 \).
Given \( \varphi : N \to \lim M_{\lambda} \), set \( m_i := \varphi(n_i) \) for \( 1 \leq i \leq r \). Repeated use of (7.13)(1) and (7.14)(1) yields \( \lambda \) and \( m_{\lambda i} \in M_{\lambda} \) with \( \alpha_\lambda m_{\lambda i} = m_i \) for all \( i \). So for all \( j \),
\[
\alpha_\lambda(f_j(m_{\lambda 1}, \ldots, m_{\lambda r})) = f_j(m_1, \ldots, m_r) = \varphi(f_j(n_1, \ldots, n_r)) = 0.
\]
Hence repeated use of (7.13)(2) and (7.14)(1), (2) yields \( \mu \) and \( \alpha_\lambda^\mu \) with, for all \( j \),
\[
\alpha_\lambda^\mu(f_j(m_{\lambda 1}, \ldots, m_{\lambda r})) = 0.
\]
Therefore, there is \( \varphi_\mu : N \to M_\mu \) with \( \varphi_\mu(n_i) := \alpha_\lambda^\mu(m_{\lambda i}) \) by (7.13)(1) and (7.14)(1). Set \( \psi := \beta_\mu(\varphi_\mu) \). Then \( \theta(\psi) = \alpha_\mu \varphi_\mu \). Hence \( \theta(\psi)(n_i) = m_i := \varphi(n_i) \) for all \( i \). So \( \theta(\psi) = \varphi \). Thus \( \theta \) is surjective. So (1) implies \( \theta \) is bijective. Thus (b) holds.

Finally, assume (c). Take \( \Lambda \) to be the directed set of finitely generated submodules \( N_\lambda \) of \( N \). Then \( N = \lim N_\lambda \) by (7.17). However, \( \theta \) is surjective. So there is \( \psi \in \lim \text{Hom}(N, N_\lambda) \) with \( \theta(\psi) = 1_N \). So (7.13)(1) yields \( \lambda \) and \( \psi_\lambda \in \text{Hom}(N, N_\lambda) \) with \( \beta_\lambda(\psi_\lambda) = \psi \). Hence \( \alpha_\lambda \psi_\lambda = \theta(\psi) \). So \( \alpha_\lambda \psi_\lambda = 1_N \). So \( \alpha_\lambda \) is surjective. But \( \alpha_\lambda : N_\lambda \to N \) is the inclusion. Thus \( N_\lambda \) is finitely generated. Say \( n_1, \ldots, n_r \) generate \( N \). Set \( F := R^r \) and let \( e_i \) be the \( i \)th standard basis vector.

Define \( \kappa : F \to N \) by \( \kappa(e_i) := n_i \) for all \( i \). Set \( N' := \text{Ker}(\kappa) \). Then \( F/N' \to N \). Let's show \( N' \) is finitely generated.

Take \( \Lambda \) to be the directed set of finitely generated submodules \( N'_\lambda \) of \( N' \). Then \( N' = \lim N'_\lambda \) by (7.17). Set \( N'_\lambda := F/N'_\lambda \). Then \( N = \lim N_\lambda \) by (7.17). Here the \( \alpha_\lambda^\mu \) and the \( \alpha_\lambda \) are the quotient maps. Since \( \theta \) is surjective, there is \( \psi \in \text{Hom}(N, N_\lambda) \) with \( \theta(\psi) = 1_N \). So (7.13)(1) yields \( \lambda \) and \( \psi_\lambda \in \text{Hom}(N, N_\lambda) \) with \( \beta_\lambda(\psi_\lambda) = \psi \). Hence \( \alpha_\lambda \psi_\lambda = \theta(\psi) \). So \( \alpha_\lambda \psi_\lambda = 1_N \). Set \( \psi_\mu := \alpha_\lambda^\mu \psi_\lambda \) for all \( \mu \); note \( \psi_\mu \) is well defined as \( \Lambda \) is directed. Then \( \alpha_\mu \psi_\mu = \alpha_\lambda \psi_\lambda = 1_N \) for all \( \mu \). Let's show there is \( \mu \) with \( \psi_\mu \alpha_\mu = 1_{N_\mu} \).

For all \( \mu \) and \( i \), let \( n_\mu i \) be the image in \( N_\mu \) of \( e_i \). Then \( \alpha_\lambda n_{\lambda i} = \alpha_\lambda(\psi_\lambda \alpha_\lambda n_{\lambda i}) = \alpha_\lambda \psi_\lambda = 1_N \). Hence repeated use of (7.13)(2) and (7.14)(1) yields \( \mu \) such that \( \alpha_\lambda^\mu n_{\lambda i} = \alpha_\lambda^\mu(\psi_\lambda \alpha_\lambda n_{\lambda i}) \) for all \( i \). Hence \( n_{\mu i} = (\psi_\mu \alpha_\mu)n_{\mu i} \). But the \( n_{\mu i} \) generate \( N_\mu \) for all \( i \). So \( 1_{N_\mu} = \psi_\mu \alpha_\mu \), as desired.

So \( \alpha_\mu : N_\mu \to N \) is an isomorphism. So \( N'_\mu = N' \). Thus \( N' \) is finitely generated. Thus (a) holds for ((R-mod)).

In the case \( C = ((R-\text{alg})) \), replace \( F \) by a polynomial ring \( R[X_1, \ldots, X_r] \), the submodule \( N' \) by the appropriate ideal \( a \), and the \( f_j \) by polynomials that generate \( a \). With these replacements, the above proof shows (a) implies (b). As to (c) implies (a), first take the \( N_\lambda \) to be the finitely generated subalgebras; then the above proof of finite generation works equally well as is. The rest of the proof works after we replace \( F \) by a polynomial ring, the \( e_i \) by the variables, \( N' \) by the appropriate ideal, and the \( N'_\lambda \) by the finitely generated subideals.

(7.13) (Finite presentations). — Let \( R \) be a ring, \( R' \) a finitely presented algebra. The proof of (7.14)(2) shows that, for any presentation \( R[X_1, \ldots, X_r]/a \) of \( R' \), where \( R[X_1, \ldots, X_r] \) is a polynomial ring and \( a \) is an ideal, necessarily \( a \) is finitely generated. Similarly, for a finitely presented module \( M \), that proof gives another solution to (7.14)(1), one not requiring Schanuel’s Lemma.

Theorem (7.14) (Exactness of Filtered Direct Limits). — Let \( R \) be a ring, \( \Lambda \) a filtered category. Let \( C \) be the category of 3-term exact sequences of \( R \)-modules: its
objects are the 3-term exact sequences, and its maps are the commutative diagrams

\[
\begin{array}{c}
L \rightarrow M \rightarrow N \\
\downarrow \downarrow \downarrow \\
L' \rightarrow M' \rightarrow N'
\end{array}
\]

Then, for any functor \( \lambda \mapsto (L_\lambda \xrightarrow{\beta_\lambda} M_\lambda \xrightarrow{\gamma_\lambda} N_\lambda) \) from \( \Lambda \) to \( \mathcal{C} \), the induced sequence

\[
\lim L_\lambda \xrightarrow{\beta} \lim M_\lambda \xrightarrow{\gamma} \lim N_\lambda
\]

is exact.

**Proof:** Abusing notation, in all three cases denote by \( \alpha_\lambda \) the transition maps and by \( \gamma_\lambda \) the insertions. Then given \( \ell_\lambda \in \lim L_\lambda \), there is \( \ell_\lambda \in L_\lambda \) with \( \alpha_\lambda \ell_\lambda = \ell \) by (7.8)(1). By hypothesis, \( \gamma_\lambda \beta_\lambda \ell_\lambda = 0 \); so \( \gamma_\beta \ell = 0 \). In sum, we have this figure:

\[
\begin{array}{c}
\ell_\lambda \downarrow \downarrow \downarrow \downarrow \\
\ell \downarrow \downarrow \downarrow \downarrow \\
0 \\
\end{array}
\]

Thus \( \text{Im}(\beta) \subseteq \text{Ker}(\gamma) \).

For the opposite inclusion, take \( m \in \lim M_\lambda \) with \( \gamma m = 0 \). By (7.8)(1), there is \( m_\lambda \in M_\lambda \) with \( \alpha_\lambda m_\lambda = m \). Now, \( \alpha_\lambda \gamma_\lambda m_\lambda = 0 \) by commutativity. So by (7.8)(3), there is \( \alpha_\mu \) with \( \alpha_\lambda \gamma_\lambda m_\lambda = 0 \). So \( \gamma_\mu \alpha_\lambda m_\lambda = 0 \) by commutativity. Hence there is \( \ell_\mu \in L_\mu \) with \( \beta_\mu \ell_\mu = \alpha_\mu \alpha_\lambda m_\lambda \) by exactness. Apply \( \alpha_\mu \) to get

\[
\beta_\mu \ell_\mu = \alpha_\mu \beta_\mu \ell_\mu = \alpha_\mu \alpha_\lambda m_\lambda = m.
\]

In sum, we have this figure:

\[
\begin{array}{c}
m_\lambda \mapsto n_\lambda \mapsto \lambda \\
\ell_\mu \mapsto m_\mu \mapsto 0 \\
\ell \mapsto m \mapsto 0 \\
\end{array}
\]

Thus \( \text{Ker}(\gamma) \subseteq \text{Im}(\beta) \). So \( \text{Ker}(\gamma) = \text{Im}(\beta) \) as asserted.

**Exercise (7.15).** Let \( R := \lim R_\lambda \) be a filtered direct limit of rings, \( a_\lambda \subset R_\lambda \) an ideal for each \( \lambda \). Assume \( \alpha_\lambda a_\lambda \subset a_\mu \) for each transition map \( \alpha_\lambda \). Set \( a := \lim a_\lambda \). If each \( a_\lambda \) is prime, show \( a \) is prime. If each \( a_\lambda \) is maximal, show \( a \) is maximal.

**Exercise (7.16).** Let \( M := \lim M_\lambda \) be a filtered direct limit of modules, with transition maps \( \alpha_\mu : M_\lambda \rightarrow M_\mu \) and insertions \( \alpha_\lambda : M_\lambda \rightarrow M \). Let \( N_\lambda \subset M_\lambda \) be a submodule for all \( \lambda \). Assume \( \alpha_\mu N_\lambda \subset N_\mu \) for all \( \alpha_\mu \). Prove \( \lim N_\lambda = \bigcup \alpha_\lambda N_\lambda \).

**Exercise (7.17).** Let \( R := \lim R_\lambda \) be a filtered direct limit of rings. Prove that \( \lim \text{nil}(R_\lambda) = \text{nil}(R) \).
Filtered Direct Limits (7.21)

Exercise (7.18). Let $R := \lim_{\to} R_\lambda$ be a filtered direct limit of rings. Assume each ring $R_\lambda$ is local, say with maximal ideal $m_\lambda$, and assume each transition map $\alpha_\lambda^\mu \colon R_\lambda \to R_\mu$ is local. Set $m := \lim_{\to} m_\lambda$. Prove that $R$ is local with maximal ideal $m$ and that each insertion $\alpha_\lambda : R_\lambda \to R$ is local.

(7.19) (Hom and direct limits again). Let a filtered category, $R$ a ring, $N$ a module, and $\lambda \mapsto M_\lambda$ a functor from $\Lambda$ to $((R\text{-mod}))$. Here is an alternative proof that the map $\theta(N)$ of (6.6.1) is injective if $N$ is finitely generated and bijective if $N$ is finitely presented.

If $N := R$, then $\theta(N)$ is bijective by (1.13). Assume $N$ is finitely generated, and take a presentation $R^{\oplus \Sigma} \to R^n \to N \to 0$ with $\Sigma$ finite if $N$ is finitely presented. It induces the following commutative diagram:

$$
\begin{array}{ccc}
0 & \to & \lim_{\Lambda} \text{Hom}(N, M_\lambda) \\
\downarrow \theta(N) & & \downarrow \theta(R^n) \\
0 & \to & \text{Hom}(N, \lim_{\Lambda} M_\lambda)
\end{array}
$$

The rows are exact owing to (5.18), the left exactness of Hom, and to (6.7), the exactness of filtered direct limits. Now, Hom preserves finite direct sums by (4.15), and direct limit does so by (6.15) and (6.7); hence, $\theta(R^n)$ is bijective, and $\theta(R^{\oplus \Sigma})$ is bijective if $\Sigma$ is finite. A diagram chase yields the assertion.

Exercise (7.20). Let $\Lambda$ and $\Lambda'$ be small categories, $C : \Lambda' \to \Lambda$ a functor. Assume $\Lambda'$ is filtered. Assume $C$ is cofinal; that is,

1. given $\lambda \in \Lambda$, there is a map $\lambda \to C\lambda'$ for some $\lambda' \in \Lambda'$, and
2. given $\psi, \varphi : \lambda \Rightarrow C\lambda'$, there is $\chi : \lambda' \to \lambda'_1$ with $(C\chi)\psi = (C\chi)\varphi$.

Let $\lambda \mapsto M_\lambda$ be a functor from $\Lambda$ to $\mathcal{C}$ whose direct limit exists. Show that

$$
\lim_{\Lambda'} \text{Hom}(C\lambda, M_{\Lambda'}) = \lim_{\Lambda} M_\lambda;
$$

more precisely, show that the right side has the UMP characterizing the left.

Exercise (7.21). Show that every $R$-module $M$ is the filtered direct limit over a directed set of finitely presented modules.
8. Tensor Products

Given two modules, their tensor product is the target of the universal bilinear map. We construct the product, and establish various properties: bifunctoriality, commutativity, associativity, cancellation, and most importantly, adjoint associativity; the latter relates the product to the module of homomorphisms. With one factor fixed, the product becomes a linear functor. We prove Watt’s Theorem; it characterizes “tensor-product” functors as those linear functors that commute with direct sums and cokernels. Lastly, we discuss the tensor product of algebras.

(8.1) (Bilinear maps). — Let $R$ be a ring, and $M$, $N$, $P$ modules. We call a map $\alpha: M \times N \to P$ bilinear if it is linear in each variable; that is, given $m \in M$ and $n \in N$, the maps $m' \mapsto \alpha(m', n)$ and $n' \mapsto \alpha(m, n')$ are $R$-linear. Denote the set of all these maps by $\text{Bil}_R(M, N; P)$. It is clearly an $R$-module, with sum and scalar multiplication performed valuewise.

(8.2) (Tensor product). — Let $R$ be a ring, and $M$, $N$ modules. Their tensor product, denoted $M \otimes_R N$ or simply $M \otimes N$, is constructed as the quotient of the free module $R^{\oplus(M \times N)}$ modulo the submodule generated by the following elements, where $(m, n)$ stands for the standard basis element $e_{(m, n)}$:

\begin{align*}
(m + m', n) - (m, n) - (m', n) \quad &\text{and} \quad (m, n + n') - (m, n) - (m, n'), \\
(xm, n) - x(m, n) \quad &\text{and} \quad (m, xn) - x(m, n)
\end{align*}

(8.2.1)

for all $m, m' \in M$ and $n, n' \in N$ and $x \in R$.

The above construction yields a canonical bilinear map

$\beta: M \times N \to M \otimes N$.

Set $m \otimes n := \beta(m, n)$.

THEOREM (8.3) (UMP of tensor product). — Let $R$ be a ring, $M$, $N$ modules. Then $\beta: M \times N \to M \otimes N$ is the universal bilinear map with source $M \times N$; in fact, $\beta$ induces, not simply a bijection, but a module isomorphism,

$\theta: \text{Hom}_R(M \otimes_R N, P) \xrightarrow{\cong} \text{Bil}_R(M, N; P).$ (8.3.1)

PROOF: Note that, if we follow any bilinear map with any linear map, then the result is bilinear; hence, $\theta$ is well defined. Clearly, $\theta$ is a module homomorphism. Further, $\theta$ is injective since $M \otimes_R N$ is generated by the image of $\beta$. Finally, given any bilinear map $\alpha: M \times N \to P$, by (8.2.1) it extends to a map $\alpha': R^{\oplus(M \times N)} \to P$, and $\alpha'$ carries all the elements in (8.2.1) to 0; hence, $\alpha'$ factors through $\beta$. Thus $\theta$ is also surjective, so an isomorphism, as asserted. □

EXERCISE (5.3). — Let $R$ be a ring, $R'$ an $R$-algebra, and $M$ an $R'$-module. Set $M' := R' \otimes_R M$. Define $\alpha: M \to M'$ by $\alpha m := 1 \otimes m$, and $\beta: M' \to M$ by $\beta(x \otimes m) := xm$. Prove $M$ is a direct summand of $M'$ with $\alpha = \iota_M$ and $\beta = \pi_M$. 48
(8.5) (Bifunctoriality). — Let \( R \) be a ring, \( \alpha : M \to M' \) and \( \alpha' : N \to N' \) module homomorphisms. Then there is a canonical commutative diagram:

\[
\begin{array}{ccc}
M \times N & \xrightarrow{\alpha \times \alpha'} & M' \times N' \\
\downarrow \beta & & \downarrow \beta' \\
M \otimes N & \xrightarrow{\alpha \otimes \alpha'} & M' \otimes N'
\end{array}
\]

Indeed, \( \beta' \circ (\alpha \times \alpha') \) is clearly bilinear; so the UMP (8.3) yields \( \alpha \otimes \alpha' \). Thus \( \bullet \otimes N \) and \( M \otimes \bullet \) are commuting linear functors — that is, linear on maps, compare (1.2).

Proposition (8.6). — Let \( R \) be a ring, \( M \) and \( N \) modules.

1. Then the switch map \( (m, n) \mapsto (n, m) \) induces an isomorphism

\[
M \otimes_R N = N \otimes_R M.
\]

(commutative law)

2. Then multiplication of \( R \) on \( M \) induces an isomorphism

\[
R \otimes_R M = M.
\]

(unitary law)

Proof: The switch map induces an isomorphism \( R^{\otimes (M \times N)} \cong R^{\otimes (N \times M)} \), and it preserves the elements of (8.7). Thus (1) holds.

Define \( \beta : R \times M \to M \) by \( \beta(x, m) := xm \). Clearly \( \beta \) is bilinear. Let’s check \( \beta \) has the requisite UMP. Given a bilinear map \( \alpha : R \times M \to P \), define \( \gamma : M \to P \) by \( \gamma(m) := \alpha(1, m) \). Then \( \gamma \) is linear as \( \alpha \) is bilinear. Also, \( \alpha = \gamma \beta \) as

\[
\alpha(x, m) = x \alpha(1, m) = \alpha(1, xm) = \gamma(xm) = \gamma \beta(x, m).
\]

Further, \( \gamma \) is unique as \( \beta \) is surjective. Thus \( b \) has the UMP, so (2) holds.

Exercise (8.7). — Let \( R \) be a domain, \( a \) a nonzero ideal. Set \( K := \text{Frac}(R) \). Show that \( a \otimes_R K = K \).

(8.8) (Bimodules). — Let \( R \) and \( R' \) be rings. An abelian group \( N \) is an \( (R, R') \)-bimodule if it is both an \( R \)-module and an \( R' \)-module and if \( x(x'n) = x'(xn) \) for all \( x \in R \), all \( x' \in R' \), and all \( n \in N \). At times, we think of \( N \) as a left \( R \)-module, with multiplication \( xn \), and as a right \( R' \)-module, with multiplication \( nx' \). Then the compatibility condition becomes the associative law: \( x(nx') = (xn)x' \). A \( (R, R') \)-homomorphism of bimodules is a map that is both \( R \)-linear and \( R' \)-linear.

Let \( M \) be an \( R \)-module, and let \( N \) be an \( (R, R') \)-bimodule. Then \( M \otimes_R N \) is an \( (R, R') \)-bimodule with \( R \)-structure as usual and with \( R' \)-structure defined by \( x' \cdot (m \otimes n) := m \otimes (x'n) \) for all \( x' \in R' \), all \( m \in M \), and all \( n \in N \). The latter multiplication is well defined and the two multiplications commute because of bifunctoriality (8.2) with \( \alpha := \mu_x \) and \( \alpha' := \mu_{x'} \).

For instance, suppose \( R' \) is an \( R \)-algebra. Then \( R' \) is an \( (R, R') \)-bimodule. So \( M \otimes_R R' \) is an \( R' \)-module. It is said to be obtained by extension of scalars.

In full generality, it is easy to check that \( \text{Hom}_R(M, N) \) is an \( (R, R') \)-bimodule under valuewise multiplication by elements of \( R' \). Further, given an \( R' \)-module \( P \), it is easy to check that \( \text{Hom}_R(N, P) \) is an \( (R, R') \)-bimodule under sourcewise multiplication by elements of \( R \).

Exercise (8.8). — Let \( R \) be a ring, \( R' \) an \( R \)-algebra, \( M, N \) two \( R' \)-modules. Show there is a canonical \( R \)-linear map \( \tau : M \otimes_R N \to M \otimes_R N \).

Let \( K \subset M \otimes_R N \) denote the \( R \)-submodule generated by all the differences \( (x'm) \otimes n - m \otimes (x'n) \) for \( x' \in R' \) and \( m \in M \) and \( n \in N \). Show \( K \) is equal to \( \text{Ker}(\tau) \), and \( \tau \) is surjective. Show \( \tau \) is an isomorphism if \( R' \) is a quotient of \( R \).
THEOREM (8.10). — Let $R$ and $R'$ be rings, $M$ an $R$-module, $P$ an $R'$-module, $N$ an $(R, R')$-bimodule. Then there are two canonical $(R, R')$-isomorphisms:
\[
M \otimes_R (N \otimes_{R'} P) = (M \otimes_R N) \otimes_{R'} P, \quad \text{(associative law)}
\]
\[
\text{Hom}_{R'}(M \otimes_R N, P) = \text{Hom}_R(M, \text{Hom}_{R'}(N, P)). \quad \text{(adjoint associativity)}
\]

PROOF: Note that $M \otimes_R (N \otimes_{R'} P)$ and $(M \otimes_R N) \otimes_{R'} P$ are $(R, R')$-bimodules. For each $(R, R')$-bimodule $Q$, call a map $\tau: M \times N \times P \to Q$ trilinear if it is $R$-bilinear in $M \times N$ and $R'$-bilinear in $N \times P$. Denote the set of all these $\tau$ by $\text{Tril}(M, N, P; Q)$. It is, clearly, an $(R, R')$-bimodule.

A trilinear map $\tau$ yields an $R$-bilinear map $M \times (N \otimes_{R'} P) \to Q$, whence a map $M \otimes_R (N \otimes_{R'} P) \to Q$, which is both $R$-linear and $R'$-linear, and vice versa. Thus
\[
\text{Tril}_{(R, R')}(M, N, P; Q) = \text{Hom}(M \otimes_R (N \otimes_{R'} P), Q).
\]

Similarly, there is a canonical isomorphism of $(R, R')$-bimodules
\[
\text{Tril}_{(R, R')}(M, N, P; Q) = \text{Hom}((M \otimes_R N) \otimes_{R'} P, Q).
\]

Hence each of $M \otimes_R (N \otimes_{R'} P)$ and $(M \otimes_R N) \otimes_{R'} P$ is the universal target of a trilinear map with source $M \times N \times P$. Thus they are equal, as asserted.

To establish the isomorphism of adjoint associativity, define a map
\[
\alpha: \text{Hom}_{R'}(M \otimes_R N, P) \to \text{Hom}_R(M, \text{Hom}_{R'}(N, P)) \quad \text{by}
\]
\[
(\alpha(\gamma)(m))(n) := \gamma(m \otimes n).
\]

Let’s check $\alpha$ is well defined. First, $\alpha(\gamma)(m)$ is $R'$-linear, because given $x' \in R'$,
\[
\gamma(m \otimes (x'n)) = \gamma(x'(m \otimes n)) = x'\gamma(m \otimes n)
\]
since $\gamma$ is $R'$-linear. Further, $\alpha(\gamma)$ is $R$-linear, because given $x \in R$,
\[
(\gamma(mx))\otimes n = m \otimes (xn) \quad \text{and so} \quad \alpha(\gamma(mx))(n) = \alpha(\gamma(m))(xn).
\]

Thus $\alpha(\gamma) \in \text{Hom}_R(M, \text{Hom}_{R'}(N, P))$. Clearly, $\alpha$ is an $(R, R')$-homomorphism.

To obtain an inverse to $\alpha$, given $\eta \in \text{Hom}_R(M, \text{Hom}_{R'}(N, P))$, define a map $\zeta: M \times N \to P$ by $\zeta(m, n) := (\eta(m))(n)$. Clearly, $\zeta$ is $Z$-bilinear, so $\zeta$ induces a $Z$-linear map $\delta: M \otimes_Z N \to P$. Given $x \in R$, clearly $(\eta(mx))(n) = (\eta(m))(xn)$; so $\delta((xm) \otimes n) = \delta(m \otimes (xn))$. Hence, $\delta$ induces a $Z$-linear map $\beta(\eta): M \otimes_R N \to P$ owing to \text{Ex} with $Z$ for $R$ and with $R'$ for $R'$. Clearly, $\beta(\eta)$ is $R'$-linear as $\eta(m)$ is so. Finally, it is easy to verify that $\alpha(\beta(\eta)) = \eta$ and $\beta(\alpha(\gamma)) = \gamma$, as desired. □

COROLLARY (8.11). — Let $R$ be a ring, and $R'$ an algebra. First, let $M$ be an $R$-module, and $P$ an $R'$-module. Then there are two canonical $R'$-isomorphisms:
\[
(M \otimes_R R') \otimes_{R'} P = M \otimes_R P, \quad \text{(cancellation law)}
\]
\[
\text{Hom}_{R'}(M \otimes_R R', P) = \text{Hom}_R(M, P). \quad \text{(left adjoint)}
\]

Instead, let $M$ be an $R'$-module, and $P$ an $R$-module. Then there is a canonical $R'$-isomorphism:
\[
\text{Hom}_R(M, P) = \text{Hom}_{R'}(M, \text{Hom}_R(P, P)). \quad \text{(right adjoint)}
\]

In other words, $\bullet \otimes_R R'$ is the left adjoint of restriction of scalars from $R'$ to $R$, and $\text{Hom}_R(R', \bullet)$ is its right adjoint.

PROOF: The cancellation law results from the associative and unitary laws; the adjoint isomorphisms, from adjoint associativity, (8.9) and the unitary law. □
By adjoint associativity, $\bullet \otimes_R N$ is the left adjoint of $\text{Hom}_R(N, \bullet)$. Thus the assertion results from (6.12) and (6.10).

**Example (8.14).** Tensor product does not preserve kernels, nor even injections. Indeed, consider the injection $\mu_2: \mathbb{Z} \to \mathbb{Z}$. Tensor it with $N := \mathbb{Z}/(2)$, obtaining $\mu_2: N \to N$. This map is zero, but not injective as $N \neq 0$.

**Exercise (8.15).** — Let $R$ be a ring, $a$ and $b$ ideals, and $M$ a module.

1. Use (8.13) to show that $(R/a) \otimes M = M/aM$.
2. Use (1) to show that $(R/a) \otimes (R/b) = R/(a + b)$.

**Exercise (8.16).** — Show $\mathbb{Z}/(m) \otimes_\mathbb{Z} \mathbb{Z}/(n) = 0$ if $m$ and $n$ are relatively prime.

**Theorem (8.18) (Watts).** — Let $F: ((R\text{-mod}) \to ((R\text{-mod})) be a linear functor. Then there is a natural transformation $\theta(\bullet): \bullet \otimes F(R) \to F(\bullet)$ with $\theta(R) = 1$, and $\theta(\bullet)$ is an isomorphism if and only if $F$ preserves direct sums and cokernels.

**Proof:** As $F$ is a linear functor, there is, by definition, a natural $R$-linear map $\theta(M): \text{Hom}(R, M) \to \text{Hom}(F(R), F(M))$. But $\text{Hom}(R, M) = M$ by (1.1). Set $N := F(R)$. Then, with $P := F(M)$, adjoint associativity yields the desired map $\theta(M) \in \text{Hom}(M, \text{Hom}(N, F(M))) = \text{Hom}(M \otimes N, F(M))$.

Explicitly, $\theta(M)(m \otimes n) = F(\rho)(n)$ where $\rho: R \to M$ is defined by $\rho(1) = m$. Alternatively, this formula can be used to construct $\theta(M)$, as $(m, n) \mapsto F(\rho)(n)$ is clearly bilinear. Either way, it’s not hard to see $\theta(M)$ is natural in $M$ and $\theta(R) = 1$.

If $\theta(\bullet)$ is an isomorphism, then $F$ preserves direct sums and cokernels by (8.13).

To prove the converse, take a presentation $R^{\oplus \Sigma} \xrightarrow{\beta} R^{\oplus \Lambda} \xrightarrow{\alpha} M \to 0$; one exists by (7.21). Applying $\theta$, we get this commutative diagram:

\[
\begin{array}{ccc}
R^{\oplus \Sigma} \otimes N & \to & R^{\oplus \Lambda} \otimes N \\
\theta(R^{\oplus \Sigma}) & \downarrow \theta(R^{\oplus \Lambda}) & \downarrow \theta(M) \\
F(R^{\oplus \Sigma}) & \to & F(R^{\oplus \Lambda}) & \to & F(M) & \to & 0
\end{array}
\] (8.18.1)

By construction, $\theta(R) = 1_N$. If $F$ preserves direct sums, then $\theta(R^{\oplus \Lambda}) = 1_{N^{\oplus \Lambda}}$ and $\theta(R^{\oplus \Sigma}) = 1_{N^{\oplus \Sigma}}$; in fact, given any natural transformation $\theta: T \to U$, let’s show that, if $T$ and $U$ preserve direct sums, then so does $\theta$.

Given a collection of modules $M_\lambda$, each inclusion $i_\lambda: M_\lambda \to \bigoplus M_\lambda$ yields, because of naturality, the following commutative diagram:

\[
\begin{array}{ccc}
T(M_\lambda) & \xrightarrow{T(i_\lambda)} & \bigoplus T(M_\lambda) \\
\theta(M_\lambda) & \downarrow & \theta(\bigoplus M_\lambda) \\
U(M_\lambda) & \xrightarrow{U(i_\lambda)} & \bigoplus U(M_\lambda)
\end{array}
\]

Hence $\theta(\bigoplus M_\lambda)T(i_\lambda) = \bigoplus \theta(M_\lambda)T(i_\lambda)$. But the UMP of direct sum says that, given any $N$, a map $\bigoplus T(M_\lambda) \to N$ is determined by its compositions with the inclusions $T(i_\lambda)$. Thus $\theta(\bigoplus M_\lambda) = \bigoplus \theta(M_\lambda)$, as desired.
Suppose $F$ preserves cokernels. Since $\bullet \otimes N$ does too, the rows of (8.18.11) are exact by (8.22). Therefore, $\theta(M)$ is an isomorphism. \hfill \Box

**Exercise (8.19).** — Let $F: ((R\text{-mod})) \to ((R\text{-mod}))$ be a linear functor. Show that $F$ always preserves finite direct sums. Show that $\theta(M): M \otimes F(R) \to F(M)$ is surjective if $F$ preserves surjections and $M$ is finitely generated, and that $\theta(M)$ is an isomorphism if $F$ preserves cokernels and $M$ is finitely presented.

(8.20) (**Additive functors**). — Let $R$ be a ring, $M$ a module, and form the diagram

$$M \xrightarrow{\delta_M} M \oplus M \xrightarrow{\sigma_M} M$$

where $\delta_M := (1_M, 1_M)$ and $\sigma_M := 1_M + 1_M$.

Let $\alpha, \beta: M \to N$ be two maps of modules. Then

$$\sigma_N(\alpha + \beta)\delta_M = \alpha + \beta,$$

(8.20.1)

because, for any $m \in M$, we have

$$(\sigma_N(\alpha + \beta)\delta_M)(m) = \sigma_N(\alpha + \beta)(m, m) = \sigma_N(\alpha(m), \beta(m)) = \alpha(m) + \beta(m).$$

Let $F: ((R\text{-mod})) \to ((R\text{-mod}))$ be a functor that preserves finite direct sums. Then $F(\alpha + \beta) = F(\alpha) \oplus F(\beta)$. Also, $F(\delta_M) = F(M)$ and $F(\sigma_M) = \sigma_{F(M)}$ as $F(1_M) = 1_{F(M)}$. Hence $F(\alpha + \beta) = F(\alpha) + F(\beta)$ by (8.20.1). Thus $F$ is additive, that is, $\mathbb{Z}$-linear.

Conversely, every additive functor preserves finite direct sums owing to (8.14).

However, not every additive functor is $R$-linear. For example, take $R := \mathbb{C}$. Define $F(M)$ to be $M$, but with the scalar product of $x \in \mathbb{C}$ and $m \in M$ to be $\overline{x}m$ where $\overline{x}$ is the conjugate. Define $F(\alpha)$ to be $\alpha$. Then $F$ is additive, but not linear.

**Lemma (8.21)** (**Equational Criterion for Vanishing**). — Let $R$ be a ring, $M$ and $N$ modules, and $\{n_\lambda\}_{\lambda \in \Lambda}$ a set of generators of $N$. Then any $t \in M \otimes N$ can be written as a finite sum $t = \sum m_\lambda \otimes n_\lambda$ with $m_\lambda \in M$. Further, $t$ is 0 if and only if there are $m_\sigma \in M$ and $x_\lambda \in R$ for $\sigma \in \Sigma$ for some $\Sigma$ such that

$$\sum_\sigma x_\lambda m_\sigma = m_\lambda \quad \text{for all } \lambda \quad \text{and} \quad \sum_\lambda x_\lambda n_\lambda = 0 \quad \text{for all } \sigma.$$

**Proof:** By (8.24), $M \otimes N$ is generated by elements of the form $m \otimes n$ with $m \in M$ and $n \in N$, and if $n = \sum x_\lambda n_\lambda$ with $x_\lambda \in R$, then $m \otimes n = \sum (x_\lambda m) \otimes n_\lambda$. It follows that $t$ can be written as a finite sum $t = \sum m_\lambda \otimes n_\lambda$ with $m_\lambda \in M$. Assume the $m_\sigma$ and the $x_\lambda$ exist. Then

$$\sum m_\lambda \otimes n_\lambda = \sum_\lambda \left( \sum_\sigma x_\lambda m_\sigma \otimes n_\lambda \right) = \sum_\sigma \left( m_\sigma \otimes \sum_\lambda x_\lambda n_\lambda \right) = 0.$$

Conversely, by (8.24), there is a presentation $R_{\oplus \Sigma} \xrightarrow{\beta} R_{\oplus \Lambda} \xrightarrow{\alpha} N \to 0$ with $\alpha(e_\lambda) = n_\lambda$ for all $\lambda$ where $\{e_\lambda\}$ is the standard basis of $R_{\oplus \Lambda}$. Then by (8.13) the following sequence is exact:

$$M \otimes R_{\oplus \Sigma} \xrightarrow{1 \otimes \beta} M \otimes R_{\oplus \Lambda} \xrightarrow{1 \otimes \alpha} M \otimes N \to 0.$$

Further, $(1 \otimes \alpha)(\sum m_\lambda \otimes e_\lambda) = 0$. So the exactness implies there is an element $s \in M \otimes R_{\oplus \Sigma}$ such that $(1 \otimes \beta)(s) \otimes n_\lambda = \sum m_\lambda \otimes n_\lambda$. Let $\{e_\sigma\}$ be the standard basis of $R_{\oplus \Sigma}$, and write $s = \sum m_\sigma \otimes e_\sigma$ with $m_\sigma \in M$. Write $\beta(e_\sigma) = \sum_\lambda x_\lambda e_\lambda$. Then clearly

$$0 = \alpha(\beta(e_\sigma)) = \sum_\lambda x_\lambda n_\lambda,$$

and

$$0 = \sum_\lambda m_\lambda \otimes e_\lambda - \sum_\sigma m_\sigma \otimes (\sum_\lambda x_\lambda m_\sigma) = \sum_\lambda (m_\lambda - \sum_\sigma x_\lambda m_\sigma) \otimes e_\lambda.$$

Since the $e_\lambda$ are independent, $m_\lambda = \sum_\sigma x_\lambda m_\sigma$, as asserted. \hfill \Box
(8.22) (Algebras). — Let $R$ be a ring, $S$ and $T$ algebras with structure maps $\sigma : R \to S$ and $\tau : R \to T$. Set $U := S \otimes_R T$; it is an $R$-module. Now, define $S \times T \times S \times T \to U$ by $(s, t, s', t') \mapsto ss' \otimes tt'$. This map is clearly linear in each factor. So it induces a bilinear map

$$\mu : U \times U \to U \quad \text{with} \quad \mu(s \otimes t, s' \otimes t')(ss' \otimes tt').$$

It is easy to check that $U$ is a ring with $\mu$ as product. In fact, $U$ is an $R$-algebra with structure map $\omega$ given by $\omega(r) := \sigma(r) \otimes 1 = 1 \otimes \tau(r)$, called the tensor product of $S$ and $T$ over $R$.

Define $\iota_S : S \to S \otimes_R T$ by $\iota_S(s) := s \otimes 1$. Clearly $\iota_S$ is an $R$-algebra homomorphism. Define $\iota_T : T \to S \otimes_R T$ similarly. Given an $R$-algebra $V$, define a map

$$\gamma : \text{Hom}_{(R\text{-alg})}(S \otimes_R T, V) \to \text{Hom}_{(R\text{-alg})}(S, V) \times \text{Hom}_{(R\text{-alg})}(T, V).$$

by $\gamma(\psi) := (\psi \iota_S, \psi \iota_T)$. Conversely, given $R$-algebra homomorphisms $\theta : S \to V$ and $\zeta : T \to V$, define $\eta : S \times T \to V$ by $\eta(s, t) := \theta(s) \cdot \zeta(t)$. Then $\eta$ is clearly bilinear, so it defines a linear map $\psi : S \otimes_R T \to V$. It is easy to see that the map $(\theta, \zeta) \mapsto \psi$ is an inverse to $\gamma$. Thus $\gamma$ is bijective.

In other words, $S \otimes_R T$ is the coproduct of $S$ and $T$ in (\((R\text{-alg})\)):

\[
\begin{array}{ccc}
S & \to & V \\
\sigma & \downarrow & \theta \\
R & \rightarrow & S \otimes_R T \\
\tau & \downarrow & \iota_T \\
T & \rightarrow & \leftarrow & \zeta \\
\end{array}
\]

Example (8.23). — Let $R$ be a ring, $S$ an algebra, and $X_1, \ldots, X_n$ variables. Then there is a canonical $S$-algebra isomorphism

$$S \otimes_R R[X_1, \ldots, X_n] = S[X_1, \ldots, X_n].$$

Indeed, given an $S$-algebra homomorphism $S \to T$ and elements $x_1, \ldots, x_n$ of $T$, there is an $R$-algebra homomorphism $R[X_1, \ldots, X_n] \to T$ by (18.22). So by (8.22), there is a unique $S$-algebra homomorphism $S \otimes_R R[X_1, \ldots, X_n] \to T$. Thus both $S \otimes_R R[X_1, \ldots, X_n] \to T$ and $S[X_1, \ldots, X_n]$ possess the same UMP.

In particular, for variables $Y_1, \ldots, Y_m$, we obtain

$$R[X_1, \ldots, X_n] \otimes_R R[Y_1, \ldots, Y_m] = R[X_1, \ldots, X_n, Y_1, \ldots, Y_m].$$

Exercise (8.24). — Let $R$ be a ring, $M$ a module, $X$ a variable. Let $M[X]$ be the set of polynomials in $X$ with coefficients in $M$, that is, expressions of the form $\sum_{i=0}^m m_iX^i$ with $m_i \in M$. Prove $M \otimes_R R[X] = M[X]$ as $R[X]$-modules.

Exercise (8.25). — Let $R$ be a ring, $(R'_\sigma)_{\sigma \in \Sigma}$ a family of algebras. For each finite subset $J$ of $\Sigma$, let $R'_J$ be the tensor product of the $R'_\sigma$ for $\sigma \in J$. Prove that the assignment $J \mapsto R'_J$ extends to a filtered direct system and that $\varinjlim R'_J$ exists and is the coproduct of the family $(R'_\sigma)_{\sigma \in \Sigma}$.

Exercise (8.26). — Let $X$ be a variable, $\omega$ a complex cubic root of 1, and $\sqrt[3]{2}$ the real cube root of 2. Set $k := \mathbb{Q}(\omega)$ and $K := k[\sqrt[3]{2}]$. Show $K = k[X]/(X^3 - 2)$ and then $K \otimes_k K = K \times K \times K$. 
9. Flatness

A module is called flat if tensor product with it is an exact functor. First, we study exact functors in general. Then we prove various properties of flat modules. Notably, we prove Lazard’s Theorem, which characterizes the flat modules as the filtered direct limits of free modules of finite rank. Lazard’s Theorem yields the Ideal Criterion for Flatness, which characterizes the flat modules as those whose tensor product with any finitely generated ideal is equal to the ordinary product.

Lemma (9.1). — Let $R$ be a ring, $\alpha: M \rightarrow N$ a homomorphism of modules. Then there is a commutative diagram with two short exact sequences involving $N'$

\[
\begin{array}{ccc}
0 & \longrightarrow & M' \longrightarrow M \longrightarrow N' \longrightarrow 0 \\
& & \alpha \downarrow \alpha' \downarrow \alpha'' \\
& & 0 \longrightarrow N' \longrightarrow 0
\end{array}
\]

if and only if $M' = \operatorname{Ker}(\alpha)$ and $N' = \operatorname{Im}(\alpha)$ and $N'' = \operatorname{Coker}(\alpha)$.

Proof: If the equations hold, then the second short sequence is exact owing to the definitions, and the first is exact since $\operatorname{Coim}(\alpha) \rightarrow \operatorname{Im}(\alpha)$ by (4.9).

Conversely, given the commutative diagram with two short exact sequences, $\alpha''$ is injective. So $\operatorname{Ker}(\alpha) = \operatorname{Ker}(\alpha')$. So $M' = \operatorname{Ker}(\alpha)$. So $N' = \operatorname{Coim}(\alpha)$ as $\alpha'$ is surjective. So $N'' = \operatorname{Im}(\alpha)$. Hence $N'' = \operatorname{Coker}(\alpha)$. Thus the equations hold. \qed

(9.2) (Exact Functors). — Let $R$ be a ring, $R'$ an algebra, $F$ a functor from $((R\text{-mod}))$ to $((R'\text{-mod}))$. Assume $F$ is $R$-linear; that is, the associated map

\[\operatorname{Hom}_R(M, N) \rightarrow \operatorname{Hom}_{R'}(FM, FN)\]

(9.2.1)
is $R$-linear. Then, if a map $\alpha: M \rightarrow N$ is 0, so is $F\alpha: FM \rightarrow FN$. But $M = 0$ if and only if $1_M = 0$. Further, $F(1_M) = 1_{FM}$. Thus if $M = 0$, then $FM = 0$.

Call $F$ faithful if (9.2.1) is injective, or equivalently, if $F\alpha = 0$ implies $\alpha = 0$.

Call $F$ exact if it preserves exact sequences. For example, $\operatorname{Hom}(P, \bullet)$ is exact if and only if $P$ is projective by (5.3.10).

Call $F$ left exact if it preserves kernels. When $F$ is contravariant, call $F$ left exact if it takes cokernels to kernels. For example, $\operatorname{Hom}(N, \bullet)$ and $\operatorname{Hom}(\bullet, N)$ are left exact covariant and contravariant functors.

Call $F$ right exact if it preserves cokernels. For example, $M \otimes \bullet$ is right exact.

Proposition (9.3). — Let $R$ be a ring, $R'$ an algebra, $F$ an $R$-linear functor from $((R\text{-mod}))$ to $((R'\text{-mod}))$. Then the following conditions are equivalent:

1. $F$ preserves exact sequences; that is, $F$ is exact.
2. $F$ preserves short exact sequences.
3. $F$ preserves kernels and surjections.
4. $F$ preserves cokernels and surjections.
5. $F$ preserves kernels and images.

Proof: Trivially, (1) implies (2). In view of (5.7.4), clearly (1) yields (3) and (4).

Assume (3). Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence. Since $F$ preserves kernels, $0 \rightarrow FM' \rightarrow FM \rightarrow FM''$ is exact; since $F$ preserves surjections, $FM \rightarrow FM'' \rightarrow 0$ is also exact. Thus (2) holds. Similarly, (4) implies (2).
Assume (2). Given $\alpha: M \to N$, form the diagram (9.10). Applying $F$ to it and using (2), we obtain a similar diagram for $F(\alpha)$. Hence (9.11) yields (5).

Finally, assume (5). Let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be exact; that is, $\text{Ker}(\beta) = \text{Im}(\alpha)$. Now, (5) yields $\text{Ker}(F(\beta)) = F(\text{Ker}(\beta))$ and $\text{Im}(F(\alpha)) = F(\text{Im}(\alpha))$. Therefore, $\text{Ker}(F(\beta)) = \text{Im}(F(\alpha))$. Thus (1) holds. \hfill \Box

**Exercise (11.3).** — Let $R$ be a ring, $R'$ an algebra, $F$ an $R$-linear functor from $((R\text{-mod}))$ to $((R'\text{-mod}))$. Assume $F$ is exact. Prove the following equivalent:

1. $F$ is faithful.
3. $F(R/m) \neq 0$ for every maximal ideal $m$ of $R$.
4. A sequence $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ is exact if $FM' \xrightarrow{F\alpha} FM \xrightarrow{F\beta} FM''$ is.

**Example (9.9).** — In (9.8), consider the second assertion. Its converse needn’t hold. For example, take a product ring $R := R_1 \times R_2$ with $R_1 \neq 0$. By (9.7), $R$ is faithfully flat over $R$. But neither $R_1$ is so as $R_1 \otimes R_2 = R_1 \otimes (R/R_1) = R_1/R_1^2 = 0$.

**Exercise (11.10).** — Let $R$ be a ring, $M$ and $N$ flat modules. Show that $M \otimes_R N$ is flat. What if “flat” is replaced everywhere by “faithfully flat”?

**Exercise (11.11).** — Let $R$ be a ring, $M$ a flat module, $R'$ an algebra. Show that $M \otimes_R R'$ is flat over $R'$. What if “flat” is replaced everywhere by “faithfully flat”?

**Exercise (11.12).** — Let $R$ be a ring, $R'$ a flat algebra, $M$ a flat $R'$-module. Show that $M$ is flat over $R$. What if “flat” is replaced everywhere by “faithfully flat”?

**Exercise (11.13).** — Show that a ring of polynomials $P$ is faithfully flat.
Exercises (9.13).

1. Let $R$ be a ring, $R'$ an algebra, $R''$ an $R'$-algebra, and $M$ an $R''$-module. Assume that $M$ is flat over $R$ and faithfully flat over $R'$. Prove that $R'$ is flat over $R$.

2. Let $R$ be a ring, $a$ an ideal. Assume $R/a$ is flat. Show $a = a^2$.

3. Let $R$ be a ring, $R'$ a flat algebra. Prove equivalent:
   1. $R'$ is faithfully flat over $R$.
   2. For every $R$-module $M$, the map $M \to M \otimes R'$ by $m \mapsto m \otimes 1$ is injective.
   3. Every ideal $a$ of $R$ is the contraction of its extension, or $a = \varphi^{-1}(aR')$.
   4. Every prime $p$ of $R$ is the contraction of some prime $q$ of $R'$, or $p = \varphi^{-1}q$.
   5. Every maximal ideal $m$ of $R$ extends to a proper ideal, or $mR' \neq R'$.
   6. Every nonzero $R$-module $M$ extends to a nonzero module, or $M \otimes R' \neq 0$.

Proposition (9.16).

Let $R$ be a ring, $0 \to M' \to M \to M'' \to 0$ an exact sequence of modules. Assume $M''$ is flat.

1. Then $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is exact for any module $N$.
2. Then $M$ is flat if and only if $M'$ is flat.

Proof: By (5.20), there is an exact sequence $0 \to K \to R \to N \to 0$. Tensor it with the given sequence to obtain the following commutative diagram:

$$
\begin{array}{c}
0 \\
M' \otimes K \longrightarrow M \otimes K \longrightarrow M'' \otimes K \rightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \alpha \\
0 \longrightarrow M' \otimes R^{\oplus A} \longrightarrow M \otimes R^{\oplus A} \longrightarrow M'' \otimes R^{\oplus A} \\
\downarrow \quad \downarrow \\
M' \otimes N \longrightarrow M \otimes N \\
\downarrow \\
0
\end{array}
$$

Here $\alpha$ and $\beta$ are injective by Definition (15.7), as $M''$ and $R^{\oplus A}$ are flat by hypothesis and by (5.20). So the rows and columns are exact, as tensor product is right exact. Finally, the Snake Lemma, (5.13), implies $\gamma$ is injective. Thus (1) holds.

To prove (2), take an injection $N' \to N$, and form this commutative diagram:

$$
\begin{array}{c}
0 \\
M' \otimes N' \longrightarrow M \otimes N' \longrightarrow M'' \otimes N' \rightarrow 0 \\
\alpha' \quad \quad \alpha \quad \quad \alpha'' \\
0 \longrightarrow M' \otimes N \longrightarrow M \otimes N \longrightarrow M'' \otimes N \rightarrow 0
\end{array}
$$

Its rows are exact by (1).

Assume $M$ is flat. Then $\alpha$ is injective. Hence $\alpha'$ is too. Thus $M'$ is flat.

Conversely, assume $M'$ is flat. Then $\alpha'$ is injective. But $\alpha''$ is injective as $M''$ is flat. Hence $\alpha$ is injective by the Snake lemma. Thus $M$ is flat. Thus (2) holds. □

Exercises (9.17).

1. Let $R$ be a ring, $0 \to M' \to M \to M'' \to 0$ an exact sequence with $M$ flat. Assume $N \otimes M' \overset{N \otimes \alpha}{\to} N \otimes M$ is injective for all $N$. Prove $M''$ is flat.
Exercise (7.18). — Prove that an $R$-algebra $R'$ is faithfully flat if and only if the structure map $\varphi : R \to R'$ is injective and the quotient $R'/\varphi R$ is flat over $R$.

Proposition (9.19). — A filtered direct limit of flat modules $\varprojlim M_\lambda$ is flat.

Proof: Let $\beta : N' \to N$ be injective. Then $M_\lambda \otimes \beta$ is injective for each $\lambda$ since $M_\lambda$ is flat. So $\varprojlim (M_\lambda \otimes \beta)$ is injective by the exactness of filtered direct limits, (7.19). So $(\varprojlim M_\lambda) \otimes \beta$ is injective by (7.19). Thus $\varprojlim M_\lambda$ is flat. \hfill $\square$

Proposition (9.20). — Let $R$ and $R'$ be rings, $M$ an $R$-module, $N$ an $(R, R')$-bimodule, and $P$ an $R'$-module. Then there is a canonical homomorphism

$$\theta : \text{Hom}_R(M, N) \otimes_{R'} P \to \text{Hom}_R(M, N \otimes_{R'} P).$$

(9.20.1)

Assume $P$ is flat. If $M$ is finitely generated, then $\theta$ is injective; if $M$ is finitely presented, then $\theta$ is an isomorphism.

Proof: The map $\theta$ exists by Watts’s Theorem, (5.18), with $R'$ for $R$, applied to $\text{Hom}_R(M, N \otimes_{R'} \bullet)$. Explicitly, $\theta(\varphi \otimes p)(m) = \varphi(m) \otimes p$.

Clearly, $\theta$ is bijective if $M = R$. So $\theta$ is bijective if $M = R^n$ for any $n$, as $\text{Hom}_R(\bullet, Q)$ preserves finite direct sums for any $Q$ by (5.18).

Assume that $M$ is finitely generated. Then from (5.20), we obtain a presentation $R^{\oplus \Sigma} \to R^n \to M \to 0$, with $\Sigma$ finite if $P$ is finitely presented. Since $\theta$ is natural, it yields this commutative diagram:

$$
\begin{array}{ccc}
0 & \to & \text{Hom}_R(M, N) \otimes_{R'} P \\
\downarrow & & \downarrow \\
0 & \to & \text{Hom}_R(M, N \otimes_{R'} P)
\end{array}
$$

Its rows are exact owing to the left exactness of Hom and to the flatness of $P$. The right-hand vertical map is bijective if $\Sigma$ is finite. The assertion follows. \hfill $\square$

Exercise (7.19). — Let $R$ be a ring, $R'$ an algebra, $M$ and $N$ modules. Show that there is a canonical map

$$\sigma : \text{Hom}_R(M, N) \otimes_R R' \to \text{Hom}_R(M \otimes_R R', N \otimes_R R').$$

Assume $R'$ is flat over $R$. Show that if $M$ is finitely generated, then $\sigma$ is injective, and that if $M$ is finitely presented, then $\sigma$ is an isomorphism.

Definition (9.22). — Let $R$ be a ring, $M$ a module. Let $\Lambda_M$ be the category whose objects are the pairs $(R^m, \alpha)$, where $\alpha : R^m \to M$ is a homomorphism, and whose maps $(R^m, \alpha) \to (R^n, \beta)$ are the homomorphisms $\varphi : R^m \to R^n$ with $\beta \varphi = \alpha$.

Proposition (9.23). — Let $R$ be a ring, $M$ a module, and $(R^m, \alpha) \to R^m$ the forgetful functor from $\Lambda_M$ to $((R\text{-mod})$. Then $M = \varprojlim_{(R^m, \alpha) \in \Lambda_M} R^m$.

Proof: By the UMP, the $\alpha : R^m \to M$ induce a map $\zeta : \varprojlim R^m \to M$. Let’s show $\zeta$ is bijective.

First, $\zeta$ is surjective, because each $x \in M$ is in the image of $(R, \alpha_x)$ where $\alpha_x(r) := r x$. For injectivity, let $y \in \text{Ker}(\zeta)$. By construction, $\bigoplus_{(R^m, \alpha)} R^m \to \varprojlim R^m$ is surjective; see the proof of (7.14). So $y$ is in the image of some finite sum $\bigoplus_{(R^m, \alpha)} R^m$. Set $m := \sum m_i$. Then $\bigoplus R^{m_i} = R^m$. Set $\alpha := \sum \alpha_i$. Then $y$ is the image of some $y' \in R^m$ under the insertion $t_m : R^m \to \varprojlim R^m$. But $y \in \text{Ker}(\zeta)$. So $\alpha(y') = 0$.

Let $\theta, \varphi : R \to R^m$ be the homomorphisms with $\theta(1) := y'$ and $\varphi(1) := 0$. They
yield maps in $\Lambda_M$. So, by definition of direct limit, they have the same compositions with the insertion $t_m$. Hence $y = t_m(y') = 0$. Thus $\zeta$ is injective, so bijective. □

**Theorem (9.24) (Lazard).** — Let $R$ be a ring, $M$ a module. Then the following conditions are equivalent:

1. $M$ is flat.
2. Given a finitely presented module $P$, this version of (9.24(1)) is surjective:
   \[ \text{Hom}_R(P, R) \otimes_R M \to \text{Hom}_R(P, M). \]
3. Given a finitely presented module $P$ and a map $\beta: P \to M$, there exists a factorization $\beta: P \xrightarrow{\phi} R^n \xrightarrow{\alpha} M$;
4. Given an $\alpha: R^n \to M$ and a $k \in \text{Ker}(\alpha)$, there exists a factorization $\alpha: R^n \xrightarrow{\phi} R^n \to M$ such that $\phi(k) = 0$.
5. Given an $\alpha: R^n \to M$ and $k_1, \ldots, k_r \in \text{Ker}(\alpha)$ there exists a factorization $\alpha: R^n \xrightarrow{\phi} R^n \to M$ such that $\phi(k_i) = 0$ for $i = 1, \ldots, r$.
6. Given $R^n \xrightarrow{\alpha} R^n \to M$ such that $\alpha \rho = 0$, there exists a factorization $\alpha: R^n \xrightarrow{\phi} R^n \to M$ such that $\phi \rho = 0$.
7. $\Lambda_M$ is filtered.
8. $M$ is a filtered direct limit of free modules of finite rank.

**Proof:** Assume (1). Then (9.24(1)) yields (2).

Assume (2). Consider (3). There are $\gamma_1, \ldots, \gamma_n \in \text{Hom}(P, R)$ and $x_1, \ldots, x_n \in M$ with $\beta(p) = \sum \gamma_i(p)x_i$ by (2). Let $\gamma: P \to R^n$ be $\gamma_1, \ldots, \gamma_n$, and let $\alpha: R^n \to M$ be given by $\alpha(r_1, \ldots, r_n) = \sum r_i x_i$. Then $\beta = \alpha \gamma$, just as (3) requires.

Assume (3), and consider (4). Set $P := R^n/\mathbf{r}$, and let $\kappa: R^n \to P$ denote the quotient map. Then $P$ is finitely presented, and there is $\beta: P \to M$ such that $\beta \kappa = \alpha$. By (3), there is a factorization $\beta: P \xrightarrow{\phi} R^n \to M$. Set $\phi := \gamma \kappa$. Then $\beta: R^n \xrightarrow{\phi} R^n \to M$ is a factorization of $\beta$ and $\phi(k) = 0$.

Assume (4), and consider (5). Set $m_0 := m$ and $\alpha_0 = \alpha$. Inductively, (4) yields

\[ \alpha_{i-1}: R^{m_{i-1}} \xrightarrow{\phi_{i-1}} R^{m_i} \xrightarrow{\alpha_i} M \quad \text{for} \quad i = 1, \ldots, r \]

such that $\phi_{i} \cdots \phi_1(k_i) = 0$. Set $\varphi := \phi_r \cdots \phi_1$ and $n := \text{m}_r$. Then (5) holds.

Assume (5), and consider (6). Let $e_1, \ldots, e_r$ be the standard basis of $R^r$, and set $k_i := \rho(e_i)$. Then $\alpha(k_i) = 0$. So (5) yields a factorization $\alpha: R^m \xrightarrow{\phi} R^n \to M$ such that $\phi(k_i) = 0$. Then $\phi \rho = 0$, as required by (6).

Assume (6). Given $(R^{m_1}, \alpha_1)$ and $(R^{m_2}, \alpha_2)$ in $\Lambda_M$, set $m := m_1 + m_2$ and $\alpha := \alpha_1 + \alpha_2$. Then the inclusions $R^{m_i} \to R^m$ induce maps in $\Lambda_M$. Thus the first condition of (6.1) is satisfied.

Given $\sigma, \tau: (R^r, \omega) \to (R^m, \alpha)$ in $\Lambda_M$, set $\rho := \sigma - \tau$. Then $\alpha \rho = 0$. So (6) yields a factorization $\alpha: R^m \xrightarrow{\phi} R^n \to M$ with $\phi \rho = 0$. Then $\phi$ is a map of $\Lambda_M$, and $\phi \sigma = \phi \tau$. Hence the second condition of (6.1) is satisfied. Thus (7) holds.

If (7) holds, then (8) does too, since $M = \lim_{\to}(R^{m, \alpha})_{\in \Lambda_M} R^m$ by (9.24).

Assume (8). Say $M = \lim_{\to} M_{\lambda}$ with the $M_{\lambda}$ free. Each $M_{\lambda}$ is flat by (5.25), and a filtered direct limit of flat modules is flat by (5.24). Thus $M$ is flat. □

**Exercise (5.25)** (Equational Criterion for Flatness). — Prove that the Condition (9.23)(4) can be reformulated as follows: Given any relation $\sum x_i y_i = 0$ with
Flatness (9.28) 59

$x_i \in R$ and $y_i \in M$, there are $x_{ij} \in R$ and $y'_j \in M$ such that

$$\sum_j x_{ij}y'_j = y_i \text{ for all } i \text{ and } \sum_i x_{ij}x_i = 0 \text{ for all } j.$$ \hspace{1cm} (9.25.1)

**Lemma (9.26) (Ideal Criterion for Flatness).** — A module $N$ is flat if and only if, given any finitely generated ideal $a$, the inclusion $a \hookrightarrow R$ induces an isomorphism:

$$a \otimes N \cong aN.$$

**Proof:** In any case, (8.16)(2) implies $R \otimes N \twoheadrightarrow N$ with $a \otimes x \mapsto ax$. If $N$ is flat, then the inclusion $a \hookrightarrow R$ yields an injection $a \otimes N \hookrightarrow R \otimes N$, and so $a \otimes N \twoheadrightarrow aN$. To prove the converse, let’s check the criterion (9.25.1). Given $\sum_{i=1}^n x_iy_i = 0$ with $x_i \in R$ and $y_i \in N$, set $a := \langle x_1, \ldots, x_n \rangle$. If $a \otimes N \twoheadrightarrow aN$, then $\sum_i x_i \otimes y_i = 0$; so the Equational Criterion for Vanishing (8.21) yields (9.25.1). Thus $N$ is flat. \hfill \Box

**Example (9.27).** — Let $R$ be a domain, and set $K := \text{Frac}(R)$. Then $K$ is flat, but $K$ is not projective unless $R = K$. Indeed, (8.7) says $a \otimes_R K = K$, with $a \otimes x = ax$, for any ideal $a$ of $R$. So $K$ is flat by (10.26).

Suppose $K$ is projective. Then $K \hookrightarrow R^\Lambda$ for some $\Lambda$ by (9.26). So there is a nonzero map $\alpha: K \rightarrow R$. So there is an $x \in K$ with $\alpha(x) \neq 0$. Set $a := \alpha(x)$. Take any nonzero $b \in R$. Then $ab \cdot \alpha(x/ab) = \alpha(x) = a$. Since $R$ is a domain, $b \cdot \alpha(x/ab) = 1$. Hence $b \in R^\times$. Thus $R$ is a field. So (2.3) yields $R = K$.

**Exercise (9.28).** — Let $R$ be a ring, $M$ a module. Prove (1) if $M$ is flat, then for $x \in R$ and $m \in M$ with $xm = 0$, necessarily $m \in \text{Ann}(x)M$, and (2) the converse holds if $R$ is a **Principal Ideal Ring** (PIR); that is, every ideal $a$ is principal.
10. Cayley–Hamilton Theorem

The Cayley–Hamilton Theorem says that a matrix satisfies its own characteristic polynomial. We prove it via a useful equivalent form, known as the “Determinant Trick.” Using the Trick, we obtain various results, including the uniqueness of the rank of a finitely generated free module. We also obtain Nakayama’s Lemma, and use it to study finitely generated modules further. Then we turn to the important notions of integral dependence and module finiteness for an algebra. Using the Trick, we relate these notions to each other, and study their properties. We end with a discussion of integral extensions and normal rings.

(10.1) (Cayley–Hamilton Theorem). — Let \( R \) be a ring, and \( M := (a_{ij}) \) an \( n \times n \) matrix with \( a_{ij} \in R \). Let \( I_n \) be the \( n \times n \) identity matrix, and \( T \) a variable. The characteristic polynomial of \( M \) is the following polynomial:

\[
p_M(T) := T^n + a_1 T^{n-1} + \cdots + a_n := \det(TI_n - M).
\]

Let \( \mathfrak{a} \) be an ideal. If \( a_{ij} \in \mathfrak{a} \) for all \( i, j \), then clearly \( a_k \in \mathfrak{a}^k \) for all \( k \).

The Cayley–Hamilton Theorem asserts that, in the ring of matrices,

\[
p_M(M) = 0.
\]

It is a special case of (10.2) below; indeed, take \( M : R^n \), take \( m_1, \ldots, m_n \) to be the standard basis, and take \( \varphi \) to be the endomorphism defined by \( M \).

Conversely, given the setup of (10.2), form the surjection \( \alpha : \mathbb{Z}^n \rightarrow M \) taking the \( i \)th standard basis element \( e_i \) to \( m_i \), and form the map \( \Phi : \mathbb{Z}^n \rightarrow \mathbb{Z}^n \) associated to the matrix \( M \). Then \( \varphi \alpha = \alpha \Phi \). Hence, given any polynomial \( p(T) \), we have \( p(\varphi)\alpha = \alpha p(\Phi) \). Hence, if \( p(\Phi) = 0 \), then \( p(\varphi) = 0 \) as \( \alpha \) is surjective. Thus the Cayley–Hamilton Theorem and the Determinant Trick (10.2) are equivalent.

Theorem (10.2) (Determinant Trick). — Let \( M \) be an \( R \)-module generated by \( m_1, \ldots, m_n \), and \( \varphi : M \rightarrow M \) an endomorphism. Say \( \varphi(m_i) = \sum_{j=1}^{n} a_{ij} m_j \) with \( a_{ij} \in R \), and form the matrix \( M := (a_{ij}) \). Then \( p_M(\varphi) = 0 \) in \( \text{End}(M) \).

Proof: Let \( \delta_{ij} \) be the Kronecker delta function, \( \mu_{a_{ij}} \) the multiplication map. Let \( \Delta \) stand for the matrix \( (\delta_{ij} \varphi - \mu_{a_{ij}}) \) with entries in the commutative subring \( R[\varphi] \) of \( \text{End}(M) \), and \( X \) for the column vector \( (m_j) \). Clearly \( \Delta X = 0 \). Multiply on the left by the matrix of cofactors \( \Gamma \) of \( \Delta \): the \( (i, j) \)th entry of \( \Gamma \) is \( (-1)^{i+j} \) times the determinant of the matrix obtained by deleting the \( j \)th row and the \( i \)th column of \( \Delta \). Then \( \Gamma \Delta X = 0 \). But \( \Gamma \Delta = \det(\Delta)I_n \). So \( \det(\Delta) m_j = 0 \) for all \( j \). Hence \( \det(\Delta) = 0 \). But \( \det(\Delta) = p_M(\varphi) \). Thus \( p_M(\varphi) = 0 \).

Proposition (10.3). — Let \( M \) be a finitely generated module, \( \mathfrak{a} \) an ideal. Then \( M = aM \) if and only if there exists \( a \in \mathfrak{a} \) such that \( (1 + a)M = 0 \).

Proof: Assume \( M = aM \). Say \( m_1, \ldots, m_n \) generate \( M \), and \( m_i = \sum_{j=1}^{n} a_{ij} m_j \) with \( a_{ij} \in \mathfrak{a} \). Set \( M := (a_{ij}) \). Say \( p_M(T) = T^n + a_1 T^{n-1} + \cdots + a_n \). Set \( a := a_1 + \cdots + a_n \in \mathfrak{a} \). Then \( (1 + a)M = 0 \) by (10.2) with \( \varphi := 1_M \).

Conversely, if there exists \( a \leq \mathfrak{a} \) such that \( (1 + a)M = 0 \), then \( m = -am \) for all \( m \in M \). So \( M \subset aM \subset M \). Thus \( M = aM \). □

60
Corollary (10.4). — Let $R$ be a ring, $M$ a finitely generated module, and $\varphi$ an endomorphism of $M$. If $\varphi$ is surjective, then $\varphi$ is an isomorphism.

Proof: Let $P := R[X]$ be the polynomial ring in one variable. By the UMP of $P$, there is an $R$-algebra homomorphism $\mu : P \to \text{End}(M)$ with $\mu(X) = \varphi$. So $M$ is a $P$-module such that $p(X)M = p(\varphi)M$ for any $p(X) \in P$ by (10.3). Set $a := \langle X \rangle$.

Since $\varphi$ is surjective, $M = aM$. By (10.3), there is $a \in a$ with $(1 + a)M = 0$. Say $a = Xq(X)$ for some polynomial $q(X)$. Then $(1 + \varphi q)(\varphi) = 0$. So the $\varphi$ is an isomorphism.

Corollary (10.5). — Let $R$ be a nonzero ring, $m$ and $n$ positive integers.

(1) Then any $n$ generators $v_1, \ldots, v_n$ of the free module $R^n$ form a free basis.

(2) If $R^n \simeq R^m$, then $m = n$.

Proof: Form the surjection $\varphi : R^n \to R^n$ taking the $i$th standard basis element to $v_i$. Then $\varphi$ is an isomorphism by (10.4). So the $v_i$ form a free basis by (10.11).

To prove (2), say $m \leq n$. Then $R^n$ has $m$ generators. Add to them $n - m$ zeros. Thus $n - m = 0$.

Exercise (10.6). — Let $R$ be a nonzero ring, $\alpha : R^m \to R^n$ a map of free modules. Assume $\alpha$ is surjective. Show that $m \geq n$.

Exercise (10.7). — Let $R$ be a ring, $a$ an ideal. Assume $a$ is finitely generated and idempotent (or $a = a^2$). Prove there is a unique idempotent $e$ with $\langle e \rangle = a$.

Exercise (10.8). — Let $R$ be a ring, $a$ an ideal. Prove the following conditions are equivalent:

1. $R/a$ is projective over $R$.
2. $R/a$ is flat over $R$, and $a$ is finitely generated.
3. $a$ is finitely generated and idempotent.
4. $a$ is generated by an idempotent.
5. $a$ is a direct summand of $R$.

Exercise (10.9). — Prove the following conditions on a ring $R$ are equivalent:

1. $R$ is absolutely flat; that is, every module is flat.
2. Every finitely generated ideal is a direct summand of $R$.
3. Every finitely generated ideal is idempotent.
4. Every principal ideal is idempotent.

Exercise (10.10). — Let $R$ be a ring.

1. Assume $R$ is Boolean. Prove $R$ is absolutely flat.
2. Assume $R$ is absolutely flat. Prove any quotient ring $R'$ is absolutely flat.
3. Assume $R$ is absolutely flat. Prove every nonunit $x$ is a zerodivisor.
4. Assume $R$ is absolutely flat and local. Prove $R$ is a field.

Lemma (10.11) (Nakayama). — Let $R$ be a ring, $m \subseteq \text{rad}(R)$ an ideal, $M$ a finitely generated module. Assume $M = mM$. Then $M = 0$.

Proof: By (10.2), there is $a \in m$ with $(1 + a)M = 0$. By (10.2), $1 + a$ is a unit. Thus $M = (1 + a)^{-1}(1 + a)M = 0$.

Alternatively, suppose $M \neq 0$. Say $m_1, \ldots, m_n$ generate $M$ with $n$ minimal. Then $n \geq 1$ and $m_1 = a_1 m_1 + \cdots + a_n m_n$ with $a_i \in m$. By (10.2), we may set $x_i := (1 - a_i)^{-1}a_i$. Then $m_1 = x_2 m_2 + \cdots + x_n m_n$, contradicting minimality of $n$. Thus $n = 0$ and so $M = 0$. □
Example (10.12). — Nakayama’s Lemma (10.14) may fail if the module is not finitely generated. For example, let $A$ be a local domain, $\mathfrak{m}$ the maximal ideal, and $K$ the fraction field. Assume $A$ is not a field, so that there’s a nonzero $x \in \mathfrak{m}$. Then any $z \in K$ can be written in the form $z = x(z/x)$. Thus $K = \mathfrak{m}K$, but $K \neq 0$.

Proposition (10.13). — Let $R$ be a ring, $\mathfrak{m} \subset \text{rad}(R)$ an ideal, $N \subset M$ modules.

1. If $M/N$ is finitely generated and if $N + \mathfrak{m}M = M$, then $N = M$.

2. Assume $M$ is finitely generated. Then elements $m_1, \ldots, m_n$ generate $M$ if and only if their images $m'_1, \ldots, m'_n$ generate $M' := M/\mathfrak{m}M$.

Proof: In (1), the second hypothesis holds if and only if $\text{m}(M/N) = M/N$. Hence (1) holds by (10.14) applied with $M/N$ for $M$.

In (2), let $N$ be the submodule generated by $m_1, \ldots, m_n$. Since $M$ is finitely generated, so is $M/N$. Hence $N = M$ if the $m'_i$ generate $M/\mathfrak{m}M$ by (1). The converse is obvious. \hfill \Box

Exercise (10.15). — Let $R$ be a ring, $\mathfrak{a}$ an ideal, and $\alpha : M \to N$ a map of modules. Assume that $\mathfrak{a} \subset \text{rad}(R)$, that $N$ is finitely generated, and that the induced map $\pi : M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective. Show that $\alpha$ is surjective.

Exercise (10.16). — Let $R$ be a ring, $\mathfrak{m} \subset \text{rad}(R)$ an ideal. Let $\alpha, \beta : M \to N$ be two maps of finitely generated modules. Assume that $\alpha$ is an isomorphism and that $\beta(M) \subset \mathfrak{m}N$. Set $\gamma := \alpha + \beta$. Show that $\gamma$ is an isomorphism.

Exercise (10.17). — Let $A$ be a local ring, $\mathfrak{m}$ the maximal ideal, $M$ a finitely generated $A$-module, and $m_1, \ldots, m_n \in M$. Set $k := A/\mathfrak{m}$ and $M' := M/\mathfrak{m}M$, and write $m'_i$ for the image of $m_i$ in $M'$. Prove that $m'_1, \ldots, m'_n \in M'$ form a basis of the $k$-vector space $M'$ if and only if $m_1, \ldots, m_n$ form a minimal generating set of $M$ (that is, no proper subset generates $M$), and prove that every minimal generating set of $M$ has the same number of elements.

Exercise (10.18). — Let $A$ be a local ring, $k$ its residue field, $M$ and $N$ finitely generated modules. (1) Show that $M = 0$ if and only if $M \otimes_A k = 0$. (2) Show that $M \otimes_A N \neq 0$ if $M \neq 0$ and $N \neq 0$.

(10.18) (Local Homomorphisms). — Let $\varphi : A \to B$ be a map of local rings, $\mathfrak{m}$ and $\mathfrak{n}$ their maximal ideals. Then the following three conditions are equivalent:

1. $\varphi^{-1}\mathfrak{n} = \mathfrak{m}$;
2. $1 \notin \mathfrak{m}\mathfrak{B}$;
3. $\mathfrak{m}\mathfrak{B} \subset \mathfrak{n}$.

Indeed, if (1) holds, then $\mathfrak{m}\mathfrak{B} = (\varphi^{-1}\mathfrak{n})\mathfrak{B} \subset \mathfrak{n}$; so (2) holds. If (2) holds, then $\mathfrak{m}\mathfrak{B}$ lies in some maximal ideal, but $\mathfrak{n}$ is the only one; thus (3) holds. If (3) holds, then $\mathfrak{m} \subset \varphi^{-1}(\mathfrak{m}\mathfrak{B}) \subset \varphi^{-1}\mathfrak{n}$; whence, (1) holds as $\mathfrak{m}$ is maximal.

If the above conditions hold, then we say $\varphi : A \to B$ is a local homomorphism.

Exercise (10.19). — Let $A \to B$ be a local homomorphism, $M$ a finitely generated $B$-module. Prove that $M$ is faithfully flat over $A$ if and only if $M$ is flat over $A$ and nonzero. Conclude that, if $B$ is flat over $A$, then $B$ is faithfully flat over $A$.

Proposition (10.20). — Consider these conditions on an $R$-module $P$:

1. $P$ is free and of finite rank;
2. $P$ is projective and finitely generated;
3. $P$ is flat and finitely presented.

Then (1) implies (2), and (2) implies (3); all three are equivalent if $R$ is local.
Proof: A free module is always projective by (17.24), and a projective module is always flat by (5.7). Further, each of the three conditions requires $P$ to be finitely generated; so assume it is. Thus (1) implies (2).

Let $p_1, \ldots, p_n \in P$ generate, and let $0 \to L \to R^n \to P \to 0$ be the short exact sequence defined by sending the $i$th standard basis element to $p_i$. Set $F := R^n$.

Assume $P$ is projective. Then the sequence splits by (17.23). So (5.3) yields a surjection $\rho: F \to L$. Hence $L$ is finitely generated. Thus (2) implies (3).

Assume $P$ is flat and $R$ is local. Denote the residue field of $R$ by $k$. Then, by (5.16)(1), the sequence $0 \to L \otimes k \to F \otimes k \to P \otimes k \to 0$ is exact. Now, $F \otimes k = (R \otimes k)^n = k^n$ by (5.9), and the unitary law; so $\dim_k F \otimes k = n$. Finally, rechoose the $p_i$ so that $n$ is minimal. Then $\dim_k P \otimes k = n$, because the $p_i \otimes 1$ form a basis by (10.16). Therefore, $\dim_k L \otimes k = 0$; so $L \otimes k = 0$.

Assume $P$ is finitely presented. Then $L$ is finitely generated by (5.24). Hence $L = 0$ by (5.17)(1). So $F = P$. Thus (3) implies (1).

Definition (10.21). — Let $R$ be a ring, $R'$ an $R$-algebra. Then $R'$ is said to be module finite over $R$ if $R'$ is a finitely generated $R$-module.

An element $x \in R'$ is said to be integral over $R$ or integrally dependent on $R$ if there exist a positive integer $n$ and elements $a_i \in R$ such that

$$x^n + a_1x^{n-1} + \cdots + a_n = 0.$$  \hfill (10.21.1)

Such an equation is called an equation of integral dependence of degree $n$.

If every $x \in R'$ is integral over $R$, then $R'$ is said to be integral over $R$.

Exercise (11.22). — Let $G$ be a finite group of automorphisms of a ring $R$. Form the subring $R^G$ of invariants. Show that every $x \in R$ is integral over $R^G$, in fact, over the subring $R'$ generated by the elementary symmetric functions in the conjugates $gx$ for $g \in G$.

Proposition (10.23). — Let $R$ be a ring, $R'$ an $R$-algebra, $n$ a positive integer, and $x \in R'$. Then the following conditions are equivalent:

1. $x$ satisfies an equation of integral dependence of degree $n$;
2. $R[x]$ is generated as an $R$-module by $1, x, \ldots, x^{n-1}$;
3. $x$ lies in a subalgebra $R''$ generated as an $R$-module by $n$ elements;
4. there is a faithful $R[x]$-module $M$ generated over $R$ by $n$ elements.

Proof: Assume (1) holds. Say $p(x)$ is a monic polynomial of degree $n$ with $p(x) = 0$. For any $m$, let $M_m \subset R[x]$ be the $R$-submodule generated by $1, \ldots, x^m$.

For $m \geq n$, clearly $x^m - x^{m-n}p(x)$ is in $M_{m-1}$. But $p(x) = 0$. So also $x^m \in M_{m-1}$. So by induction, $M_m = M_{m-1}$. Hence $M_{n-1} = R[x]$. Thus (2) holds.

If (2) holds, then trivially (3) holds with $R'' := R[x]$.

If (3) holds, then (4) holds with $M := R''$, as $xM = 0$ implies $x = x \cdot 1 = 0$.

Assume (4) holds. In (11.22), take $\varphi := \mu_x$. We obtain a monic polynomial $p$ of degree $n$ with $p(x)M = 0$. Since $M$ is faithful, $p(x) = 0$. Thus (1) holds.

Exercise (11.23). — Let $k$ be a field, $P := k[X]$ the polynomial ring in one variable, $f \in P$. Set $R := k[X^2] \subset P$. Using the free basis $1, X$ of $P$ over $R$, find an explicit equation of integral dependence of degree 2 on $R$ for $f$.

Corollary (10.25). — Let $R$ be a ring, $P = R[X]$ the polynomial ring in one variable, and $a$ an ideal of $P$. Set $R' := P/a$, and let $x$ be the image of $X$ in $R'$. Let $n$ be a positive integer. Then the following conditions are equivalent:
Trivially, (1) implies (2).

Assume (1) holds. Then
\[ b_1 x^{n-1} + \cdots + b_n = 0 \]
with the \( b_i \in R \). Set \( q(X) := b_1 X^{n-1} + \cdots + b_n \). Then \( q(x) = 0 \). So \( q \in \mathfrak{a} \). Hence \( q = fp \) for some \( f \in P \). But \( p \) is monic of degree \( n \). Hence \( q = 0 \). Thus (2) holds.

Trivially, (2) implies (3).

Finally, assume (3) holds. Then (3)\(\Rightarrow\)(1) of (10.23) yields a monic polynomial \( p \in \mathfrak{a} \) of degree \( n \). Form the induced homomorphism \( \psi : P/\langle p \rangle \to R' \). It is obviously surjective. Since (1) implies (3), the quotient \( P/\langle p \rangle \) is free of rank \( n \). So \( \psi \) is an isomorphism by (10.23). Hence \( \langle p \rangle = \mathfrak{a} \). Thus (1) holds. \( \square \)

**Lemma (10.26).** — Let \( R \) be a ring, \( R' \) a module-finite \( R \)-algebra, and \( M \) a finitely generated \( R' \)-module. Then \( M \) is a finitely generated \( R \)-module.

**Proof:** Say elements \( x_i \) generate \( R' \) as a module over \( R \), and say elements \( m_j \) generate \( M \) over \( R' \). Then clearly the products \( x_i m_j \) generate \( M \) over \( R \). \( \square \)

**Theorem (10.27) (Tower Law for Integrality).** — Let \( R \) be a ring, \( R' \) an algebra, and \( R'' \) an \( R' \)-algebra. If \( x \in R'' \) is integral over \( R' \) and if \( R' \) is integral over \( R \), then \( x \) is integral over \( R \).

**Proof:** Say \( x^n + a_1 x^{n-1} + \cdots + a_n = 0 \) with \( a_i \in R' \). For \( m = 1, \ldots, n \), set \( R_m := R[a_1, \ldots, a_m] \subset R'' \). Then \( R_m \) is module finite over \( R_{m-1} \) by (1)\(\Rightarrow\)(2) of (10.23). So \( R_m \) is module finite over \( R \) by (10.23) and induction on \( m \).

Moreover, \( x \) is integral over \( R_n \). So \( R_n[x] \) is module finite over \( R_n \) by (1)\(\Rightarrow\)(2) of (10.23). Hence \( R_n[x] \) is module finite over \( R \) by (10.23). So \( x \) is integral over \( R \) by (3)\(\Rightarrow\)(1) of (10.23), as desired. \( \square \)

**Theorem (10.28).** — Let \( R \) be a ring, \( R' \) an \( R \)-algebra. Then the following conditions are equivalent:

1. \( R' \) is finitely generated as an \( R \)-algebra and is integral over \( R \);
2. \( R' = R[x_1, \ldots, x_n] \) with all \( x_i \) integral over \( R \);
3. \( R' \) is module finite over \( R \).

**Proof:** Trivially, (1) implies (2).

Assume (2) holds. To prove (3), set \( R'' := R[x_1] \subset R' \). Then \( R'' \) is module finite over \( R \) by (1)\(\Rightarrow\)(2) of (10.23). We may assume \( R' \) is module finite over \( R'' \) by induction on \( n \). So (10.23) yields (3).

If (3) holds, then \( R' \) is integral over \( R \) by (3)\(\Rightarrow\)(1) of (10.23); so (1) holds. \( \square \)

**Exercise (10.24).** — Let \( R_1, \ldots, R_n \) be \( R \)-algebras, integral over \( R \). Show that their product \( \prod R_i \) is an integral over \( R \).

**Definition (10.30).** — Let \( R \) be a ring, \( R' \) an algebra. The **integral closure** or **normalization** of \( R \) in \( R' \) is the subset \( \overline{R} \) of elements that are integral over \( R \). If \( R \subset R' \) and \( R = \overline{R} \), then \( R \) is said to be **integral closed** in \( R' \).

If \( R \) is a domain, then its integral closure \( \overline{R} \) in its fraction field \( \text{Frac}(R) \) is called simply its **normalization**, and \( R \) is said to be **normal** if \( R = \overline{R} \).
Let $Z$ which provides us with another description of $Z$.

### Exercise (10.31)

- For $1 \leq i \leq r$, let $R_i$ be a ring, $R'_i$ an extension of $R_i$, and $x_i \in R'_i$. Set $R := \prod R_i$, set $R' := \prod R'_i$, and set $x := (x_1, \ldots, x_r)$. Prove
  - (1) $x$ is integral over $R$ if and only if $x_i$ is integral over $R_i$ for each $i$;
  - (2) $R$ is integrally closed in $R'$ if and only if each $R_i$ is integrally closed in $R'_i$.

### Theorem (10.32)

Let $R$ be a ring, $R'$ an $R$-algebra, $\overline{R}$ the integral closure of $R$ in $R'$. Then $\overline{R}$ is an $R$-algebra, and is integrally closed in $R'$.

**Proof:** Take $a \in R$ and $x, y \in \overline{R}$. Then the ring $R[x, y]$ is integral over $R$ by (2)$\Rightarrow$(1) of (10.27). So $ax$ and $x + y$ and $xy$ are integral over $R$. Thus $\overline{R}$ is an $R$-algebra. Finally, $\overline{R}$ is integrally closed in $R'$ owing to (10.27).

### Theorem (10.33) (Gauss)

A UFD is normal.

**Proof:** Let $R$ be the UFD. Given $x \in \text{Frac}(R)$, say $x = r/s$ with $r, s \in R$ relatively prime. Suppose $x$ satisfies (10.24.1). Then

$$r^n = -(a_1r^{n-1} + \cdots + a_nr^{n-1})s.$$ 

So any prime element dividing $s$ also divides $r$. Hence $s$ is a unit. Thus $x \in R$. $\square$

### Example (10.34)

(1) A polynomial ring in $n$ variables over a field is a UFD, so normal by (10.31).

(2) The ring $R := \mathbb{Z}[(\sqrt{5})]$ is not a UFD, since

$$(1 + \sqrt{5})(1 - \sqrt{5}) = -4 = -2 \cdot 2,$$

and $1 + \sqrt{5}$ and $1 - \sqrt{5}$ and $2$ are irreducible, but not associates. However, set $r := (1 + \sqrt{5})/2$, the “golden ratio.” The ring $\mathbb{Z}[r]$ is known to be a PID; see [4], p. 292. Hence, $\mathbb{Z}[r]$ is a UFD, so normal by (10.31); hence, $\mathbb{Z}[r]$ contains the normalization $\overline{R}$ of $R$. On the other hand, $r^2 - r - 1 = 0$; hence, $\mathbb{Z}[r] \subset \overline{R}$. Thus $\mathbb{Z}[r] = \overline{R}.$

(3) Let $d \in \mathbb{Z}$ be square-free. In the field $K := \mathbb{Q}((\sqrt{d}))$, form $R := \mathbb{Z} + \mathbb{Z}\delta$ where

$$\delta := \begin{cases} (1 + \sqrt{d})/2, & \text{if } d \equiv 1 \pmod{4}; \\ \sqrt{d}, & \text{if not.} \end{cases}$$

Then $R$ is the normalization $\overline{R}$ of $\mathbb{Z}$ in $K$; see [4], pp. 412–3.

(4) Let $k$ be a field, $k[t]$ the polynomial ring in one variable. Set $R := k[t^2, t^3]$. Then $\text{Frac}(R) = k(t)$. Further, $t$ is integral over $R$ as $t$ satisfies $X^2 - t^2 = 0$; hence, $k[t] \subset \overline{R}$. However, $k[t]$ is normal by (1); hence, $k[t] \supset \overline{R}$. Thus $k[t] = \overline{R}$.

Let $k[X, Y]$ be the polynomial ring in two variables, and $\varphi : k[X, Y] \to R$ the $k$-algebra homomorphism defined by $\varphi(X) := t^2$ and $\varphi(Y) := t^3$. Clearly $\varphi$ is surjective. Set $\mathfrak{p} := \text{Ker} \varphi$. Since $R$ is a domain, but not a field, $\mathfrak{p}$ is prime by (10.32), but not maximal by (20.17). Clearly $\mathfrak{p} \supset (Y^2 - X^3)$. Since $Y^2 - X^3$ is irreducible, (20.19) implies that $\mathfrak{p} = (Y^2 - X^3)$. So $k[X, Y]/(Y^2 - X^3) \cong R$, which provides us with another description of $R$.

### Exercise (10.35)

Let $k$ be a field, $X$ and $Y$ variables. Set

$$R := k[X, Y]/(Y^2 - X^2 - X^3),$$

and let $x, y \in R$ be the residues of $X, Y$. Prove that $R$ is a domain, but not a field. Set $t := y/x \in \text{Frac}(R)$. Prove that $k[t]$ is the integral closure of $R$ in $\text{Frac}(R)$. 

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Cayley–Hamilton Theorem (10.35) 65
11. Localization of Rings

Localization generalizes construction of the fraction field of a domain. We localize an arbitrary ring using as denominators the elements of any given multiplicative subset. The result is universal among algebras rendering all these elements units. When the multiplicative subset is the complement of a prime ideal, we obtain a local ring. We relate the ideals in the original ring to those in the localized ring. We finish by localizing algebras and then varying the set of denominators.

(11.1) (Localization). — Let \( R \) be a ring, and \( S \) a multiplicative subset. Define a relation on \( R \times S \) by \((x,s) \sim (y,t)\) if there is \( u \in S \) such that \( xtu = ysu \).

This relation is an equivalence relation. Indeed, it is reflexive as \( 1 \in S \) and is trivially symmetric. As to transitivity, let \((y,t) \sim (z,r)\). Say \( yrv = ztv \) with \( v \in S \).

Then \( xturv = ysurv = ztusu \). Thus \((x,s) \sim (z,t)\).

Denote by \( S^{-1}R \) the set of equivalence classes, and by \( x/s \) the class of \((x,s)\).

Define \( x/s \cdot y/t := xy/st \). This product is well defined. Indeed, say \( y/t = z/r \). Then there is \( v \in S \) such that \( yrv = ztv \). So \( xsyrv = xszrv \). Thus \( xy/st = xz/sr \).

Define \( x/s + y/t := (tx + sy)/(st) \). Then, similarly, this sum is well defined.

It is easy to check that \( S^{-1}R \) is a ring, with \( 0/1 \) for \( 0 \) and \( 1/1 \) for \( 1 \). It is called the ring of fractions with respect to \( S \) or the localization at \( S \).

Let \( \varphi_S: R \to S^{-1}R \) be the map given by \( \varphi_S(x) := x/1 \). Then \( \varphi_S \) is a ring map, and it carries elements of \( S \) to units in \( S^{-1}R \) as \( s/1 \cdot 1/s = 1 \).

Exercise (11.2). — Let \( R \) be a ring, \( S \) a multiplicative subset. Prove \( S^{-1}R = 0 \) if and only if \( S \) contains a nilpotent element.

(11.3) (Total quotient ring). — Let \( R \) be a ring, and \( S_0 \) the set of nonzerodivisors. Then \( S_0 \) is a saturated multiplicative subset, as noted in (11.1). The map \( \varphi_{S_0}: R \to S_0^{-1}R \) is injective, because if \( \varphi_{S_0}x = 0 \), then \( sx = 0 \) for some \( s \in S \), and so \( x = 0 \). We call \( S_0^{-1}R \) the total quotient ring of \( R \), and view \( R \) as a subring.

Let \( S \subseteq S_0 \) be a multiplicative subset. Clearly, \( R \subseteq S^{-1}R \subseteq S_0^{-1}R \).

Suppose \( R \) is a domain. Then \( S_0 = R \setminus \{0\} \); so the total quotient ring is just the fraction field \( \text{Frac}(R) \), and \( \varphi_{S_0} \) is just the natural inclusion of \( R \) into \( \text{Frac}(R) \).

Further, \( S^{-1}R \) is a domain by (10.3) as \( S^{-1}R \subseteq S_0^{-1}R = \text{Frac}(R) \).

Exercise (11.4). — Find all intermediate rings \( Z \subset R \subset \mathbb{Q} \), and describe each \( R \) as a localization of \( Z \). As a starter, prove \( \mathbb{Z}[2/3] = S^{-1}\mathbb{Z} \) where \( S = \{3^i \mid i \geq 0\} \).

Theorem (11.5) (UMP). — Let \( R \) be a ring, \( S \) a multiplicative subset. Then \( S^{-1}R \) is the \( R \)-algebra universal among algebras rendering all the \( s \in S \) units. In fact, given a ring map \( \psi: R \to R' \), then \( \psi(S) \subset R'^\times \) if and only if there is a ring map \( \rho: S^{-1}R \to R' \) with \( \rho \varphi_S = \psi \); that is, this diagram commutes:

\[
\begin{array}{ccc}
R & \xrightarrow{\varphi_S} & S^{-1}R \\
\downarrow{\psi} & & \downarrow{\rho} \\
R' & & \\
\end{array}
\]

Further, there is at most one \( \rho \). Moreover, \( R' \) may be noncommutative.
Localization of Rings (11.13) 67

Proof: First, suppose that ρ exists. Let s ∈ S. Then ψ(s) = ρ(s/1). Hence ψ(s)/ρ(1/s) = ρ(s/1 · 1/s) = 1. Thus ψ(S) ⊂ R×.

Next, note that ρ is determined by ψ as follows:

$$ρ(x/s) = ρ(x/1)ρ(1/s) = ψ(x)ψ(s)^{-1}.$$

Conversely, suppose ψ(S) ⊂ R×. Set ρ(x/s) := ψ(s)^{-1}ψ(x). Let’s check that ρ is well defined. Say x/s = y/t. Then there is u ∈ S such that xtu = ysu. Hence

$$ψ(x)ψ(t)ψ(u) = ψ(y)ψ(s)ψ(u).$$

Since ψ(u) is a unit, ψ(x)ψ(t) = ψ(y)ψ(s). Now, st = ts, so

$$ψ(t)^{-1}ψ(s)^{-1} = ψ(s)^{-1}ψ(t)^{-1}.$$

Hence ψ(x)ψ(s)^{-1} = ψ(y)ψ(t)^{-1}. Thus ρ is well defined. Clearly, ρ is a ring map. Clearly, ψ = ρφ.

Corollary (11.6). — Let R be a ring, and S a multiplicative subset. Then the canonical map φS: R → S^{-1}R is an isomorphism if and only if S consists of units.

Proof: If φS is an isomorphism, then S consists of units, because φS(S) does so. Conversely, if S consists of units, then the identity map R → R has the UMP that characterizes φS; whence, φS is an isomorphism.

Exercise (11.4). — Let R’ and R” be rings. Consider R := R’ × R” and set S := { (1, 1), (1, 0) }. Prove R’ = S^{-1}R.

Exercise (11.5). — Take R and S as in (11.7). On R × S, impose this relation:

$$(x, s) ∼ (y, t) \text{ if } xt = ys.$$ Show that it is not an equivalence relation.

Exercise (11.6). — Let R be a ring, S ⊂ T a multiplicative subsets, S and T their saturations; see (11.2). Set U := (S^{-1}R)^×. Show the following:

1. U = \{ x/s | x ∈ S and s ∈ S \}. (2) φ^{-1}U = S.
2. S^{-1}R = T^{-1}R if and only if S = T.
3. S^{-1}R = S^{-1}R.

Exercise (11.7). — Let R be a ring, S ⊂ T ⊂ U and W multiplicative subsets.

1. Show there’s a unique R-algebra map φT: S^{-1}R → T^{-1}R and φTφS ≡ φW. (2) Given a map φ: S^{-1}R → W^{-1}R, show S ⊂ S ⊂ W and φ = φSφW.
3. Let A be a set, Sλ ⊂ S a multiplicative subset for all λ ∈ A. Assume ∪Sλ = S. Assume given λ, μ ∈ A, there is ν such that Sλ, Sμ ⊂ Sν. Order A by inclusion: λ ≤ μ if Sλ ⊂ Sμ. Using (1), show lim Sλ^{-1}R = S^{-1}R.

Exercise (11.8). — Let R be a ring, S₀ the set of nonzerodivisors.

1. Show S₀ is the largest multiplicative subset S with φS: R → S^{-1}R injective.
2. Show every element x/s of S₀^{-1}R is either a zerodivisor or a unit.
3. Suppose every element of R is either a zerodivisor or a unit. Show R = S₀^{-1}R.

Definition (11.9). — Let R be a ring, f ∈ R. Set S := \{ fn | n ≥ 0 \}. We call the ring S^{-1}R the localization of R at f, and set R_f := S^{-1}R and φ_f := φ_S.

Proposition (11.10). — Let R be a ring, f ∈ R, and X a variable. Then

$$R_f = R[X]/(1 - fX).$$
Let $R' := R[X]/(1-fX)$, and let $\varphi: R \to R'$ be the canonical map. Let’s show that $R'$ has the UMP characterizing localization (11.5).

First, let $x \in R'$ be the residue of $X$. Then $1-x\varphi(f) = 0$. So $\varphi(f)$ is a unit. So $\varphi(f^n)$ is a unit for $n \geq 0$.

Second, let $\psi: R \to R''$ be a homomorphism carrying $f$ to a unit. Define $\theta: R[X] \to R''$ by $\theta R = \psi$ and $\theta X = \psi(f)^{-1}$. Then $\theta(1-fX) = 0$. So $\theta$ factors via a homomorphism $\rho: R' \to R''$, and $\psi = \rho \varphi$. Further, $\rho$ is unique, since every element of $R'$ is a polynomial in $x$ and since $\rho x = \psi(f)^{-1}$ as $1 - (\rho x)(\rho f) = 0$. □

**Proposition (11.14).** — Let $R$ be a ring, $S$ a multiplicative subset, $a$ an ideal.

1. Then $aS^{-1}R = \{a/s \in S^{-1}R \mid a \in a$ and $s \in S\}$.
2. Then $a \cap S \neq \emptyset$ if and only if $aS^{-1}R = S^{-1}R$ if and only if $\varphi_S^{-1}(aS^{-1}R) = R$.

**Proof:** Let $a, b \in a$ and $x/s, y/t \in S^{-1}R$. Then $ax/s + by/t = (axt + bys)/st$; further, $axt + bys \in a$ and $st \in S$. So $aS^{-1}R \subset \{a/s \mid a \in a$ and $s \in S\}$. But the opposite inclusion is trivial. Thus (1) holds.

As to (2), if $a \cap S \ni s$, then $aS^{-1}R \ni s/s = 1$, so $aS^{-1}R = S^{-1}R$; whence, $\varphi_S^{-1}(aS^{-1}R) = R$. Finally, suppose $\varphi_S^{-1}(aS^{-1}R) = R$. Then $aS^{-1}R \ni 1$. So (1) yields $a \in a$ and $s \in S$ such that $a/s = 1$. So there exists a $t \in S$ such that $at = st$. But $at \in a$ and $st \in S$. So $a \cap S \neq \emptyset$. Thus (2) holds. □

**Definition (11.15).** — Let $R$ be a ring, $S$ a multiplicative subset, $a$ a subset of $R$. The **saturation** of $a$ with respect to $S$ is the set denoted by $a^S$ and defined by

$$a^S := \{a \in R \mid \text{there is } s \in S \text{ with } as \in a\}.$$ 

If $a = a^S$, then we say $a$ is **saturated**.

**Proposition (11.16).** — Let $R$ be a ring, $S$ a multiplicative subset, $a$ an ideal.

1. Then $\ker(\varphi_S) = (0)^S$.
2. Then $a \subset a^S$.
3. Then $a^S$ is an ideal.

**Proof:** Clearly, (1) holds, for $a/1 = 0$ if and only if there is $s \in S$ with $as = 0$. Clearly, (2) holds as $1 \in S$. Clearly, (3) holds, for if $as, bt \in a$, then $(a + b)st \in a$, and if $x \in R$, then $xts \in a$. □

**Exercise (11.17).** — Let $R$ be a ring, $S$ a multiplicative subset, $a$ and $b$ ideals. Show (1) if $a \subset b$, then $a^S \subset b^S$; (2) $(a^S)^S = a^S$; and (3) $(a^Sb^S)^S = (ab)^S$.

**Exercise (11.18).** — Let $R$ be a ring, $S$ a multiplicative subset. Prove that

$$\text{nil}(R)(S^{-1}R) = \text{nil}(S^{-1}R).$$

**Proposition (11.19).** — Let $R$ be a ring, $S$ a multiplicative subset.

1. Let $b$ be an ideal of $S^{-1}R$. Then

   (a) $\varphi_S^{-1}b = (\varphi_S^{-1}b)^S$ and (b) $b = (\varphi_S^{-1}b)(S^{-1}R)$.

2. Let $a$ be an ideal of $R$. Then $\varphi_S^{-1}(aS^{-1}R) = a^S$.

3. Let $p$ be a prime ideal of $R$, and assume $p \cap S = \emptyset$. Then

   (a) $p = p^S$ and (b) $pS^{-1}R$ is prime.
To prove (1)(a), take \( a \in R \) and \( s \in S \) with \( as \in \varphi^{-1}_S b \). Then \( as/1 \in b \); so \( a/1 \in b \) because \( 1/s \in S^{-1}R \). Hence \( a \in \varphi^{-1}_S b \). Therefore, \((\varphi^{-1}_S b)^S \subset \varphi^{-1}_S b\). The opposite inclusion holds as \( s \in 1 \). Thus (1)(a) holds.

To prove (1)(b), take \( a/s \in b \). Then \( a/1 \in b \). So \( a \in \varphi^{-1}_S b \). Hence \( a/1 \cdot 1/s \) is in \((\varphi^{-1}_S b)(S^{-1}R)\). Thus \( b \subset (\varphi^{-1}_S b)(S^{-1}R) \). Now, take \( a \in \varphi^{-1}_S b \). Then \( a/1 \in b \). So \( b \supset (\varphi^{-1}_S b)(S^{-1}R) \). Thus (1)(b) holds too.

To prove (2), take \( a \in a^S \). Then there is \( s \in S \) with \( a/s \in a^S \). But \( a/1 = as/1 \cdot 1/s \). Thus \( a \in \varphi^{-1}_S (aS^{-1}R) \supset a^S \). Now, take \( x \in \varphi^{-1}_S (aS^{-1}R) \). Then \( x/1 = a/s \) with \( a \in a \) and \( s \in S \) by (11.14)(1). Hence there is \( t \in S \) such that \( xst = at \in a \). So \( x \in a^S \). Thus \( \varphi^{-1}_S (aS^{-1}R) \subset a^S \). Thus (2) holds.

To prove (3), note \( p \subset p^S \) as \( 1 \in S \). Conversely, if \( sa \in p \) with \( s \in S \subset R - p \), then \( a \in p \) as \( p \) is prime. Thus (a) holds.

As for (b), first note \( pS^{-1}R \neq S^{-1}R \) as \( \varphi^{-1}_S (pS^{-1}R) = p^S = p \) by (2) and (3)(a) and as \( 1 \notin p \). Second, say \( a/s \cdot b/t \in pS^{-1}R \). Then \( ab \in \varphi^{-1}_S (pS^{-1}R) \), and the latter is equal to \( p^S \) by (2), so to \( p \) by (a). Hence \( ab \in p \), so either \( a \in p \) or \( b \in p \). So either \( a/s \in pS^{-1}R \) or \( b/t \in pS^{-1}R \). Thus \( pS^{-1}R \) is prime. Thus (3) holds. 

**Corollary (11.20).** — Let \( R \) be a ring, \( S \) a multiplicative subset.

1. Then \( a \mapsto aS^{-1}R \) is an inclusion-preserving bijection from the set of all ideals \( a \) of \( R \) with \( a = a^S \) to the set of all ideals \( b \) of \( S^{-1}R \). The inverse is \( b \mapsto \varphi^{-1}_S b \).

2. Then \( p \mapsto pS^{-1}R \) is an inclusion-preserving bijection from the set of all primes \( p \) with \( p \cap S = \emptyset \) to the set of all primes \( q \) of \( S^{-1}R \). The inverse is \( q \mapsto \varphi^{-1}_S q \).

**Proof:** In (1), the maps are inverses by (11.14)(1), (2); clearly, they preserve inclusions. Further, (1) implies (2) by (11.14)(3), by (4.2), and by (11.14)(2).

**Definition (11.21).** — Let \( R \) be a ring, \( p \) a prime ideal. Set \( S := R - p \). We call the ring \( S^{-1}R \) the localization of \( R \) at \( p \), and set \( R_p := S^{-1}R \) and \( \varphi_p := \varphi_S \).

**Proposition (11.22).** — Let \( R \) be a ring, \( p \) a prime ideal. Then \( R_p \) is local with maximal ideal \( pR_p \).

**Proof:** Let \( b \) be a proper ideal of \( R_p \). Then \( \varphi^{-1}_p b \subset p \) owing to (11.14)(2). Hence (11.20)(1) yields \( b \subset pR_p \). Thus \( pR_p \) is a maximal ideal, and the only one.

Alternatively, let \( x/s \in R_p \). Suppose \( x/s \) is a unit. Then there is a \( y/t \) with \( xy/st = 1 \). So there is a \( u \notin p \) with \( yu = stu \). But \( stu \notin p \). Hence \( x \notin p \).

Conversely, let \( x \notin p \). Then \( s/x \in R_p \). So \( x/s \) is a unit in \( R_p \) if and only if \( x \notin p \), so if and only if \( x/s \notin pR_p \). Thus by (11.14)(1), the nonunits of \( R_p \) form \( pR_p \), which is an ideal. Hence (11.20) yields the assertion. 

**Localization of Rings (11.23)** — Let \( R \) be a ring, \( S \) a multiplicative subset, \( R' \) an \( R \)-algebra. It is easy to generalize (11.14) as follows. Define a relation on \( R' \times S \) by \( (x, s) \sim (y, t) \) if there is \( u \in S \) with \( xu = ysu \). It is easy to check, as in (11.14), that this relation is an equivalence relation.

Denote by \( S^{-1}R' \) the set of equivalence classes, and by \( x/s \) the class of \( (x, s) \). Clearly, \( S^{-1}R' \) is an \( S^{-1}R \)-algebra with addition and multiplication given by 
\[
x/s + y/t := (xt + ys)/(st) \quad \text{and} \quad x/s \cdot y/t := xy/st.
\]
We call \( S^{-1}R' \) the localization of \( R' \) with respect to \( S \).

Let \( \varphi_S : R' \to S^{-1}R' \) be the map given by \( \varphi_S(x) := x/1 \). Then \( \varphi_S \) makes \( S^{-1}R' \
into an $R'$-algebra, so also into an $R$-algebra, and $\varphi'_S$ is an $R$-algebra map.

Note that elements of $S$ become units in $S^{-1}R'$. Moreover, it is easy to check, as in (11.23), that $S^{-1}R'$ has the following UMP: $\varphi'_S$ is an algebra map, and elements of $S$ become units in $S^{-1}R'$; further, given an algebra map $\psi: R' \to R''$ such that elements of $S$ become units in $R''$, there is a unique $R$-algebra map $\rho: S^{-1}R' \to R''$ such that $\rho \varphi'_S = \psi$; that is, the following diagram is commutative:

$$
\begin{array}{c}
R' \\
\downarrow \psi \\
R''
\end{array}
\xrightarrow{\varphi'_S}
\begin{array}{c}
S^{-1}R' \\
\downarrow \rho \\
S^{-1}R''
\end{array}
$$

In other words, $S^{-1}R'$ is universal among $R'$-algebras rendering the $s \in S$ units.

Let $\tau: R' \to R''$ be an $R$-algebra map. Then there is a commutative diagram of $R$-algebra maps

$$
\begin{array}{c}
R' \\
\downarrow \varphi_S \\
S^{-1}R'
\end{array}
\xrightarrow{\tau}
\begin{array}{c}
R'' \\
\downarrow \varphi'_S \\
S^{-1}R''
\end{array}
$$

Further, $S^{-1}\tau$ is an $S^{-1}R$-algebra map.

Let $T \subset R'$ be the image of $S \subset R$. Then $T$ is multiplicative. Further,

$$
S^{-1}R' = T^{-1}R',
$$

(11.23.1)
even though $R' \times S$ and $R' \times T$ are rarely equal, because the two UMPs are essentially the same; indeed, any ring map $R' \to R''$ may be viewed as an $R$-algebra map, and trivially the elements of $S$ become units in $R''$ if and only if the elements of $T$ do.

**Exercise (11.24).** — Let $R'/R$ be an integral extension of rings, $S$ a multiplicative subset of $R$. Show that $S^{-1}R'$ is integral over $S^{-1}R$.

**Exercise (11.25).** — Let $R$ be a domain, $K$ its fraction field, $L$ a finite extension field, and $\overline{R}$ the integral closure of $R$ in $L$. Show $L = \text{Frac}(\overline{R})$. Show every element of $L$ can, in fact, be expressed as a fraction $b/a$ with $b \in \overline{R}$ and $a \in R$.

**Exercise (11.26).** — Let $R \subset R'$ be domains, $K$ and $L$ their fraction fields. Assume that $R'$ is a finitely generated $R$-algebra, and that $L$ is a finite dimensional $K$-vector space. Find an $f \in R$ such that $R'_f$ is module finite over $R_f$.

**Proposition (11.27).** — Let $R$ be a ring, $S$ a multiplicative subset. Let $T'$ be a multiplicative subset of $S^{-1}R$, and set $T := \varphi_S^{-1}(T')$. Assume $S \subset T$. Then

$$(T')^{-1}(S^{-1}R) = T^{-1}R.$$

**Proof:** Let’s check $(T')^{-1}(S^{-1}R)$ has the UMP characterizing $T^{-1}R$. Clearly $\varphi_T \varphi_S$ carries $T$ into $((T')^{-1}(S^{-1}R))^\times$. Next, let $\psi: R \to R'$ be a map carrying $T$ into $R'^\times$. We must show $\psi$ factors uniquely through $((T')^{-1}(S^{-1}R))$.

First, $\psi$ carries $S$ into $R'^\times$ since $S \subset T$. So $\psi$ factors through a unique map $\rho: S^{-1}R \to R'$. Now, given $r \in T'$, write $r = x/s$. Then $x/1 = s/1 \cdot r \in T'$ since $S \subset T$. So $x \in T$. Hence $\rho(r) = \psi(x) \cdot \rho(1/s) \in (R')^\times$. So $\rho$ factors through a unique map $\rho': (T')^{-1}(S^{-1}R) \to R'$. Hence $\psi = \rho' \varphi_T \varphi_S$, and $\rho'$ is clearly unique, as required. $\square$
Corollary (11.28). — Let $R$ be a ring, $p \subset q$ prime ideals. Then $R_p$ is the localization of $R_q$ at the prime ideal $pR_q$.

Proof: Set $S := R - q$ and $T' := R_q - pR_q$. Set $T := \varphi_S^{-1}(T')$. Then $T = R - p$ by (11.20)(2). So $S \subset T$, and (11.27) yields the assertion. □

Exercise (11.29). — Let $R$ be a ring, $S$ and $T$ multiplicative subsets.

1. Set $T' := \varphi_S(T)$ and assume $S \subset T$. Prove

$$T^{-1}R = T'^{-1}(S^{-1}R) = T^{-1}(S^{-1}R).$$

2. Set $U := \{st \in R \mid s \in S \text{ and } t \in T\}$. Prove

$$T^{-1}(S^{-1}R) = S^{-1}(T^{-1}R) = U^{-1}R.$$

Proposition (11.30). — Let $R$ be a ring, $S$ a multiplicative subset, $X$ a variable. Then $(S^{-1}R)[X] = S^{-1}(R[X]).$

Proof: In spirit, the proof is like that of (11.7): the two rings are equal, as each is universal among $R$-algebras with a distinguished element and where the $s \in S$ become units. □

Corollary (11.31). — Let $R$ be a ring, $S$ a multiplicative subset, $X$ a variable, $p$ an ideal of $R[X]$. Set $R' := S^{-1}R$, and let $\varphi: R[X] \to R'[X]$ be the canonical map. Then $p$ is prime and $p \cap S = \emptyset$ if and only if $pR'[X]$ is prime and $p\varphi^{-1}(pR'[X])$.

Proof: The assertion results directly from (11.30) and (11.24)(2). □

Exercise (11.32) (Localization and normalization commute). — Given a domain $R$ and a multiplicative subset $S$ with $0 \notin S$. Show that the localization of the normalization $S^{-1}R$ is equal to the normalization of the localization $S^{-1}R$. 


12. Localization of Modules

Formally, we localize a module just as we do a ring. The result is a module over the localized ring, and comes equipped with a linear map from the original module; in fact, the result is universal among modules with those two properties. Consequently, Localization is a functor; in fact, it is the left adjoint of Restriction of Scalars from the localized ring to the base ring. So Localization preserves direct limits, or equivalently, direct sums and cokernels. Further, by uniqueness of left adjoints or by Watts’s Theorem, Localization is naturally isomorphic to Tensor Product with the localized ring. Moreover, Localization is exact; so the localized ring is flat. We end the section by discussing various compatibilities and examples.

Proposition (12.1). — Let $R$ be a ring, $S$ a multiplicative subset. Then a module $M$ has a compatible $S^{-1}R$-module structure if and only if, for all $s \in S$, the multiplication map $\mu_s: M \to M$ is bijective; if so, then the $S^{-1}R$-structure is unique.

Proof: Assume $M$ has a compatible $S^{-1}R$-structure, and take $s \in S$. Then $\mu_s = \mu_s/1$. So $\mu_s \cdot \mu_{1/s} = \mu_{(s/1)(1/s)} = 1$. Similarly, $\mu_{1/s} \cdot \mu_s = 1$. So $\mu_s$ is bijective.

Conversely, assume $\mu_s$ is bijective for all $s \in S$. Then $\mu_R: R \to \text{End}_2(M)$ sends $S$ into the units of $\text{End}_2(M)$. Hence $\mu_R$ factors through a unique ring map $\mu_{S^{-1}R}: S^{-1}R \to \text{End}_2(M)$ by (12.2). Thus $M$ has a unique compatible $S^{-1}R$-structure by (9.3). □

(12.2) (Localization of modules). — Let $R$ be a ring, $S$ a multiplicative subset, $M$ a module. Define a relation on $M \times S$ by $(m, s) \sim (n, t)$ if there is $u \in S$ such that $utm = usn$. As in (11.11), this relation is an equivalence relation.

Denote by $S^{-1}M$ the set of equivalence classes, and by $m/s$ the class of $(m, s)$. Then $S^{-1}M$ is an $S^{-1}R$-module with addition given by $m/s + n/t := (tm + sn)/st$ and scalar multiplication by $a/s \cdot m/t := am/st$ similar to (11.11). We call $S^{-1}M$ the localization of $M$ at $S$.

For example, let $a$ be an ideal. Then $S^{-1}a = aS^{-1}R$ by (11.12)(1). Similarly, $S^{-1}(aM) = S^{-1}aS^{-1}M = aS^{-1}M$. Further, given an $R$-algebra $R'$, the $S^{-1}R$-module $S^{-1}R'$ constructed here underlies the $S^{-1}R$-algebra $S^{-1}R'$ of (11.22).

Define $\varphi_S: M \to S^{-1}M$ by $\varphi_S(a) := m/1$. Clearly, $\varphi_S$ is $R$-linear.

Note that $\mu_s: S^{-1}M \to S^{-1}M$ is bijective for all $s \in S$ by (12.1).

If $S = \{f^n \mid n \geq 0\}$ for some $f \in R$, then we call $S^{-1}M$ the localization of $M$ at $f$, and set $M_f := S^{-1}M$ and $\varphi_f := \varphi_S$.

Similarly, if $S = R - p$ for some prime ideal $p$, then we call $S^{-1}M$ the localization of $M$ at $p$, and set $M_p := S^{-1}M$ and $\varphi_p := \varphi_S$.

Theorem (12.3) (UMP). — Let $R$ be a ring, $S$ a multiplicative subset, and $M$ a module. Then $S^{-1}M$ is universal among $S^{-1}R$-modules equipped with an $R$-linear map from $M$.

Proof: The proof is like that of (11.15): given an $R$-linear map $\psi: M \to N$ where $N$ is an $S^{-1}R$-module, it is easy to prove that $\psi$ factors uniquely via the $S^{-1}R$-linear map $\rho: S^{-1}M \to N$ well defined by $\rho(m/s) := 1/s \cdot \psi(m)$. □

Exercise (12.2). — Let $R$ be a ring, $S$ a multiplicative subset, and $M$ a module. Show that $M = S^{-1}M$ if and only if $M$ is an $S^{-1}R$-module.


**Exercise (12.8).** — Let $R$ be a ring, $S \subset T$ multiplicative subsets, $M$ a module. Set $T_1 := \varphi_S(T) \subset S^{-1}R$. Show $T^{-1}M = T^{-1}(S^{-1}M) = T_1^{-1}(S^{-1}M)$.

**Exercise (12.9).** — Let $R$ be a ring, $S$ a multiplicative subset. Show that $S$ becomes a filtered category when equipped as follows: given $s, t \in S$, set

$$\text{Hom}(s, t) := \{x \in R \mid xs = t\}.$$ 

Given a module $M$, define a functor $S \to ((R\text{-mod}))$ as follows: for $s \in S$, set $M_s := M$; to each $x \in \text{Hom}(s, t)$, associate $\mu_xt : M_s \to M_t$. Define $\beta_s : M_s \to S^{-1}M$ by $\beta_s(m) := m/s$. Show the $\beta_s$ induce an isomorphism $\lim M_s \cong S^{-1}M$.

**Exercise (12.10).** — Let $R$ be a ring, $S$ a multiplicative subset, $M$ a module. Prove $S^{-1}M = 0$ if $\text{Ann}(M) \cap S \neq \emptyset$. Prove the converse if $M$ is finitely generated.

**Exercise (12.11).** — Let $R$ be a ring, $M$ a finitely generated module, $a$ an ideal.

1. Set $S := 1 + a$. Show that $S^{-1}a$ lies in the radical of $S^{-1}R$.

2. Use (1), Nakayama’s Lemma (11.14), and (12.7), but not the determinant trick (12.8), to prove this part of (12.11): if $M = aM$, then $sM = 0$ for an $s \in S$.

**Theorem (12.12).** — Let $R$ be a ring, $S$ a multiplicative subset, $\alpha : M \to N$ an $R$-linear map. Then $\varphi_S \alpha$ carries $M$ to the $S^{-1}R$-module $S^{-1}N$. So (12.1) yields a unique $S^{-1}R$-linear map $S^{-1}\alpha$ making the following diagram commutative:

$$
\begin{array}{ccc}
M & \overset{\varphi_S}{\to} & S^{-1}M \\
\downarrow{}^{\alpha} & & \downarrow{}^{S^{-1}\alpha} \\
N & \overset{\varphi_S}{\to} & S^{-1}N
\end{array}
$$

The construction in the proof of (12.1) yields

$$(S^{-1}\alpha)(m/s) = \alpha(m)/s. \quad (12.9.1)$$

Thus, canonically, we obtain the following map, and clearly, it is $R$-linear:

$$\text{Hom}_R(M, N) \to \text{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N). \quad (12.9.2)$$

Any $R$-linear map $\beta : N \to P$ yields $S^{-1}(\beta \alpha) = (S^{-1}\beta)(S^{-1}\alpha)$ owing to uniqueness or to (12.9.1). Thus $S^{-1}(\bullet)$ is a linear functor from $((R\text{-mod}))$ to $((S^{-1}R\text{-mod}))$.

**Theorem (12.10).** — Let $R$ be a ring, $S$ a multiplicative subset. Then the functor $S^{-1}(\bullet)$ is the left adjoint of the functor of restriction of scalars.

**Proof:** Let $N$ be an $S^{-1}R$-module. Then $N = S^{-1}N$ by (12.11), and the map (12.12) is bijective with inverse taking $\beta : S^{-1}M \to N$ to $\beta \varphi_S : M \to N$. And (12.12) is natural in $M$ and $N$ by (12.3). Thus the assertion holds. 

**Corollary (12.11).** — Let $R$ be a ring, $S$ a multiplicative subset. Then the functor $S^{-1}(\bullet)$ preserves direct limits, or equivalently, direct sums and cokernels.

**Proof:** By (12.10), the functor is a left adjoint. Hence it preserves direct limits by (12.4); equivalently, it preserves direct sums and cokernels by (12.11).

**Exercise (12.12).** — Let $R$ be a ring, $S$ a multiplicative subset, $P$ a projective module. Then $S^{-1}P$ is a projective $S^{-1}R$-module.

**Corollary (12.13).** — Let $R$ be a ring, $S$ a multiplicative subset. Then the functors $S^{-1}(\bullet)$ and $S^{-1}R \otimes_R \bullet$ are canonically isomorphic.
PROOF: As $S^{-1}(\bullet)$ preserves direct sums and cokernels by (12.3), the assertion is an immediate consequence of Watts Theorem (12.11).

Alternatively, both functors are left adjoints of the same functor by (12.4) and by (12.11). So they are canonically isomorphic by (12.3).

EXERCISE (12.13). — Let $R$ be a ring, $S$ a multiplicative subset, $M$ and $N$ modules. Show $S^{-1}(M \otimes_R N) = S^{-1}M \otimes_R S^{-1}N = S^{-1}M \otimes_S S^{-1}N = S^{-1}M \otimes_R S^{-1}N$.

EXERCISE (12.14). — Let $R$ be a ring, $S$ a multiplicative subset, $M$ a finitely presented module, and $r$ an integer. Show

$$F_r(M \otimes_R R') = F_r(M)R' \quad \text{and} \quad F_r(S^{-1}M) = F_r(M)S^{-1}R = S^{-1}F_r(M).$$

DEFINITION (12.16). — Let $R$ be a ring, $S$ a multiplicative subset, $M$ a module. Given a submodule $N$, its saturation $N^S$ is defined by

$$N^S := \{ m \in M \mid \text{there is } s \in S \text{ with } sm \in N \}.$$

If $N = N^S$, then we say $N$ is saturated.

PROPOSITION (12.17). — Let $R$ be a ring, $M$ a module, $N$ and $P$ submodules. Let $S$ be a multiplicative subset, and $K$ an $S^{-1}R$-submodule of $S^{-1}M$.

1. Then (a) $N^S$ is a submodule of $M$, and (b) $S^{-1}N$ is a submodule of $S^{-1}M$.
2. Then (a) $\varphi_S^2 K = (\varphi_S^2 K)^S$ and (b) $K = S^{-1}(\varphi_S^2 K)$.
3. Then $\varphi_S^{-1}(S^{-1}N) = N^S$; in particular, $\text{Ker}(\varphi_S) = 0^S$.
4. Then (a) $(N^S)^S = N^S$ and (b) $S^{-1}(S^{-1}N) = S^{-1}N$.
5. If $N \subset P$, then (a) $N^S \subset P^S$ and (b) $S^{-1}N \subset S^{-1}P$.
6. Then (a) $(N \cap P)^S = N^S \cap P^S$ and (b) $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$.
7. Then (a) $(N + P)^S = N^S + P^S$ and (b) $S^{-1}(N + P) = S^{-1}N + S^{-1}P$.

PROOF: Assertion (1)(b) holds because $N \times S$ is a subset of $M \times S$ and is equipped with the induced equivalence relation. Assertion (5)(b) follows by taking $M := P$. Assertion (4)(b) follows from (12.3) with $M := S^{-1}M$.

Assertions (1)(a), (2), (3) can be proved as in (12.4) and (12.14)(1), (2).Assertions (4)(a) and (5)(a) can be proved as in (12.14)(1) and (2).

As to (6)(a), clearly $(N \cap P)^S \subset N^S \cap P^S$. Conversely, given $n \in N^S \cap P^S$, there are $s,t \in S$ with $sn \in N$ and $tn \in P$. Then $stn \in N \cap P$ and $st \in S$. So $n \in (N \cap P)^S$. Thus (a) holds. Alternatively, (6)(b) and (3) yield (6)(a).

As to (6)(b), since $N \cap P \subset N, P$, using (1) yields $S^{-1}(N \cap P) \subset S^{-1}N \cap S^{-1}P$. But, given $n/s = p/t \in S^{-1}N \cap S^{-1}P$, there is a $u \in S$ with $utn = usp \in N \cap P$. Hence $utn/uts = usp/uts \in S^{-1}(N \cap P)$. Thus (b) holds.

As to (7)(a), given $n \in N^S$ and $p \in P^S$, there are $s,t \in S$ with $sn \in N$ and $tp \in P$. Then $st \in S$ and $st(n + p) \in N + P$. Thus (7)(a) holds.

As to (7)(b), note $N, P \subset N + P$. So (1)(b) yields $S^{-1}(N + P) \subset S^{-1}N + S^{-1}P$. But the opposite inclusion holds as $(n + p)/s = n/s + p/s$. Thus (7)(b) holds.

EXERCISE (12.18). — Let $R$ be a ring, $S$ a multiplicative subset.

1. Let $M_1 \rightarrow M_2$ be a map of modules, which restricts to a map $N_1 \rightarrow N_2$ of submodules. Show $\alpha(N_1^S) \subset N_2^S$; that is, there is an induced map $N_1^S \rightarrow N_2^S$.

2. Let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3$ be a left exact sequence, which restricts to a left exact sequence $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3$ of submodules. Show there is an induced left exact sequence of saturations: $0 \rightarrow N_1^S \rightarrow N_2^S \rightarrow N_3^S$. 
EXERCISE (12.13). — Let \( R \) be a ring, \( M \) a module, and \( S \) a multiplicative subset.
Set \( T^S M := \{0\} \). We call it the \textbf{\( S \)-torsion submodule of \( M \)}. Prove the following:
(1) \( T^S(M/T^S M) = 0 \).
(2) \( T^S M = \text{Ker}(\varphi_S) \).
(3) Let \( \alpha : M \to N \) be a map. Then \( \alpha(T^S M) \subseteq T^S N \).
(4) Let \( 0 \to M' \to M \to M'' \) be exact. Then so is \( 0 \to T^S M' \to T^S M \to T^S M'' \).
(5) Let \( S_1 \subseteq S \) be a multiplicative subset. Then \( T^S(S_1^{-1} M) = S_1^{-1}(T^S M) \).

\textbf{Theorem (12.20) (Exactness of Localization).} — Let \( R \) be a ring, and \( S \) a multiplicative subset. Then the functor \( S^{-1}(\bullet) \) is exact.

\textbf{Proof:} As \( S^{-1}(\bullet) \) preserves injections by (12.14)(1) and cokernels by (12.14), it is exact by (12.16).
Alternatively, given an exact sequence \( M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \), for each \( s \in S \), take a copy \( M'_s \to M_s \to M''_s \). Using (12.14), make \( S \) into a filtered category, and make these copies into a functor from \( S \) to the category of 3-term exact sequences; its limit is the following sequence, which is exact by (12.14), as desired:
\[ S^{-1}M' \xrightarrow{S^{-1}\alpha} S^{-1}M \xrightarrow{S^{-1}\beta} S^{-1}M''. \]

The latter argument can be made more direct as follows. Since \( \beta\alpha = 0 \), we have \( (S^{-1}\beta)(S^{-1}\alpha) = S^{-1}(\beta\alpha) = 0 \). Hence \( \text{Ker}(S^{-1}\beta) \supset \text{Im}(S^{-1}\alpha) \). Conversely, given \( m/s \in \text{Ker}(S^{-1}\beta) \), there is \( t \in S \) with \( t\beta(m) = 0 \). So \( \beta(tm) = 0 \). So exactness yields \( m'/m' \in M' \) with \( \alpha(m') = tm \). So \( (S^{-1}\alpha')(m'/ts) = m/s \). Hence \( \text{Ker}(S^{-1}\beta) \subset \text{Im}(S^{-1}\alpha) \). Thus \( \text{Ker}(S^{-1}\beta) = \text{Im}(S^{-1}\alpha) \), as desired. \( \square \)

\textbf{Corollary (12.21).} — Let \( R \) be a ring, \( S \) a multiplicative subset. Then \( S^{-1}R \) is flat over \( R \).
\textbf{Proof:} The functor \( S^{-1}(\bullet) \) is exact by (12.14), and is isomorphic to \( S^{-1}R \otimes_R \bullet \) by (12.16). Thus \( S^{-1}R \) is flat.
Alternatively, using (12.14), write \( S^{-1}R \) as a filtered direct limit of copies of \( R \).
But \( R \) is flat by (7.1). Thus \( S^{-1}R \) is flat by (12.18). \( \square \)

\textbf{Corollary (12.22).} — Let \( R \) be a ring, \( S \) a multiplicative subset, \( a \) an ideal, and \( M \) a module. Then \( S^{-1}(M/aM) = S^{-1}M/S^{-1}(aM) = S^{-1}M/aS^{-1}M \).
\textbf{Proof:} The assertion results from (12.14) and (12.20). \( \square \)

\textbf{Corollary (12.23).} — Let \( R \) be a ring, \( p \) a prime. Then \( \text{Frac}(R/p) = R_p/pR_p \).
\textbf{Proof:} We have \( \text{Frac}(R/p) \approx (R/p)_p = R_p/pR_p \) by (12.24) and (12.33). \( \square \)

\textbf{Proposition (12.24).} — Let \( R \) be a ring, \( M \) a module, \( S \) a multiplicative subset.
(1) Let \( m_1, \ldots, m_n \in M \). If \( M \) is finitely generated and if the \( m_i/1 \in S^{-1}M \) generate over \( S^{-1}R \), then there’s \( f \in S \) so that the \( m_i/1 \in M_f \) generate over \( R_f \).
(2) Assume \( M \) is finitely presented and \( S^{-1}M \) is a free \( S^{-1}R \)-module of rank \( n \). Then there is \( h \in S \) such that \( M_h \) is a free \( R_h \)-module of rank \( n \).
\textbf{Proof:} To prove (1), define \( \alpha : R^n \to M \) by \( \alpha(e_i) := m_i \) with \( e_i \) the \( i \)th standard basis vector. Set \( C := \text{Coker}(\alpha) \). Then \( S^{-1}C = \text{Coker}(S^{-1}\alpha) \) by (12.31). Assume the \( m_i/1 \in S^{-1}M \) generate over \( S^{-1}R \). Then \( S^{-1}\alpha \) is surjective by (12.33)(1) as \( S^{-1}(R^n) = (S^{-1}R)^n \) by (12.33). Hence \( S^{-1}C = 0 \).

In addition, assume \( M \) is finitely generated. Then so is \( C \). Hence, (12.27) yields \( f \in S \) such that \( C_f = 0 \). Hence \( \alpha_f \) is surjective. So the \( m_i/1 \) generate \( M_f \) over \( R_f \) again by (12.31)(1). Thus (1) holds.
For (2), let \(m_1/s_1, \ldots, m_n/s_n\) be a free basis of \(S^{-1}M\) over \(S^{-1}R\). Then so is \(m_1/1, \ldots, m_n/1\) as the 1\(s_i\) are units. Form \(\alpha\) and \(C\) as above, and set \(K := \text{Ker}(\alpha)\). Then (12.26) yields \(S^{-1}K = \text{Ker}(S^{-1}\alpha)\) and \(S^{-1}C = \text{Coker}(S^{-1}\alpha)\). But \(S^{-1}\alpha\) is bijective. Hence \(S^{-1}K = 0\) and \(S^{-1}C = 0\).

Since \(M\) is finitely generated, \(C\) is too. Hence, as above, there is \(f \in S\) such that \(C_f = 0\). Then \(0 \rightarrow K_f \rightarrow R_f^n \overset{\alpha_f}{\rightarrow} M_f \rightarrow 0\) is exact by (12.26). Take a finite presentation \(R^n \rightarrow R^q \rightarrow M \rightarrow 0\). By (12.24), it yields a finite presentation \(R^n_f \rightarrow R^q_f \rightarrow M_f \rightarrow 0\). Hence \(K_f\) is a finitely generated \(R_f\)-module by (12.4).

Let \(S_1 \subset R_f\) be the image of \(S\). Then (12.23) yields \(S^{-1}_1(K_f) = S^{-1}_1K\). But \(S^{-1}_1K = 0\). Hence there is \(g/1 \in S_1\) such that \((K_f)g/1 = 0\). Set \(h := fg\). Let's show \(K_h = 0\). Let \(x \in K\). Then there is \(a\) such that \((g^a x)/1 = 0\) in \(K_f\). Hence there is \(b\) such that \(f^b g^a x = 0\) in \(K\). Take \(c \geq a, b\). Then \(h^c x = 0\). Thus \(K_h = 0\). But \(C_f = 0\) implies \(C_h = 0\). Hence \(\alpha_h : R^n_h \rightarrow M_h\) is an isomorphism, as desired. □

**Proposition (12.25).** — Let \(R\) be a ring, \(S\) a multiplicative subset, \(M\) and \(N\) modules. Then there is a canonical homomorphism

\[
\sigma : S^{-1}\text{Hom}_R(M, N) \rightarrow \text{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N).
\]

Further, \(\sigma\) is injective if \(M\) is finitely generated, and \(\sigma\) is an isomorphism if \(M\) is finitely presented.

**Proof:** The assertions result from (12.74) with \(R' := S^{-1}R\), since \(S^{-1}R\) is flat by (12.41) and since \(S^{-1}R \otimes P = S^{-1}P\) for every \(R\)-module \(P\) by (12.43).

**Example (12.26).** — Set \(R := \mathbb{Z}\) and \(S := \mathbb{Z} - \{0\}\) and \(M := \mathbb{Q}/\mathbb{Z}\). Then \(M\) is faithful since \(z \in S\) implies \(z \cdot (1/2z) = 1/2 \neq 0\); thus, \(\mu_R : R \rightarrow \text{Hom}_R(M, M)\) is injective. But \(S^{-1}R = \mathbb{Q}\). So (12.74) yields \(S^{-1}\text{Hom}_R(M, M) \neq 0\). On the other hand, \(S^{-1}M = 0\) as \(s \cdot r/s = 0\) for any \(r/s \in M\). So the map \(\sigma(M, M)\) of (12.74) (2) can fail if \(M\) is not finitely generated.

**Example (12.27).** — Take \(R := \mathbb{Z}\) and \(S := \mathbb{Z} - 0\) and \(M_n := \mathbb{Z}/\langle n \rangle\) for \(n \geq 2\). Then \(S^{-1}M_n = 0\) for all \(n\) as \(nm = 0\) (mod \(n\)) for all \(m\). On the other hand, \((1,1,\ldots)/1\) is nonzero in \(S^{-1}(\prod M_n)\) as the \(k\)th component of \(m \cdot (1,1,\ldots)/1\) is nonzero in \(\prod M_n\) for \(k > m\) if \(m\) is nonzero. Thus \(S^{-1}(\prod M_n) \neq \prod(S^{-1}M_n)\).

Also \(S^{-1}\mathbb{Z} = \mathbb{Q}\). So (12.74) yields \(\mathbb{Q} \otimes (\prod M_n) \neq \prod(\mathbb{Q} \otimes M_n)\), whereas (12.43) yields \(\mathbb{Q} \otimes (\bigoplus M_n) = \bigoplus(\mathbb{Q} \otimes M_n)\).

**Exercise (12.28).** — Set \(R := \mathbb{Z}\) and \(S := \mathbb{Z} - \{0\}\). Set \(M := \bigoplus_{n \geq 2} \mathbb{Z}/\langle n \rangle\) and \(N := M\). Show that the map \(\sigma\) of (12.74) is not injective.
13. Support

The spectrum of a ring is the following topological space: its points are the prime ideals, and each closed set consists of those primes containing a given ideal. The support of a module is the following subset: its points are the primes at which the localized module is nonzero. We prove that a sequence is exact if and only if it is exact after localizing at every maximal ideal. We end this section by proving that the following conditions on a module are equivalent: it is finitely generated and projective; it is finitely presented and flat; and it is locally free of finite rank.

(13.1) (Spectrum of a ring). — Let $R$ be a ring. Its set of prime ideals is denoted $\text{Spec}(R)$, and is called the (prime) spectrum of $R$.

Let $a$ be an ideal. Let $V(a)$ denote the subset of $\text{Spec}(R)$ consisting of those primes that contain $a$. We call $V(a)$ the variety of $a$.

Let $b$ be a second ideal. Obviously, if $a \subseteq b$, then $V(b) \subseteq V(a)$. Conversely, if $V(b) \subseteq V(a)$, then $a \subseteq \sqrt{b}$, owing to the Scheinnullstellensatz (3.29). Therefore, $V(a) = V(b)$ if and only if $\sqrt{a} = \sqrt{b}$. Further, (2.2) yields $V(a) = V(b) = V(a \setminus b) = V(ab)$.

A prime ideal $p$ contains the ideals $\sum a_i$ in an arbitrary collection if and only if $p$ contains their sum $\sum a_i$; hence,

$$\bigcap V(a_i) = V(\sum a_i).$$

Finally, $V(R) = \emptyset$, and $V(\langle 0 \rangle) = \text{Spec}(R)$. Thus the subsets $V(a)$ of $\text{Spec}(R)$ are the closed sets of a topology; it is called the Zariski topology.

Given an element $f \in R$, we call the open set $D(f) := \text{Spec}(R) - V(\langle f \rangle)$ a principal open set. These sets form a basis for the topology of $\text{Spec}(R)$; indeed, given any prime $p \not\supseteq a$, there is an $f \in a - p$, and so $p \in D(f) \subseteq \text{Spec}(R) - V(a)$. Further, $f, g \not\in p$ if and only if $fg \not\in p$, for any $f, g \in R$ and prime $p$; in other words,

$$D(f) \cap D(g) = D(fg).$$  \hspace{1cm} (13.1.1)

A ring map $\varphi: R \to R'$ induces a set map

$$\text{Spec}(\varphi): \text{Spec}(R') \to \text{Spec}(R) \text{ by } \text{Spec}(\varphi)(p') := \varphi^{-1}(p').$$  \hspace{1cm} (13.1.2)

Notice $\varphi^{-1}(p') \supseteq a$ if and only if $p' \supseteq aR'$; so $\text{Spec}(\varphi)^{-1} V(a) = V(aR')$. Hence $\text{Spec}(\varphi)$ is continuous. Thus $\text{Spec}(\bullet)$ is a contravariant functor from $\text{(Rings)}$ to $\text{(Top spaces)}$.

For example, the quotient map $R \to R/a$ induces a topological embedding

$$\text{Spec}(R/a) \to \text{Spec}(R),$$  \hspace{1cm} (13.1.3)

whose image is $V(a)$, owing to (13.1) and (12.8). Furthermore, the localization map $R \to R_f$ induces a topological embedding

$$\text{Spec}(R_f) \to \text{Spec}(R),$$  \hspace{1cm} (13.1.4)

whose image is $D(f)$, owing to (12.24).
Let $X$ be a ring, $p \in \text{Spec}(R)$. Show that $p$ is a closed point — that is, $\{p\}$ is a closed set — if and only if $p$ is a maximal ideal.

**Exercise (13.9)**. Let $R$ be a ring, and set $X := \text{Spec}(R)$. Let $X_1, X_2 \subseteq X$ be closed subsets. Show that the following three conditions are equivalent:

1. $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = \emptyset$.
2. There are complementary idempotents $e_1, e_2 \in R$ with $V(\langle e_i \rangle) = X_i$.
3. There are comaximal ideals $a_1, a_2 \subseteq R$ with $a_1a_2 = 0$ and $V(a_1) = X_i$.
4. There are ideals $a_1, a_2 \subseteq R$ with $a_1 \oplus a_2 = R$ and $V(a_i) = X_i$.

Finally, given any $e_i$ and $a_i$ satisfying (2) and either (3) or (4), necessarily $e_i \in a_i$.

**Exercise (13.10)**. Let $\varphi: R \to R'$ be a map of rings, $a$ an ideal of $R$, and $b$ an ideal of $R'$. Set $\varphi^* := \text{Spec}(\varphi)$. Prove these two statements:

1. Every prime of $R$ is a contraction of a prime if and only if $\varphi^*$ is surjective.
2. If every prime of $R'$ is an extension of a prime, then $\varphi^*$ is injective.

Is the converse of (2) true?

**Exercise (13.11)**. Let $R$ be a ring, $S$ a multiplicative subset. Set $X := \text{Spec}(R)$ and $Y := \text{Spec}(S^{-1}R)$. Set $\varphi^*_S := \text{Spec}(\varphi_S)$ and $S^{-1}X := \text{Im} \varphi^*_S \subseteq X$. Show (1) that $S^{-1}X$ consists of the primes $p$ of $R$ with $p \cap S = \emptyset$ and (2) that $\varphi^*_S$ is a homeomorphism of $Y$ onto $S^{-1}X$.

**Exercise (13.12)**. Let $\theta: R \to R'$ be a ring map, $S \subseteq R$ a multiplicative subset. Set $X := \text{Spec}(R)$ and $Y := \text{Spec}(R')$ and $\theta^* := \text{Spec}(\theta)$. Via (13.10)(2) and (11.20), identify $\text{Spec}(S^{-1}R)$ and $\text{Spec}(S^{-1}R')$ with their images $S^{-1}X \subseteq X$ and $S^{-1}Y \subseteq Y$. Show (1) that $S^{-1}Y = \theta^{-1}(S^{-1}X)$ and (2) $S^{-1}\text{Spec}(S^{-1}\theta) = \theta^* S^{-1}Y$.

**Exercise (13.13)**. Let $\theta: R \to R'$ be a ring map, $a \subseteq R$ an ideal. Set $b := aR'$. Let $\theta: R/a \to R'/b$ be the induced map. Set $X := \text{Spec}(R)$ and $Y := \text{Spec}(R')$. Set $\theta^* := \text{Spec}(\theta)$ and $\theta^* := \text{Spec}(\theta)$. Via (13.11), identify $\text{Spec}(R/a)$ and $\text{Spec}(R'/b)$ with $V(a) \subseteq X$ and $V(b) \subseteq Y$. Show (1) $V(b) = \theta^{-1}(V(a))$ and (2) $\theta^* = \theta|_{V(b)}$.

**Exercise (13.14)**. Let $R \to R'$ be a ring map, $p \subseteq R$ a prime, $k$ the residue field of $R_p$. Set $\theta^* := \text{Spec}(\theta)$. Show (1) that $\theta^{-1}(p)$ is canonically homeomorphic to $\text{Spec}(R' \otimes_R k)$ and (2) that $p \in \text{Im} \theta^*$ if and only if $R' \otimes_R k \neq 0$.

**Exercise (13.15)**. Let $R$ be a ring, $p$ a prime ideal. Show that the image of $\text{Spec}(R_p)$ in $\text{Spec}(R)$ is the intersection of all open neighborhoods of $p$ in $\text{Spec}(R)$.

**Exercise (13.16)**. Let $\varphi: R \to R'$ and $\psi: R \to R''$ be ring maps, and define $\theta: R \to R' \otimes_R R''$ by $\theta(x) := \varphi(x) \otimes \psi(x)$. Show
\[
\text{Im} \text{Spec}(\theta) = \text{Im} \text{Spec}(\varphi) \cap \text{Im} \text{Spec}(\psi).
\]

**Exercise (13.17)**. Let $R$ be a filtered direct limit of rings $R_\lambda$ with transition maps $\alpha^\lambda_\mu$ and insertions $\alpha_\lambda$. For each $\lambda$, let $\varphi^\lambda_\mu: R' \to R_\lambda$ be a ring map with $\varphi^\lambda_\mu = \alpha^\lambda_\mu \varphi_\lambda$ for all $\alpha^\lambda_\mu$, so that $\varphi := \alpha_\lambda \varphi_\lambda$ is independent of $\lambda$. Show
\[
\text{Im} \text{Spec}(\varphi) = \bigcap_\lambda \text{Im} \text{Spec}(\varphi_\lambda).
\]

**Exercise (13.18)**. Let $A$ be a domain with just one nonzero prime $p$. Set $K := \text{Frac}(A)$ and $R := (A/p) \times K$. Define $\varphi: A \to R$ by $\varphi(x) := (x', x)$ with $x'$ the residue of $x$. Set $\varphi^* := \text{Spec}(\varphi)$. Show $\varphi^*$ is bijective, but not a homeomorphism.
Let $\varphi : R \to R'$ be a ring map, and $b$ an ideal of $R'$. Set $\varphi^* := \text{Spec}(\varphi)$. Show (1) that the closure $\varphi^*(\text{V}(b))$ in $\text{Spec}(R)$ is equal to $\text{V}(\varphi^{-1}b)$ and (2) that $\varphi^*(\text{Spec}(R'))$ is dense in $\text{Spec}(R)$ if and only if $\ker(\varphi) \subset \text{nil}(R)$.

**Exercise (13.14).** — Let $R$ be a ring, $R'$ a flat algebra with structure map $\varphi$. Show that $R'$ is faithfully flat if and only if $\text{Spec}(\varphi)$ is surjective.

**Exercise (13.15).** — Let $\varphi : R \to R'$ be a flat map of rings, $q$ a prime of $R'$, and $p = \varphi^{-1}(q)$. Show that the induced map $\text{Spec}(R'_q) \to \text{Spec}(R_p)$ is surjective.

**Exercise (13.16).** — Let $R$ be a ring. Given $f \in R$, set $S_f := \{f^n \mid n \geq 0\}$, and let $\overline{S}_f$ denote its saturation; see (6.17). Given $f, g \in R$, show that the following conditions are equivalent:

1. $D(g) \subset D(f)$.
2. $V((g)) \supset V((f))$.
3. $\sqrt{(g)} \subset \sqrt{(f)}$.
4. $\overline{S}_f \subset \overline{S}_g$.
5. $g \in \sqrt{(f)}$.
6. $f \in \overline{S}_g$.
7. There is a unique $R$-algebra map $\varphi'_g : \overline{S}_f^{-1}R \to \overline{S}_g^{-1}R$.
8. There is an $R$-algebra map $R_f \to R_g$.

Show that, if these conditions hold, then the map in (8) is equal to $\varphi'_g$.

**Exercise (13.17).** — Let $R$ be a ring. (1) Show that $D(f) \to R_f$ is a well-defined contravariant functor from the category of principal open sets and inclusions to $(\text{R-alg})$. (2) Given $p \in \text{Spec}(R)$, show $\lim_{f(D(f)) \ni p} R_f = R_p$.

**Exercise (13.18).** — A topological space is called irreducible if it’s nonempty and if every pair of nonempty open subsets meet. Let $R$ be a ring. Set $X := \text{Spec}(R)$ and $n := \text{nil}(R)$. Show that $X$ is irreducible if and only if $n$ is prime.

**Exercise (13.19).** — Let $X$ be a topological space, $Y$ an irreducible subspace.

1. Show that the closure $\overline{Y}$ of $Y$ is also irreducible.
2. Show that $Y$ is contained in a maximal irreducible subspace.
3. Show that the maximal irreducible subspaces of $X$ are closed, and cover $X$.

They are called its irreducible components. What are they if $X$ is Hausdorff?

4. Let $R$ be a ring, and take $X := \text{Spec}(R)$. Show that its irreducible components are the closed sets $V(p)$ where $p$ is a minimal prime.

**Proposition (13.20).** — Let $R$ be a ring, $X := \text{Spec}(R)$. Then $X$ is quasi-compact: if $X = \bigcup_{\lambda \in \Lambda} U_{\lambda}$ with $U_{\lambda}$ open, then $X = \bigcup_{i=1}^n U_{\lambda_i}$ for some $\lambda_i \in \Lambda$.

**Proof:** Say $U_{\lambda} = X - V(a_{\lambda})$. As $X = \bigcup_{\lambda \in \Lambda} U_{\lambda}$, then $\emptyset = \bigcap V(a_{\lambda}) = V(\sum a_{\lambda})$. So $\sum a_{\lambda}$ lies in no prime ideal. Hence there are $\lambda_1, \ldots, \lambda_n \in \Lambda$ and $f_{\lambda_i} \in a_{\lambda_i}$ with $1 = \sum f_{\lambda_i}$. So $R = \sum a_{\lambda_i}$. So $\emptyset = V(a_{\lambda_i}) = V(\sum a_{\lambda_i})$. Thus $X = \bigcup U_{\lambda_i}$. \[\square\]

**Exercise (13.21).** — Let $R$ be a ring, $X := \text{Spec}(R)$, and $U$ an open subset. Show $U$ is quasi-compact if and only if $X - U = V(a)$ where $a$ is finitely generated.

**Exercise (13.22).** — Let $R$ be a ring, $M$ a module, $m \in M$. Set $X := \text{Spec}(R)$. Assume $X = \bigcup D(f_{\lambda})$ for some $f_{\lambda}$, and $m/1 = 0$ in $M_{f_{\lambda}}$ for all $\lambda$. Show $m = 0$.

**Exercise (13.23).** — Let $R$ be a ring; set $X := \text{Spec}(R)$. Prove that the four following conditions are equivalent:

1. $R/\text{nil}(R)$ is absolutely flat.
2. $X$ is Hausdorff.
80  Support (13.31)

(3) \( X \) is \( T_1 \); that is, every point is closed.

(4) Every prime \( p \) of \( R \) is maximal.

Assume (1) holds. Prove that \( X \) is \textit{totally disconnected}; namely, no two distinct points lie in the same connected component.

**Exercise (13.25)**. — Let \( B \) be a Boolean ring, and set \( X := \text{Spec}(B) \). Show a subset \( U \subset X \) is both open and closed if and only if \( U = \text{D}(f) \) for some \( f \in B \). Further, show \( X \) is a compact Hausdorff space. (Following Bourbaki, we shorten “quasi-compact” to “compact” when the space is Hausdorff.)

**Exercise (13.26)** (Stone’s Theorem). — Show every Boolean ring \( B \) is isomorphic to the ring of continuous functions from a compact Hausdorff space \( X \) to \( \mathbb{F}_2 \) with the discrete topology. Equivalently, show \( B \) is isomorphic to the ring \( R \) of open and closed subsets of \( X \); in fact, \( X := \text{Spec}(B) \), and \( B \rightarrow R \) is given by \( f \mapsto \text{D}(f) \).

**Definition (13.26)**. — Let \( R \) be a ring, \( M \) a module. Its \textit{support} is the set

\[
\text{Supp}(M) := \{ p \in \text{Spec}(R) \mid M_p \neq 0 \}.
\]

**Proposition (13.27)**. — Let \( R \) be a ring, \( M \) a module.

(1) Let \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) be exact. Then \( \text{Supp}(L) \cup \text{Supp}(N) = \text{Supp}(M) \).

(2) Let \( M_\lambda \) be submodules with \( \sum M_\lambda = M \). Then \( \bigcup \text{Supp}(M_\lambda) = \text{Supp}(M) \).

(3) Then \( \text{Supp}(M) \subseteq \text{V}(\text{Ann}(M)) \), with equality if \( M \) is finitely generated.

**Proof**: Consider (1). For every prime \( p \), the sequence \( 0 \rightarrow L_p \rightarrow M_p \rightarrow N_p \rightarrow 0 \) is exact by (13.21). So \( M_p \neq 0 \) if and only if \( L_p \neq 0 \) or \( N_p \neq 0 \). Thus (1) holds.

In (2), \( M_\lambda \subset M \). So (1) yields \( \bigcup \text{Supp}(M_\lambda) \subset \text{Supp}(M) \). To prove the opposite inclusion, take \( p \notin \bigcup \text{Supp}(M_\lambda) \). Then \( (M_\lambda)_p = 0 \) for all \( \lambda \). By hypothesis, the natural map \( \bigoplus M_\lambda \rightarrow M \) is surjective. So \( \bigoplus (M_\lambda)_p \rightarrow M_p \) is surjective by (13.21). Hence \( M_p = 0 \). Alternatively, given \( m/s \in M_p \), express \( m \) as a finite sum \( m = \sum m_\lambda \) with \( m_\lambda \in M_\lambda \). For each such \( \lambda \), there is \( t_\lambda \in R - p \) with \( t_\lambda m_\lambda = 0 \). Set \( t := \prod t_\lambda \). Then \( tm = 0 \) and \( t \notin p \). So \( m/s = 0 \) in \( M_p \). Hence again, \( M_p = 0 \). Thus \( p \notin \text{Supp}(M) \), and so (2) holds.

Consider (3). Let \( p \) be a prime. By (13.27), \( M_p = 0 \) if \( \text{Ann}(M) \cap (R - p) \neq 0 \), and the converse holds if \( M \) is finitely generated. But \( \text{Ann}(M) \cap (R - p) \neq 0 \) if and only if \( \text{Ann}(M) \not\subset p \). Thus (3) holds.

**Definition (13.28)**. — Let \( R \) be a ring, \( x \in R \). We say \( x \) is \textit{nilpotent} on a module \( M \) if there is \( n \geq 1 \) with \( x^n m = 0 \) for all \( m \in M \); that is, \( x \in \sqrt{\text{Ann}(M)} \).

We denote the set of nilpotents on \( M \) by \( \text{nil}(M) \); that is, \( \text{nil}(M) := \sqrt{\text{Ann}(M)} \).

**Proposition (13.29)**. — Let \( R \) be a ring, \( M \) a finitely generated module. Then

\[
\text{nil}(M) = \bigcap_{p \supseteq \text{Ann}(M)} p.
\]

**Proof**: First, \( \text{nil}(M) = \bigcap_{p \supseteq \text{Ann}(M)} p \) by the Scheinnullstellensatz (13.23). But \( p \supset \text{Ann}(M) \) if and only if \( p \in \text{Supp}(M) \) by (13.27)(3).

**Proposition (13.30)**. — Let \( R \) be a ring, \( M \) and \( N \) modules. Then

\[
\text{Supp}(M \otimes_R N) \subseteq \text{Supp}(M) \cap \text{Supp}(N), \tag{13.30.1}
\]

with equality if \( M \) and \( N \) are finitely generated.

**Proof**: First, \( (M \otimes_R N)_p = M_p \otimes_{R_p} N_p \) by (13.13); whence, (13.30.1) holds. The opposite inclusion follows from (10.1.7) if \( M \) and \( N \) are finitely generated.
EXERCISE (13.31). — Let $R$ be a ring, $a$ an ideal, $M$ a module. Prove that

$$\text{Supp}(M/aM) \subseteq \text{Supp}(M) \cap V(a),$$

with equality if $M$ is finitely generated.

EXERCISE (13.32). — Let $\phi: R \to R'$ be a map of rings, $M$ an $R$-module. Prove

$$\text{Supp}(M \otimes_R R') \subseteq \text{Spec}(\phi)^{-1}(\text{Supp}(M)),$$

with equality if $M$ is finitely generated.

EXERCISE (13.33). — Let $R$ be a ring, $M$ a module, $p \in \text{Supp}(M)$. Prove

$$V(p) \subseteq \text{Supp}(M).$$

EXERCISE (13.34). — Let $Z$ be the integers, $Q$ the rational numbers, and set $M := Q/Z$. Find $\text{Supp}(M)$, and show that it is not Zariski closed.

PROPOSITION (13.35). — Let $R$ be a ring, $M$ a module. These conditions are equivalent: (1) $M = 0$; (2) $\text{Supp}(M) = \emptyset$; (3) $M_m = 0$ for every maximal ideal $m$.

PROOF: Trivially, if (1) holds, then $S^{-1}M = 0$ for any multiplicative subset $S$. In particular, (2) holds. Trivially, (2) implies (3).

Finally, assume $M \neq 0$, and take a nonzero $m \in M$, and set $a := \text{Ann}(m)$. Then $1 \notin a$, so $a$ lies in some maximal ideal $m$. Then, for all $f \in R - m$, we have $fm \neq 0$.

Hence $m/1 \neq 0$ in $M_m$. Thus (3) implies (1). $\square$

EXERCISE (13.36). — Let $R$ be a domain, and $M$ a module. Set $S := R - 0$ and $T(M) := T^S(M)$. We call $T(M)$ the torsion submodule of $M$, and we say $M$ is torsionfree if $T(M) = 0$.

Prove $M$ is torsionfree if and only if $M_m$ is torsionfree for all maximal ideals $m$.

EXERCISE (13.37). — Let $R$ be a ring, $P$ a module, $M$, $N$ submodules. Assume $M_m = N_m$ for every maximal ideal $m$. Show $M = N$. First assume $M \subseteq N$.

EXERCISE (13.38). — Let $R$ be a ring, $a$ an ideal. Suppose $M_m = 0$ for all maximal ideals $m \supset a$. Show that $M = aM$.

EXERCISE (13.39). — Let $R$ be a ring, $P$ a module, $M$ a submodule, and $p \in P$ an element. Assume $p/1 \in M_m$ for every maximal ideal $m$. Show $p \in M$.

EXERCISE (13.40). — Let $R$ be a domain, $a$ an ideal. Show $a = \bigcap_m aR_m$ where $m$ runs through the maximal ideals and the intersection takes place in $\text{Frac}(R)$.

EXERCISE (13.41). — Prove these three conditions on a ring $R$ are equivalent:

1. $R$ is reduced.
2. $S^{-1}R$ is reduced for all multiplicative subsets $S$.
3. $R_m$ is reduced for all maximal ideals $m$.

If $R_m$ is a domain for all maximal ideals $m$, is $R$ necessarily a domain?

EXERCISE (13.42). — Let $R$ be a ring, $\Sigma$ the set of minimal primes. Prove this:

1. If $R_p$ is a domain for any prime $p$, then the $p \in \Sigma$ are pairwise comaximal.
2. $R = \prod_{i=1}^n R_i$ where $R_i$ is a domain if and only if $R_p$ is a domain for any prime $p$ and $\Sigma$ is finite. If so, then $R_i = R/p_i$ with $\{p_1, \ldots, p_n\} = \Sigma$.

PROPOSITION (13.43). — A sequence of modules $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is exact if and only if its localization $L_m \xrightarrow{\alpha_m} M_m \xrightarrow{\beta_m} N_m$ is exact at each maximal ideal $m$. 

Proof: If the sequence is exact, then so is its localization by (13.50). Consider the converse. First $\text{Im}(\beta_m \alpha_m) = 0$. But $\text{Im}(\beta_m \alpha_m) = (\text{Im}(\beta \alpha))_m$ by (13.49) and (13.48). Hence $\text{Im}(\beta \alpha) = 0$ by (13.39). So $\beta \alpha = 0$. Thus $\text{Im}(\alpha) \subseteq \text{Ker}(\beta)$.

Set $H := \text{Ker}(\beta)/\text{Im}(\alpha)$. Then $H_m = \text{Ker}(\beta_m)/\text{Im}(\alpha_m)$ by (13.44) and (13.43). So $H_m = 0$ owing to the hypothesis. Hence $H = 0$ by (13.38), as required.

Exercise (13.50). — Let $R$ be a ring, $M$ a module. Prove elements $m_\lambda \in M$ generate $M$ if and only if, at every maximal ideal $m$, their images $m_\lambda$ generate $M_m$.

Proposition (13.49). — Let $A$ be a semilocal ring, $m_1, \ldots, m_n$ its maximal ideals, $M, N$ finitely presented modules. Assume $M_{m_i} \cong N_{m_i}$ for each $i$. Then $M \cong N$.

Proof: For each $i$, take an isomorphism $\psi_i : M_{m_i} \cong N_{m_i}$. Then (13.49) yields $s_i \in A - m_i$ and $\varphi_i : M \to N$ with $(\varphi_i)_{m_i} = s_i \psi_i$. However, (2.22) implies $\bigcap_{j \neq i} m_j \not\subseteq m_i$, so there’s $x_i \in \bigcap_{j \neq i} m_j$ with $x_i \notin m_i$. Set $\gamma := \sum x_i s_i \psi_i$.

For each $j$, set $\alpha_j := x_j s_j \psi_j$. Then $\alpha_m : M_m \cong N_m$, as $x_j$ and $s_j$ are units. Set $\beta_j : (M_m)_j \subseteq m_j N_m$ as $x_i \in m_j$, for $i \neq j$. Further, $\gamma = \alpha_j + \beta_j$. So $\gamma_{m_i}$ is an isomorphism by (13.48). Hence (13.39) implies $\gamma : M \cong N$.

Proposition (13.49). — Let $R$ be a ring, $M$ a module. Then $M$ is flat over $R$ if and only if, at every maximal ideal $m$, the localization $M_m$ is flat over $R_m$.

Proof: If $M$ is flat over $R$, then $M \otimes_R R_m$ is flat over $R_m$ by (5.11). But $M \otimes_R R_m = M_m$ by (13.49). Thus $M_m$ is flat over $R_m$.

Conversely, assume $M_m$ is flat over $R_m$ for every $m$. Let $\alpha : N' \to N$ be an injection of $R$-modules. Then $\alpha_m$ is injective by (13.48). Hence $M_m \otimes_R m\alpha$ is injective. But that map is equal to $(M \otimes \alpha)_m$ by (13.44). So $(M \otimes \alpha)_m$ is injective. Hence $M \otimes \alpha$ is injective by (13.49). Thus $M$ is flat over $R$.

Exercise (13.49). — Let $R$ be a ring, $R'$ a flat algebra, $p'$ a prime in $R'$, and $p$ its contraction in $R$. Prove that $R'_p$ is a faithfully flat $R_p$-algebra.

Exercise (13.49). — Let $R$ be a ring, $S$ a multiplicative subset.

1. Assume $R$ is absolutely flat. Show $S^{-1} R$ is absolutely flat.

2. Show $R$ is absolutely flat if and only if $R_m$ is a field for each maximal $m$.

Definition (13.49). — Let $R$ be a ring, $M$ a module. We say $M$ is locally finitely generated if each $p \in \text{Spec}(R)$ has a neighborhood on which $M$ becomes finitely generated; more precisely, there exists $f \in R - p$ such that $M_f$ is finitely generated over $R_f$. It is enough that an $f$ exist for each maximal ideal $m$ as every $p$ lies in some $m$ by (2.30). Similarly, we define the properties locally finitely presented, locally free of finite rank, and locally free of rank $n$.

Proposition (13.50). — Let $R$ be a ring, $M$ a module.

1. If $M$ is locally finitely generated, then it is finitely generated.

2. If $M$ is locally finitely presented, then it is finitely presented.

Proof: By (13.24), there are $f_1, \ldots, f_n \in R$ with $\bigcup D(f_i) = \text{Spec}(R)$ and finitely many $m_{i,j} \in M$ such that, for some $n_{i,j} \geq 0$, the $m_{i,j}/f_i^{n_{i,j}}$ generate $M_{f_i}$. Clearly, for each $i$, the $m_{i,j}/1$ also generate $M_{f_i}$.

Given any maximal ideal $m$, there is $i$ such that $f_i \notin m$. Let $S_i$ be the image of $R - m$ in $R_{f_i}$. Then (13.48) yields $M_m = S_i^{-1}(M_{f_i})$. Hence the $m_{i,j}/1$ generate
\[ M_m. \text{ Thus (13.53) yields (1).} \]

Assume \( M \) is locally finitely presented. Then \( M \) is finitely generated by (1). So there is a surjection \( R^k \to M \). Let \( K \) be its kernel. Then \( K \) is locally finitely generated owing to (13.44). Hence \( K \) too is finitely generated by (1). So there is a surjection \( R^t \to K \). It yields the desired finite presentation \( R^k \to R^t \to M \to 0 \). Thus (2) holds. □

**Theorem (13.51).** — These conditions on an \( R \)-module \( P \) are equivalent:

1. \( P \) is finitely generated and projective.
2. \( P \) is finitely presented and flat.
3. \( P \) is finitely presented, and \( P_m \) is free over \( R_m \) at each maximal ideal \( m \).
4. \( P \) is locally free of finite rank.
5. \( P \) is finitely generated, and for each \( \mathfrak{p} \in \text{Spec}(R) \), there are \( f \) and \( n \) such that \( \mathfrak{p} \in D(f) \) and \( P_\mathfrak{p} \) is free of rank \( n \) over \( R_\mathfrak{p} \) at each \( \mathfrak{q} \in D(f) \).

**Proof.** Condition (1) implies (2) by (13.46).

Let \( m \) be a maximal ideal. Then \( R_m \) is local by (13.22). If \( P \) is finitely presented, then \( P_m \) is finitely presented, because localization preserves direct sums and cokernels by (13.21).

Assume (2). Then \( P_m \) is flat by (13.46), so free by (13.41). Thus (3) holds.

Assume (3). Fix a surjective map \( \alpha : M \to N \). Then \( \alpha_m : M_m \to N_m \) is surjective. So \( \text{Hom}(P_m, \alpha_m) : \text{Hom}(P_m, M_m) \to \text{Hom}(P_m, N_m) \) is surjective by (13.50) and (13.47). But \( \text{Hom}(P_m, \alpha_m) = \text{Hom}(P, \alpha)_m \) by (13.48) as \( P \) is finitely presented. Further, \( m \) is arbitrary. Hence \( \text{Hom}(P, \alpha) \) is surjective by (13.45). Therefore, \( P \) is projective by (13.23). Thus (1) holds.

Again assume (3). Given any prime \( \mathfrak{p} \), take a maximal ideal \( \mathfrak{m} \) containing it. By hypothesis, \( P_\mathfrak{m} \) is free; its rank is finite as \( P_m \) is finitely generated. By (13.22)(2), there is \( f \in R - m \) such that \( P_f \) is free of finite rank over \( R_f \). Thus (4) holds.

Assume (4). Then \( P \) is locally finitely presented. So \( P \) is finitely presented by (13.51)(2). Further, given \( \mathfrak{p} \in \text{Spec}(R) \), there are \( f \in R - \mathfrak{p} \) and \( n \) such that \( P_f \) is free of rank \( n \) over \( R_f \). Given \( \mathfrak{q} \in D(f) \), let \( S \) be the image of \( R - \mathfrak{q} \) in \( R_f \). Then (13.41) yields \( P_\mathfrak{q} = S^{-1}(P_f) \). Hence \( P_\mathfrak{q} \) is free of rank \( n \) over \( R_\mathfrak{q} \). Thus (5) holds. Further, (3) results from taking \( \mathfrak{p} := m \) and \( \mathfrak{q} := m \).

Finally, assume (5), and let’s prove (4). Given \( \mathfrak{p} \in \text{Spec}(R) \), let \( f \) and \( n \) be provided by (5). Take a free basis \( p_1/f^{k_1}, \ldots, p_n/f^{k_n} \) of \( P_\mathfrak{p} \) over \( R_\mathfrak{p} \). The \( p_i \) define a map \( \alpha : R^n \to P \), and \( \alpha_\mathfrak{p} : R^n_\mathfrak{p} \to P_\mathfrak{p} \) is injective, in particular, surjective.

As \( P \) is finitely generated, (13.22)(1) provides \( g \in R - \mathfrak{p} \) such that \( \alpha_\mathfrak{p} : R^n_\mathfrak{p} \to P_\mathfrak{p} \) is surjective. It follows that \( \alpha_\mathfrak{q} : R^n_\mathfrak{q} \to P_\mathfrak{q} \) is surjective for every \( \mathfrak{q} \in D(f) \). If also \( \mathfrak{q} \in D(f) \), then by hypothesis \( P_\mathfrak{q} \cong R^n_\mathfrak{q} \). So \( \alpha_\mathfrak{q} \) is bijective by (13.3). Set \( h := fg \). Clearly, \( D(f) \cap D(g) = D(h) \). By (13.3), \( D(h) = \text{Spec}(R_h) \). Clearly, \( \alpha_\mathfrak{q} = (\alpha_\mathfrak{h})_{(g_R_h)} \) for all \( \mathfrak{q} \in D(h) \). Hence \( \alpha_\mathfrak{h} : R^n_\mathfrak{h} \to P_\mathfrak{h} \) is bijective owing to (13.3) with \( R_h \) for \( R \). Thus (4) holds.

**Exercise (13.57).** — Given \( n \), prove an \( R \)-module \( P \) is locally free of rank \( n \) if and only if \( P \) is finitely generated and \( P_m \cong \bigoplus_{i=1}^n R_{\mathfrak{m}} \) holds at each maximal ideal \( \mathfrak{m} \).

**Exercise (13.58).** — Let \( A \) be a semilocal ring, \( P \) a locally free module of rank \( n \). Show that \( P \) is free of rank \( n \).

**Exercise (13.59).** — Let \( R \) be a ring, \( M \) a finitely presented module, \( n \geq 0 \). Show that \( M \) is locally free of rank \( n \) if and only if \( F_{n-1}(M) = 0 \) and \( F_n(M) = R \).

Krull–Cohen–Seidenberg Theory relates the prime ideals in a ring to those in an integral extension. We prove each prime has at least one prime lying over it—that is, contracting to it. The overprime can be taken to contain any ideal that contracts to an ideal contained in the given prime; this stronger statement is known as the Going-up Theorem. Further, one prime is maximal if and only if the other is, and two overprimes cannot be nested. On the other hand, the Going-down Theorem asserts that, given nested primes in the subring and a prime lying over the larger, there is a subprime lying over the smaller, either if the subring is normal and the overring is a domain or if the extension is flat even if it’s not integral.

Lemma (14.1). — Let \( R \subseteq R' \) be an integral extension of domains. Then \( R' \) is a field if and only if \( R \) is.

**Proof:** First, suppose \( R' \) is a field. Let \( x \in R \) be nonzero. Then \( 1/x \in R' \), so satisfies an equation of integral dependence:

\[
(1/x)^n + a_1(1/x)^{n-1} + \cdots + a_n = 0
\]

with \( n \geq 1 \) and \( a_i \in R \). Multiplying the equation by \( x^{n-1} \), we obtain

\[
1/x = -(a_1 + a_{n-2}x + \cdots + a_n x^{n-1}) \in R.
\]

Conversely, suppose \( R \) is a field. Let \( y \in R' \) be nonzero. Then \( y \) satisfies an equation of integral dependence

\[
y^n + a_1 y^{n-1} + \cdots + a_{n-1} y + a_n = 0
\]

with \( n \geq 1 \) and \( a_i \in R \). Rewriting the equation, we obtain

\[
y(y^{n-1} + \cdots + a_{n-1}) = -a_n.
\]

Take \( n \) minimal. Then \( a_n \neq 0 \) as \( R' \) is a domain. So dividing by \( -a_n y \), we obtain

\[
1/y = (-1/a_n)(y^{n-1} + \cdots + a_{n-1}) \in R'.\]

\[\square\]

Definition (14.2). — Let \( R \) be a ring, \( R' \) an \( R \)-algebra, \( p \) a prime of \( R \), and \( p' \) a prime of \( R' \). We say \( p' \) **lies over** \( p \) if \( p' \) contracts to \( p \).

Theorem (14.3). — Let \( R \subseteq R' \) be an integral extension of rings, and \( p \) a prime of \( R \). Let \( p' \subseteq q' \) be nested primes of \( R' \), and \( a' \) an arbitrary ideal of \( R' \).

1. **(Maximality)** Suppose \( p' \) lies over \( p \). Then \( p' \) is maximal if and only if \( p \) is.
2. **(Incomparability)** Suppose both \( p' \) and \( q' \) lie over \( p \). Then \( p' = q' \).
3. **(Lying over)** Then there is a prime \( \mathfrak{c}' \) of \( R' \) lying over \( p \).
4. **(Going up)** Suppose \( a' \cap R \subseteq p \). Then in (3) we can take \( \mathfrak{c}' \) to contain \( a' \).

**Proof:** Assertion (1) follows from (14.1) applied to the extension \( R/p \subseteq R'/p' \), which is integral as \( R \subseteq R' \) is, since, if \( y \in R' \) satisfies \( y^n + a_1 y^{n-1} + \cdots + a_n = 0 \), then reduction modulo \( p' \) yields an equation of integral dependence over \( R/p \).

To prove (2), localize at \( R - p \), and form this commutative diagram:

\[
\begin{array}{ccc}
R' & \to & R'_p \\
\uparrow & & \uparrow \\
R & \to & R_p
\end{array}
\]
Here \( R_p \to R'_p \) is injective by (14.29), and the extension is integral by (14.30).

Here \( p'R'_p \) and \( q'R'_p \) are nested primes of \( R'_p \) by (14.31). By the same token, both lie over \( pR_p \), because both their contractions in \( R_p \) contract to \( p \) in \( R \). Thus we may replace \( R \) by \( R_p \) and \( R' \) by \( R'_p \), and so assume \( R \) is local with \( p \) as maximal ideal by (14.32). Then \( p' \) is maximal by (1); whence, \( p' = q' \).

To prove (3), again we may replace \( R \) by \( R_p \) and \( R' \) by \( R'_p \); if \( t'' \) is a prime ideal of \( R'_p \), lying over \( pR_p \), then the contraction \( t' \) of \( t'' \) in \( R' \) lies over \( p \). So we may assume \( R \) is local with \( p \) as unique maximal ideal. Now, \( R' \) has a maximal ideal \( t' \) by (14.34); further, \( t' \) contracts to a maximal ideal \( r \) of \( R \) by (1). Thus \( r = p \).

Finally, (4) follows from (3) applied to the extension \( R/(a' \cap R) \subset R'/a' \). □

Exercise (14.33). — Let \( R \subset R' \) be an integral extension of rings, and \( p \) a prime of \( R \). Suppose \( R' \) has just one prime \( p' \) over \( p \). Show (a) that \( p'R'_p \) is the only maximal ideal of \( R'_p \), (b) that \( R'_p \), and (c) that \( R'_p \) is integral over \( R_p \).

Exercise (14.34). — Let \( R \subset R' \) be an integral extension of domains, and \( p \) a prime of \( R \). Suppose \( R' \) has at least two distinct primes \( p' \) and \( q' \) over \( p \). Show that \( R'_p \) is not integral over \( R_p \). Show that, in fact, if \( y \) lies in \( q' \), but not in \( p' \), then \( 1/y \in R'_p \) is not integral over \( R_p \).

Exercise (14.35). — Let \( k \) be a field, and \( X \) an indeterminate. Set \( R' := k[X] \), and \( R := k[Y] \). Set \( p := (Y - 1)R \) and \( p' := (X - 1)R' \). Is \( R'_p \), integral over \( R_p \)? Explain.

Lemma (14.7). — Let \( R \subset R' \) be a ring extension, \( X \) a variable, \( f \in R[X] \) a monic polynomial. Suppose \( f = gh \) with \( g, h \in R'[X] \) monic. Then the coefficients of \( g \) and \( h \) are integral over \( R \).

Proof: Set \( R_1 := R'[X]/(g) \). Let \( x_1 \) be the residue of \( X \). Then \( 1, x_1, x_1^2, \ldots \) form a free basis of \( R_1 \) over \( R' \) by (14.25) as \( g \) is monic; hence, \( R' \subset R_1 \). Now, \( g(x_1) = 0 \); so \( g \) factors as \( (X - x_1)g_1 \) with \( g_1 \in R_1[X] \) monic of degree 1 less than \( g \). Repeat this process, extending \( R_1 \). Continuing, obtain \( g(X) = \prod(X - x_i) \) and \( h(X) = \prod(X - y_j) \) with all \( x_i \) and \( y_j \) in an extension of \( R' \). The \( x_i \) and \( y_j \) are integral over \( R \) as they are roots of \( f \). But the coefficients of \( g \) and \( h \) are polynomials in the \( x_i \) and \( y_j \); so they too are integral over \( R \). □

Proposition (14.8). — Let \( R \) be a normal domain, \( K := \text{Frac}(R) \), and \( L/K \) a field extension. Let \( y \in L \) be integral over \( R \), and \( p \in K[X] \) its monic minimal polynomial. Then \( p \in R[X] \), and \( p(y) = 0 \) is an equation of integral dependence.

Proof: Since \( y \) is integral, there is a monic polynomial \( f \in R[X] \) with \( f(y) = 0 \). Write \( f = pq \) with \( q \in K[X] \). Then by (14.37) the coefficients of \( p \) are integral over \( R \), so in \( R \) since \( R \) is normal. □

Theorem (14.9) (Going down for integral extensions). — Let \( R \subset R' \) be an integral extension of domains with \( R \) normal, \( p \not\subset q \) nested primes of \( R \), and \( q' \) a prime of \( R' \) lying over \( q \). Then there is a prime \( p' \) lying over \( p \) and contained in \( q' \).

Proof: First, let us show \( pR'_q \cap R = p \). Given \( y \in pR'_q \cap R \), say \( y = x/s \) with \( x \in pR' \) and \( s \in R' - q' \). Say \( x = \sum_{i=1}^n y_i x_i \) with \( y_i \in p \) and \( x_i \in R' \), and set \( R'' := R[x_1, \ldots, x_n] \). Then \( R'' \) is module finite by (14.28) and \( xR'' \subset pR'' \). Let \( f(X) = X^n + a_1 X^{n-1} + \cdots + a_n \) be the characteristic polynomial of \( \mu_x \colon R'' \to R'' \).
Let the canonical map \( K = R \) and thus assume (by \( q \)) that \( z \) trivially. Thus (Exercise (14.11)) \( x \) is a prime of \( R \) and contained in \( p \). Then \( f \) is the minimal polynomial of \( x \) over \( K \).

Recall \( s = x/y \). So \( s \) satisfies the equation

\[
s^n + b_1 s^{n-1} + \cdots + b_n = 0 \quad \text{with} \quad b_i := a_i/y^i \in K. \tag{14.9.1}
\]

Conversely, any such equation yields one of the same degree for \( x \) as \( y \in R \subset K \). So (14.9.1) is the minimal polynomial of \( s \) over \( K \). So all \( b_i \) are in \( R \) by (13.8).

Suppose \( y \notin p \). Then \( b_i \in p \) as \( a_i = b_i y^i \in p \). So \( s^n \in pR' \subset qR' \subset q' \). So \( s \in q' \), a contradiction. Hence \( y \in p \). Thus \( pR'_q \cap R \subset p \). But the opposite inclusion holds trivially. Thus \( pR'_q \cap R = p \).

Hence, there is a prime \( p'' \) of \( R'_q \) with \( p'' \cap R = p \) by (13.8). Then \( p'' \) lies in \( q'R'_q \) as it is the only maximal ideal. Set \( p' := p'' \cap R' \). Then \( p' \cap R = p \), and \( p' \subset q' \) by (13.20) (2), as desired. \( \square \)

**Lemma (14.10).** Always, a minimal prime consists entirely of zerodivisors.

**Proof:** Let \( R \) be the ring, \( p \) the minimal prime. Then \( R_p \) has only one prime \( pR_p \) by (13.20) (2). So by the Scheinnullstellensatz, \( pR_p \) consists entirely of nilpotents. Hence, given \( x \in p \), there is \( s \in R - p \) with \( sx^n = 0 \) for some \( n \geq 1 \). Take \( n \) minimal. Then \( sx^{n-1} \neq 0 \), but \( (sx^{n-1})x = 0 \). Thus \( x \) is a zerodivisor. \( \square \)

**Theorem (14.11) (Going down for Flat Algebras).** Let \( R \) be a ring, \( R' \) a flat algebra, \( p \triangleleft q \) nested primes of \( R \), and \( q' \) a prime of \( R' \) lying over \( q \). Then there is a prime \( p' \) lying over \( p \) and contained in \( q' \).

**Proof:** The canonical map \( R_q \to R'_q \) is faithfully flat by (13.20). Therefore, \( \text{Spec}(R'_q) \to \text{Spec}(R_q) \) is surjective by (13.20). Thus (13.20) yields the desired \( p' \).

Alternatively, \( R' \otimes_R (R/p) \) is flat over \( R/p \) by (13.40). Also, \( R'/pR' = R' \otimes_R R/p \) by (13.40) (1). Hence, owing to (13.6), we may replace \( R \) by \( R/p \) and \( R' \) by \( R'/pR' \), and thus assume \( R \) is a domain and \( p = 0 \).

By (13.3), \( q' \) contains a minimal prime \( p' \) of \( R' \). Let’s show that \( p' \) lies over \( (0) \). Let \( x \in R \) be nonzero. Then the multiplication map \( \mu_x: R \to R' \) is injective. Since \( R' \) is flat, \( \mu_x: R' \to R' \) is also injective. Hence, (13.40) implies that \( x \) does not belong to the contraction of \( p' \), as desired. \( \square \)

**Exercise (14.12).** Let \( R \) be a reduced ring, \( \Sigma \) the set of minimal primes. Prove that \( z\text{div}(R) = \bigcup_{p \in \Sigma} p \) and that \( R_p = \text{Frac}(R/p) \) for any \( p \in \Sigma \).

**Exercise (14.13).** Let \( R \) be a ring, \( \Sigma \) the set of minimal primes, and \( K \) the total quotient ring. Assume \( \Sigma \) is finite. Prove these three conditions are equivalent:

1. \( R \) is reduced.
2. \( z\text{div}(R) = \bigcup_{p \in \Sigma} p \), and \( R_p = \text{Frac}(R/p) \) for each \( p \in \Sigma \).
3. \( K/pK = \text{Frac}(R/p) \) for each \( p \in \Sigma \), and \( K = \prod_{p \in \Sigma} K/pK \).
Exercise (14.14). — Let $A$ be a reduced local ring with residue field $k$ and finite set $\Sigma$ of minimal primes. For each $p \in \Sigma$, set $K(p) := \text{Frac}(A/p)$. Let $P$ be a finitely generated module. Show that $P$ is free of rank $r$ if and only if $\dim_k(P \otimes_A k) = r$ and $\dim_{K(p)}(P \otimes_A K(p)) = r$ for each $p \in \Sigma$.

Exercise (14.15). — Let $A$ be a reduced local ring with residue field $k$ and a finite set of minimal primes. Let $P$ be a finitely generated module, $B$ an $A$-algebra with $\text{Spec}(B) \to \text{Spec}(A)$ surjective. Show that $P$ is a free $A$-module of rank $r$ if and only if $P \otimes B$ is a free $B$-module of rank $r$.

(14.16) (Arbitrary normal rings). — An arbitrary ring $R$ is said to be normal if $R_p$ is a normal domain for every prime $p$. If $R$ is a domain, then this definition recovers that in (10.30), owing to (11.32).

Exercise (14.17). — Let $R$ be a ring, $p_1, \ldots, p_r$ all its minimal primes, and $K$ the total quotient ring. Prove that these three conditions are equivalent:

1. $R$ is normal.
2. $R$ is reduced and integrally closed in $K$.
3. $R$ is a finite product of normal domains $R_i$.

Assume the conditions hold. Prove the $R_i$ are equal to the $R/p_j$ in some order.
15. Noether Normalization

The Noether Normalization Lemma describes the basic structure of a finitely generated algebra over a field; namely, given a chain of ideals, there is a polynomial subring over which the algebra is module finite, and the ideals contract to ideals generated by initial segments of variables. After proving this lemma, we derive several versions of the Nullstellensatz. The most famous is Hilbert’s; namely, the radical of any ideal is the intersection of all the maximal ideals containing it.

Then we study the (Krull) dimension: the maximal length of any chain of primes. We prove our algebra is catenary; that is, if two chains have the same ends and maximal lengths, then the lengths are the same. Further, if the algebra is a domain, then its dimension is equal to the transcendence degree of its fraction field.

In an appendix, we give a simple direct proof of the Hilbert Nullstellensatz. At the same time, we prove it in significantly greater generality: for Jacobson rings.

Lemma (15.1) (Noether Normalization). — Let $k$ be a field, $R := k[x_1, \ldots, x_n]$ a finitely generated $k$-algebra, and $a_1 \subseteq \cdots \subseteq a_r$ a chain of proper ideals of $R$. Then there are algebraically independent elements $t_1, \ldots, t_r \in R$ such that

1. $R$ is module finite over $P := k[t_1, \ldots, t_r]$ and
2. for $i = 1, \ldots, r$, there is an $h_i$ such that $a_i \cap P = \langle t_1, \ldots, t_{i-1}, h_i \rangle$.

If $k$ is infinite, then we may choose the $t_i$ to be $k$-linear combinations of the $x_i$.

Proof: Let $R' := k[X_1, \ldots, X_n]$ be the polynomial ring, and $\varphi: R' \to R$ the $k$-algebra map with $\varphi X_i := x_i$. Set $a'_0 := \text{Ker} \varphi$ and $a'_i := \varphi^{-1} a_i$ for $i = 1, \ldots, r$. It suffices to prove the lemma for $R'$ and $a'_0 \subseteq \cdots \subseteq a'_r$: if $t'_i \in R'$ and $h'_i$ work here, then $t_i := \varphi t'_i + h'_i$ and $h := h'_i - h'_0$ work for $R$ and the $a_i$, because the $t_i$ are algebraically independent by (15.1), and clearly (1) and (2) hold. Thus we may assume the $x_i$ are algebraically independent.

The proof proceeds by induction on $r$ (and shows $v := n$ works now).

First, assume $r = 1$ and $a_1 = t_1 R$ for some nonzero $t_1$. Then $t_1 \notin k$ because $a_1$ is proper. Suppose we have found $t_2, \ldots, t_n \in R$ so that $x_1$ is integral over $P := k[t_1, t_2, \ldots, t_r]$ and so that $P[x_1] = R$. Then (15.28) yields (1).

Further, by the theory of transcendence bases [8, (8.3), p. 526], [10, Thm. 1.1, p. 356], the elements $t_1, \ldots, t_n$ are algebraically independent. Now, take $x \in a_1 \cap P$. Then $x = t_1 x'$ where $x' \in R \cap \text{Frac}(P)$. Also, $R \cap \text{Frac}(P) = P$, for $P$ is normal by (15.29) as $P$ is a polynomial algebra. Hence $a_1 \cap P = t_1 P$. Thus (2) holds too.

To find $t_2, \ldots, t_n$, we are going to choose $\ell_i$ and set $t_i := x_i - x_1^{\ell_i}$. Then clearly $P[x_1] = R$. Now, say $t_1 = \sum a_{ij} x_1^{\ell_1} \cdots x_n^{\ell_n}$ with $(j) := (j_1, \ldots, j_n)$ and $a_{ij} \in k$. Recall $t_1 \notin k$, and note that $x_1$ satisfies this equation:

$$\sum a_{ij} x_1^{\ell_1} (t_2 + x_1^{\ell_2})^{j_2} \cdots (t_n + x_1^{\ell_n})^{j_n} = t_1.$$  

Set $e(j) := j_1 + \ell_2 j_2 + \cdots + \ell_n j_n$. Take $\ell > \max \{j_i\}$ and $\ell_i := \ell$. Then the $e(j)$ are distinct. Let $e(j')$ be largest among the $e(j)$ with $a_{ij} \neq 0$. Then $e(j') > 0$, and the above equation may be rewritten as follows:

$$a_{ij'} x_1^{e(j')} + \sum_{e < e(j')} p_i x_1^e = 0$$

where $p_e \in P$. Thus $x_1$ is integral over $P$, as desired.
Suppose \( k \) is infinite. We are going to reorder the \( x_i \), choose \( a_i \in k \), and set \( t_i := x_i - a_ix_1 \). Then \( P[x_i] = R \). Now, say \( t_1 = H_d + \cdots + H_0 \) where \( H_d \neq 0 \) and where \( H_i \) is \textbf{homogeneous of degree} \( i \) in \( x_1, \ldots, x_n \); that is, \( H_i \) is a linear combination of monomials of degree \( i \). Then \( d > 0 \) as \( t_1 \not\in k \). As \( k \) is infinite, \( (5.21) \) yields \( a_i \in k \) with \( H_d(a_1, a_2, \ldots, a_n) \neq 0 \). Since \( H_d \) is homogeneous, \( a_i \neq 0 \) for some \( i \); reordering the \( x_i \), we may assume \( a_1 \neq 0 \). Again since \( H_d \) is homogeneous, we may replace \( a_i \) by \( a_i/a_1 \). Then \( H_d(1, a_2, \ldots, a_n) \neq 0 \). But \( H_d(1, a_2, \ldots, a_n) \) is the coefficient of \( x_1^d \) in \( H_d(x_1, t_2 + a_2x_1, \ldots, t_n + a_nx_1) \). So after we collect like powers of \( x_1 \), the equation

\[
H_d(x_1, t_2 + a_2x_1, \ldots, t_n + a_nx_1) + \cdots + H_0(x_1, t_2 + a_2x_1, \ldots, t_n + a_nx_1) + t_1 = 0
\]

becomes an equation of integral dependence for \( x_1 \) over \( P \), as desired.

Second, assume \( r = 1 \) and \( a_1 \) is arbitrary. We may assume \( a_1 \neq 0 \). The proof proceeds by induction on \( n \). The case \( n = 1 \) follows from the first case (but is simpler) because \( k[x_1] \) is a PID. Let \( t_1 \in a_1 \) be nonzero. By the first case, there exist elements \( u_2, \ldots, u_n \) such that \( t_1, u_2, \ldots, u_n \) are algebraically independent and satisfy (1) and (2) with respect to \( R \) and \( t_1R \). By induction, there are \( t_2, \ldots, t_n \) satisfying (1) and (2) with respect to \( k[u_2, \ldots, u_n] \) and \( a_1 \cap k[u_2, \ldots, u_n] \).

Set \( P := k[t_1, \ldots, t_n] \). Since \( R \) is module finite over \( k[t_1, u_2, \ldots, u_n] \) and the latter is module finite over \( P \), the former is module finite over \( P \) by (15.27). Thus (1) holds, and so \( t_1, \ldots, t_n \) are algebraically independent. Further, by assumption,

\[
a_1 \cap k[t_2, \ldots, t_n] = \langle t_2, \ldots, t_h \rangle
\]

for some \( h \). But \( t_1 \in a_1 \). So \( a_1 \cap P \supset \langle t_1, \ldots, t_h \rangle \).

Conversely, given \( x \in a_1 \cap P \), say \( x = \sum_{i=0}^d f_i t_i^1 \) with \( f_i \in k[t_2, \ldots, t_n] \). Since \( t_1 \in a_1 \), we have \( f_0 \in a_1 \cap k[t_2, \ldots, t_n] \); so \( f_0 \in \langle t_2, \ldots, t_h \rangle \). Hence \( x \in \langle t_1, \ldots, t_h \rangle \). Thus \( a_1 \cap P = \langle t_1, \ldots, t_h \rangle \). Thus (2) holds for \( r = 1 \).

Finally, assume the lemma holds for \( r - 1 \). Let \( u_1, \ldots, u_n \in R \) be algebraically independent elements satisfying (1) and (2) for the sequence \( a_1 \subset \cdots \subset a_{r-1} \), and set \( h := h_{r-1} \). By the second case, there exist elements \( t_{h+1}, \ldots, t_n \) satisfying (1) and (2) for \( k[u_{h+1}, \ldots, u_n] \) and \( a_r \cap k[u_{h+1}, \ldots, u_n] \). Then, for some \( h_r \),

\[
a_r \cap k[t_{h+1}, \ldots, t_n] = \langle t_{h+1}, \ldots, t_{h_r} \rangle.
\]

Set \( t_i := u_i \) for \( 1 \leq i \leq h \). Set \( P := k[t_1, \ldots, t_n] \). Then, by assumption, \( R \) is module finite over \( k[u_1, \ldots, u_n] \), and \( k[u_1, \ldots, u_n] \) is module finite over \( P \); hence, \( R \) is module finite over \( P \) by (15.27). Thus (1) holds, and \( t_1, \ldots, t_n \) are algebraically independent over \( k \).

Fix \( i \) with \( 1 \leq i \leq r \). Set \( m := h_i \). Then \( t_1, \ldots, t_m \in a_i \). Given \( x \in a_i \cap P \), say \( x = \sum f(v) t_1^{v_1} \cdots t_m^{v_m} \) with \( (v) = (v_1, \ldots, v_m) \) and \( f(v) \in k[t_{m+1}, \ldots, t_n] \). Then \( f(0) \) lies in \( a_i \cap k[t_{m+1}, \ldots, t_n] \). We are going to see the latter intersection is equal to \( \langle 0 \rangle \). It is so if \( i \leq r - 1 \) because it lies in \( a_i \cap k[u_{m+1}, \ldots, u_n] \), which is equal to \( \langle 0 \rangle \). Further, if \( i = r \), then, by assumption, \( a_i \cap k[t_{m+1}, \ldots, t_n] = \langle t_{m+1}, \ldots, t_m \rangle = 0 \). Thus \( f(0) = 0 \). Hence \( x \in \langle t_1, \ldots, t_{h_r} \rangle \). Thus \( a_i \cap P \supset \langle t_1, \ldots, t_{h_r} \rangle \). So the two are equal. Thus (2) holds, and the proof is complete.

Exercise (15.4). — Let \( k := \mathbb{F}_q \) be the finite field with \( q \) elements, and \( k[X, Y] \) the polynomial ring. Set \( f := XY - X^q \) and \( R := k[X, Y] / \langle f \rangle \). Let \( x, y \in R \) be the residues of \( X, Y \). For every \( a \in k \), show that \( R \) is not module finite over \( P := k[y - ax] \). (Thus, in (15.30), no \( k \)-linear combination works.) First, take \( a = 0 \).
Exercise (15.7). — Let $k$ be a field, and $X, Y, Z$ variables. Set

$$R := k[X, Y, Z]/\langle X^2 - Y^3 - 1, XZ - 1 \rangle,$$

and let $x, y, z \in R$ be the residues of $X, Y, Z$. Fix $a, b \in k$, and set $t := x + ay + bz$ and $P := k[t]$. Show that $x$ and $y$ are integral over $P$ for any $a, b$ and that $z$ is integral over $P$ if and only if $b \neq 0$.

Theorem (15.4) (Zariski Nullstellensatz). — Let $k$ be a field, $R$ an algebra-finite extension. Assume $R$ is a field. Then $R/k$ is finite.

Proof: By the Noether Normalization Lemma (15.1), $R$ is module finite over a polynomial subring $P := k[t_1, \ldots, t_n]$. Then $R/P$ is integral by (15.2). As $R$ is a field, so is $P$ by (14.1). Hence $\nu = 0$. So $P = k$. Thus $R/k$ is finite, as asserted.

Alternatively, here’s a short proof, not using (15.1). Say $R = k[x_1, \ldots, x_n]$. Set $P := k[x_1]$ and $K := \text{Frac}(P)$. Then $R = K[x_2, \ldots, x_n]$. By induction on $n$, assume $R/K$ is finite. Suppose $x_1$ is transcendental over $k$, so $P$ is a polynomial ring.

Note $R = P[x_2, \ldots, x_n]$. Hence (15.2) yields $f \in P$ with $R_f/P_f$ module finite, so integral by (14.2). But $R_f = R$. Thus $P_f$ is a field by (14.1). So $f \notin k$.

Set $g := 1 + f$. Then $1/g \in P_f$. So $1/g = h/f^r$ for some $h \in P$ and $r \geq 1$. Then $f^r = gh$. But $f$ and $g$ are relatively prime, a contradiction. Thus $x_1$ is algebraic over $k$. Hence $P = K$, and $K/k$ is finite. But $R/K$ is finite. Thus $R/k$ is too. \(\square\)

Corollary (15.5). — Let $k$ be a field, $R := k[x_1, \ldots, x_n]$ an algebra-finite extension, and $m$ a maximal ideal of $R$. Assume $k$ is algebraically closed. Then there are $a_1, \ldots, a_n \in k$ such that $m = (x_1 - a_1, \ldots, x_n - a_n)$.

Proof: Set $K := R/m$. Then $K$ is a finite extension field of $k$ by the Zariski Nullstellensatz (15.3). But $k$ is algebraically closed. Hence $k = K$. Let $a_i \in k$ be the residue of $x_i$, and set $n := (x_1 - a_1, \ldots, x_n - a_n)$. Then $n \subset m$.

Let $R' := k[X_1, \ldots, X_n]$ be the polynomial ring, and $\varphi: R' \to R$ the $k$-algebra map with $\varphi X_i := x_i$. Set $n' := (X_1 - a_1, \ldots, X_n - a_n)$. Then $\varphi(n') = n$. But $n'$ is maximal by (7.21). So $n$ is maximal. Hence $n = m$, as desired. \(\square\)

Corollary (15.6). — Let $k$ be any field, $P := k[X_1, \ldots, X_n]$ the polynomial ring, and $m$ a maximal ideal of $P$. Then $m$ is generated by $n$ elements.

Proof: Set $K := P/m$. Then $K$ is a field. So $K/k$ is finite by (15.3).

Induct on $n$. If $n = 0$, then $m = 0$. Assume $n \geq 1$. Set $R := k[X_1]$ and $p := m \cap R$. Then $p = (f_1)$ for some $f_1 \in R$ as $R$ is a PID. Set $k_1 := R/p$. Then $k_1$ is isomorphic to the image of $R$ in $K$. But $K$ is a finite-dimensional $k$-vector space. So $k_1$ is too. So $k \subset k_1$ is an integral extension by (14.23). Since $k$ is a field, so is $k_1$ by (14.1).

Note $P/pP = k_1[X_2, \ldots, X_n]$ by (14.1). But $m/p$ is a maximal ideal. So by induction $m/p$ is generated by $n - 1$ elements, say the residues of $f_2, \ldots, f_n \in m$. Then $m = (f_1, \ldots, f_n)$, as desired. \(\square\)

Theorem (15.7) (Hilbert Nullstellensatz). — Let $k$ be a field, and $R$ a finitely generated $k$-algebra. Let $\mathfrak{a}$ be a proper ideal of $R$. Then

$$\sqrt{\mathfrak{a}} = \bigcap_{m \supseteq \mathfrak{a}} m$$

where $m$ runs through all maximal ideals containing $\mathfrak{a}$. 

We may assume \( a = 0 \) by replacing \( R \) by \( R/a \). Clearly \( \sqrt{0} \subset \mathfrak{m} \). Conversely, take \( f \notin \sqrt{0} \). Then \( R_f \neq 0 \) by (10.23). So \( R_f \) has a maximal ideal \( \mathfrak{n} \) by (2.30). Let \( \mathfrak{m} \) be its contraction in \( R \). Now, \( R \) is a finitely generated \( k \)-algebra by hypothesis; hence, \( R_f \) is one too owing to (10.12). Therefore, by the weak Nullstellensatz, \( R_f/\mathfrak{n} \) is a finite extension field of \( k \).

Set \( K := R/\mathfrak{m} \). By construction, \( K \) is a \( k \)-subalgebra of \( R_f/\mathfrak{n} \). Therefore, \( K \) is a finite-dimensional \( k \)-vector space. So \( k \subset K \) is an integral extension by (10.23). Since \( k \) is a field, so is \( K \) by (10.11). Thus \( \mathfrak{m} \) is maximal. But \( f/1 \) is a unit in \( R_f \); so \( f/1 \notin \mathfrak{n} \). Hence \( f \notin \mathfrak{m} \). So \( f \notin \bigcap \mathfrak{m} \). Thus \( \sqrt{0} = \bigcap \mathfrak{m} \).

**Exercise (15.5).** — Let \( k \) be a field, \( K \) an algebraically closed extension field. (So \( K \) contains a copy of every finite extension field.) Let \( P := k[\{X_1, \ldots, X_n\}] \) be the polynomial ring, and \( f, f_1, \ldots, f_r \in P \). Assume \( f \) vanishes at every zero in \( K^n \) of \( f, f_1, \ldots, f_r \); in other words, if \( (a) := (a_1, \ldots, a_n) \in K^n \) and \( f_1(a) = 0, \ldots, f_r(a) = 0 \), then \( f(a) = 0 \). Prove that there are polynomials \( g_1, \ldots, g_r \in P \) and an integer \( N \) such that \( f^N = g_1f_1 + \cdots + g_rf_r \).

**Lemma (15.9).** — Let \( k \) be a field, \( R \) a finitely generated \( k \)-algebra. Assume \( R \) is a domain. Let \( \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r \) be a chain of primes. Set \( K := \text{Frac}(R) \) and \( d := \text{tr.deg}_k K \). Then \( r \leq d \), with equality if and only if the chain is maximal, that is, it is not a proper subchain of a longer chain.

**Proof:** By the Noether Normalization Lemma (15.1), \( R \) is module finite over a polynomial ring \( P := k[t_1, \ldots, t_r] \) such that \( \mathfrak{p}_i \cap P = (t_1, \ldots, t_{h_i}) \) for suitable \( h_i \). Set \( L := \text{Frac}(P) \). Then \( \nu := \text{tr.deg}_k L \). But \( P \subset R \) is an integral extension by (10.23). So \( L \subset K \) is an algebraic extension. Hence \( \nu = d \). Now, Incomparability (14.8)(2) yields \( h_i < h_{i+1} \) for all \( i \). Hence \( r \leq h_r \). But \( h_r \leq \nu \) and \( \nu = d \). Thus \( r \leq d \).

If \( r = d \), then \( r \) is maximal, as it was just proved that no chain can be longer. Conversely, assume \( r \) is maximal. Then \( \mathfrak{p}_0 = \{0\} \) since \( R \) is a domain. So \( h_0 = 0 \). Further, \( \mathfrak{p}_r \) is maximal since \( \mathfrak{p}_r \) is contained in some maximal ideal and it is prime. So \( \mathfrak{p}_r \cap P \) is maximal by Maximality (14.8)(1). Hence \( h_r = \nu \).

Suppose there is an \( i \) such that \( h_i + 1 < h_{i+1} \). Then\( (\mathfrak{p}_i \cap P) \subsetneq (t_1, \ldots, t_{h_i+1}) \subsetneq (\mathfrak{p}_{i+1} \cap P) \).

Now, \( P/(\mathfrak{p}_i \cap P) \) is, by (10.44), equal to \( k[t_{h_i+1}, \ldots, t_r] \); the latter is a polynomial ring, so normal by (10.25)(1). Also, the extension \( P/(\mathfrak{p}_i \cap P) \subset R/\mathfrak{p}_i \) is integral as \( P \subset R \) is. Hence, the Going-down Theorem (14.3) yields a prime \( \mathfrak{p} \) with \( \mathfrak{p}_i \subset \mathfrak{p} \subset \mathfrak{p}_{i+1} \) and \( \mathfrak{p} \cap P = (t_1, \ldots, t_{h_{i+1}}) \). Then \( \mathfrak{p}_i \subsetneq \mathfrak{p} \subsetneq \mathfrak{p}_{i+1} \), contradicting the maximality of \( r \). Thus \( h_i + 1 = h_{i+1} \) for all \( i \). But \( h_0 = 0 \). Hence \( h_r = \nu \). But \( h_r = \nu \) and \( \nu = d \). Thus \( r = d \), as desired.

(15.10) (Krull Dimension). — Given a ring \( R \), its (Krull) dimension \( \dim(R) \) is the supremum of the lengths \( r \) of all strictly ascending chains of primes:

\[
\dim(R) := \sup \{ r \mid \text{there's a chain of primes } \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r \text{ in } R \}.
\]

For example, if \( R \) is a field, then \( \dim(R) = 0 \); more generally, \( \dim(R) = 0 \) if and only if every minimal prime is maximal. If \( R \) is a PID, but not a field, then \( \dim(R) = 1 \), as every nonzero prime is maximal by (2.24).

**Exercise (15.14).** — Let \( R \) be a domain of (finite) dimension \( r \), and \( \mathfrak{p} \) a nonzero prime. Prove that \( \dim(R/\mathfrak{p}) < r \).
Exercise (15.19). — Let $R'/R$ be an integral extension of rings. Prove that $\dim(R) = \dim(R')$.

Theorem (15.13). — Let $k$ be a field, $R$ a finitely generated $k$-algebra. If $R$ is a domain, then $\dim(R) = \text{tr.deg}_k(\text{Frac}(R))$.

Proof: The assertion is an immediate consequence of (15.9) $\square$

Theorem (15.14). — Let $k$ be a field, $R$ a finitely generated $k$-algebra, $\mathfrak{p}$ a prime ideal, and $\mathfrak{m}$ a maximal ideal. Suppose $R$ is a domain. Then

$$\dim(R_\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R) \quad \text{and} \quad \dim(R_\mathfrak{m}) = \dim(R).$$

Proof: A chain of primes $\mathfrak{p}_0 \subseteq \cdots \subseteq \mathfrak{p} \subseteq \cdots \subseteq \mathfrak{p}_r$ in $R$ gives rise to a pair of chains of primes, one in $R_\mathfrak{p}$ and one in $R/\mathfrak{p}$,

$$\mathfrak{p}_0 R_\mathfrak{p} \subseteq \cdots \subseteq \mathfrak{p} R_\mathfrak{p} \quad \text{and} \quad 0 = \mathfrak{p}/\mathfrak{p} \subseteq \cdots \subseteq \mathfrak{p}_r/\mathfrak{p},$$

owing to (15.21) and to (15.4) and (2.7); conversely, every such pair arises from a unique chain in $R$ through $\mathfrak{p}$. But by (15.14), every maximal strictly ascending chain through $\mathfrak{p}$ is of length $\dim(R)$. The first equation follows.

Clearly $\dim(R/\mathfrak{m}) = 0$, and so $\dim(R_\mathfrak{p}) = \dim(R)$ $\square$

Definition (15.15). — We call a ring catenary if, given any two nested primes, all maximal chains of primes between the two have the same (finite) length.

Theorem (15.16). — Over a field, a finitely generated algebra is catenary.

Proof: Let $R$ be the algebra, and $\mathfrak{q} \subseteq \mathfrak{p}$ two nested primes. Replacing $R$ by $R/\mathfrak{q}$, we may assume $R$ is a domain. Then the proof of (15.14) shows that any maximal chain of primes $\langle 0 \rangle \subseteq \cdots \subseteq \mathfrak{p}$ is of length $\dim(R) - \dim(R/\mathfrak{p})$ $\square$

Exercise (15.17). — Let $k$ be a field, $R$ a finitely generated $k$-algebra, $f \in R$ nonzero. Assume $R$ is a domain. Prove that $\dim(R) = \dim(R_f)$.

Exercise (15.18). — Let $k$ be a field, $P := k[f]$ the polynomial ring in one variable $f$. Set $\mathfrak{p} := \langle f \rangle$ and $R := P_\mathfrak{p}$. Find $\dim(R)$ and $\dim(R_f)$.

Exercise (15.19). — Let $R$ be a ring, $R[X]$ the polynomial ring. Prove

$$1 + \dim(R) \leq \dim(R[X]) \leq 1 + 2 \dim(R).$$

(In particular, $\dim(R[X]) = \infty$ if and only if $\dim(R) = \infty$.)
**15. Appendix: Jacobson Rings**

(15.20) (*Jacobson Rings*). — We call a ring $R$ **Jacobson** if, given any ideal $a$, its radical is equal to the intersection of all maximal ideals containing it; that is,

$$
\sqrt{a} = \bigcap_{m \supseteq a} m. \quad (15.20.1)
$$

Plainly, the nilradical of a Jacobson ring is equal to its Jacobson radical. Also, any quotient ring of a Jacobson ring is Jacobson too. In fact, a ring is Jacobson if and only if the the nilradical of every quotient ring is equal to its Jacobson radical.

In general, the right-hand side of (15.20.1) contains the left. So (15.20.1) holds if and only if every $f$ outside $\sqrt{a}$ lies outside some maximal ideal $m$ containing $a$.

Recall the Scheinnullstellensatz, (4.34): it says $\sqrt{a} = \bigcap_{p \supseteq a} p$ with $p$ prime. Thus $R$ is Jacobson if and only if $p = \bigcap_{m \supseteq p} m$ for every prime $p$.

For example, a field $k$ is Jacobson; in fact, a local ring $A$ is Jacobson if and only if its maximal ideal is its only prime. Further, a Boolean ring $B$ is Jacobson, as every prime is maximal by (15.21), and so trivially $p = \bigcap_{m \supseteq p} m$ for every prime $p$. Finally, owing to (15.11) and (2.10) and the next lemma, a PID is Jacobson.

**Lemma (15.21).** — Let $R$ be a 1-dimensional domain, $\{m_\lambda\}_{\lambda \in \Lambda}$ its set of maximal ideals. Assume every nonzero element lies in only finitely many $m_\lambda$. Then $R$ is Jacobson if and only if $\Lambda$ is infinite.

**Proof:** If $\Lambda$ is finite, take a nonzero $x_\lambda \in m_\lambda$ for each $\lambda$, and set $x := \prod x_\lambda$. Then $x \neq 0$ and $x \in \bigcap m_\lambda$. But $\sqrt{0} = \{0\}$ as $R$ is a domain. So $\sqrt{0} \neq \bigcap m_\lambda$. Thus $R$ is not Jacobson.

If $\Lambda$ is infinite, then $\bigcap m_\lambda = \{0\}$ by hypothesis. But every nonzero prime is maximal as $R$ is 1-dimensional. Thus $p = \bigcap m_\lambda$ for every prime $p$. □

**Proposition (15.22).** — A ring $R$ is Jacobson if and only if, for any nonmaximal prime $p$ and any $f \notin p$, the extension $pR_f$ is not maximal.

**Proof:** Assume $R$ is Jacobson. Take a nonmaximal prime $p$ and an $f \notin p$. Then $f \notin m$ for some maximal ideal $m$ containing $p$. So $pR_f$ is not maximal by (11.20).

Conversely, let $a$ be an ideal, $f \notin \sqrt{a}$. Then $(R/a)_f \neq 0$. So there is a maximal ideal $n$ in $(R/a)_f$. Let $m$ be its contraction in $R$. Then $m \supseteq a$ and $f \notin m$. Further, (8.35) and (12.22) yield $R_f/mR_f = (R/a/m/a)_f = (R/a)_f/n$. Since $n$ is maximal, $R_f/mR_f$ is a field. So $m$ is maximal by hypothesis. Thus $R$ is Jacobson. □

**Exercise (15.24).** — Let $X$ be a topological space. We say a subset $Y$ is **locally closed** if $Y$ is the intersection of an open set and a closed set; equivalently, $Y$ is open in its closure $\overline{Y}$; equivalently, $Y$ is closed in an open set containing it.

We say a subset $X_0$ of $X$ is **very dense** if $X_0$ meets every nonempty locally closed subset $Y$. We say $X$ is Jacobson if its set of closed points is very dense.

Show that the following conditions on a subset $X_0$ of $X$ are equivalent:

1. $X_0$ is very dense.
2. Every closed set $F$ of $X$ satisfies $\overline{F} \cap X_0 = F$.
3. The map $U \mapsto U \cap X_0$ from the open sets of $X$ to those of $X_0$ is bijective.

**Exercise (15.25).** — Let $R$ be a ring, $X := \text{Spec}(R)$, and $X_0$ the set of closed points of $X$. Show that the following conditions are equivalent:
(1) \( R \) is a Jacobson ring.
(2) \( X \) is a Jacobson space.
(3) If \( y \in X \) is a point such that \( \{ y \} \) is locally closed, then \( y \in X_0 \).

**Lemma (15.25).** — Let \( R \subset R' \) be domains. Assume that \( R' = R[x] \) for some \( x \in R' \) and that there is \( y \in R' \) with \( R'_y \) a field. Then there is \( z \in R \) with \( R_z \) a field and \( x \) algebraic over \( R_z \). Further, if \( R \) is Jacobson, then \( R \) and \( R' \) are fields.

**Proof:** Set \( Q := \text{Frac}(R) \). Then \( Q \subset R'_y \), so \( R'_y = R[x]_y \subset Q'_y \). Hence \( Q[x]_y = R'_y \). So \( Q[x]_y \) is a field. Now, if \( x \) is transcendental over \( Q \), then \( Q[x] \) is a polynomial ring, so Jacobson by (15.24); whence, \( Q[x]_y \) is not a field by (15.22), a contradiction. Thus \( x \) is algebraic over \( Q \). Hence \( y \) is algebraic over \( Q \) too.

Let \( a_0 x^n + \cdots + a_n = 0 \) and \( b_0 y^m + \cdots + b_m = 0 \) be equations of minimal degree with \( a_i, b_j \in R \). Set \( z := a_0 b_m \). Then \( z \neq 0 \). Further,

\[
1/y = -a_0(b_0 y^{-m} + \cdots + b_{m-1})/z \in R_z[x].
\]

Hence \( R[x]_y \subset R_z[x] \subset R'_y \). Hence \( R_z[x] = R'_y \). Therefore \( R_z[x] \) is a field too. But \( x^n + (a_1 b_m/z)x^{n-1} + \cdots + (a_n b_0/z) = 0 \), so is an equation of integral dependence of \( x \) on \( R_z \). So \( R_z[x] \) is integral over \( R_z \) (15.28). Hence \( R_z \) is a field by (15.11).

Further, if \( R \) is Jacobson, then \( (0) \) is a maximal ideal by (15.22), and so \( R \) is a field. Hence \( R = R_z \). Thus \( R' \) is a field by (15.11).

**Theorem (15.26)** (Generalized Hilbert Nullstellensatz). — Let \( R \) be a Jacobson ring, \( R' \) a finitely generated algebra, and \( \mathfrak{m}' \) a maximal ideal of \( R' \). Set \( \mathfrak{m} := \mathfrak{m}' \cap R \). Then (1) \( \mathfrak{m} \) is maximal, and \( R'/\mathfrak{m}' \) is algebraic over \( R/\mathfrak{m} \), and (2) \( R' \) is Jacobson.

**Proof:** First, assume \( R' = R[x] \) for some \( x \in R' \). Given a prime \( \mathfrak{p} \subset \mathfrak{p}' \in \mathfrak{P} \), and \( y \in R' - \mathfrak{p} \), set \( \mathfrak{p} := \mathfrak{p} \cap R \) and \( R_1 := R/\mathfrak{p} \) and \( R'_1 := R'/\mathfrak{p}' \). Then \( R \) is Jacobson by (15.24). Suppose \( (R_1')_y \) is a field. Then by (15.26), \( R_1 \subset R'_1 \) is a finite extension of fields. Thus \( \mathfrak{p} \) and \( \mathfrak{p}' \) are maximal. To obtain (1), simply take \( \mathfrak{p} := \mathfrak{m} \) and \( y := 1 \). To obtain (2), take \( \mathfrak{p} \) nonmaximal, so \( R_1 \) is not a field; conclude \( (R_1')_y \) is not a field; whence, (15.26) yields (2).

Second, assume \( R' = R[x_1, \ldots, x_n] \) with \( n \geq 2 \). Set \( R'' := R[x_1, \ldots, x_{n-1}] \) and \( \mathfrak{m}'' := \mathfrak{m}' \cap R'' \). Then \( R''/R'' \) is maximal, and \( R'/\mathfrak{m}' \) is algebraic over \( R''/\mathfrak{m}'' \). Hence, \( \mathfrak{m} \) is maximal, and \( R''/\mathfrak{m}'' \) is algebraic over \( R/\mathfrak{m} \) by (1) for \( R''/R'' \). Finally, the Tower Law (15.27) implies that \( R''/\mathfrak{m}'' \) is algebraic over \( R/\mathfrak{m} \), as desired.

**Example (15.27).** — Part (1) of (15.26) may fail if \( R \) is not Jacobson, even if \( R' := R[Y] \) is the polynomial ring in one variable \( Y \) over \( R \). For example, let \( k \) be a field, and \( R := k[[X]] \) the formal power series ring. According to (15.11), the ideal \( \mathfrak{N} := (1 - XY) \) is maximal, but \( \mathfrak{M} \cap R \) is (0), not \( (X) \).

**Exercise (15.4.2).** — Let \( P := Z[X_1, \ldots, X_n] \) be the polynomial ring. Assume \( f \in P \) vanishes at every zero in \( K^n \) of \( f_1, \ldots, f_r \in P \) for every finite field \( K \); that is, if \( \{ a \} := (a_1, \ldots, a_n) \in K^n \) and \( f_1(a) = 0, \ldots, f_r(a) = 0 \) in \( F \), then \( f(a) = 0 \) too. Prove there are \( g_1, \ldots, g_r \in P \) and \( N \geq 1 \) such that \( f^N = g_1 f_1 + \cdots + g_r f_r \).

**Exercise (15.4.4).** — Let \( R \) be a ring, \( R' \) an algebra. Prove that if \( R' \) is integral over \( R \) and \( R \) is Jacobson, then \( R' \) is Jacobson.
EXERCISE (15.31). — Let $R$ be a Jacobson ring, $S$ a multiplicative subset, $f \in R$. True or false: prove or give a counterexample to each of the following statements.

1. The localized ring $R_f$ is Jacobson.
2. The localized ring $S^{-1}R$ is Jacobson.
3. The filtered direct limit $\varprojlim R_\lambda$ of Jacobson rings is Jacobson.
4. In a filtered direct limit of rings $R_\lambda$, necessarily $\varprojlim \text{rad}(R_\lambda) = \text{rad}(\varprojlim R_\lambda)$.

EXERCISE (15.31). — Let $R$ be a reduced Jacobson ring with a finite set $\Sigma$ of minimal primes, and $P$ a finitely generated module. Show that $P$ is locally free of rank $r$ if and only if $\dim_{R/m}(P/mP) = r$ for any maximal ideal $m$. 

16. Chain Conditions

In a ring, often every ideal is finitely generated; if so, we call the ring *Noetherian*. Examples include the ring of integers and any field. We characterize Noetherian rings as those in which every ascending chain of ideals stabilizes, or equivalently, in which every nonempty set of ideals has a member maximal under inclusion. We prove the Hilbert Basis Theorem: if a ring is Noetherian, then so is any finitely generated algebra over it. We define and characterize Noetherian modules similarly, and we prove that, over a Noetherian ring, it is equivalent for a module to be Noetherian, to be finitely generated, or to be finitely presented.

Lastly, we study Artinian rings and modules; in them, by definition, every descending chain of ideals or of submodules, stabilizes.

(16.1) (Noetherian rings). — We call a ring *Noetherian* if every ideal is finitely generated. For example, a Principal Ideal Ring (PIR) is, trivially, Noetherian.

Here are two standard examples of non-Noetherian rings. A third is given below in (16.6), and a fourth later in (18.31).

First, form the polynomial ring \( k[X_1, X_2, \ldots] \) in infinitely many variables. It is non-Noetherian as \( \langle X_1, X_2, \ldots \rangle \) is not finitely generated (but the ring is a UFD).

Second, in the polynomial ring \( k[ X, Y] \), form this subring and its ideal \( a \):

\[
R := \{ f := a + Xg \mid a \in k \text{ and } g \in k[X,Y] \} \quad \text{and} \quad a := \langle X, XY, XY^2, \ldots \rangle.
\]

Then \( a \) is not generated by any \( f_1, \ldots, f_m \in a \). Indeed, let \( n \) be the highest power of \( Y \) occurring in any \( f_i \). Then \( XY^{n+1} \notin \langle f_1, \ldots, f_m \rangle \). Thus \( R \) is non-Noetherian.

Exercise (16.2). — Let \( M \) be a finitely generated module over an arbitrary ring. Show every set that generates \( M \) contains a finite subset that generates.

Definition (16.3). — We say the ascending chain condition (acc) is satisfied if every ascending chain of ideals \( a_0 \subseteq a_1 \subseteq \cdots \) stabilizes; that is, there is a \( j \geq 0 \) such that \( a_j = a_{j+1} = \cdots \).

We say the maximal condition (maxc) is satisfied if every nonempty set of ideals \( S \) contains ones maximal for inclusion, that is, properly contained in no other in \( S \).

Lemma (16.4). — Acc is satisfied if and only if maxc is.

Proof: Let \( a_0 \subseteq a_1 \subseteq \cdots \) be a chain of ideals. If \( a_j \) is maximal, then trivially \( a_j = a_{j+1} = \cdots \). Thus maxc implies acc.

Conversely, given a nonempty set of ideals \( S \) with no maximal member, there’s \( a_0 \in S \); for each \( j \geq 0 \), there’s \( a_{j+1} \in S \) with \( a_j \subsetneq a_{j+1} \). So the Axiom of Countable Choice provides an infinite chain \( a_0 \subsetneq a_1 \subsetneq \cdots \). Thus acc implies maxc.

□

Proposition (16.5). — Given a ring \( R \), the following conditions are equivalent:

1. \( R \) is Noetherian;
2. acc is satisfied;
3. maxc is satisfied.

Proof: Assume (1) holds. Let \( a_0 \subseteq a_1 \subseteq \cdots \) be a chain of ideals. Set \( a := \bigcup a_n \). Clearly, \( a \) is an ideal. So by hypothesis, \( a \) is finitely generated, say by \( x_1, \ldots, x_r \). For each \( i \), there is a \( j_i \) with \( x_i \in a_{j_i} \). Set \( j := \max \{ j_i \} \). Then \( x_i \in a_j \) for all \( i \). So \( a \subseteq a_j \subseteq a_{j+1} \subseteq \cdots \subseteq a \). So \( a_j = a_{j+1} = \cdots \). Thus (2) holds.
Assume (2) holds. Then (3) holds by (16.10).

Assume (3) holds. Let \( a \) be an ideal, \( x_\lambda \) for \( \lambda \in \Lambda \) generators, \( S \) the set of ideals generated by finitely many \( x_\lambda \). Let \( b \) be a maximal element of \( S \); say \( b \) is generated by \( x_{\lambda_1}, \ldots, x_{\lambda_m} \). Then \( b \subset b + \langle x_\lambda \rangle \) for any \( \lambda \). So by maximality, \( b = b + \langle x_\lambda \rangle \). Hence \( x_\lambda \in b \). So \( b = a \); whence, \( a \) is finitely generated. Thus (1) holds.

**Example (16.6).** — In the field of rational functions \( k(X, Y) \), form this ring:

\[
R := k[X, Y, X/Y, X/Y^2, X/Y^3, \ldots].
\]

Then \( R \) is non-Noetherian by (16.8). Indeed, \( X \) does not factor into irreducibles:

\( X = (X/Y) \cdot Y \) and \( X/Y = (X/Y^2) \cdot Y \) and so on. Correspondingly, there is an ascending chain of ideals that does not stabilize:

\[
\langle X \rangle \subsetneq \langle X/Y \rangle \subsetneq \langle X/Y^2 \rangle \subsetneq \cdots.
\]

**Proposition (16.7).** — Let \( R \) be a Noetherian ring, \( S \) a multiplicative subset, \( a \) an ideal. Then \( R/a \) and \( S^{-1}R \) are Noetherian.

**Proof:** If \( R \) satisfies the acc, so do \( R/a \) and \( S^{-1}R \) by (11.5) and by (16.21)(1). Alternatively, any ideal \( b/a \) of \( R/a \) is, clearly, generated by the images of generators of \( b \). Similarly, any ideal \( b \) of \( S^{-1}R \) is generated by the images of generators of \( b \) by (16.21)(1)(b).

**Exercise (16.8).** — Let \( R \) be a ring, \( X \) a variable, \( R[X] \) the polynomial ring. Prove this statement or find a counterexample: if \( R[X] \) is Noetherian, then so is \( R \).

**Exercise (16.9).** — Let \( R \subset R' \) be a ring extension with an \( R \)-linear retraction \( \rho: R' \to R \). Assume \( R' \) is Noetherian, and prove \( R \) is too.

**Theorem (16.10) (Cohen).** — A ring \( R \) is Noetherian if every prime \( p \) is finitely generated.

**Proof:** Suppose there are no finitely-generated ideals. Given a nonempty set of them \( \{a_\lambda\} \) that is linearly ordered by inclusion, set \( a := \bigcup a_\lambda \). If \( a \) is finitely generated, then all the generators lie in some \( a_\lambda \), so generate \( a_\lambda \) as \( a_\lambda = a \), a contradiction. Thus \( a \) is non-finitely-generated. Hence, by Zorn’s Lemma, there is a maximal non-finitely-generated ideal \( p \). In particular, \( p \neq R \).

Assume every prime is finitely generated. Then there are \( a, b \in R - p \) with \( ab \in p \). So \( p + \langle a \rangle \) is finitely generated, say by \( x_1 + w_1 a, \ldots, x_n + w_n a \) with \( x_i \in p \). Then \( \{x_1, \ldots, x_n, a\} \) generate \( p + \langle a \rangle \).

Set \( b = \operatorname{Ann}(\langle p + \langle a \rangle \rangle / p) \). Then \( b \supset p + \langle b \rangle \) and \( b \notin p \). So \( b \) is finitely generated, say by \( y_1, \ldots, y_m \). Take \( z \in p \). Then \( z \in p + \langle a \rangle \), so write

\[
z = a_1 x_1 + \cdots + a_n x_n + y a
\]

with \( a_i, y \in R \). Then \( ya \in p \). So \( y \in b \). Hence \( y = b_1 y_1 + \cdots + b_m y_m \) with \( b_j \in R \). Thus \( p \) is generated by \( \{x_1, \ldots, x_n, a y_1, \ldots, a y_m\} \), a contradiction. Thus there are no non-finitely-generated ideals; in other words, \( R \) is Noetherian.

**Lemma (16.11).** — If a ring \( R \) is Noetherian, then so is the polynomial ring \( R[X] \).

**Proof:** By way of contradiction, assume there is an ideal \( a \) of \( R[X] \) that is not finitely generated. Set \( a_0 := \langle 0 \rangle \). For each \( i \geq 1 \), choose inductively \( f_i \in a - a_{i-1} \) of least degree \( d_i \), and set \( a_i := \langle f_1, \ldots, f_i \rangle \). Let \( a_i \) be the leading coefficient of \( f_i \), and \( b \) the ideal generated by all the \( a_i \). Since \( R \) is Noetherian, \( b = \langle a_1, \ldots, a_n \rangle \) for
some $n$ by (16.12). Then $a_{n+1} = a_1 + \cdots + a_n$ with $r_i \in R$.

By construction, $d_i \leq d_{i+1}$ for all $i$. Set

$$f := f_{n+1} - (r_1 f_1 X^{d_{n+1} - d_1} + \cdots + r_n X^{d_{n+1} - d_n}).$$

Then $\deg(f) < d_{n+1}$, so $f \in a_n$. Therefore, $f_{n+1} \in a_n$, a contradiction. □

**Theorem (16.12)** (Hilbert Basis). — Let $R$ be a Noetherian ring, $R'$ a finitely generated algebra. Then $R'$ is Noetherian.

**Proof:** Say $x_1, \ldots, x_r$ generate $R'$ over $R$, and let $P := R[X_1, \ldots, X_r]$ be the polynomial ring in $r$ variables. Then $P$ is Noetherian by (16.11) and induction on $r$. Assigning $x_i$ to $X_i$ defines an $R$-algebra map $P \to R'$, and obviously, it is surjective. Hence $R'$ is Noetherian by (16.7). □

**Lemma (16.13)** (Noetherian modules). — We call a module $M$ Noetherian if every submodule is finitely generated. In particular, a ring is Noetherian as a ring if and only if it is Noetherian as a module, because its submodules are just the ideals.

We say the ascending chain condition (acc) is satisfied in $M$ if every ascending chain of submodules $M_0 \subset M_1 \subset \cdots$ stabilizes. We say the maximal condition (maxc) is satisfied in $M$ if every nonempty set of submodules contains ones maximal under inclusion. It is simple to generalize (16.12): These conditions are equivalent:

1. $M$ is Noetherian;
2. acc is satisfied in $M$;
3. maxc is satisfied in $M$.

**Exercise (16.14).** — Let $R$ be a ring, $M$ a module. Nested submodules $M_1 \subset M_2$ of $M$ are equal if both these equations hold:

$$M_1 \cap N = M_2 \cap N \quad \text{and} \quad (M_1 + N)/N = (M_2 + N)/N.$$

**Proof:** Given $m_2 \in M_2$, there is $m_1 \in M_1$ with $n := m_2 - m_1 \in N$. Then $n \in M_2 \cap N = M_1 \cap N$. Hence $m_2 \in M_1$. Thus $M_1 = M_2$. □

**Proposition (16.15).** — Let $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ be a short exact sequence of $R$-modules, and $M_1, M_2$ two submodules of $M$. Prove or give a counterexample to this statement: if $\beta(M_1) = \beta(M_2)$ and $\alpha^{-1}(M_1) = \alpha^{-1}(M_2)$, then $M_1 = M_2$.

**Proof:** Assertion (1) is equivalent to (3.10) owing to (3.24).

To prove (2), first assume $M$ is Noetherian. A submodule $N'$ of $N$ is also a submodule of $M$, so $N'$ is finitely generated; thus $N$ is Noetherian. A submodule of $M/N$ is finitely generated as its inverse image in $M$ is so; thus $M/N$ is Noetherian.

Conversely, assume $N$ and $M/N$ are Noetherian. Let $P$ be a submodule of $M$. Then $P \cap N$ and $(P + N)/N$ are finitely generated. But $P/(P \cap N) \leftrightarrow (P + N)/N$ by (3.8.4). So (1) implies $P$ is finitely generated. Thus $M$ is Noetherian.

Here is a second proof of (2). First assume $M$ is Noetherian. Then any ascending chain in $N$ is also a chain in $M$, so it stabilizes. And any chain in $M/N$ is the image of a chain in $M$, so it too stabilizes. Thus $N$ and $M/N$ are Noetherian.

Conversely, assume $N$ and $M/N$ are Noetherian. Given $M_1 \subset M_2 \subset \cdots \subset M$, both $(M_1 \cap N) \subset (M_2 \cap N) \subset \cdots$ and $(M_1 + N)/N \subset (M_2 + N)/N \subset \cdots$ stabilize, say $M_j \cap N = M_{j+1} \cap N = \cdots$ and $(M_j + N)/N = (M_{j+1} + N)/N = \cdots$. Then $M_j = M_{j+1} = \cdots$ by (16.12). Thus $M$ is Noetherian. □
COROLLARY (16.17). — Modules $M_1, \ldots, M_r$ are Noetherian if and only if their direct sum $M_1 \oplus \cdots \oplus M_r$ is Noetherian.

Proof: The sequence $0 \rightarrow M_1 \rightarrow M_1 \oplus (M_2 \oplus \cdots \oplus M_r) \rightarrow M_2 \oplus \cdots \oplus M_r \rightarrow 0$ is exact. So the assertion results from (16.16)(2) by induction on $r$. □

Exercise (16.18). — Let $R$ be a ring, $a_1, \ldots, a_r$ ideals such that each $R/a_i$ is a Noetherian ring. Prove (1) that $\bigoplus R/a_i$ is a Noetherian $R$-module, and (2) that, if $\bigcap a_i = 0$, then $R$ too is a Noetherian ring.

Theorem (16.19). — Let $R$ be a Noetherian ring, and $M$ a module. Then the following conditions on $M$ are equivalent:

1. $M$ is Noetherian;
2. $M$ is finitely generated;
3. $M$ is finitely presented.

Proof: Assume (2). Then there is an exact sequence $0 \rightarrow K \rightarrow R^n \rightarrow M \rightarrow 0$. Since (16.21.1) $K$ is finitely generated, so (3) holds; further, (1) holds by (16.16)(2). Trivially, (1) or (3) implies (2). □

Exercise (16.20). — Let $R$ be a Noetherian ring, $M$ and $N$ finitely generated modules. Show that $\text{Hom}(M,N)$ is finitely generated.

Lemma (16.21) (Artin–Tate [11, Thm. 1]). — Let $R \subset R' \subset R''$ be rings. Assume that $R$ is Noetherian, that $R''/R$ is algebra finite, and that $R''/R'$ either is module finite or is integral. Then $R'/R$ is algebra finite.

Proof: Since $R''/R$ is algebra finite, so is $R''/R'$. Hence, the two conditions on $R''/R'$ are equivalent by (16.28). Say $x_1, \ldots, x_m$ generate $R''$ as an $R$-algebra, and $y_1, \ldots, y_n$ generate $R''$ as an $R'$-module. Then there exist $z_{ij} \in R'$ and $z_{ijk} \in R'$ with

$$x_i = \sum_j z_{ij}y_j \quad \text{and} \quad y_iy_j = \sum_k z_{ijk}y_k. \quad (16.21.1)$$

Let $R'_0$ be the $R$-algebra generated by the $z_{ij}$ and the $z_{ijk}$. Since $R$ is Noetherian, so is $R'_0$ by the Hilbert Basis Theorem, (16.22).

Any $x \in R''$ is a polynomial in the $x_i$ with coefficients in $R$. So (16.21.1) implies $x$ is a linear combination of the $y_j$ with coefficients in $R'_0$. Thus $R'/R'_0$ is module finite. But $R'_0$ is a Noetherian ring, and $R'$ is an $R'_0$-submodule of $R''$. So $R'/R'_0$ is module finite by (16.16). Since $R'_0/R$ is algebra finite, $R'/R$ is too. □

Theorem (16.22) (Noether on Invariants). — Let $R$ be a Noetherian ring, $R'$ an algebra-finite extension, and $G$ a finite group of $R$-automorphisms of $R'$. Then the subring of invariants $R'^G$ is also algebra finite; in other words, every invariant can be expressed as a polynomial in a certain finite number of “fundamental” invariants.

Proof: By (16.22), $R'$ is integral over $R'^G$. So (16.24) yields the assertion. □

(16.23) (Artin–Tate proof [11, Thm. 2] of the Zariski Nullstellensatz (15.20)). — In the setup of (16.23), take a transcendence base $x_1, \ldots, x_r$ of $R/k$. Then $R$ is integral over $k(x_1, \ldots, x_r)$ by definition of transcendence basis [2, (8.3), p. 526]. So $k(x_1, \ldots, x_r)$ is algebra finite over $k$ by (16.21), say $k(x_1, \ldots, x_r)k[y_1, \ldots, y_s]$.

Suppose $r \geq 1$. Write $y_i = F_i/G_i$ with $F_i, G_i \in k[x_1, \ldots, x_r]$. Let $H$ be an irreducible factor of $G_1 \cdots G_s + 1$. Plainly $H \nmid G_i$ for all $i$.

Say $H^{-1} = P(y_1, \ldots, y_s)$ where $P$ is a polynomial. Then $H^{-1} = Q/(G_1 \cdots G_s)^m$ for some $Q \in k[x_1, \ldots, x_r]$ and $m \geq 1$. But $H \nmid G_i$ for all $i$, a contradiction. Thus $r = 0$. So (16.28) implies $R/k$ is module finite, as desired.
EXERCISE (16.27). — Let $R$ be a domain, $R'$ an algebra, and set $K := \text{Frac}(R)$. Assume $R$ is Noetherian.

1. [16. Thm. 3] Assume $R'$ is a field containing $R$. Show $R'/R$ is algebra finite if and only if $K/R$ is algebra finite and $R'/K$ is (module) finite.

2. [16. bot. p. 77] Let $K' \supset R$ be a field that embeds in $R'$. Assume $R'/R$ is algebra finite. Show $K/R$ is algebra finite and $K'/K$ is finite.

EXAMPLE (16.25). — Set $\delta := \sqrt{-5}$, set $R := \mathbb{Z}[\delta]$, and set $p := (2, 1 + \delta)$. Let’s prove that $p$ is finitely presented and that $pR_q$ is free of rank 1 over $R_q$ for every maximal ideal $q$ of $R$, but that $p$ is not free. Thus the equivalent conditions of (16.31) do not imply that $P$ is free.

Since $Z$ is Noetherian and since $R$ is generated over $Z$, the Hilbert Basis Theorem (16.14) yields that $R$ is Noetherian. So since $p$ is generated by two elements, (16.14) yields that $p$ is finitely presented.

Recall from [2] pp. 417, 421, 425] that $p$ is maximal in $R$, but not principal. Now, $3 \notin p$; otherwise, $1 \in p$ as $2 \in p$, but $p \neq R$. So $(1 - \delta)/3 \in R_p$. Hence $(1 + \delta)R_p$ contains $(1 + \delta)(1 - \delta)/3$, or 2. So $(1 + \delta)R_p = pR_p$.

Since $R_p$ is a domain, the map $\mu_{1,\delta}: R_p \to pR_q$ is injective, so bijective. Thus $pR_q$ is free of rank 1.

Let $q$ be a maximal ideal distinct from $p$. Then $p \cap (R - q) \neq \emptyset$; so, $pR_q = R_q$ by (16.4)(2). Thus $pR_q$ is free of rank 1.

Finally, suppose $p \simeq R'$. Set $S := R - 0$. Then $S^{-1}R$ is the fraction field, $K$ say, of $R$. So $S^{-1}p \simeq K^n$. But the inclusion $p \hookrightarrow R$ yields an injection $S^{-1}p \hookrightarrow K$. Hence $S^{-1}p \hookrightarrow K$, since $S^{-1}p$ is a nonzero $K$-vector space. Therefore, $n = 1$. So $p \simeq R$. Hence $p$ is generated by one element. But $p$ is not principal. So there is a contradiction. Thus $p$ is not free.

DEFINITION (16.26). — We say a module is Artinian or the descending chain condition (dcc) is satisfied if every descending chain of submodules stabilizes.

We say the ring itself is Artinian if it is an Artinian module.

We say the minimal condition (minc) is satisfied in a module if every nonempty set of submodules has a minimal member.

PROPOSITION (16.27). — Let $M_1, \ldots, M_r, M$ be modules, $N$ a submodule of $M$.

1. Then $M$ is Artinian if and only if minc is satisfied in $M$.

2. Then $M$ is Artinian if and only if $N$ and $M/N$ are Artinian.

3. Then $M_1, \ldots, M_r$ are Artinian if and only if $M_1 \oplus \cdots \oplus M_r$ is Artinian.

Proof: It is easy to adapt the proof of (16.4), the second proof of (16.14)(2), and the proof of (16.17).

EXERCISE (16.28). — Let $k$ be a field, $R$ an algebra. Assume that $R$ is finite dimensional as a $k$-vector space. Prove that $R$ is Noetherian and Artinian.

EXERCISE (16.29). — Let $p$ be a prime number, and set $M := \mathbb{Z}[1/p]/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$.

Prove that any $\mathbb{Z}$-submodule $N \subset M$ is either finite or all of $M$. Deduce that $M$ is an Artinian $\mathbb{Z}$-module, and that it is not Noetherian.

EXERCISE (16.30). — Let $R$ be an Artinian ring. Prove that $R$ is a field if it is a domain. Deduce that, in general, every prime ideal $p$ of $R$ is maximal.
17. Associated Primes

Given a module, a prime is associated to it if the prime is equal to the annihilator of an element. Given a subset of the set of all associated primes, we prove there is a submodule whose own associated primes constitute that subset. If the ring is Noetherian, then the set of annihilators of elements has maximal members; we prove the latter are prime, so associated. Then the union of all the associated primes is the set of zerodivisors on the module. If also the module is finitely generated, then the intersection is the set of nilpotents. Lastly, we prove there is then a finite chain of submodules whose successive quotients are cyclic with prime annihilators; these primes include all associated primes, which are, therefore, finite in number.

Definition (17.1). — Let $R$ be a ring, $M$ a module. A prime ideal $p$ is said to be associated to $M$ if there is a (nonzero) $m \in M$ with $p = \text{Ann}(m)$. The set of associated primes is denoted by $\text{Ass}(M)$ or $\text{Ass}_R(M)$.

The primes that are minimal in $\text{Ass}(M)$ are called the minimal primes of $M$; the others, the embedded primes.

Warning: following a old custom, we mean by the associated primes of an ideal $a$ not those of $a$ viewed as an abstract module, but rather those of $R/a$.

Lemma (17.2). — Let $R$ be a ring, $M$ a module, and $p$ a prime ideal. Then $p \in \text{Ass}(M)$ if and only if there is an $R$-injection $R/p \hookrightarrow M$.

Proof: Assume $p = \text{Ann}(m)$ with $m \in M$. Define a map $R \to M$ by $x \mapsto xm$. This map induces an $R$-injection $R/p \hookrightarrow M$.

Conversely, suppose there is an $R$-injection $R/p \hookrightarrow M$, and let $m \in M$ be the image of 1. Then $p = \text{Ann}(m)$, so $p \in \text{Ass}(M)$.

Proposition (17.3). — Let $M$ be a module. Then $\text{Ass}(M) \subseteq \text{Supp}(M)$.

Proof: Let $p \in \text{Ass}_R(M)$. Say $p = \text{Ann}(m)$. Then $m/1 \in M_p$ is nonzero as no $x \in (R - p)$ satisfies $xm = 0$. Thus $M_p \neq 0$ and so $p \in \text{Supp}(M)$.

Alternatively, $(R/p)_p \twoheadrightarrow M_p$ by (17.2). But $(R/p)_p = \text{Frac}(R/p)$ by (17.2). Thus $M_p \neq 0$.

Lemma (17.4). — Let $R$ be a ring, $p$ a prime ideal, $m \in R/p$ a nonzero element. Then (1) $\text{Ann}(m) = p$ and (2) $\text{Ass}(R/p) = \{p\}$.

Proof: To prove (1), say $m$ is the residue of $y \in R$. Let $x \in R$. Then $xm = 0$ if and only if $xy \in p$, so if and only if $x \in p$, as $p$ is prime and $m \neq 0$. Thus (1) holds.

Trivially, (1) implies (2).

Proposition (17.5). — Let $M$ be a module, $N$ a submodule. Then

$$
\text{Ass}(N) \subset \text{Ass}(M) \subset \text{Ass}(N) \cup \text{Ass}(M/N).
$$

Proof: Take $m \in N$. Then the annihilator of $m$ is the same whether $m$ is regarded as an element of $N$ or of $M$. So $\text{Ass}(N) \subset \text{Ass}(M)$.

Let $p \in \text{Ass}(M)$. Then (17.2) yields an $R$-injection $R/p \to M$. Denote its image by $E$. If $E \cap N = 0$, then the composition $R/p \to M \to M/N$ is injective; hence, $p \in \text{Ass}(M/N)$ by (17.2). Else, take a nonzero $m \in E \cap N$. Then $\text{Ann}(m) = p$ by (17.2)(1). Thus $p \in \text{Ass}(N)$.

101
Exercises (17.4). — Given modules \( M_1, \ldots, M_r \), set \( M := M_1 \oplus \cdots \oplus M_r \). Prove 
\[
\text{Ass}(M) = \text{Ass}(M_1) \cup \cdots \cup \text{Ass}(M_r).
\]

Exercise (17.7). — Take \( R := \mathbb{Z} \) and \( M := \mathbb{Z}/(2) \oplus \mathbb{Z} \). Find \( \text{Ass}(M) \) and find two submodules \( L, N \subseteq M \) with \( L + N = M \) but \( \text{Ass}(L) \cup \text{Ass}(N) \nsubseteq \text{Ass}(M) \).

Exercise (17.8). — If a prime \( p \) is sandwiched between two primes in \( \text{Ass}(M) \), is \( p \) necessarily in \( \text{Ass}(M) \) too?

Proposition (17.9). — Let \( M \) be a module, and \( \Psi \) a subset of \( \text{Ass}(M) \). Then there is a submodule \( N \) of \( M \) with \( \text{Ass}(M/N) = \Psi \) and \( \text{Ass}(N) = \text{Ass}(M) - \Psi \).

Proof: Given submodules \( N_\lambda \) of \( M \) totally ordered by inclusion, set \( N := \bigcup N_\lambda \). Given \( p \in \text{Ass}(N) \), say \( p = \text{Ann}(m) \). Then \( m \in N_\lambda \) for some \( \lambda \); so \( p \in \text{Ass}(N_\lambda) \). Conversely, \( \text{Ass}(N_\lambda) \subseteq \text{Ass}(N) \) for all \( \lambda \) by (17.7). Thus \( \text{Ass}(N) = \bigcup \text{Ass}(N_\lambda) \).

So we may apply Zorn’s Lemma to obtain a submodule \( N \) of \( M \) that is maximal with \( \text{Ass}(N) \subseteq \text{Ass}(M) - \Psi \). By (17.7), it suffices to show that \( \text{Ass}(M/N) \subseteq \Psi \).

Take \( p \in \text{Ass}(M/N) \). Then \( M/N \) has a submodule \( N'/N \) isomorphic to \( R/p \) by (17.7). So \( \text{Ass}(N') \subseteq \text{Ass}(N) \cup \{ p \} \) by (17.7) and (17.8). Now, \( N' \subseteq N \) and \( N \) is maximal with \( \text{Ass}(N) \subseteq \text{Ass}(M) - \Psi \). Hence \( p \in \text{Ass}(N') \subseteq \text{Ass}(M) \), but \( p \notin \text{Ass}(M) - \Psi \). Thus \( p \in \Psi \).

Proposition (17.10). — Let \( R \) be a ring, \( S \) a multiplicative subset, \( M \) a module, and \( p \) a prime ideal. If \( p \cap S = \emptyset \) and \( p \in \text{Ass}(M) \), then \( S^{-1}p \in \text{Ass}(S^{-1}M) \); the converse holds if \( p \) is finitely generated.

Proof: Assume \( p \in \text{Ass}(M) \). Then (17.7) yields an injection \( R/p \rightarrow M \). It induces an injection \( S^{-1}(R/p) \rightarrow S^{-1}M \) by (17.20). But \( S^{-1}(R/p) = S^{-1}R/S^{-1}p \) by (17.20). Assume \( p \cap S = \emptyset \) also. Then \( pS^{-1}R \) is prime by (17.11)(3)(b). But \( pS^{-1}R = S^{-1}p \) by (17.20). Thus \( S^{-1}p \in \text{Ass}(S^{-1}M) \).

Conversely, assume \( S^{-1}p \in \text{Ass}(S^{-1}M) \). Then there are \( m \in M \) and \( t \in S \) with \( S^{-1}p = \text{Ann}(m/t) \). Say \( p = (x_1, \ldots, x_n) \). Fix \( i \). Then \( x_i m/t = 0 \). So there is \( s_i \in S \) with \( s_i x_i m = 0 \). Set \( s := \prod s_i \). Then \( x_i \in \text{Ann}(sm) \). Thus \( p \subseteq \text{Ann}(sm) \).

Take \( b \in \text{Ann}(sm) \). Then \( bsm/st = 0 \). So \( b/1 \in S^{-1}p \). So \( b \in p \) by (17.11)(1)(a) and (17.13)(3)(a). Thus \( p \supseteq \text{Ann}(sm) \). So \( p = \text{Ann}(sm) \). Thus \( p \subseteq \text{Ass}(M) \).

Finally, \( p \cap S = \emptyset \) by (17.22)(2), as \( S^{-1}p \) is prime.

Exercise (17.11). — Let \( R \) be a ring, and suppose \( R_p \) is a domain for every prime \( p \). Prove every associated prime of \( R \) is minimal.

Lemma (17.12). — Let \( R \) be a ring, \( M \) a module, and \( p \) an ideal. Suppose \( p \) is maximal in the set of annihilators of nonzero elements \( m \) of \( M \). Then \( p \in \text{Ass}(M) \).

Proof: Say \( p := \text{Ann}(m) \) with \( m \neq 0 \). Then \( 1 \notin p \) as \( m \neq 0 \). Now, take \( b, c \in R \) with \( bc \in p \), but \( c \notin p \). Then \( bcm = 0 \), but \( cm \neq 0 \). Plainly, \( p \subseteq \text{Ann}(cm) \). So \( p = \text{Ann}(cm) \) by maximality. But \( b \in \text{Ann}(cm) \), so \( b \in p \). Thus \( p \) is prime.

Proposition (17.13). — Let \( R \) be a Noetherian ring, \( M \) a module. Then \( M = 0 \) if and only if \( \text{Ass}(M) = \emptyset \).

Proof: Obviously, if \( M = 0 \), then \( \text{Ass}(M) = \emptyset \). Conversely, suppose \( M \neq 0 \). Let \( S \) be the set of annihilators of nonzero elements of \( M \). Then \( S \) has a maximal element \( p \) by (17.14). By (17.12), \( p \in \text{Ass}(M) \). Thus \( \text{Ass}(M) \neq \emptyset \).
Definition (17.14). — Let $R$ be a ring, $M$ a module, $x \in R$. We say $x$ is a \textbf{zerodivisor} on $M$ if there is a nonzero $m \in M$ with $xm = 0$; otherwise, we say $x$ is a \textbf{nonzerodivisor}. We denote the set of zerodivisors by $\text{z.div}(M)$.

Proposition (17.15). — Let $R$ be a Noetherian ring, $M$ a module. Then
\[
\text{z.div}(M) = \bigcup_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p}.
\]

Proof: Given $x \in \text{z.div}(M)$, say $xm = 0$ where $m \in M$ and $m \neq 0$. Then $x \in \text{Ann}(m)$. But $\text{Ann}(m)$ is contained in an ideal $\mathfrak{p}$ that is maximal among annihilators of nonzero elements because of (15.5); hence, $\mathfrak{p} \in \text{Ass}(M)$ by (17.12). Thus $\text{z.div}(M) \subseteq \bigcup \mathfrak{p}$. The opposite inclusion results from the definitions. \hfill $\Box$

Exercise (17.16). — Let $R$ be a Noetherian ring, $M$ a module, $N$ a submodule, $x \in R$. Show that, if $x \notin \mathfrak{p}$ for any $\mathfrak{p} \in \text{Ass}(M/N)$, then $xM \cap N = xN$.

Lemma (17.17). — Let $R$ be a Noetherian ring, $M$ a module. Then
\[
\text{Supp}(M) = \bigcup_{q \notin \text{Ass}(M)} V(q) \supseteq \text{Ass}(M).
\]

Proof: Let $\mathfrak{p}$ be a prime. Then $R_{\mathfrak{p}}$ is Noetherian by (11.7) as $R$ is. So $M_{\mathfrak{p}} \neq 0$ if and only if $\text{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq \emptyset$ by (17.13). But $R$ is Noetherian; so $\text{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq \emptyset$ if and only if there is a $q \in \text{Ass}(M)$ with $q \cap (R - \mathfrak{p}) = \emptyset$, or $q \subseteq \mathfrak{p}$, owing to (11.41)(2) and (17.10). Thus $\mathfrak{p} \in \text{Supp}(M)$ if and only if $\mathfrak{p} \in V(q)$ for some $q \in \text{Ass}(M)$. \hfill $\Box$

Theorem (17.18). — Let $R$ be a Noetherian ring, $M$ a module, $\mathfrak{p} \in \text{Supp}(M)$. Then $\mathfrak{p}$ contains some $q \in \text{Ass}(M)$; if $\mathfrak{p}$ is minimal in $\text{Supp}(M)$, then $\mathfrak{p} \in \text{Ass}(M)$.

Proof: By (17.17), $q$ exists. Also, $q \in \text{Supp}(M)$; so $q = \mathfrak{p}$ if $\mathfrak{p}$ is minimal. \hfill $\Box$

Theorem (17.19). — Let $R$ be a Noetherian ring, and $M$ a finitely generated module. Then
\[
\text{nil}(M) = \bigcap_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p}.
\]

Proof: Since $M$ is finitely generated, $\text{nil}(M) = \bigcap_{\mathfrak{p} \in \text{Supp}(M)} \mathfrak{p}$ by (13.29). Since $R$ is Noetherian, given $\mathfrak{p} \in \text{Supp}(M)$, there is $q \in \text{Ass}(M)$ with $q \subseteq \mathfrak{p}$ by (17.17). The assertion follows.

Lemma (17.20). — Let $R$ be a Noetherian ring, $M$ a finitely generated module. Then there exists a chain of submodules
\[
0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M
\]
with $M_i/M_{i-1} \cong R/p_i$ for some prime $p_i$ for $i = 1, \ldots, n$. For any such chain,
\[
\text{Ass}(M) \subset \{p_1, \ldots, p_n\} \subset \text{Supp}(M).
\] \hfill (17.20.1)

Proof: Among all submodules of $M$ having such a chain, there is a maximal submodule $N$ by (16.13) and (16.13). Suppose $M/N \neq 0$. Then by (17.13), the quotient $M/N$ contains a submodule $N'/N$ isomorphic to $R/p$ for some prime $p$. Then $N \nsubseteq N'$, contradicting maximality. Hence $N = M$. Thus a chain exists.

The first inclusion of (17.20) follows by induction from (17.13) and (17.21)\((2)\). Now, $p_i \in \text{Supp}(R/p_i)$ owing to (12.28). Thus (13.27)(1) yields (17.20.1). \hfill $\Box$

Theorem (17.21). — Let $R$ be a Noetherian ring, and $M$ a finitely generated module. Then the set $\text{Ass}(M)$ is finite.

Proof: The assertion follows directly from (17.21). \hfill $\Box$
**Exercise (17.25).** — Let \( R \) be a Noetherian ring, \( \mathfrak{a} \) an ideal. Prove the primes minimal containing \( \mathfrak{a} \) are associated to \( \mathfrak{a} \). Prove such primes are finite in number.

**Exercise (17.26).** — Take \( R := \mathbb{Z} \) and \( M := \mathbb{Z} \) in (17.25.1). Determine when a chain \( 0 \subset M_1 \subset \cdots \subset M \) is acceptable, and show that then \( p_2 \not\in \text{Ass}(M) \).

**Exercise (17.27).** — Take \( R := \mathbb{Z} \) and \( M := \mathbb{Z}/(12) \) in (17.25.1). Find all three acceptable chains, and show that, in each case, \( \{ p_1 \} = \text{Ass}(M) \).

**Proposition (17.25).** — Let \( R \) be a Noetherian ring, and \( M \) and \( N \) finitely generated modules. Then

\[
\text{Ass}(\text{Hom}(M, N)) = \text{Supp}(M) \cap \text{Ass}(N).
\]

**Proof:** Take \( p \in \text{Ass}(\text{Hom}(M, N)) \). Then (17.25) yields an injective \( R \)-map \( R / p \hookrightarrow \text{Hom}(M, N) \). Set \( k(p) := \text{Frac}(R / p) \). Then \( k(p) = (R / pR)_p \) by (17.25). Now, \( M \) is finitely presented by \((17.24)\) as \( R \) is Noetherian; hence,

\[
\text{Hom}(M, N)_p = \text{Hom}_{R_p}(M_p, N_p)
\]

by (17.25)(2). Therefore, by exactness, localizing yields an injection

\[
\varphi : k(p) \hookrightarrow \text{Hom}_{R_p}(M_p, N_p).
\]

Thus \( M_p \neq 0 \); so \( p \in \text{Supp}(M) \).

For any \( m \in M_p \) with \( \varphi(1)(m) \neq 0 \), the map \( k(p) \rightarrow N_p \) given by \( x \mapsto \varphi(x)(m) \) is nonzero, so an injection. But \( k(p) = R_p / pR_p \) by (17.22). Hence by (17.2), we have \( pR_p \in \text{Ass}(N_p) \). Thus also \( p \in \text{Ass}(N) \) by (17.21).

Conversely, take \( p \in \text{Supp}(M) \cap \text{Ass}(N) \). Then \( M_p \neq 0 \). So by Nakayama’s Lemma, \( M_p / pM_p \) is a nonzero vector space over \( k(p) \). Take any nonzero \( R \)-map \( M_p / pM_p \rightarrow k(p) \), precede it by the canonical map \( M_p \rightarrow M_p / pM_p \), and follow it by an \( R \)-injection \( k(p) \hookrightarrow N_p \); the latter exists by (17.2) and (17.21) since \( p \in \text{Ass}(N) \). We obtain a nonzero element of \( \text{Hom}_{R_p}(M_p, N_p) \), annihilated by \( pR_p \). But \( pR_p \) is maximal, so is the entire annihilator. So \( pR_p \in \text{Ass}(\text{Hom}_{R_p}(M_p, N_p)) \).

Hence \( p \in \text{Ass}(\text{Hom}(M, N)) \) by (17.25.1) and (17.21). \( \square \)

**Exercise (17.28).** — Let \( R \) be a Noetherian ring, \( \mathfrak{a} \) an ideal, and \( M \) a finitely generated module. Show that the following conditions are equivalent:

1. \( \mathbf{V}(\mathfrak{a}) \cap \text{Ass}(M) = \emptyset \);
2. \( \text{Hom}(N, M) = 0 \) for all finitely generated modules \( N \) with \( \text{Supp}(N) \subset \mathbf{V}(\mathfrak{a}) \);
3. \( \text{Hom}(N, M) = 0 \) for some finitely generated module \( N \) with \( \text{Supp}(N) = \mathbf{V}(\mathfrak{a}) \);
4. \( \mathfrak{a} \not
\mathfrak{a} \).
5. \( \mathfrak{a} \not
\mathfrak{p} \) for any \( \mathfrak{p} \in \text{Ass}(M) \).

**Proposition (17.27).** — Let \( R \) be a Noetherian ring, \( \mathfrak{p} \) a prime, \( M \) a finitely generated module, and \( x, y \in \mathfrak{p} \) nonzerodivisors on \( M \). Then \( \mathfrak{p} \in \text{Ass}(M / xM) \) if and only if \( \mathfrak{p} \in \text{Ass}(M / yM) \).

**Proof:** Form the sequence \( 0 \rightarrow K \rightarrow M / xM \xrightarrow{\mu_y} M / xM \) with \( K := \text{Ker}(\mu_y) \). Apply the functor \( \text{Hom}(R / \mathfrak{p}, \bullet) \) to that sequence, and get the following one:

\[
0 \rightarrow \text{Hom}(R / \mathfrak{p}, K) \rightarrow \text{Hom}(R / \mathfrak{p}, M / xM) \xrightarrow{\mu_y} \text{Hom}(R / \mathfrak{p}, M / xM).
\]

It is exact by (17.18). But \( y \in \mathfrak{p} \); so the right-hand map vanishes. Thus

\[
\text{Hom}(R / \mathfrak{p}, K) \rightarrow \text{Hom}(R / \mathfrak{p}, M / xM).
\]
Form the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & \to & M \\
& \xrightarrow{\mu_x} & M \\
\mu_y & \downarrow & \mu_y \\
0 & \to & M \\
\end{array}
\]

The Snake Lemma \((\text{5.13})\) yields an exact sequence \(0 \to K \to M/yM \xrightarrow{\mu_y} M/yM\) as \(\text{Ker}(\mu_y) = 0\). Hence, similarly, \(\text{Hom}(R/p, K) \twoheadrightarrow \text{Hom}(R/p, M/yM)\). Therefore,

\[
\text{Hom}(R/p, M/yM) = \text{Hom}(R/p, M/xM). \tag{17.27.1}
\]

Finally, \(p \in \text{Supp}(R/p)\) by \((\text{13.27})(3)\). Thus \((\text{17.27})\) yields the assertion. \(\square\)
18. Primary Decomposition

Primary decomposition of a submodule generalizes factorization of an integer into powers of primes. A submodule is called primary if the quotient module has only one associated prime. We characterize these submodules in various ways over a Noetherian ring, emphasizing the case of ideals. A primary decomposition is a representation of a submodule as a finite intersection of primary submodules. The decomposition is called irredundant, or minimal, if it cannot be shortened. We consider several illustrative examples in a polynomial ring.

Then we prove existence and uniqueness theorems for a proper submodule of a finitely generated module over a Noetherian ring. The celebrated Lasker–Noether Theorem asserts the existence of an irredundant primary decomposition. The First Uniqueness Theorem asserts the uniqueness of the primes that arise; they are just the associated primes of the quotient. The Second Uniqueness Theorem asserts the uniqueness of the primary components whose primes are minimal among these associated primes; the other primary components may vary.

Definition (18.1). — Let \( R \) be a ring, \( M \) a module, \( Q \) a submodule. If \( \text{Ass}(M/Q) \) consists of a single prime \( p \), we say \( Q \) is primary or \( p \)-primary in \( M \).

Example (18.2). — A prime \( p \) is \( p \)-primary, as \( \text{Ass}(R/p) = \{p\} \) by (17.4) (2).

Proposition (18.3). — Let \( R \) be a Noetherian ring, \( M \) a finitely generated module, \( Q \) a submodule. If \( Q \) is \( p \)-primary, then \( p = \text{nil}(M/Q) \).

Proof: The assertion holds as \( \text{nil}(M/Q) = \bigcap_{q \in \text{Ass}(M/Q)} q \) by (17.11). \( \square \)

Theorem (18.4). — Let \( R \) be a Noetherian ring, \( M \) a nonzero finitely generated module, \( Q \) a submodule. Set \( p := \text{nil}(M/Q) \). Then these conditions are equivalent:

1. \( p \) is prime and \( Q \) is \( p \)-primary.
2. \( p = \text{z.div}(M/Q) \).
3. Given \( x \in R \) and \( m \in M \) with \( xm \in Q \) but \( m \notin Q \), necessarily \( x \in p \).

Proof: Recall \( p = \bigcap_{q \in \text{Ass}(M/Q)} q \) by (17.13), and \( \text{z.div}(M/Q) = \bigcup_{q \in \text{Ass}(M/Q)} q \) by (17.15). Thus \( p \subseteq \text{z.div}(M/Q) \).

Further, (2) holds if \( \text{Ass}(M/Q) = \{p\} \), that is, if (1) holds.

Conversely, if \( x \in q \in \text{Ass}(M/Q) \), but \( x \notin q' \in \text{Ass}(M/Q) \), then \( x \notin p \), but \( x \in \text{z.div}(M/Q) \); hence, (2) implies (1). Thus (1) and (2) are equivalent.

Clearly, (3) means every zerodivisor on \( M/Q \) is nilpotent, or \( p \supset \text{z.div}(M/Q) \).

But the opposite inclusion always holds. Thus (2) and (3) are equivalent. \( \square \)

Corollary (18.5). — Let \( R \) be a Noetherian ring, and \( q \) a proper ideal. Set \( p := \sqrt{q} \). Then \( q \) is primary in \( R \) if and only if, given \( x, y \in R \) with \( xy \in q \) but \( x \notin q \), necessarily \( y \in p \); if so, then \( p \) is prime and \( q \) is \( p \)-primary.

Proof: Clearly \( q = \text{Ann}(R/q) \). So \( p = \text{nil}(R/q) \). So the assertions result directly from (18.3) and (18.3). \( \square \)

Exercise (18.6). — Let \( R \) be a ring, and \( p = \langle p \rangle \) a principal prime generated by a nonzerodivisor \( p \). Show every positive power \( p^n \) is \( p \)-primary. Show conversely, if \( R \) is Noetherian, then every \( p \)-primary ideal \( q \) is equal to some power \( p^n \).
Since

Say

Form the canonical map

The condition

(18.14.1)

Let

\( N \)

are clearly irredundant. Note:

the

assertion results from

\( q \)

condition:

and then discarding those of them that are not needed.

M \subseteq Q_1 \oplus M/Q_2.

Its kernel is \( Q \), so it

induces an injection \( M/Q \hookrightarrow M/Q_1 \oplus M/Q_2 \). Hence (17.1.3) and (17.7.6) yield

\[ \emptyset \neq \text{Ass}(M/Q) \subseteq \text{Ass}(M/Q_1) \cup \text{Ass}(M/Q_2). \]

Since the latter two sets are each equal to \( \{ p \} \), so is \( \text{Ass}(M/Q) \), as desired. \( \square \)

(18.13) (Primary decomposition). — Let \( R \) be a ring, \( M \) a module, and \( N \) a submodule. A primary decomposition of \( N \) is a decomposition

\[ N = Q_1 \cap \cdots \cap Q_r \quad \text{with the } Q_i \text{ primary.} \]

We call the decomposition irredundant or minimal if these conditions hold:

1. \( N \neq \bigcap_{i \neq j} Q_j \), or equivalently, \( \bigcap_{i \neq j} Q_j \nsubseteq Q_i \) for \( i = 1, \ldots, r \).
2. Say \( Q_i \) is \( p_i \)-primary for \( i = 1, \ldots, r \). Then \( p_1, \ldots, p_r \) are distinct.

If so, then we call \( Q_i \) the \( p_i \)-primary component of the decomposition.

If \( R \) is Noetherian, then owing to (18.14.2), any primary decomposition can be made irredundant by intersecting all the primary submodules with the same prime and then discarding those of them that are not needed.

Example (18.14). — Let \( k \) be a field, \( R := k[X,Y] \) the polynomial ring. Set

\[ a := \langle X^2, XY \rangle. \]

Below, it is proved that, for any \( n \geq 1 \),

\[ a = \langle X \rangle \cap \langle X^2, XY, Y^n \rangle = \langle X \rangle \cap \langle X^2, Y \rangle. \tag{18.14.1} \]

Here \( \langle X^2, XY, Y^n \rangle \) and \( \langle X^2, Y \rangle \) contain \( \langle X, Y \rangle^n \); so they are \( \langle X, Y \rangle \)-primary by (18.14.4). Thus (18.14.4) gives infinitely many primary decompositions of \( a \). They are clearly irredundant. Note: the \( \langle X, Y \rangle \)-primary component is not unique!

Plainly, \( a \subseteq \langle X \rangle \) and \( a \subset \langle X^2, XY, Y^n \rangle \subset \langle X^2, Y \rangle \). To see \( a \supset \langle X \rangle \cap \langle X^2, Y \rangle \),
take $F \in \langle X \rangle \cap \langle X^2, Y \rangle$. Then $F = GX = AX^2 + BY$ where $A, B, G \in R$. Then $X(G - AX) = BY$. So $X | B$. Say $B = B'X$. Then $F = AX^2 + B'XY \in a$.

**Example (18.15).** — Let $k$ be a field, $R := k[X, Y]$ the polynomial ring, $a \in k$. Set $a := \langle X^2, XY \rangle$. Define an automorphism $\alpha$ of $R$ by $X \mapsto X$ and $Y \mapsto aX + Y$.

Then $\alpha$ preserves $a$ and $\langle X \rangle$, and carries $(X^2, Y)$ onto $\langle X^2, aX + Y \rangle$. So (18.14) implies that $a = \langle X \rangle \cap \langle X^2, aX + Y \rangle$ is an irredundant primary decomposition. Moreover, if $a \neq b$, then $(X^2, aX + Y, bX + Y) = \langle X, Y \rangle$. Thus two $\langle X, Y \rangle$-primary components are not always contained in a third, although their intersection is one by (18.12).

**Example (18.16).** — Let $k$ be a field, $P := k[X, Y, Z]$ the polynomial ring. Set $\langle x, y, z \rangle$ the residues of $X, Y, Z$ in $R$. Set $p := \langle x, y \rangle$.

Clearly $p^2 = \langle x^2, xy, y^2 \rangle = \langle x, y, z \rangle$. Let’s show that $p^2 = \langle x \rangle \cap \langle x^2, y, z \rangle$ is an irredundant primary decomposition.

First note the inclusions $x(x, y, z) \subseteq \langle x \rangle \cap \langle x, y, z \rangle^2 \subseteq \langle x \rangle \cap \langle x^2, y, z \rangle$.

Conversely, given $f \in \langle x \rangle \cap \langle x^2, y, z \rangle$, represent $f$ by $GX$ with $G \in P$. Then

$$GX = AX^2 + BY + C_Z + D(XZ - Y^2) \quad \text{with} \quad A, B, C, D \in P.$$ 

So $(G - AX)X = B'Y + C'Z$ with $B', C' \in P$. Say $G - AX = A'' + B''Y + C''Z$ with $A'' \in k[X]$ and $B'', C'' \in P$. Then

$$A''X = -B''XY - C''XZ + B'Y + C'Z = (B' - B''X)Y + (C' - C''X)Z;$$

whence, $A'' = 0$. Therefore, $GX \in \langle X, Y, Z \rangle$. Thus $p^2 = \langle x \rangle \cap \langle x^2, y, z \rangle$.

The ideal $\langle x \rangle$ is $\langle x, y \rangle$-primary in $R$ by (18.8). Indeed, the preimage in $P$ of $\langle x \rangle$ is $\langle X, Y^2 \rangle$ and of $\langle x, y \rangle$ is $\langle X, Y \rangle$. Further, $\langle X, Y^2 \rangle$ is $\langle X, Y \rangle$-primary, as under the map $\varphi: P \to k[Y, Z]$ with $\varphi(X) = 0$, clearly $\langle X, Y^2 \rangle = \varphi^{-1}(Y^2)$ and $\langle X, Y \rangle = \varphi^{-1}(Y)$; moreover, $(Y^2)$ is $\langle Y \rangle$-primary by (18.8), or by (18.6).

Finally $\langle x, y, z \rangle^2 \subseteq \langle x^2, y, z \rangle \subseteq \langle x, y, z \rangle$ and $\langle x, y, z \rangle$ is maximal. So $\langle x^2, y, z \rangle$ is $\langle x, y, z \rangle$-primary by (18.9).

Thus $p^2 = \langle x \rangle \cap \langle x^2, y, z \rangle$ is a primary decomposition. It is clearly irredundant. Moreover, $\langle x \rangle$ is the $p$-primary component of $p^2$.

**Exercise (185.4).** — Let $k$ be a field, $R := k[X, Y, Z]$ be the polynomial ring. Set $a := \langle XY, X - YZ \rangle$, set $q_1 := \langle X, Z \rangle$ and set $q_2 := \langle Y^2, X - YZ \rangle$. Show that $a = q_1 \cap q_2$ and that this expression is an irredundant primary decomposition.

**Exercise (185.5).** — Let $R := R' \times R''$ be a product of two domains. Find an irredundant primary decomposition of $\langle 0 \rangle$.

**Lemma (18.19).** — Let $R$ be a ring, $M$ a module, $N = Q_1 \cap \cdots \cap Q_r$ a primary decomposition in $M$. Say $Q_i$ is $p_i$-primary for $i = 1, \ldots, r$. Then

$$\text{Ass}(M/N) \subseteq \{p_1, \ldots, p_r\}.$$  

(18.19.1)

If equality holds and if $p_1, \ldots, p_r$ are distinct, then the decomposition is irredundant; the converse holds if $R$ is Noetherian.

**Proof:** Since $N = \bigcap Q_i$, the canonical map is injective: $M/N \hookrightarrow \bigoplus M/Q_i$. So (17.12) and (17.14) yield $\text{Ass}(M/N) \subseteq \bigcup \text{Ass}(M/Q_i)$. Thus (18.19.1) holds.

If $N = Q_2 \cap \cdots \cap Q_r$, then $\text{Ass}(M/N) \subseteq \{p_2, \ldots, p_r\}$ too. Thus if equality holds in (18.19.1) and if $p_1, \ldots, p_r$ are distinct, then $N = Q_1 \cap \cdots \cap Q_r$ is irredundant. Conversely, assume $N = Q_1 \cap \cdots \cap Q_r$ is irredundant. Given $i$, set $P_i := \bigcap_{j \neq i} Q_j$.  


Then \( P_i \cap Q_i = N \) and \( P_i/N \neq 0 \). Consider these two canonical injections:

\[
P_i/N \hookrightarrow M/Q_i \quad \text{and} \quad P_i/N \hookrightarrow M/N.
\]

Assume \( R \) is Noetherian. Then \( \text{Ass}(P_i/N) \neq \emptyset \) by (17.4.12). So the first injection yields \( \text{Ass}(P_i/N) = \{p_i\} \) by (17.4.3); then the second yields \( p_i \in \text{Ass}(M/N) \). Thus \( \text{Ass}(M/N) \supseteq \{p_1, \ldots, p_r\} \), and (18.19.1) yields equality, as desired. \( \square \)

**Theorem (18.20) (First Uniqueness).** — Let \( R \) be a Noetherian ring, and \( M \) a module. Let \( N = Q_1 \cap \cdots \cap Q_r \) be an irredundant primary decomposition in \( M \); say \( Q_i \) is \( p_i \)-primary for \( i = 1, \ldots, r \). Then \( p_1, \ldots, p_r \) are uniquely determined; in fact, they are just the distinct associated primes of \( M/N \).

**Proof:** The assertion is just part of (18.19). \( \square \)

**Theorem (18.21) (Lasker–Noether).** — Over a Noetherian ring, each proper submodule of a finitely generated module has an irredundant primary decomposition.

**Proof:** Let \( M \) be the module, \( N \) the submodule. By (18.21), \( M/N \) has finitely many distinct associated primes, say \( p_1, \ldots, p_r \). Owing to (18.23), for each \( i \), there is a \( p_i \)-primary submodule \( Q_i \) of \( M \) with \( \text{Ass}(Q_i/N) = \text{Ass}(M/N) - \{p_i\} \). Set \( P := \bigcap Q_i \). Fix \( i \). Then \( P/N \subseteq Q_i/N \). So \( \text{Ass}(P/N) \subseteq \text{Ass}(Q_i/N) \) by (17.4.5). But \( i \) is arbitrary. Hence \( \text{Ass}(P/N) = \emptyset \). Therefore, \( P/N = 0 \) by (17.5.20). Finally, the decomposition \( N = \bigcap Q_i \) is irredundant by (18.19). \( \square \)

**Exercise (18.22).** — Let \( R \) be a Noetherian ring, \( a \) an ideal, and \( M \) a finitely generated module. Consider the following submodule of \( M \):

\[
\Gamma_a(M) := \bigcup_{n \geq 1}\{m \in M \mid a^nm = 0\}.
\]

1. For any decomposition \( 0 = \bigcap Q_i \) with \( Q_i \) \( p_i \)-primary, show \( \Gamma_a(M) = \bigcap_{a \not\in p_i} Q_i \).
   (By convention, if \( a \subseteq p_i \) for all \( i \), then \( \bigcap_{a \not\in p_i} Q_i = M \).)

2. Show \( \Gamma_a(M) \) is the set of all \( m \in M \) such that \( m/1 \in M_p \) vanishes for every prime \( p \) with \( a \not\in p \). (Thus \( \Gamma_a(M) \) is the set of all \( m \) whose support lies in \( \text{V}(a) \).)

**Lemma (18.23).** — Let \( R \) be a Noetherian ring, \( S \) a multiplicative subset, \( p \) a prime ideal, \( M \) a module, and \( Q \) a \( p \)-primary submodule. If \( S \cap p \neq \emptyset \), then \( S^{-1}Q = S^{-1}M \) and \( Q^S = M \). If \( S \cap p = \emptyset \), then \( S^{-1}Q \) is \( S^{-1}p \)-primary and \( Q^S = \varphi^{-1}_S(S^{-1}Q) = Q \).

**Proof:** Every prime of \( S^{-1}R \) is of the form \( S^{-1}q \) where \( q \) is a prime of \( R \) with \( S \cap q = \emptyset \) by (17.6.10) and (17.6.20). And \( S^{-1}q \in \text{Ass}(S^{-1}(M/Q)) \) if and only if \( q \in \text{Ass}(M/Q) \), that is, \( q = p \), by (17.6.11).

However, \( S^{-1}(M/Q) = S^{-1}M/S^{-1}Q \) by (17.6.11). Therefore, if \( S \cap p \neq \emptyset \), then \( \text{Ass}(S^{-1}M/S^{-1}Q) = \emptyset \); whence, (17.6.3) yields \( S^{-1}M/S^{-1}Q = 0 \). Otherwise, if \( S \cap p = \emptyset \), then \( \text{Ass}(S^{-1}M/S^{-1}Q) = \{S^{-1}p\} \); whence, \( S^{-1}Q \) is \( S^{-1}p \)-primary.

Finally, \( Q^S = \varphi^{-1}_S(S^{-1}Q) \) by (17.6.5). So if \( S^{-1}Q = S^{-1}M \), then \( Q^S = M \). Now, suppose \( S \cap p = \emptyset \). Given \( m \in Q^S \), there is \( s \in S \) with \( sm \in Q \). But \( s \not\in p \). Further, \( p = z.\text{div}(M/Q) \) owing to (17.6.10). Therefore, \( m \in Q \). Thus \( Q^S \subseteq Q \). But \( Q^S \cap Q = 1 \in S \). Thus \( Q^S = Q \). \( \square \)

**Proposition (18.24).** — Let \( R \) be a Noetherian ring, \( S \) a multiplicative subset, \( M \) a finitely generated module. Let \( N = Q_1 \cap \cdots \cap Q_r \subset M \) be an irredundant primary decomposition. Say \( Q_i \) is \( p_i \)-primary for all \( i \), and \( S \cap p_i = \emptyset \) just for
i \leq h. Then
\[ S^{-1}N = S^{-1}Q_1 \cap \cdots \cap S^{-1}Q_h \subset S^{-1}M \quad \text{and} \quad N^S = Q_1 \cap \cdots \cap Q_h \subset M \]
are irredundant primary decompositions.

Proof: By (18.24) (4)(b), \( S^{-1}N = S^{-1}Q_1 \cap \cdots \cap S^{-1}Q_r \). Further, by (18.28), \( S^{-1}Q_i \) is \( S^{-1}p_i \)-primary for \( i \leq h \), and \( S^{-1}Q_i = S^{-1}M \) for \( i > h \). Therefore, \( S^{-1}N = S^{-1}Q_1 \cap \cdots \cap S^{-1}Q_h \) is a primary decomposition.

It is irredundant by (18.12). Indeed, \( \text{Ass}(S^{-1}M/S^{-1}N)\{S^{-1}p_1, \ldots, S^{-1}p_h\} \) by an argument like that in the first part of (18.28). Further, \( S^{-1}p_1, \ldots, S^{-1}p_h \) are distinct by (18.24) (2) as the \( p_i \) are distinct.

Apply \( \varphi_S^{-1} \) to \( S^{-1}N = S^{-1}Q_1 \cap \cdots \cap S^{-1}Q_h \). Owing to (18.24) (3), we get \( N^S = Q_1^S \cap \cdots \cap Q_h^S \). But \( Q_i^S = Q_i \) by (18.23). So \( N^S = Q_1 \cap \cdots \cap Q_h \) is a primary decomposition. It is irredundant as, clearly, (18.13) (1) and (2) hold for it, since they hold for \( N = Q_1 \cap \cdots \cap Q_r \).

Theorem (18.25) (Second Uniqueness). — Let \( R \) be a ring, \( M \) a module, \( N \) a submodule. Assume \( R \) is Noetherian and \( M \) is finitely generated. Let \( p \) be a minimal prime of \( M/N \). Then, in any irredundant primary decomposition of \( N \) in \( M \), the \( p \)-primary component \( Q \) is uniquely determined; in fact, \( Q = N^S \) where \( S := R - p \).

Proof: In (18.24), take \( S := R - p \). Then \( h = 1 \) as \( p \) is minimal.

Exercise (18.26). — Let \( R \) be a Noetherian ring, \( M \) a finitely generated module, \( N \) a submodule. Prove \( N = \bigcap_{p \in \text{Ass}(M/N)} \varphi_p^{-1}(N_p) \).

Exercise (18.27). — Let \( R \) be a Noetherian ring, \( p \) a prime. Its \( n \)th symbolic power \( p^{(n)} \) is defined as the saturation \((p^n)^S\) where \( S := R - p \).

(1) Show \( p^{(n)} \) is the \( p \)-primary component of \( p^n \).

(2) Show \( p^{(m+n)} \) is the \( p \)-primary component of \( p^{(m)}p^{(n)} \).

(3) Show \( p^{(n)} = p^n \) if and only if \( p^n \) is \( p \)-primary.

(4) Given a \( p \)-primary ideal \( q \), show \( q \supset p^{(n)} \) for all large \( n \).

Exercise (18.28). — Let \( R \) be a Noetherian ring, \( 0 = q_1 \cap \cdots \cap q_n \) an irredundant primary decomposition. Set \( p_i := \sqrt{q_i} \) for \( i = 1, \ldots, n \).

(1) Suppose \( p_i \) is minimal for some \( i \). Show \( q_i = p_i^{(r)} \) for all large \( r \).

(2) Suppose \( p_i \) is not minimal for some \( i \). Show that replacing \( q_i \) by \( p_i^{(r)} \) for large \( r \) gives infinitely many distinct irredundant primary decompositions of \( 0 \).

Theorem (18.29) (Krull Intersection). — Let \( R \) be a Noetherian ring, \( a \) an ideal, and \( M \) a finitely generated module. Set \( N := \bigcap_{n \geq 0} a^nM. \) Then there exists \( x \in a \) such that \((1+x)N = 0\).

Proof: By (18.12), \( N \) is finitely generated. So the desired \( x \in a \) exists by (18.20) provided \( N = aN. \) Clearly \( N \supset aN. \) To prove \( N \supset aN \), use (18.24): take a decomposition \( aN = \bigcap Q_i \) with \( Q_i \) \( p_i \)-primary. Fix \( i. \) If there’s \( a \in a - p_i \), then \( aN \subset Q_i \), and so (18.23) yields \( N \subset Q_i. \) If \( a \subset p_i \), then there’s \( n_i \) with \( a^{n_i}M \subset Q_i \) by (18.8) and (18.7), and so again \( N \subset Q_i. \) Thus \( N \subset \bigcap Q_i = aN \) as desired.

Exercise (18.30). — Let \( R \) be a Noetherian ring, \( m \subset \text{rad}(R) \) an ideal, \( M \) a finitely generated module, and \( M' \) a submodule. Considering \( M/M' \), show that \( M' = \bigcap_{n \geq 0}(m^nM + M'). \)
Example (18.31) (Another non-Noetherian ring). — Let \( R \) denote the ring of \( C^\infty \) functions on the real line, \( \mathfrak{m} \) the ideal of all \( f \in R \) that vanish at the origin. Note that \( \mathfrak{m} \) is maximal, as \( f \mapsto f(0) \) defines an isomorphism \( R/\mathfrak{m} \rightarrow \mathbb{R} \).

Let \( f \in R \) and \( n \geq 1 \). Then, Taylor’s Theorem yields

\[
f(x) = f(0) + f'(0)x + \cdots + \left( \frac{f^{(n-1)}(0)}{(n-1)!} \right) x^{n-1} + x^n f_n(x)
\]

where \( f_n(x) := \int_0^1 \left( \frac{1-t}{n-1} \right)^n f^{(n)}(tx) \, dt. \)

Here \( f_n \) is \( C^\infty \) too, since we can differentiate under the integral sign by [3], (7.1), p. 276. So, if \( f \in \mathfrak{m} \), then \( f(x) = xf_1(x) \). Thus \( \mathfrak{m} \subset \langle x \rangle \). But, obviously, \( \mathfrak{m} \supset \langle x \rangle \). Hence \( \mathfrak{m} = \langle x \rangle \). Therefore, \( \mathfrak{m}^n = \langle x^n \rangle \).

If the first \( n-1 \) derivatives of \( f \) vanish at 0, then Taylor’s Theorem yields \( f \in \langle x^n \rangle \). Conversely, assume \( f(x) = x^ng(x) \) for some \( g \in R \). By Leibniz’s Rule,

\[
f^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} \frac{x^{n-j+1}}{(n-j+1)!} g^{(k-j)}(x).
\]

Hence \( f^{(k)} \) vanishes at 0 if \( n > k \). Thus \( \langle x^n \rangle \) consists of the \( f \in R \) whose first \( n-1 \) derivatives vanish at 0. But \( \langle x^n \rangle = \mathfrak{m}^n \). Thus \( \bigcap_{n \geq 0} \mathfrak{m}^n \) consists of those \( f \in R \) all of whose derivatives vanish at 0.

There is a well-known nonzero \( C^\infty \)-function all of whose derivatives vanish at 0:

\[
h(x) := \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0; \end{cases}
\]

see [3], Ex. 7, p. 82. Thus \( \bigcap_{n \geq 0} \mathfrak{m}^n \neq 0 \).

Given \( g \in \mathfrak{m} \), let’s show \( (1 + g)h \neq 0 \). Since \( g(0) = 0 \) and \( g \) is continuous, there is \( \delta > 0 \) such that \( |g(x)| < 1/2 \) if \( |x| < \delta \). Hence \( 1 + g(x) \geq 1/2 \) if \( |x| < \delta \). Hence \( (1 + g(x))h(x) > (1/2)h(x) > 0 \) if \( 0 < |x| < \delta \). Thus \( (1 + g)(\bigcap \mathfrak{m}^n) \neq 0 \). Thus the Krull Intersection Theorem (18.32) fails for \( R \), and so \( R \) is non-Noetherian.
19. Length

The length of a module is a generalization of the dimension of a vector space. The length is the number of links in a composition series, which is a finite chain of submodules whose successive quotients are simple—that is, their only proper submodules are zero. Our main result is the Jordan–Hölder Theorem: any two composition series do have the same length and even the same successive quotients; further, their annihilators are just the primes in the support of the module, and the module is equal to the product of its localizations at these primes. Consequently, the length is finite if and only if the module is both Artinian and Noetherian. We also prove the Akizuki–Hopkins Theorem: a ring is Artinian if and only if it is Noetherian and every prime is maximal. Consequently, a ring is Artinian if and only if its length is finite; if so, then it is the product of Artinian local rings.

(19.1) (Length). — Let $R$ be a ring, and $M$ a module. We call $M$ simple if it is nonzero and its only proper submodule is 0. We call a chain of submodules,

$$M = M_0 \supset M_1 \supset \cdots \supset M_m = 0$$

(19.1.1)

a composition series of length $m$ if each successive quotient $M_{i-1}/M_i$ is simple.

Finally, we define the length $\ell(M)$ to be the infimum of all those lengths:

$$\ell(M) := \inf \{ m \mid M \text{ has a composition series of length } m \}. \quad (19.1.2)$$

By convention, if $M$ has no composition series, then $\ell(M) := 1$. Further, $\ell(M) = 0$ if and only if $M = 0$.

For example, if $R$ is a field, then $M$ is a vector space and $\ell(M) = \dim_R(M)$. Also, the chains in (17.24) are composition series, but those in (17.23) are not.

Exercise (19.2). — Let $R$ be a ring, $M$ a module. Prove these statements:

1. If $M$ is simple, then any nonzero element $m \in M$ generates $M$.
2. $M$ is simple if and only if $M \cong R/m$ for some maximal ideal $m$, and if so, then $m = \text{Ann}(M)$.
3. If $M$ has finite length, then $M$ is finitely generated.

Theorem (19.3) (Jordan–Hölder). — Let $R$ be a ring, and $M$ a module with a composition series (19.1.1). Then any chain of submodules can be refined to a composition series, and every composition series is of the same length $\ell(M)$. Also,

$$\text{Supp}(M) = \{ m \in \text{Spec}(R) \mid m = \text{Ann}(M_{i-1}/M_i) \text{ for some } i \};$$

the $m \in \text{Supp}(M)$ are maximal; there is a canonical isomorphism

$$M \rightarrow \prod_{m \in \text{Supp}(M)} M_m;$$

and $\ell(M_m)$ is equal to the number of $i$ with $m = \text{Ann}(M_{i-1}/M_i)$.

Proof: First, let $M'$ be a proper submodule of $M$. Let’s show that

$$\ell(M') < \ell(M). \quad (19.3.1)$$

To do so, set $M' := M_i \cap M'$. Then $M'_{i-1} \cap M_i = M_i'$. So

$$M'_{i-1}/M'_{i} = (M'_{i-1} + M_i)/M_i \subset M_{i-1}/M_i.$$
Since $M_{i-1}/M_i$ is simple, either $M'_{i-1}/M'_i = 0$, or $M'_{i-1}/M'_i = M_{i-1}/M_i$ and so
\[ M'_{i-1} + M_i = M_{i-1}. \] (19.3.2)
If (19.3.2) holds and if $M_i \subseteq M'$, then $M_{i-1} \subseteq M'$. Hence, if (19.3.2) holds for all $i$, then $M \subseteq M'$, a contradiction. Therefore, there is an $i$ with $M'_{i-1}/M'_i = 0$.

Now, $M' = M'_0 \supseteq \cdots \supseteq M'_m = 0$. Omit $M'_i$ whenever $M'_{i-1}/M'_i = 0$. Thus $M'$ has a composition series of length strictly less than $m$. Therefore, $\ell(M') < m$ for any choice of (19.3.1). Thus (19.3.1) holds.

Next, given a chain $N_0 \supseteq \cdots \supseteq N_n = 0$, let’s prove $n \leq \ell(M)$ by induction on $\ell(M)$. If $\ell(M) = 0$, then $M = 0$; so also $n = 0$. Assume $\ell(M) \geq 1$. If $n = 0$, then we’re done. If $n \geq 1$, then $\ell(N_1) < \ell(M)$ by (19.3.1); so $n - 1 \leq \ell(N_1)$ by induction. Thus $n \leq \ell(M)$.

If $N_{i-1}/N_i$ is not simple, then there is $N'$ with $N_{i-1} \nsubseteq N' \supsetneq N_i$. The new chain can have length at most $\ell(M)$ by the previous paragraph. Repeating, we can refine the given chain into a composition series in at most $\ell(M) - n$ steps.

Suppose the given chain is a composition series. Then $\ell(M) \leq n$ by (19.3.2). But we proved $n \leq \ell(M)$ above. Thus $n = \ell(M)$, and the first assertion is proved.

To proceed, fix a prime $p$. Exactness of Localization, (12.20), yields this chain:
\[ M_p = (M_0)_p \supseteq (M_1)_p \supseteq \cdots \supseteq (M_m)_p = 0. \] (19.3.3)
Now, consider a maximal ideal $m$. If $p = m$, then $(R/m)_p \simeq R/m$ by (12.2) and (12.20). If $p \neq m$, then there is $s \in m - p$; so $(R/m)_p = 0$.

Set $m_i := \text{Ann}(M_{i-1}/M_i)$. So $M_{i-1}/M_i \simeq R/m_i$ and $m_i$ is maximal by (12.2) (2). Then Exactness of Localization yields $(M_{i-1}/M_i)_p = (M_{i-1})_p/(M_i)_p$. Hence
\[ (M_{i-1})_p/(M_i)_p \begin{cases} 0, & \text{if } p \neq m_i; \\ M_{i-1}/M_i \simeq R/m_i, & \text{if } p = m_i. \end{cases} \]
Thus $\text{Supp}(M) = \{m_1, \ldots, m_m\}$.

If we omit the duplicates from the chain (19.3.3), then we get a composition series from the $(M_i)_p$ with $M_{i-1}/M_i \simeq R/p$. Thus the number of such $i$ is $\ell(M_p)$.

Finally, consider the canonical map $\varphi : M \to \prod_{m \in \text{Supp}(M)} M_m$. To prove $\varphi$ is an isomorphism, it suffices, by (12.20), to prove $\varphi_p$ is for each maximal ideal $p$. Now, localization commutes with finite product by (12.20). Therefore,
\[ \varphi_p : M_p \to \left( \prod_{m} M_m \right)_p = \prod_{m} (M_m)_p = M_p \]
as $(M_m)_p = 0$ if $m \neq p$ and $(M_m)_p = M_p$ if $m = p$ by the above. Thus $\varphi_p = 1$. □

**EXERCISE (19.3).** — Let $R$ be a Noetherian ring, $M$ a finitely generated module. Prove the equivalence of the following three conditions:

1. that $M$ has finite length;
2. that $\text{Supp}(M)$ consists entirely of maximal ideals;
3. that $\text{Ass}(M)$ consists entirely of maximal ideals.

Prove that, if the conditions hold, then $\text{Ass}(M)$ and $\text{Supp}(M)$ are equal and finite.

**EXERCISE (19.2).** — Let $R$ be a Noetherian ring, $q$ a $p$-primary ideal. Consider chains of primary ideals from $q$ to $p$. Show (1) all such chains have length at most $\ell(A)-1$ where $A := (R/q)_p$ and (2) all maximal chains have length exactly $\ell(A)-1$.

**COROLLARY (19.6).** — A module $M$ is both Artinian and Noetherian if and only if $M$ is of finite length.
Any chain $M \supset N_0 \supset \cdots \supset N_n = 0$ has $n < \ell(M)$ by the Jordan–Hölder Theorem, (19.9). So if $\ell(M) < \infty$, then $M$ satisfies both the dcc and the acc.

Conversely, assume $M$ is both Artinian and Noetherian. Form a chain as follows. Set $M_0 := M$. For $i \geq 1$, if $M_{i-1} \neq 0$, take a maximal $M_i \supset M_{i-1}$ by the maxc. By the dcc, this recursion terminates. Then the chain is a composition series. □

Example (19.7). — Any simple $\mathbb{Z}$-module is finite owing to (19.4)(2). Hence, a $\mathbb{Z}$-module is of finite length if and only if it is finite. In particular, $\ell(\mathbb{Z}) = \infty$.

Of course, $\mathbb{Z}$ is Noetherian, but not Artinian.

Let $p \in \mathbb{Z}$ be a prime, and set $M := \mathbb{Z}[1/p]/\mathbb{Z}$. Then $M$ is an Artinian $\mathbb{Z}$-module, but not Noetherian by (19.9). Since $M$ is infinite, $\ell(M) = \infty$.

Exercise (19.3). — Let $k$ be a field, $R$ an algebra-finite extension. Prove that $R$ is Artinian if and only if $R$ is a finite-dimensional $k$-vector space.

Theorem (19.9) (Additivity of Length). — Let $M$ be a module, and $M'$ a submodule. Then $\ell(M) = \ell(M') + \ell(M/M')$.

Proof: If $M$ has a composition series, then the Jordan–Hölder Theorem yields another one of the form $M = M_0 \supset \cdots \supset M' \supset \cdots \supset M_m = 0$. The latter yields a pair of composition series: $M/M' = M_0/M' \supset \cdots \supset M'/M' = 0$ and $M' \supset \cdots \supset M_m = 0$. Conversely, every such pair arises from a unique composition series in $M$ through $M'$. Therefore, $\ell(M) < \infty$ if and only if $\ell(M/M') < \infty$ and $\ell(M') < \infty$; furthermore, if so, then $\ell(M) = \ell(M') + \ell(M/M')$, as desired. □

Exercise (19.11). — Let $k$ be a field, $A$ a local $k$-algebra. Assume the map from $k$ to the residue field is bijective. Given an $A$-module $M$, prove $\ell(M) = \dim_k(M)$.

Theorem (19.11) (Akizuki–Hopkins). — A ring $R$ is Artinian if and only if $R$ is Noetherian and $\dim(R) = 0$. If so, then $R$ has only finitely many primes.

Proof: If $\dim(R) = 0$, then every prime is maximal. If also $R$ is Noetherian, then $R$ has finite length by (19.4). Thus $R$ is Artinian by (19.9).

Conversely, suppose $R$ is Artinian. Let $m$ be a minimal product of maximal ideals of $R$. Then $m^2 = m$. Let $S$ be the set of ideals $a$ contained in $m$ such that $am \neq 0$. If $S \neq \emptyset$, take $a \in S$ minimal. Then $am^2 = am \neq 0$; hence, $am = a$ by minimality of $a$. For any $x \in a$, if $xm \neq 0$, then $a = \langle x \rangle$ by minimality of $a$.

Let $n$ be any maximal ideal. Then $nn = m$ by minimality of $m$. But $nn \subset n$. Thus $m \subset \text{rad}(R)$. But $a = \langle x \rangle$. So Nakayama’s Lemma yields $a = 0$, a contradiction. So $xm = 0$ for any $x \in a$. Thus $am = 0$, a contradiction. Hence $S = \emptyset$. Therefore, $m^2 = 0$. But $m^2 = m$. Thus $m = 0$. Say $m = m_1 \cdots m_r$ with $m_i$ maximal.

Set $n_i := m_1 \cdots m_i$ for $1 \leq i \leq r$. Consider the chain

$$R =: n_0 \supset n_1 \supset \cdots \supset n_r = 0.$$  

Fix $i$. Set $V_i := n_{i-1}/n_i$. Then $V_i$ is a vector space over $R/m_i$.

Suppose $\dim(V_i) = \infty$. Take linearly independent elements $x_1, x_2, \ldots \in V_i$, let $W_j \subset V_i$ be the subspace spanned by $x_j, x_{j+1}, \ldots$. The $W_j$ form a strictly descending chain, a contradiction as $R$ is Artinian. Thus $\dim(V_i) < \infty$. Hence $\ell(R) < \infty$ by (19.9). So $R$ is Noetherian by (19.6). Now, $\text{Ann}(R) = 0$; so (19.9) yields $\text{Supp}(R) = \text{Spec}(R)$. Thus, by (19.3), every prime is maximal, and there are only finitely many primes. □

Exercise (19.12). — Prove these conditions on a Noetherian ring $R$ equivalent:
(1) that $R$ is Artinian;
(2) that $\text{Spec}(R)$ is discrete and finite;
(3) that $\text{Spec}(R)$ is discrete.

EXERCISE (19.13). — Let $R$ be an Artinian ring. Show that $\text{rad}(R)$ is nilpotent.

COROLLARY (19.14). — Let $R$ be an Artinian ring, and $M$ a finitely generated module. Then $M$ has finite length, and $\text{Ass}(M)$ and $\text{Supp}(M)$ are equal and finite.

PROOF: By (19.11) every prime is maximal, so $\text{Supp}(M)$ consists of maximal ideals. Also $R$ is Noetherian by (19.11). Hence (19.2) yields the assertions. □

COROLLARY (19.15). — A ring $R$ is Artinian if and only if $\ell(R) < \infty$.

PROOF: Simply take $M := R$ in (19.13) and (19.16). □

EXERCISE (19.16). — Let $R$ be a ring, $p$ a prime ideal, and $R'$ a module-finite $R$-algebra. Show that $R'$ has only finitely many primes $p'$ over $p$, as follows: reduce to the case that $R$ is a field by localizing at $p$ and passing to the residue rings.

COROLLARY (19.17). — A ring $R$ is Artinian if and only if $R$ is a finite product of Artinian local rings; if so, then $R = \prod_{m \in \text{Spec}(R)} R_m$.

PROOF: A finite product of rings is Artinian if and only if each factor is Artinian by (16.27)(3). If $R$ is Artinian, then $\ell(R) < \infty$ by (19.15); whence, $R = \prod R_m$ by the Jordan–Hölder Theorem. Thus the assertion holds. □

EXERCISE (19.18). — Let $R$ be a Noetherian ring, and $M$ a finitely generated module. Prove the following four conditions are equivalent:

(1) that $M$ has finite length;
(2) that $M$ is annihilated by some finite product of maximal ideals $\prod m_i$;
(3) that every prime $p$ containing $\text{Ann}(M)$ is maximal;
(4) that $R/\text{Ann}(M)$ is Artinian.
20. Hilbert Functions

The **Hilbert Function** of a graded module lists the lengths of its components. The corresponding generating function is called the **Hilbert Series**. This series is, under suitable hypotheses, a rational function, according to the Hilbert–Serre Theorem, which we prove. Passing to an arbitrary module, we study its **Hilbert–Samuel Series**, namely, the generating function of the colengths of the submodules in a filtration. We prove Samuel’s Theorem: if the ring is Noetherian, if the module is finitely generated, and if the filtration is stable, then the Hilbert–Samuel Series is a rational function with poles just at 0 and 1. In the same setup, we prove the Artin–Rees Lemma: given any submodule, its induced filtration is stable.

In a brief appendix, we study further one notion that arose: homogeneity.

(20.1) (Graded rings and modules). — We call a ring $R$ **graded** if there are additive subgroups $R_n$ for $n \geq 0$ with $R = \bigoplus R_n$ and $R_m R_n \subseteq R_{m+n}$ for all $m, n$.

For example, a polynomial ring $R$ with coefficient ring $R_0$ is graded if $R_n$ is the $R_0$-submodule generated by the monomials of (total) degree $n$.

In general, $R_0$ is a **subring**. Obviously, $R_0$ is closed under addition and under multiplication, but we must check $1 \in R_0$. So say $1 = \sum x_m$ with $x_m \in R_m$. Given $z \in R$, say $z = \sum z_n$ with $z_n \in R_n$. Fix $n$. Then $z_n = 1 \cdot z_n = \sum x_m z_n$ with $x_m z_n \in R_{m+n}$. So $\sum_{m>0} x_m z_n = z_n - x_0 z_n \in R_n$. Hence $x_m z_n = 0$ for $m > 0$.

But $n$ is arbitrary. So $x_m z = 0$ for $m > 0$. But $z$ is arbitrary. Taking $z := 1$ yields $x_m = x_m \cdot 1 = 0$ for $m > 0$. Thus $1 = x_0 \in R_0$.

We call an $R$-module $M$ (compatibly) **graded** if there are additive subgroups $M_n$ for $n \in \mathbb{Z}$ with $M = \bigoplus M_n$ and $R_m M_n \subseteq M_{m+n}$ for all $m, n$. We call $M_n$ the $n$th **homogeneous component**; we say its elements are **homogeneous**. Obviously, $M_n$ is an $R_0$-module.

Given $m \in \mathbb{Z}$, set $M(m) := \bigoplus M_{m+n}$. Then $M(m)$ is another graded module; its $n$th graded component $M(m)_n$ is $M_{m+n}$. Thus $M(m)$ is obtained from $M$ by **shifting** $m$ places to the left.

**Lemma** (20.2). — Let $R = \bigoplus R_n$ be a graded ring, and $M = \bigoplus M_n$ a graded $R$-module. If $R$ is a finitely generated $R_0$-algebra and if $M$ is a finitely generated $R$-module, then each $M_n$ is a finitely generated $R_0$-module.

**Proof:** Say $R = R_0[x_1, \ldots, x_r]$. If $x_i = \sum_j x_{ij}$ with $x_{ij} \in R_j$, then replace the $x_i$ by the nonzero $x_{ij}$. Similarly, say $M$ is generated over $R$ by $m_1, \ldots, m_s$ with $m_i \in M_{l_i}$. Then any $m \in M_n$ is a sum $m = \sum f_i m_i$ where $f_i \in R$. Say $f_i = \sum f_{ij}$ with $f_{ij} \in R_j$, and replace $f_i$ by $f_{ik}$ with $k := n - l_i$ or by 0 if $n < l_i$. Then $f_i$ is an $R_0$-linear combination of monomials $x_1^{i_1} \cdots x_r^{i_r} \in R_k$; hence, $m$ is an $R_0$-linear combination of the products $x_1^{i_1} \cdots x_r^{i_r} m_i \in M_n$, as desired. \hfill $\square$

(20.3) (Hilbert functions). — Let $R = \bigoplus R_n$ be a graded ring, and $M = \bigoplus M_n$ a graded $R$-module. Assume $R_0$ is Artinian, $R$ is a finitely generated $R_0$-algebra, and $M$ is a finitely generated $R$-module. Then each $M_n$ is a finitely generated $R_0$-module by (20.2), so is of finite length $\ell(M_n)$ by (19.7, 13). We call $n \mapsto \ell(M_n)$ the **Hilbert Function** of $M$ and its generating function

$$H(M, t) := \sum_{n \in \mathbb{Z}} \ell(M_n) t^n$$

116
the Hilbert Series of $M$. This series is a rational function by (20.7) below.

If $R = R_0[x_1, \ldots, x_r]$ with $x_i \in R_1$, then by (20.8) below, the Hilbert Function
is, for $n \gg 0$, a polynomial $h(M, n)$, called the Hilbert Polynomial of $M$.

**Example (20.4).** — Let $R := R_0[X_1, \ldots, X_r]$ be the polynomial ring, graded by
degree. Then $R_n$ is free over $R_0$ on the monomials of degree $n$, so of rank \((r-1+n)\).

Assume $R_0$ is Artinian. Then $\ell(R_n) = \ell(R_0)(r-1+n)$ by Additivity of Length, (19.5a).
Thus the Hilbert Function is, for $n \geq 0$, a polynomial of degree $r - 1$.

Formal manipulation yields \((r-1+n)/(r-1) = (-1)^n(r-1)^n\). Therefore, Newton’s binomial theorem for negative exponents yields this computation for the Hilbert Series:

$$H(R, t) = \sum_{n \geq 0} \ell(R_n) \left(\frac{t}{r-1}\right)^n = \sum_{n \geq 0} \ell(R_0) \left(\frac{t}{r-1}\right)^n = \ell(R_0)/\left(1 - \frac{t}{r-1}\right)^r.$$ 

**Exercise (20.9).** — Let $k$ be a field, $k[X, Y]$ the polynomial ring. Show $\langle X, Y^2\rangle$
and $\langle X^2, Y^2\rangle$ have different Hilbert Series, but the same Hilbert Polynomial.

**Exercise (20.10).** — Let $R = \bigoplus R_n$ be a graded ring, $M = \bigoplus M_n$ a graded
module. Let $N = \bigoplus N_n$ be a homogeneous submodule: that is, $N_n = N \cap M_n$.
Assume $R_0$ is Artinian, $R$ is a finitely generated $R_0$-algebra, and $M$ is a finitely
generated $R$-module. Set

$$N' := \{ m \in M \mid \text{there is } k_0 \text{ such that } R_k m \subseteq N \text{ for all } k \geq k_0 \}.$$ 

(1) Prove that $N'$ is a homogeneous submodule of $M$ with the same Hilbert Polynomial as $N$, and that $N'$ is the largest such submodule containing $N$.

(2) Let $N = \bigcap Q_i$ be a decomposition with $Q_i$ $p_i$-primary. Set $R_+ := \bigoplus_{n > 0} R_n$.
Prove that $N' = \bigcap_{p_i \notdivides R_+} Q_i$.

**Theorem (20.7) (Hilbert–Serre).** — Let $R = \bigoplus R_n$ be a graded ring, and let $M = \bigoplus M_n$ a graded $R$-module. Assume $R_0$ is Artinian, $R$ is a finitely generated $R_0$-algebra, and $M$ is a finitely generated $R$-module. Then

$$H(M, t) = e(t)/t! \left(1 - t^{k_2}\right) \cdots \left(1 - t^{k_r}\right)$$

with $e(t) \in \mathbb{Z}[t]$, with $l \geq 0$, and with $k_1, \ldots, k_r \geq 1$.

**Proof:** Say $R = R_0[x_1, \ldots, x_r]$ with $x_i \in R_{k_i}$. First, assume $r = 0$. Say $M$ is
generated over $R$ by $m_1, \ldots, m_s$ with $m_i \in M_{l_i}$. Then $R = R_0$. So $M_n = 0$ for
$n < l_0 := \min\{l_i\}$ and for $n > \max\{l_i\}$. Hence $t^{-l_0}H(M, t)$ is a polynomial.

Next, assume $r \geq 1$ and form the exact sequence

$$0 \to K \to M(-k_1) \overset{\mu_{x_1}}\to M \to L \to 0$$

where $\mu_{x_1}$ is the map of multiplication by $x_1$. Since $x_1 \in R_{k_1}$, the grading on $M$
induces a grading on $K$ and on $L$. Further, $\mu_{x_1}$ acts as 0 on both $K$ and $L$.

As $R_0$ is Artinian, $R_0$ is Noetherian by the Akizuki–Hopkins Theorem, (19.1a).
So, since $R$ is a finitely generated $R_0$-algebra, $R$ is Noetherian by (16.7a). Since
$M$ is a finitely generated $R$-module, obviously so is $M(-k_1)$. Hence, so are both
$K$ and $L$ by (16.4a)(2). Set $R' := R_0[x_2, \ldots, x_r]$. Since $x_1$ acts as 0 on $K$ and $L$, they are finitely generated $R'$-modules. Therefore, $H(K, t)$ and $H(L, t)$ may be
written in the desired form by induction on $r$.

By definition, $M(-k_1)_n := M_{n-k_1}$; hence, $H(M(-k_1), t) = t^{k_1} H(M, t)$. Therefore, Additivity of Length, (19.5a), and the previous paragraph yield

$$\left(1 - t^{k_2}\right) H(M, t) = H(L, t) - H(K, t) = e(t)/t! \left(1 - t^{k_2}\right) \cdots \left(1 - t^{k_r}\right).$$

Thus the assertion holds. \(\square\)
COROLLARY (20.8). — Under the conditions of (20.7), say \( R = R_0[x_1, \ldots, x_r] \) with \( x_i \in R_1 \). Assume \( M \neq 0 \). Then \( H(M, t) \) can be written uniquely in the form

\[
H(M, t) = e(t)/t^d
\]

(20.8.1)

with \( e(t) \in \mathbb{Z}[t] \) and \( e(0), e(1) \neq 0 \) and \( l \in \mathbb{Z} \) and \( r \geq d \geq 0 \); also, there is a polynomial \( h(M, n) \in \mathbb{Q}[n] \) with degree \( d - 1 \), leading coefficient \( e(1)/(d - 1)! \) and

\[
\ell(M_n) = h(M, n) \quad \text{for} \quad n \geq \deg(e(t)) - l.
\]

(20.8.2)

PROOF: We may take \( k_i = 1 \) for all \( i \) in the proof of (20.7). Hence \( H(M, t) \) has the form \( e(t)/(1 - t)^d / t^d (1 - t)^r \) with \( e(0) \neq 0 \) and \( e(1) \neq 0 \) and \( l \in \mathbb{Z} \). Set \( d := r - s \).

Then \( d \geq 0 \) since \( H(M, 1) > 0 \) as \( M \neq 0 \). Thus \( H(M, t) \) has the asserted form. This form is unique owing to the uniqueness of factorization of polynomials.

Say \( e(t) = \sum_{i=0}^N e_i t^i \). Now, \( (1 - t)^d = \sum \binom{d}{n} (-t)^n = \sum (d-1+n) t^n \). Hence

\[
\ell(M_n) = \sum_{i=0}^N e_i \left( \frac{d-1+n+l}{d-1} \right) \text{ for } n + l \geq N. \quad \text{But } \left( \frac{d-1+n-l}{d-1} \right) = \frac{n-1}{(d-1)!} + \cdots.
\]

Therefore, \( \ell(M_n) = e(1) n^{d-1}/(d-1)! + \cdots \), as asserted.

\( \square \)

EXERCISE (20.9). — Let \( k \) be a field, \( P := k[X, Y, Z] \) the polynomial ring in three variables, \( f \in P \) a homogeneous polynomial of degree \( d \geq 1 \). Set \( R := f/(f) \). Find the coefficients of the Hilbert Polynomial \( h(R, n) \) explicitly in terms of \( d \).

EXERCISE (20.10). — Under the conditions of (20.8), assume there is a homogeneous nonzerodivisor \( f \in R \) with \( M_f = 0 \). Prove \( \deg h(R, n) > \deg h(M, n) \); start with the case \( M := R/(f^k) \).

(20.11) (Filtrations). — Let \( R \) be an arbitrary ring, \( q \) an ideal, and \( M \) a module. A filtraton \( F^* M \) of \( M \) is an infinite descending chain of submodules:

\[
M \supset \cdots \supset F^n M \supset F^{n+1} M \supset \cdots.
\]

Call it a \( q \)-filtration if \( q F^n M \subset F^{n+1} M \) for all \( n \), and a stable \( q \)-filtration if also \( M = F^n M \) for \( n = 0 \) and \( q F^n M = F^{n+1} M \) for \( n > 0 \). This condition means that there are \( \mu \) and \( \nu \) with \( M = F^\mu \) and \( q F^\nu M = F^{\nu+1} M \) for \( n > 0 \).

For example, setting \( F^0 M := M \) for \( n \leq 0 \) and \( F^n M := q^n M \) for \( n \geq 0 \), we get a stable \( q \)-filtration. It is called the \( q \)-adic filtration.

The \( q \)-adic filtration of \( R \) yields a graded ring \( G^* R \), defined by

\[
G^* R := \bigoplus_{n \geq 0} G^n R \quad \text{where} \quad G^n R := q^n / q^{n+1}.
\]

We form the product of an element in \( q^i / q^{i+1} \) and one in \( q^j / q^{j+1} \) by choosing representatives, forming their product, and taking its residue in \( q^{i+j} / q^{i+j+1} \). We call \( G^* R \) the associated graded ring.

As each \( F^n M \) is an \( R \)-module, so is the direct sum

\[
G^* M := \bigoplus_{n \in \mathbb{Z}} G^n M \quad \text{where} \quad G^n M := F^n M / F^{n+1} M.
\]

If \( F^* M \) is a \( q \)-filtration, then this \( R \)-structure amounts to an \( R/q \)-structure; further, \( G^* R \) is a graded \( G^* \)-module.

Given \( m \in \mathbb{Z} \), let \( M[m] \) denote \( M \) with the filtration \( F^* M \) reindexed by shifting it \( m \) places to the left; that is, \( F^n(M[m]) := F^{n+m} M \) for all \( n \). Then

\[
G^n(M[m]) = F^{n+m} M / F^{n+m+1} M = (G^n M)(m).
\]

If the quotients \( M / F^n M \) have finite length, call \( n \mapsto \ell(M / F^n M) \) the Hilbert–Samuel function, and call the generating function

\[
P(F^* M, t) := \sum_{n \geq 0} \ell(M / F^n M) t^n
\]
the Hilbert–Samuel Series. If the function \( n \mapsto \ell(M/F^nM) \) is, for \( n \gg 0 \), a polynomial \( p(F^*M, n) \), then call it the Hilbert–Samuel Polynomial. If the filtration is the \( q \)-adic filtration, we also denote \( P(F^*M, t) \), and \( p(F^*M, n) \) by \( P_q(M, t) \) and \( p_q(M, n) \).

**Lemma (20.12).** — Let \( R \) be a Noetherian ring, \( q \) an ideal, \( M \) a finitely generated module with a stable \( q \)-filtration. Then \( G^*R \) is generated as an \( R/q \)-algebra by finitely many elements of \( q/q^2 \), and \( G^*M \) is a finitely generated \( G^*R \)-module.

**Proof:** Since \( R \) is Noetherian, \( q \) is a finitely generated ideal, say by \( x_1, \ldots, x_r \). Then, clearly, the residues of the \( x_i \) in \( q/q^2 \) generate \( G^*R \) as an \( R/q \)-algebra. By stability, there are \( \mu \) and \( \nu \) with \( F^\mu M = M \) and \( q^nF^\nu M = F^{n+1}M \) for \( n \geq 0 \). Hence \( G^*M \) is generated by \( F^\mu M/F^\mu+1M, \ldots, F^\nu M/F^\nu+1M \) over \( G^*R \). But \( R \) is Noetherian and \( M \) is finitely generated over \( R \); hence, every \( F^nM \) is finitely generated over \( R \). Therefore, every \( F^nM/F^{n+1}M \) is finitely generated over \( R/q \). Thus \( G^*M \) is a finitely generated \( G^*R \)-module.

**Theorem (20.13) (Samuel).** — Let \( R \) be a Noetherian ring, \( q \) an ideal, and \( M \) a finitely generated module with a stable \( q \)-filtration \( F^*M \). Assume \( \ell(M/qM) < \infty \). Then \( \ell(F^nM/F^{n+1}M) < \infty \) and \( \ell(M/F^nM) < \infty \) for every \( n \geq 0 \), further,

\[
P(F^*M, t) = H(G^*M, t) t/(1-t).
\]

**Proof:** Set \( a := \text{Ann}(M) \). Set \( R' := R/a \) and \( q' := (a+q)/a \). Then \( R'/q' \) is Noetherian as \( R \) is. Also, \( M \) can be viewed as a finitely generated \( R' \)-module, and \( F^*M \) as a stable \( q' \)-filtration. So \( G^*R' \) is generated as an \( R'/q' \)-algebra by finitely many elements of degree 1, and \( G^*M \) is a finitely generated \( G^*R' \)-module by (20.12). Therefore, each \( F^nM/F^{n+1}M \) is a finitely generated \( R'/q' \)-module by (20.12) or by the proof of (20.12).

On the other hand, (13.31) and (13.37) yield, respectively,

\[
V(a+q) = V(a) \cap V(q) = \text{Supp}(M) \cap V(q) = \text{Supp}(M/qM).
\]

Hence \( V(a+q) \) consists entirely of maximal ideals, because \( \text{Supp}(M/qM) \) does by (13.33) as \( \ell(M/qM) < \infty \). Thus \( \dim(R'/q') = 0 \). But \( R'/q' \) is Noetherian. Therefore, \( R'/q' \) is Artinian by the Akizuki–Hopkins Theorem, (19.11).

Hence \( \ell(F^nM/F^{n+1}M) < \infty \) for every \( n \) by (19.13). Form the exact sequence

\[
0 \to F^nM/F^{n+1}M \to M/F^{n+1}M \to M/F^nM \to 0.
\]

Then Additivity of Length, (19.14), yields

\[
\ell(F^nM/F^{n+1}M) = \ell(M/F^nM) - \ell(M/F^{n+1}M).
\]

So induction on \( n \) yields \( \ell(M/F^nM) < \infty \) for every \( n \). Further, multiplying that equation by \( t^n \) and summing over \( n \) yields the desired expression in another form:

\[
H(G^*M, t) = (t^{-1} - 1)P(F^*M, t) = P(F^*M, t) (1-t)/t.
\]

**Corollary (20.14).** — Under the conditions of (20.13), assume \( q \) is generated by \( r \) elements and \( M \neq 0 \). Then \( P(F^*M, t) \) can be written uniquely in the form

\[
P(F^*M, t) = e(t)/t^{d+1}(1-t)^{d+1}
\]

with \( e(t) \in \mathbb{Z}[t] \) and \( e(0), e(1) \neq 0 \) and \( l \in \mathbb{Z} \) and \( r \geq d \geq 0 \); also, there is a polynomial \( p(F^*M, n) \in \mathbb{Q}[n] \) with degree \( d \) and leading coefficient \( e(1)/d! \) such that

\[
\ell(M/F^nM) = p(F^*M, n) \quad \text{for } n \geq \deg e(t) - l.
\]

(20.14.2)
Finally, \( p_q(M, n) - p(F^*M, n) \) is a polynomial with degree at most \( d-1 \) and positive leading coefficient; also, \( d \) and \( e(1) \) are the same for every stable \( q \)-filtration.

**Proof:** The proof of (20.18) shows that \( G^*R' \) and \( G^*M \) satisfy the hypotheses of (20.8). So (20.8.11) and (20.8.14) yield (20.14.11). In turn, (20.14.11) yields (20.14.2) by the argument in the second paragraph of the proof of (20.8).

Finally, as \( F^*M \) is a stable \( q \)-filtration, there is an \( m \) such that

\[
F^n M \supset q^n M \supset q^n F^m M = F^{n+m} M
\]

for all \( n \geq 0 \). Dividing into \( M \) and extracting lengths, we get

\[
\ell(M/F^n M) \leq \ell(M/q^n M) \leq \ell(M/F^{n+m} M).
\]

Therefore, (20.14.7) yields

\[
p(F^*M, n) \leq p_q(M, n) \leq p(F^*M, n + m) \quad \text{for } n \gg 0.
\]

The two extremes are polynomials in \( n \) with the same degree \( d \) and the same leading coefficient \( c \) where \( c := e(1)/d! \). Dividing by \( n^d \) and letting \( n \to \infty \), we conclude that the polynomial \( p_q(M, n) \) also has degree \( d \) and leading coefficient \( c \).

Thus the degree and leading coefficient are the same for every stable \( q \)-filtration. Also \( p_q(M, n) - p(F^*M, n) \) has degree at most \( d-1 \) and positive leading coefficient, owing to cancellation of the two leading terms and to the first inequality. \( \Box \)

**Exercise (20.15).** — Let \( R \) be a Noetherian ring, \( q \) an ideal, and \( M \) a finitely generated module. Assume \( \ell(M/q^n M) < \infty \). Set \( m := \sqrt{q} \). Show

\[
\deg p_m(M, n) = \deg p_q(M, n).
\]

(20.16) (Rees Algebras). — Let \( R \) be an arbitrary ring, \( q \) an ideal. The sum

\[
\bigoplus_{n \in \mathbb{Z}} R_n(q) \quad \text{with } R_n(q) := \begin{cases} R & \text{if } n \leq 0, \\ q^n & \text{if } n > 0 \end{cases}
\]

is canonically an \( R \)-algebra, known as the extended Rees Algebra of \( q \).

Let \( M \) be a module with a \( q \)-filtration \( F^*M \). Then the sum

\[
\bigoplus_{n \in \mathbb{Z}} F^n M
\]

is canonically an \( R(q) \)-module, known as the Rees Module of \( F^*M \).

**Lemma (20.17).** — Let \( R \) be a Noetherian ring, \( q \) an ideal, \( M \) a finitely generated module with a \( q \)-filtration \( F^*M \). Then \( R(q) \) is algebra finite over \( R \). Also, \( F^*M \) is stable if and only if \( R(F^*M) \) is module finite over \( R(q) \) and \( \bigcup F^n M = M \).

**Proof:** As \( R \) is Noetherian, \( q \) is finitely generated, say by \( x_1, \ldots, x_r \). View the \( x_i \) as in \( R_1(q) \) and 1 in \( R_{-1}(q) \). These \( r+1 \) elements generate \( R(q) \) over \( R \).

Suppose that \( F^*M \) is stable: say \( F^n M = M \) and \( q^n F^n MF^{n+\nu} M \) for \( n > 0 \). Then \( \bigcup F^n M = M \). Further, \( R(F^*M) \) is generated by \( F^n M \). But \( R \) is Noetherian and \( M \) is finitely generated over \( R \); hence, every \( F^n M \) is finitely generated over \( R \). Thus \( R(F^*M) \) is a finitely generated \( R(q) \)-module.

Conversely, suppose that \( R(F^*M) \) is generated over \( R(q) \) by \( m_1, \ldots, m_s \). Say \( m_i = \sum_{j=\mu}^{\nu} m_{ij} \) with \( m_{ij} \in F^j M \) for some uniform \( \mu \leq \nu \). Then given \( n \), any \( m \in F^n M \) can be written as \( m = \sum f_{ij} m_{ij} \) with \( f_{ij} \in R_{n-j}(q) \). Hence if \( n \leq \mu \), then \( F^n M \subset F^\mu M \). Suppose \( \bigcup F^n M = M \). Then \( F^n M = M \). But if \( j \leq \nu \leq n \), then \( f_{ij} \in q^{n-j} = q^{\nu-j} \). Thus \( q^{\nu-j} F^n M = F^n M \). Thus \( F^*M \) is stable. \( \Box \)
Lemmas 20.18 (Artin–Rees). — Let $R$ be a Noetherian ring, $M$ a finitely generated module, $N$ a submodule, $q$ an ideal, $F^*M$ a stable $q$-filtration. Set

$$F^nN := N \cap F^nM \quad \text{for } n \in \mathbb{Z}.$$  

Then the $F^nN$ form a stable $q$-filtration $F^*N$.

Proof: By (20.14.1), the extended Rees Algebra $\mathcal{R}(q)$ is finitely generated over $R$, so Noetherian by the Hilbert Basis Theorem (11.11.2). By (20.17), the module $\mathcal{R}(F^*M)$ is finitely generated over $\mathcal{R}(q)$, so Noetherian by (11.11.1). Clearly, $F^*N$ is a $q$-filtration; hence, $\mathcal{R}(F^*N)$ is a submodule of $\mathcal{R}(F^*M)$, so finitely generated. But $\bigcup F^nM = M$, so $\bigcup F^nN = N$. Thus $F^*N$ is stable by (20.14.2). 


Proposition 20.20. — Let $R$ be a Noetherian ring, $q$ an ideal, and

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

an exact sequence of finitely generated modules. Then $M/qM$ has finite length if and only if $M'/qM'$ and $M''/qM''$ do. If so, then the polynomial

$$p_q(M', n) - p_q(M, n) + p_q(M'', n)$$

has degree at most $\deg p_q(M', n) - 1$ and has positive leading coefficient; also then

$$\deg p_q(M, n) = \max\{ \deg p_q(M', n), \deg p_q(M'', n) \}.$$  

Proof: First off, (13.37) and (13.27)1 and (13.37) again yield

$$\text{Supp}(M/qM) = \text{Supp}(M) \cap \mathbb{V}(q) = (\text{Supp}(M') \cup \text{Supp}(M'')) \cap \mathbb{V}(q)$$

$$= (\text{Supp}(M') \cap \mathbb{V}(q)) \cup (\text{Supp}(M'') \cap \mathbb{V}(q))$$

$$= \text{Supp}(M'/qM') \cup \text{Supp}(M''/qM'').$$

Hence $M/qM$ has finite length if and only if $M'/qM'$ and $M''/qM''$ do by (13.27).

For $n \in \mathbb{Z}$, set $F^nM' := M' \cap q^nM$. Then the $F^nM'$ form a stable $q$-filtration $F^*M'$ by the Artin–Rees Lemma. Form this canonical commutative diagram:

$$0 \rightarrow F^nM' \rightarrow q^nM \rightarrow q^nM'' \rightarrow 0$$

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

Its rows are exact. So the Nine Lemma yields this exact sequence:

$$0 \rightarrow M'/F^nM' \rightarrow M/q^nM \rightarrow M''/q^nM'' \rightarrow 0.$$  

Assume $M/qM$ has finite length. Then Additivity of Length and (20.13) yield

$$p(F^*M', n) - p_q(M, n) + p_q(M'', n) = 0. \quad (20.20.1)$$

Hence $p_q(M', n) - p_q(M, n) + p_q(M'', n)$ is equal to $p_q(M', n) - p(F^*M', n)$. But by (20.13) again, the latter is a polynomial with degree at most $\deg p_q(M', n) - 1$ and positive leading coefficient.

Finally, $\deg p_q(M, n) = \max\{ \deg p(M'_*, n), \deg p_q(M'', n) \}$ owing to (20.20.1), as the leading coefficients of $p(M'_*, n)$ and $p_q(M'', n)$ are both positive, so cannot cancel. But $\deg p(M'_*, n) = \deg p_q(M', n)$ by (20.14), completing the proof. 

□
20. Appendix: Homogeneity

(20.21) (Homogeneity). — Let \( R \) be a graded ring, and \( M = \bigoplus M_n \) a graded module. We call the \( M_n \), the \textbf{homogeneous components} of \( M \).

Given \( m \in M \), write \( m = \sum m_n \) with \( m_n \in M_n \). Call the finitely many nonzero \( m_n \) the \textbf{homogeneous components} of \( m \). Say that a component \( m_n \) is \textbf{homogeneous of degree} \( n \). If \( n \) is lowest, call \( m_n \) the \textbf{initial component} of \( m \).

Call a submodule \( N \subset M \) \textbf{homogeneous} if, whenever \( m \in N \), also \( m_n \in N \), or equivalently, \( N = \bigoplus (M_n \cap N) \).

Call a map \( \alpha : M' \to M \) of graded modules with components \( M'_n \) and \( M_n \) \textbf{homogeneous of degree} \( r \) if \( \alpha(M'_n) \subset M_{n+r} \) for all \( n \). If so, then clearly \( \text{Ker}(\alpha) \) is a homogeneous submodule of \( M \). Further, \( \text{Coker}(\alpha) \) is canonically graded, and the quotient map \( M \to \text{Coker}(\alpha) \) is homogeneous of degree 0.

Exercise (20.22). — Let \( R = \bigoplus R_n \) be a graded ring, \( M = \bigoplus_{n \geq n_0} M_n \) a graded module, \( \mathfrak{a} \subset \bigoplus_{n \geq 0} R_n \) a homogeneous ideal. Assume \( M = \mathfrak{a} M \). Show \( M = 0 \).

Exercise (20.23). — Let \( R = \bigoplus R_n \) be a Noetherian graded ring, \( M = \bigoplus M_n \) a finitely generated graded \( R \)-module, \( N = \bigoplus N_n \) a homogeneous submodule. Set \( N' := \{ m \in M \mid R_n m \in N \text{ for all } n \gg 0 \} \).

Show that \( N' \) is the largest homogeneous submodule of \( M \) containing \( N \) and having, for all \( n \gg 0 \), its degree-\( n \) homogeneous component \( N'_n \) equal to \( N_n \).

Proposition (20.24). — Let \( R \) be a Noetherian graded ring, \( M \) a nonzero finitely generated graded module, \( Q \) a homogeneous submodule. Suppose \( Q \) possesses this property: given any homogeneous \( x \in R \) and homogeneous \( m \in M \) with \( x m \in Q \) but \( m \notin Q \), necessarily \( x \in \mathfrak{p} := \text{nil}(M/Q) \). Then \( \mathfrak{p} \) is prime, and \( Q \) is \( \mathfrak{p} \)-primary.

Proof: Given \( x \in R \) and \( m \in M \), decompose them into their homogeneous components: \( x = \sum_{i \geq r} x_i \) and \( m = \sum_{j \geq s} m_j \). Suppose \( x m \in Q \), but \( m \notin Q \). Then \( m_t \notin Q \) for some \( t \); take \( t \) minimal. Set \( m' := \sum_{j < t} m_j \). Then \( m' \in Q \). Set \( m'' := m - m' \). Then \( x m'' \in Q \).

Either \( x_s m_t \) vanishes or it’s the initial component of \( x m'' \). But \( Q \) is homogeneous. So \( x_s m_t \in Q \). But \( m_t \notin Q \). Hence \( x_s \notin \mathfrak{p} \) by the hypothesis. Say \( x_{s+1}, \ldots, x_u \in \mathfrak{p} \) with \( u \) maximal. Set \( x' := \sum_{i = s}^u x_i \). Then \( x' \in \mathfrak{p} \). \( x' \in x m' \), so \( x' \in \text{Ann}(M/Q) \).

Suppose \( x \notin \mathfrak{p} \). Then \( x'' \notin \mathfrak{p} \), and its initial component is \( x_v \) with \( v > u \). Either \( x'' m'' \) vanishes or it is the initial component of \( x m \). But \( Q \) is homogeneous. So \( x_v m_t \in Q \). But \( m_t \notin Q \). Hence \( x_v \notin \mathfrak{p} \) by the hypothesis, contradicting \( v > u \).

Thus \( x \in \mathfrak{p} \). Thus \( Q \) is \( \mathfrak{p} \)-primary by (18.7.3).

Exercise (20.25). — Let \( R \) be a graded ring, \( \mathfrak{a} \) a homogeneous ideal, and \( M \) a graded module. Prove that \( \sqrt{\mathfrak{a}} \) and \( \text{Ann}(M) \) and \( \text{nil}(M) \) are homogeneous.

Exercise (20.26). — Let \( R \) be a graded ring, \( M \) a graded module, and \( Q \) a primary submodule. Let \( Q^* \subset Q \) be the submodule generated by the homogeneous elements of \( Q \). Then \( Q^* \) is primary.
Theorem (20.27). — Let \( R \) be a Noetherian graded ring, \( M \) a finitely generated graded module, \( N \) a homogeneous submodule. Then all the associated primes of \( M/N \) are homogeneous, and \( N \) admits an irredundant primary decomposition in which all the primary submodules are homogeneous.

Proof: Let \( N = \bigcap Q_j \) be any primary decomposition; one exists by (18.24). Let \( Q_j^* \subset Q_j \) be the submodule generated by the homogeneous elements of \( Q_j \). Trivially, \( \bigcap Q_j^* \subset \bigcap Q_j = N \subset \bigcap Q_j^* \). Further, each \( Q_j^* \) is clearly homogeneous, and is primary by (20.24). Thus \( N = \bigcap Q_j^* \) is a primary decomposition into homogeneous primary submodules. And, owing to (18.14), it is irredundant if \( N = \bigcap Q_j \) is, as both decompositions have minimal length. Finally, \( M/Q_j^* \) is graded by (20.24); so each associated prime is homogeneous by (18.24) and (20.25). \( \square \)

(20.28) (Graded Domains). — Let \( R = \bigoplus_{n \geq 0} R_n \) be a graded domain, and set \( K := \text{Frac}(R) \). We call \( z \in K \) homogeneous of degree \( n \in \mathbb{Z} \) if \( z = x/y \) with \( x \in R_n \) and \( y \in R_{m-n} \). Clearly, \( n \) is well defined.

Let \( K_n \) be the set of all such \( z \), plus 0. Then \( K_m K_n \subset K_{m+n} \). Clearly, the canonical map \( \bigoplus_{n \in \mathbb{Z}} K_n \rightarrow K \) is injective. Thus \( \bigoplus_{n \geq 0} K_n \) is a graded subring of \( K \). Further, \( K_0 \) is a field.

The \( n \) with \( K_n \neq 0 \) form a subgroup of \( \mathbb{Z} \). So by renumbering, we may assume \( K_1 \neq 0 \). Fix any nonzero \( x \in K_1 \). Clearly, \( x \) is transcendental over \( K_0 \). If \( z \in K_n \), then \( z/x^n \in K_0 \). Hence \( R \subset K_0[x] \). So (18.3) yields \( K = K_0(x) \).

Any \( w \in \bigoplus K_n \) can be written \( w = a/b \) with \( a, b \in R \) and \( b \) homogeneous: say \( w = \sum (a_n/b_n) \) with \( a_n, b_n \in R \) homogeneous; set \( b := \prod b_n \) and \( a := \sum (a_n b/b_n) \).

Theorem (20.29). — Let \( R \) be a Noetherian graded domain, \( K := \text{Frac}(R) \), and \( \overline{R} \) the integral closure of \( R \) in \( K \). Then \( \overline{R} \) is a graded subring of \( K \).

Proof: Use the setup of (20.28). Since \( K_0[x] \) is a polynomial ring over a field, it is normal by (18.43). Hence \( R \subset K_0[x] \). So every \( y \in R \) can be written as \( y = \sum y_i \), with \( y_i \) homogeneous and nonzero. Let’s show \( y_i \in \overline{R} \) for all \( i \).

Since \( y \) is integral over \( R \), the \( R \)-algebra \( R[y] \) is module finite by (10.29). So (20.29) yields a homogeneous \( b \in R \) with \( bR[y] \subset R \). Hence \( b y_i \in R \) for all \( y \geq 0 \). But \( R \) is graded. Hence \( b y_i \in R \). Set \( z := 1/b \). Then \( y_i \) lies in \( R_z \). Since \( R \) is Noetherian, the \( R \)-algebra \( R[y_i] \) is module finite. Hence \( y_i \in \overline{R} \). Thus \( y = y_i \in \overline{R} \) for all \( i \) by induction on \( n \). Thus \( \overline{R} \) is graded. \( \square \)

Exercise (20.30). — Under the conditions of (20.8), assume that \( R \) is a domain and that its integral closure \( \overline{R} \) in \( \text{Frac}(R) \) is a finitely generated \( R \)-module.

1. Prove that there is a homogeneous \( f \in R \) with \( R_f = \overline{R}_f \).

2. Prove that the Hilbert Polynomials of \( R \) and \( \overline{R} \) have the same degree and same leading coefficient.
21. Dimension

The dimension of a module is defined as the sup of the lengths of the chains of primes in its support. The Dimension Theorem, which we prove, characterizes the dimension of a nonzero finitely generated semilocal module over a Noetherian ring in two ways. First, the dimension is the degree of the Hilbert–Samuel Polynomial formed with the radical of the ring. Second, the dimension is the smallest number of elements in the radical that span a submodule of finite colength.

Next, in an arbitrary Noetherian ring, we study the height of a prime: the length of the longest chain of subprimes. We bound the height by the minimal number of generators of an ideal over which the prime is minimal. In particular, when this number is 1, we obtain Krull’s Principal Ideal Theorem. Finally, we study regular local rings: Noetherian local rings whose maximal ideal has the minimum number of generators, namely, the dimension.

\[(21.1)\] (Dimension of a module). — Let \( R \) be a ring, and \( M \) a nonzero module. The \textbf{dimension} of \( M \), denoted \( \dim(M) \), is defined by this formula:

\[
\dim(M) := \sup \{ r \mid \text{there's a chain of primes } p_0 \subseteq \cdots \subseteq p_r \text{ in } \text{Supp}(M) \}. 
\]

Assume \( R \) is Noetherian, and \( M \) is finitely generated. Then \( M \) has finitely many minimal (associated) primes by \((17.20)\). They are also the minimal primes \( p_0 \in \text{Supp}(M) \) by \((14.7.7)\). Thus \((14.5)\) yields

\[
\dim(M) = \max \{ \dim(R/p_0) \mid p_0 \in \text{Supp}(M) \text{ is minimal} \}. \quad (21.1.1)
\]

\[(21.2)\] (Parameters). — Let \( R \) be a ring, \( M \) a nonzero module. Denote the intersection of the maximal ideals in \( \text{Supp}(M) \) by \( \text{rad}(M) \), and call it the \textbf{radical} of \( M \). If there are only finitely many such maximal ideals, call \( M \) \textbf{semilocal}. Call an ideal \( q \) a \textbf{parameter ideal} of \( M \) if \( q \subseteq \text{rad}(M) \) and \( M/qM \) is Artinian.

Assume \( M \) is finitely generated. Then \( \text{Supp}(M) = \text{V}(\text{Ann}(M)) \) by \((13.27)\) (3). Hence \( M \) is semilocal if and only if \( R/\text{Ann}(M) \) is a semilocal ring.

Assume, in addition, \( R \) is Noetherian; so \( M \) is Noetherian by \((16.14)\). Fix an ideal \( q \). Then by \((13.4.7)\), \( M/qM \) is Artinian if and only if \( \ell(M/qM) < \infty \).

However, \( \ell(M/qM) < \infty \) if and only if \( \text{Supp}(M/qM) \) consists of finitely many maximal ideals by \((14.29)\) and \((14.27)\). Also, by \((13.5.5)\), \((13.5.7)\) (3), and \((13.5.1)\),

\[
\text{Supp}(M/qM) = \text{Supp}(M) \cap \text{V}(q) = \text{V}(\text{Ann}(M)) \cap \text{V}(q) = \text{V}(\text{Ann}(M) + q).
\]

Set \( q' := \text{Ann}(M) + q \). Thus \( M/qM \) is Artinian if and only if \( \text{V}(q') \) consists of finitely many maximal ideals; so by \((13.11)\), if and only if \( R/q' \) is Artinian. But \((13.4.8)\) implies that \( R/q' \) is Artinian if and only if \( q' \) contains a product of maximal ideals each containing \( q \). Then each lies in \( \text{Supp}(M) \), so contains \( \text{rad}(M) \).

Set \( m := \text{rad}(M) \). Thus if \( R/q' \) is Artinian, then \( q' \supseteq m^n \) for some \( n > 0 \).

Assume, in addition, \( M \) is semilocal, so that \( \text{Supp}(M) \) contains only finitely many maximal ideals. Then their product is contained in \( m \). Thus, conversely, if \( q' \supseteq m^n \) for some \( n > 0 \), then \( R/q' \) is Artinian. Thus \( q \) is a \textit{parameter ideal if and only if}

\[
m \supseteq q' \supseteq m^n \quad \text{for some } n. \quad (21.2.1)
\]

or by \((13.5.2)\) if and only if \( m = \sqrt{q'} \), or by \((13.5.1)\) if and only if \( \text{V}(m) = \text{V}(q') \). In particular, \( m^n \) is a parameter ideal for any \( n \).
Assume \( q \) is a parameter ideal. Then the Hilbert–Samuel polynomial \( p_q(M, n) \) exists by (21.14). Similarly, \( p_m(M, n) \) exists, and the two polynomials have the same degree by (21.15) since \( m = \sqrt{q} \) and \( p_q(M, n) = p_q(M, n) \). Thus the degree is the same for every parameter ideal. Denote this common degree by \( d(M) \).

Alternatively, \( d(M) \) can be viewed as the order of pole at 1 of the Hilbert series \( H(G^*M, t) \). Indeed, that order is 1 less than the order of pole at 1 of the Hilbert–Samuel series \( P_q(M, t) \) by (20.13). In turn, the latter order is \( d(M)+1 \) by (20.13).

Denote by \( s(M) \) the smallest \( s \) such that there are \( x_1, \ldots, x_s \in m \) with

\[
\ell(M/(x_1, \ldots, x_s)M) < \infty.
\]

By convention, if \( \ell(M) < \infty \), then \( s(M) = 0 \). We say that \( x_1, \ldots, x_s \in m \) form a system of parameters (sop) for \( M \) if \( s = s(M) \) and (21.2.2) holds. Note that a sop generates a parameter ideal.

**Lemma (21.3).** — Let \( R \) be a Noetherian ring, \( M \) a nonzero Noetherian semilocal module, \( q \) a parameter ideal of \( M \), and \( x \in \rad(M) \). Set \( K := \Ker(M \xrightarrow{\mu_x} M) \).

1. Then \( s(M) \leq s(M/xM) + 1 \).
2. Then \( \dim(M/xM) \leq \dim(M) - 1 \) if \( x \notin p \) for any \( p \in \Supp(M) \) with \( \dim(M/p) = \dim(M) \).
3. Then \( \deg(p_q(K, n) - p_q(M/xM, n)) \leq d(M) - 1 \).

**Proof:** For (1), set \( s := s(M/xM) \). There are \( x_1, \ldots, x_s \in \rad(M/xM) \) with

\[
\ell(M/(x_1, \ldots, x_s)M) < \infty.
\]

Now, \( \Supp(M/xM) = \Supp(M) \cap V(\langle x \rangle) \) by (15.5.1). However, \( x \in \rad(M) \). Hence, \( \Supp(M/xM) \) and \( \Supp(M) \) have the same maximal ideals. Therefore, \( \rad(M/xM) = \rad(M) \). Hence \( s(M) \leq s + 1 \). Thus (1) holds.

To prove (2), take a chain of primes \( p_0 \subset \cdots \subset p_r \) in \( \Supp(M/xM) \). Again, \( \Supp(M/xM) = \Supp(M) \cap V(\langle x \rangle) \) by (15.5.1). So \( x \in p_0 \in \Supp(M) \). By hypothesis, \( \dim(R/p_0) < \dim(M) \). Hence \( r \leq \dim(M) - 1 \). Thus (2) holds.

To prove (3), note that \( xM := \Im(\mu_x) \), and form these two exact sequences:

\[
0 \to K \to M \to xM \to 0, \quad \text{and} \quad 0 \to xM \to M \to M/xM \to 0.
\]

Then (20.20) yields \( d(K) \leq d(M) \) and \( d(xM) \leq d(M) \). So by (20.20) again, both \( p_q(K, n) + p_q(xM, n) - p_q(M, n) \) and \( p_q(xM, n) + p_q(M/xM, n) - p_q(M, n) \) are of degree at most \( d(M) - 1 \). So their difference is too. Thus (3) holds.

**Theorem (21.4) (Dimension).** — Let \( R \) be a Noetherian ring, \( M \) a nonzero finitely generated semilocal module. Then

\[
\dim(M) = d(M) = s(M) < \infty.
\]

**Proof:** Let’s prove a cycle of inequalities. Set \( m := \rad(M) \). First, let’s prove \( \dim(M) \leq d(M) \). We proceed by induction on \( d(M) \). Suppose \( d(M) = 0 \). Then \( \ell(M/m^aM) \) stabilizes. So \( m^aM = m^{a+1}M \) for some \( n \). Hence \( m^aM = 0 \) by Nakayama’s Lemma (11.1.4) applied over the semilocal ring \( R/\Ann(M) \). Hence \( \ell(M) < \infty \). So \( \dim(M) = 0 \) by (11.1.3).

Suppose \( d(M) \geq 1 \). By (21.1.1), \( \dim(R/p_0) = \dim(M) \) for some \( p_0 \in \Supp(M) \). Then \( p_0 \) is minimal. So \( p_0 \in \Ass(M) \) by (17.1.8). Hence \( M \) has a submodule \( N \) isomorphic to \( R/p_0 \) by (17.1.9). Further, by (20.4.1), \( d(N) \leq d(M) \).

Take a chain of primes \( p_0 \subset \cdots \subset p_r \) in \( \Supp(N) \). If \( r = 0 \), then \( r \leq d(M) \). Suppose \( r \geq 1 \). Then there’s an \( x_1 \in p_1 - p_0 \). Further, since \( p_0 \) is not maximal, for
each maximal ideal \( n \) in \( \text{Supp}(M) \), there is an \( x_n \in n - p_0 \). Set \( x := x_1 \prod x_n \). Then \( x \in (p_1 \cap m) - p_0 \). Then \( p_1 \subseteq \cdots \subseteq p_r \) lies in \( \text{Supp}(N) \cap V(x) \). But the latter is equal to \( \text{Supp}(N/xN) \) by (13.31). So \( r - 1 \leq \dim(N/xN) \).

However, \( \mu_x \) is injective on \( N \) as \( N \cong R/p_0 \) and \( x \notin p_0 \). So (21.8) (3) yields \( d(N/xN) \leq d(N) - 1 \). But \( d(N) \leq d(M) \). So \( \dim(N/xN) \leq d(N/xN) \) by the induction hypothesis. Therefore, \( r \leq d(M) \). Thus \( \dim(M) \leq d(M) \).

Second, let’s prove \( d(M) \leq s(M) \). Let \( q \) be a parameter ideal of \( M \) with \( s(M) \) generators. Then \( d(M) := \deg p_q(M, n) \). But \( \deg p_q(M, n) \leq s(M) \) owing to (20.8.1). Thus \( d(M) \leq s(M) \).

Finally, let’s prove \( s(M) \leq \dim(M) \). Set \( r := \dim(M) \), which is finite since \( r \leq d(M) \) by the first step. The proof proceeds by induction on \( r \). If \( r = 0 \), then \( M \) has finite length by (11.3): so by convention \( s(M) = 0 \).

Suppose \( r \geq 1 \). Let \( p_1, \ldots, p_k \) be the primes of \( \text{Supp}(M) \) with \( \dim(R/p_i) = r \). No \( p_i \) is maximal as \( r \geq 1 \). So \( m \) lies in no \( p_i \). Hence, by Prime Avoidance (19.14), there is an \( x \in m \) such that \( x \notin p_i \) for all \( i \). So (21.3) (1), (2) yield \( s(M) \leq s(M/xM) + 1 \) and \( \dim(M/xM) + 1 \leq r \). By the induction hypothesis, \( s(M/xM) \leq \dim(M/xM) \). Hence \( s(M) \leq r \), as desired.

**Corollary (21.5).** — Let \( R \) be a Noetherian ring, \( M \) a nonzero Noetherian semi-local module, \( x \in \text{rad}(M) \). Then \( \dim(M/xM) \geq \dim(M) - 1 \), with equality if \( x \notin p \) for \( p \in \text{Supp}(M) \) with \( \dim(R/p) = \dim(M) \): equality holds if \( x \notin z\text{div}(M) \).

**Proof:** By (21.3) (1), we have \( s(M/xM) \geq s(M) - 1 \). So the asserted inequality holds by (21.2). If \( x \notin p \in \text{Supp}(M) \) when \( \dim(R/p) = \dim(M) \), then (21.3) (2) yields the opposite inequality, so equality. Finally, if \( x \notin z\text{div}(M) \), then \( x \notin p \) for any \( p \in \text{Supp}(M) \) with \( \dim(R/p) = \dim(M) \) owing to (17.4.13) and (17.4.12).

**Exercise (21.4).** — Let \( A \) be a Noetherian local ring, \( N \) a finitely generated module, \( y_1, \ldots, y_r \) a sop for \( N \). Set \( N_i := N/\langle y_1, \ldots, y_i \rangle \). Show \( \dim(N_i) = r - i \).

(21.7) **(Height).** — Let \( R \) be a ring, and \( p \) a prime. The **height** of \( p \), denoted \( \text{ht}(p) \), is defined by this formula:

\[
\text{ht}(p) := \sup \{ r \mid \text{there's a chain of primes } p_0 \subseteq \cdots \subseteq p_r = p \}.
\]

The bijective correspondence \( p \mapsto pR_p \) of (18.4.1) (2) yields this formula:

\[
\text{ht}(p) = \dim(R_p).
\]

(21.7.1)

If \( \text{ht}(p) = h \), then we say that \( p \) is a **height-\( h \)** prime.

**Corollary (21.8).** — Let \( R \) be a Noetherian ring, \( p \) a prime. Then \( \text{ht}(p) \leq r \) if and only if \( p \) is a minimal prime of some ideal generated by \( r \) elements.

**Proof:** Assume \( p \) is minimal containing an ideal \( a \) generated by \( r \) elements. Now, any prime of \( R_p \) containing \( aR_p \) is of the form \( qR_p \) where \( q \) is a prime of \( R \) with \( a \subseteq q \subseteq p \) by (18.4.11). So \( q = p \). Hence \( pR_p = \sqrt{aR_p} \) by the Scheinnullstellensatz. Hence \( r \geq s(R_p) \) by (21.7.1). But \( s(R_p) = \dim(R_p) \) by (21.4.13), and \( \dim(R_p) = \text{ht}(p) \) by (21.4.13). Thus \( \text{ht}(p) \leq r \).

Conversely, assume \( \text{ht}(p) \leq r \). Then \( R_p \) has a parameter ideal \( b \) generated by \( r \) elements, say \( y_1, \ldots, y_r \) by (21.4.11) and (21.4.20). Say \( y_i = x_i/s_i \) with \( s_i \notin p \). Set \( a := \langle x_1, \ldots, x_r \rangle \). Then \( aR_p = b \).

Suppose there is a prime \( q \) with \( a \subseteq q \subseteq p \). Then \( b = aR_p \subseteq qR_p \subseteq pR_p \), and \( qR_p \) is prime by (18.4.1) (2). But \( \sqrt{b} = pR_p \). So \( qR_p = pR_p \). Hence \( q = p \) by...
Let $R$ be a Noetherian ring, and $p$ a prime minimal containing $x_1, \ldots, x_r$. Given $r'$ with $1 \leq r' \leq r$, set $R' := R/(x_1, \ldots, x_{r'})$ and $p' := p/(x_1, \ldots, x_{r'})$. Assume $\text{ht}(p) = r$. Prove $\text{ht}(p') = r - r'$.

**Theorem (21.10) (Krull Principal Ideal).** — Let $R$ be a Noetherian ring, $x \in R$, and $p$ a minimal prime of $\langle x \rangle$. If $x \notin z \text{.div}(R)$, then $\text{ht}(p) = 1$.

**Proof:** By (21.8), $\text{ht}(p) \leq 1$. But by (13.11), $x \in z \text{.div}(R)$ if $\text{ht}(p) = 0$. □

**Exercise (21.11).** — Let $R$ be a Noetherian ring, $p$ a prime of height at least 2. Prove that $p$ is the union of height-1 primes, but not of finitely many.

**Exercise (21.12).** — Let $R$ be a Noetherian ring. Prove the following equivalent:

1. $R$ has only finitely many primes.
2. $R$ has only finitely many height-1 primes.
3. $R$ is semilocal of dimension 1.

**Exercise (21.13) (Artin–Tate [1, Thm. 4]).** — Let $R$ be a Noetherian domain, and set $K := \text{Frac}(R)$. Prove the following statements are equivalent:

1. $K = R_f$ for some nonzero $f \in R$.
2. $K$ is algebra finite over $R$.
3. Some nonzero $f \in R$ lies in every nonzero prime.
4. $R$ has only finitely many height-1 primes.
5. $R$ is semilocal of dimension 1.

**Exercise (21.14).** — Let $R$ be a domain. Prove that, if $R$ is a UFD, then every height-1 prime is principal, and that the converse holds if $R$ is Noetherian.

**Exercise (21.15).** — (1) Let $A$ be a Noetherian local ring with a principal prime $p$ of height at least 1. Prove $A$ is a domain by showing any prime $q \subseteq p$ is $(0)$.

(2) Let $k$ be a field, $P := k[[X]]$ the formal power series ring in one variable. Set $R := P \times P$. Prove that $R$ is Noetherian and semilocal, and that $R$ contains a principal prime $p$ of height 1, but that $R$ is not a domain.

**Exercise (21.16).** — Let $R$ be a finitely generated algebra over a field. Assume $R$ is a domain of dimension $r$. Let $x \in R$ be neither 0 nor a unit. Set $R' := R/(x)$. Prove that $r - 1$ is the length of any chain of primes in $R'$ of maximal length.

**Corollary (21.17).** — Let $A$ and $B$ be Noetherian local rings, $\mathfrak{m}$ and $\mathfrak{n}$ their maximal ideals. Let $\varphi : A \to B$ be a local homomorphism. Then

$$\dim(B) \leq \dim(A) + \dim(B/\mathfrak{m}B),$$

with equality if $B$ is flat over $A$.

**Proof:** Set $s := \dim(A)$. By (21.3), there is a parameter ideal $q$ generated by $s$ elements. Then $\mathfrak{m}/q$ is nilpotent by (21.7). Hence $\mathfrak{m}B/qB$ is nilpotent. It follows that $\dim(B/\mathfrak{m}B) = \dim(B/qB)$. But (21.5) yields $\dim(B/qB) \geq \dim(B) - s$. Thus the inequality holds.

Assume $B$ is flat over $A$. Let $p \supseteq \mathfrak{m}B$ be a prime with $\dim(B/p) = \dim(B/\mathfrak{m}B)$. Then $\dim(B) \geq \dim(B/p) + \text{ht}(p)$ because the concatenation of a chain of primes containing $p$ of length $\dim(B/p)$ with a chain of primes contained in $p$ of length $\text{ht}(p)$ is a chain of primes of $B$ of length $\text{ht}(p) + \dim(B/p)$. Hence it suffices to show
that $\text{ht}(p) \geq \dim(A)$.

As $n \supset p \supset mB$ and as $\varphi$ is local, $\varphi^{-1}(p) = m$. Since $B$ is flat over $A$, (14.1.22) and induction yield a chain of primes of $B$ descending from $p$ and lying over any given chain in $A$. Thus $\text{ht}(p) \geq \dim(A)$, as desired.

**Exercise (21.22).** — Let $R$ be a Noetherian ring. Prove that
\[ \dim(R[X]) = \dim(R) + 1. \]

**Exercise (21.21).** — Let $A$ be a Noetherian local ring of dimension $r$. Let $m$ be the maximal ideal, and $k := A/m$ the residue class field. Prove that
\[ r \leq \dim_k(m/m^2), \]
with equality if and only if $m$ is generated by $r$ elements.

(21.20) (Regular local rings). — Let $A$ be a Noetherian local ring of dimension $r$. We say $A$ is **regular** if its maximal ideal is generated by $r$ elements. Then any $r$ generators are said to form a **regular** system of parameters.

By (21.1.22), $A$ is regular if and only if $r = \dim_k(m/m^2)$.

For example, a field is a regular local ring of dimension $0$, and conversely. An example of a regular local ring of given dimension $n$ is the localization $P_m$ of a polynomial ring $P$ in $n$ variables over a field at any maximal ideal $m$, as $\dim(P_m) = n$ by (15.13) and (15.14) and as $m$ is generated by $n$ elements by (15.6).

**Lemma (21.21).** — Let $A$ be a Noetherian semifield of dimension $r$, and $q$ a parameter ideal. Then $\deg h(G^*A, n) = r - 1$.

**Proof:** By (20.8), $\deg h(G^*A, n)$ is equal to $1$ less than the order of pole at 1 of the Hilbert series $H(G^*A, t)$. But that order is equal to $d(A)$ by (21.7). Also, $d(A) = r$ by the Dimension Theorem, (21.3). Thus the assertion holds.

**Proposition (21.22).** — Let $A$ be a Noetherian local ring of dimension $r$, and $m$ its maximal ideal. Then $A$ is regular if and only if its associated graded ring $G^*A$ is a polynomial ring; if so, then the number of variables is $r$.

**Proof:** Say $G^*A$ is a polynomial ring in $s$ variables. Then $\dim(m/m^2) = s$. By (21.21), $\deg h(G^*A, n) = s - 1$. So $s = r$ by (21.21). So $A$ is regular by (21.22). Conversely, assume $A$ is regular. Let $x_1, \ldots, x_r$ be a regular sop, and $x_i' \in m/m^2$ the residue of $x_i$. Set $k := A/m$, and let $P := k[X_1, \ldots, X_r]$ be the polynomial ring. Form the $k$-algebra homomorphism $\varphi: P \to G^*A$ with $\varphi(X_i) = x_i'$.

Then $\varphi$ is surjective as the $x_i'$ generate $G^*A$. Set $a := \ker \varphi$. Let $P = \bigoplus P_n$ be the grading by total degree. Then $\varphi$ preserves the gradings of $P$ and $G^*A$. So $a$ inherits a grading: $a = \bigoplus a_n$. So for $n \geq 0$, there’s this canonical exact sequence:

$$0 \to a_n \to P_n \to m^n/m^{n+1} \to 0.$$  \hfill (22.21)

Suppose $a \neq 0$. Then there’s a nonzero $f \in a_m$ for some $m$. Take $n \geq m$. Then $P_n - m \subset a_n$. Since $P$ is a domain, $P_{n-m} \to P_{n-m}f$. Therefore, (21.22) yields

$$\dim_k(m^n/m^{n+1}) = \dim_k(P_n) - \dim_k(a_n) \leq \dim_k(P_n) - \dim_k(P_{n-m}) = \left(\frac{r-1+n}{r-1}\right) - \left(\frac{r-1+n}{r-1-m}\right).$$

The expression on the right is a polynomial in $n$ of degree $r - 2$.

On the other hand, $\dim_k(m^n/m^{n+1}) = h(G^*A, n)$ for $n \geq 0$ by (20.8). Further, $\deg h(G^*A, n) = r - 1$ by (20.8). However, it follows from the conclusion of the
preceding paragraph that deg $h(G^\bullet A, n) \leq r - 2$. We have a contradiction! Hence $a = 0$. Thus $\varphi$ is injective, so bijective, as desired. \hfill \Box

Exercise (21.23). — Let $A$ be a Noetherian local ring of dimension $r$, and let $x_1, \ldots, x_s \in A$ with $s \leq r$. Set $a := \langle x_1, \ldots, x_s \rangle$ and $B := A/a$. Prove equivalent:

1. $A$ is regular, and there are $x_{s+1}, \ldots, x_r \in A$ with $x_1, \ldots, x_r$ a regular sop.
2. $B$ is regular of dimension $r - s$.

Theorem (21.24). — A regular local ring $A$ is a domain.

Proof: Use induction on $r := \dim A$. If $r = 0$, then $A$ is a field, so a domain.
Assume $r \geq 1$. Let $x$ be a member of a regular sop. Then $A/(x)$ is regular of dimension $r - 1$ by (21.23). By induction, $A/(x)$ is a domain. So $(x)$ is prime. Thus $A$ is a domain by (21.15). \hfill \Box

Lemma (21.25). — Let $A$ be a local ring, $m$ its maximal ideal, $a$ a proper ideal. Set $n := m/a$ and $k := A/m$. Then this sequence of $k$-vector spaces is exact:

$$0 \to (m^2 + a)/m^2 \to m/m^2 \to n/n^2 \to 0.$$  

Proof: The assertion is very easy to check. \hfill \Box

Proposition (21.26). — Let $A$ be a regular local ring of dimension $r$, and $a$ an ideal. Set $B := A/a$, and assume $B$ is regular of dimension $r - s$. Then $a$ is generated by $s$ elements, and any such $s$ elements form part of a regular sop.

Proof: In its notation, (21.22) yields $\dim((m^2 + a)/m^2) = s$. Hence, any set of generators of $a$ includes $s$ members of a regular sop of $A$. Let $b$ be the ideal the $s$ generate. Then $A/b$ is regular of dimension $r - s$ by (21.23). By (21.23), both $A/b$ and $B$ are domains of dimension $r - s$; whence, (15.11) implies $a = b$. \hfill \Box
22. Completion

Completion is used to simplify a ring and its modules beyond localization. First, we discuss the topology of a filtration, and use Cauchy sequences to construct the completion. Then we discuss the inverse limit, the dual notion of the direct limit; thus we obtain an alternative construction. We conclude that, if we use the adic filtration of an ideal, then the functor of completion is exact on finitely generated modules over a Noetherian ring. Further, then the completion of a Noetherian ring is Noetherian; if the ideal is maximal, then the completion is local. We end with a useful version of the Cohen Structure Theorem for complete Noetherian local rings.

(22.1) (Topology and completion). — Let \( R \) be a ring, \( M \) a module equipped with a filtration \( F^n M \). Then \( M \) has a topology: the open sets are the arbitrary unions of sets of the form \( m + F^n M \) for various \( m \) and \( n \). Indeed, the intersection of two open sets is open, as the intersection of two unions is the union of the pairwise intersections; further, if the intersection \( U \) of \( m + F^n M \) and \( m' + F^n M \) is nonempty and if \( n \geq n' \), then \( U = m + F^n M \), because, if say \( m'' \in U \), then

\[
m + F^n M = m'' + F^n M \subseteq m'' + F^n M = m' + F^n M.
\]

The addition map \( M \times M \to M \), given by \( (m, m') \mapsto m + m' \), is continuous, as

\[
(m + F^n M) + (m' + F^n M) \subset (m + m' + F^n M).
\]

So, with \( m' \) fixed, the translation \( m \mapsto m + m' \) is a homeomorphism \( M \to M \).
(Similarly, inversion \( m \mapsto -m \) is a homeomorphism; so \( M \) is a topological group.)

Let \( \mathfrak{a} \) be an ideal, and give \( R \) the \( \mathfrak{a} \)-adic filtration. If the filtration on \( M \) is an \( \mathfrak{a} \)-adic filtration, then scalar multiplication \( (x, m) \mapsto x m \) too is continuous, because

\[
(x + a^n)(m + F^n M) \subset x m + F^n M.
\]

Further, if the filtration is \( \mathfrak{a} \)-stable, then it yields the same topology as the \( \mathfrak{a} \)-adic filtration, because for some \( n' \) and any \( n \),

\[
F^n M \supset a^n M \supset a^n F^{n'} M = F^{n+n'} M.
\]

Thus any two stable \( \mathfrak{a} \)-filtrations give the same topology: the \( \mathfrak{a} \)-adic topology.

When \( \mathfrak{a} \) is given, it is conventional to use the \( \mathfrak{a} \)-adic filtration and \( \mathfrak{a} \)-adic topology unless there’s explicit mention to the contrary. Further, if \( R \) is semi-local, then it is conventional to take \( \mathfrak{a} := \text{rad}(R) \).

Let \( N \subset M \) be a submodule. Its closure \( \overline{N} \) is equal to \( \bigcap_n (N + F^n M) \), as \( m \notin \overline{N} \) means there’s \( n \) with \( (m + F^n M) \cap N = \emptyset \), or equivalently \( m \notin (N + F^n M) \). In particular, each \( F^n M \) is closed, and \( \{0\} \) is closed if and only if \( \bigcap F^n M = \{0\} \).

Also, \( M \) is separated — that is, Hausdorff — if and only if \( \{0\} \) is closed. For, if \( \{0\} \) is closed, so is each \( \{m\} \). So given \( m' \neq m \), there’s \( n' \) with \( m \notin (m' + F^{n'} M) \). Take \( n \geq n' \). Then \( (m + F^n M) \cap (m' + F^n M) = \emptyset \) owing to (Hausdorff).

Finally, \( M \) is discrete — that is, every \( \{m\} \) is both open and closed — if and only if \( \{0\} \) is just open.

A sequence \( (m_n)_{n \geq 0} \) in \( M \) is called Cauchy if, given \( n_0 \), there’s \( n_1 \) with

\[
m_n - m_{n_1} \in F^{n_0} M, \quad \text{or simply } m_n - m_{n+1} \in F^{n_0} M, \quad \text{for all } n, n' \geq n_1;
\]

the two conditions are equivalent because \( F^{n_0} M \) is a subgroup and
Recall from \( \prod \) linear maps \( n \rightarrow M \) homomorphism, which carries \([\text{Exercise } (22.2)]\)

The Cauchy sequences form a module under termwise addition and scalar multiplication. The sequences with 0 as a limit form a submodule. The quotient module is denoted \( \tilde{M} \) and called the (separated) completion. There is a canonical homomorphism, which carries \( m \in M \) to the class of the constant sequence \( (m) \):

\[
\kappa: M \rightarrow \tilde{M} \quad \text{by} \quad \kappa m := (m).
\]

If \( M \) is complete, but not separated, then \( \kappa \) is surjective, but not bijective.

It is easy to check that the notions of Cauchy sequence and limit depend only on the topology. Further, \( \tilde{M} \) is separated and complete with respect to the filtration \( F^k \tilde{M} := (F^k M) \) where \( (F^k M) \) is the completion of \( F^k M \) arising from the intersections \( F^k M \cap F^n M \) for all \( n \). In addition, \( \kappa \) is the universal continuous \( R \)-linear map from \( M \) into a separated and complete, filtered \( \tilde{R} \)-module.

Again, let \( a \) be an ideal. Under termwise multiplication of Cauchy sequences, \( \tilde{R} \) is a ring, \( \kappa: R \rightarrow \tilde{R} \) is a ring homomorphism, and \( \tilde{M} \) is an \( \tilde{R} \)-module. Further, \( M \rightarrow \tilde{M} \) is a linear functor from \((\tilde{R} \text{-mod})\) to \((\tilde{R} \text{-mod})\).

For example, let \( R' \) be a ring, and \( \tilde{R} := R'[X_1, \ldots, X_r] \) the polynomial ring in \( r \) variables. Set \( a := \langle X_1, \ldots, X_r \rangle \). Then a sequence \( (m_n)_{n \geq 0} \) of polynomials is Cauchy if and only if, given \( n_0 \), there’s \( n_1 \) such that, for all \( n \geq n_1 \), the \( m_n \) agree in degree less than \( n_0 \). Thus \( \tilde{R} \) is just the power series ring \( R'[[X_1, \ldots, X_r]] \).

For another example, take a prime integer \( p \), and set \( a := \langle p \rangle \). Then a sequence \( (m_n)_{n \geq 0} \) of integers is Cauchy if and only if, given \( n_0 \), there’s \( n_1 \) such that, for all \( n, n' \geq n_1 \), the difference \( m_n - m_{n'} \) is a multiple of \( p^n \). The completion of \( \mathbb{Z} \) is called the \( p \)-adic integers, and consists of the sums \( \sum_{i=0}^{\infty} z_i p^i \) with \( 0 \leq z_i < p \).

**Proposition (22.2).** — Let \( R \) be a ring, and \( a \) an ideal. Then \( \tilde{a} \subset \text{rad}(\tilde{R}) \).

**Proof:** Recall from \((22.1)\) that \( \tilde{R} \) is complete in the \( \tilde{a} \)-adic topology. Hence for \( x \in \tilde{a} \), we have \( 1/(1 - x) = 1 + x + x^2 + \cdots \) in \( \tilde{R} \). Thus \( \tilde{a} \subset \text{rad}(\tilde{R}) \) by \((22.2)\). \( \Box \)

**Exercise (22.3).** — In the 2-adic integers, evaluate the sum \( 1 + 2 + 4 + 8 + \cdots \).

**Exercise (22.4).** — Let \( R \) be a ring, \( a \) an ideal, and \( M \) an \( a \)-module. Prove that the following three conditions are equivalent:

1. \( \kappa: M \rightarrow \tilde{M} \) is injective;
2. \( \bigcap a^n M = (0) \);
3. \( M \) is separated.

Assume \( R \) is Noetherian and \( M \) finitely generated. Assume either (a) \( a \subset \text{rad}(R) \) or (b) \( R \) is a domain, \( a \) is proper, and \( M \) is torsionfree. Conclude \( M \subset \tilde{M} \).

**Exercise (22.5) (Inverse limits).** — Let \( R \) be a ring. Given \( R \)-modules \( Q_n \) equipped with linear maps \( \alpha_n^{n+1}: Q_{n+1} \rightarrow Q_n \) for \( n \), their inverse limit \( \varprojlim Q_n \) is the submodule of \( \prod Q_n \) of all vectors \( (q_n) \) with \( \alpha_n^{n+1} q_{n+1} = q_n \) for all \( n \).

Given \( Q_n \) and \( \alpha_n^{n+1} \) for all \( n \), use only those for \( n \) in the present context. Define \( \theta: \prod Q_n \rightarrow \prod Q_n \) by \( \theta(q_n) := (q_n - \alpha_n^{n+1} q_{n+1}) \). Then

\[
\varprojlim Q_n = \ker \theta.
\]

Set \( \varprojlim^1 Q_n := \text{coker} \theta. \) (22.5.1)

Plainly, \( \varprojlim Q_n \) has this UMP: given maps \( \beta_n: P \rightarrow Q_n \) with \( \alpha_n^{n+1} \beta_{n+1} = \beta_n \), there’s a unique map \( \beta: P \rightarrow \varprojlim Q_n \) with \( \pi_n \beta = \beta_n \) for all \( n \).
Further, the UMP yields the following natural $R$-linear isomorphism:

$$\varprojlim \text{Hom}(P, Q_n) = \text{Hom}(P, \varprojlim Q_n).$$

(The notion of inverse limit is formally dual to that of direct limit.)

For example, let $R'$ be a ring, and $R := R'[X_1, \ldots, X_r]$ the polynomial ring in $r$ variables. Set $m := (X_1, \ldots, X_r)$ and $R_n := R/m^{n+1}$. Then $R_n$ is just the $R$-algebra of polynomials of degree at most $n$, and the canonical map $\alpha^{n+1}_n : R_{n+1} \to R_n$ is just truncation. Thus $\varprojlim R_n$ is equal to the power series ring $R[[X_1, \ldots, X_r]]$.

For another example, take a prime integer $p$, and set $\mathbb{Z}_n := \mathbb{Z}/(p^{n+1})$. Then $\mathbb{Z}_n$ is just the ring of sums $\sum_{i=0}^{n} z_i p^i$ with $0 \leq z_i < p$, and the canonical map $\alpha^{n+1}_n : \mathbb{Z}_{n+1} \to \mathbb{Z}_n$ is just truncation. Thus $\varprojlim \mathbb{Z}_n$ is just the ring of $p$-adic integers.

**Exercise (22.8).** — Let $R$ be a ring. Given $R$-modules $Q_n$ equipped with linear maps $\alpha^{n+1}_n : Q_{n+1} \to Q_n$ for $n \geq 0$, set $\alpha^m_n := \alpha^{n+1}_n \cdots \alpha^{m+1}_n$ for $m > n$. We say the $Q_n$ satisfy the **Mittag-Leffler Condition** if the descending chain

$$Q_n \supset \alpha^{n+1}_n Q_{n+1} \supset \alpha^{n+2}_n Q_{n+2} \supset \cdots \supset \alpha^m_n Q_m \supset \cdots$$

stabilizes; that is, $\alpha^m_n Q_m = \alpha^m_n Q_{m+k}$ for all $k > 0$.

1. Assume for each $n$, there is $m > n$ with $\alpha^m_n = 0$. Show $\varprojlim Q_n = 0$.

2. Assume $\alpha^{n+1}_n$ is surjective for all $n$. Show $\varprojlim Q_n = 0$.

3. Assume the $Q_n$ satisfy the Mittag-Leffler Condition. Set $P_n := \bigcap_{m \geq n} \alpha^m_n Q_m$, which is the stable submodule. Show $\alpha^{n+1}_n P_{n+1} = P_n$.

4. Assume the $Q_n$ satisfy the Mittag-Leffler Condition. Show $\varprojlim Q_n = 0$.

**Lemma (22.7).** — For $n \geq 0$, consider commutative diagrams with exact rows

$$0 \to Q_n' \xrightarrow{\gamma'_n} Q_{n+1} \xrightarrow{\gamma_{n+1}} Q''_{n+1} \to 0$$

$$0 \to Q_n' \xrightarrow{\gamma'_n} Q_n \xrightarrow{\gamma_n} Q_n'' \to 0$$

Then the induced sequence

$$0 \to \varprojlim Q'_n \xrightarrow{\hat{\gamma}} \varprojlim Q_n \xrightarrow{\hat{\gamma}} \varprojlim Q''_n$$

is exact; further, $\hat{\gamma}$ is surjective if the $Q'_n$ satisfy the Mittag-Leffler Condition.

**Proof:** The given commutative diagrams yield the following one:

$$0 \to \prod Q'_n \xrightarrow{\prod \gamma'_n} \prod Q_n \xrightarrow{\prod \gamma_n} \prod Q''_n \to 0$$

Owing to (22.7.1), the Snake Lemma (5.13) yields the exact sequence (22.7.1) and an injection $\text{Coker} \hat{\gamma} \hookrightarrow \varprojlim Q'_n$. Assume the $Q'_n$ satisfy the Mittag-Leffler Condition. Then $\varprojlim Q'_n = 0$ by (22.6). So $\text{Coker} \hat{\gamma} = 0$. Thus $\hat{\gamma}$ is surjective. □

**Proposition (22.8).** — Let $R$ be a ring, $M$ a module, $F^\bullet M$ a filtration. Then

$$\widehat{M} \to \varprojlim (M/F^n M).$$
PROOF: First, let us define a map \( \alpha: \widehat{M} \to \varprojlim (M/F^n M) \). Given a Cauchy sequence \((m_\nu)\), let \( q_\nu \) be the residue of \( m_\nu \) in \( M/F^n M \) for \( \nu \gg 0 \). Then \( q_\nu \) is independent of \( \nu \), because the sequence is Cauchy. Clearly, \( q_\nu \) is the residue of \( q_{\nu+1} \) in \( M/F^n M \). Also, \((m_\nu)\) has 0 as a limit if and only if \( q_n = 0 \) for all \( n \). Define \( \alpha \) by \( \alpha m_\nu := (q_\nu) \). It is easy to check that \( \alpha \) is well defined, linear, and injective.

As to surjectivity, given \((q_n) \in \varprojlim (M/F^n M)\), for each \( \nu \) lift \( q_\nu \in M/F^n M \) up to \( m_\nu \in M \). Then \( m_\mu - m_\nu \in F^n M \) for \( \mu \geq \nu \), as \( q_\nu \in M/F^n M \) maps to \( q_\nu \in M/F^n M \). Hence \((m_\nu)\) is Cauchy. Thus \( \alpha \) is surjective, so an isomorphism. □

**Example (22.9).** — Let \( R \) be a ring, \( M \) a module, \( F^\bullet M \) a filtration. For \( n \geq 0 \), consider the following natural commutative diagrams with exact rows:

\[
\begin{array}{cccc}
0 & \to & F^{n+1} M & \to & M & \to & M/F^{n+1} M & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F^n M & \to & M & \to & M/F^n M & \to & 0
\end{array}
\]

with vertical maps, respectively, the inclusion, the identity, and the quotient map. By (22.23), the left-exact sequence of inverse limits is

\[
0 \to \varprojlim F^n M \to M \xrightarrow{\alpha} \widehat{M}.
\]

But \( \kappa \) is not surjective when \( M \) is not complete; for examples of such \( M \), see the end of (22.11). Thus \( \varprojlim F^n M \) is not always exact, nor \( \varprojlim^1 F^n M \) always 0.

**Exercise (22.11).** — Let \( A \) be a ring, and \( m_1, \ldots, m_m \) be maximal ideals. Set \( m := \bigcap m_i \), and give \( A \) the \( m \)-adic topology. Prove that \( \widehat{A} = \prod \widehat{A}_{m_i} \).

**Exercise (22.11).** — Let \( R \) be a ring, \( M \) a module, \( F^\bullet M \) a filtration, and \( N \subseteq M \) a submodule. Give \( N \) and \( M/N \) the induced filtrations:

\[
F^n N := N \cap F^n M \quad \text{and} \quad F^n (M/N) := F^n M/F^n N.
\]

1. Prove \( \widehat{N} \subseteq \widehat{M} \) and \( \widehat{M}/\widehat{N} = (M/N)^\wedge \).

2. Also assume \( N \supseteq F^n M \) for \( n \gg 0 \). Prove \( \widehat{M}/\widehat{N} = M/N \) and \( G^\bullet \widehat{M} = G^\bullet M \).

**Exercise (22.12).** — (1) Let \( R \) be a ring, \( a \) an ideal. If \( G^\bullet R \) is a domain, show \( \widehat{R} \) is a domain. If also \( \bigcap_{n \geq 0} a^n = 0 \), show \( \widehat{R} \) is a domain.

2. Use (1) to give an alternative proof that a regular local ring is a domain.

**Proposition (22.13).** — Let \( A \) be a ring, \( m \) a maximal ideal. Then \( \widehat{A} \) is a local ring with maximal ideal \( \widehat{m} \).

**Proof:** First, \( \widehat{A}/\widehat{m} = A/m \) by (22.11): so \( \widehat{m} \) is maximal. Next, \( \mathrm{rad}(\widehat{A}) \supseteq \widehat{m} \) by (22.2). Finally, let \( m' \) be any maximal ideal of \( \widehat{A} \). Then \( m' \supseteq \mathrm{rad}(\widehat{A}) \). Hence \( m' = \widehat{m} \). Thus \( \widehat{m} \) is the only maximal ideal. □

**Exercise (22.13).** — Let \( A \) be a Noetherian local ring, \( m \) the maximal ideal, \( M \) a finitely generated module. Prove (1) that \( \widehat{A} \) is a Noetherian local ring with \( \widehat{m} \) as maximal ideal, (2) that \( \dim(M) = \dim(\widehat{M}) \), and (3) that \( A \) is regular if and only if \( \widehat{A} \) is regular.

**Exercise (22.15).** — Let \( A \) be a ring, and \( m_1, \ldots, m_m \) maximal ideals. Set \( m := \bigcap m_i \), and give \( A \) the \( m \)-adic topology. Prove that \( \widehat{A} \) is a semilocal ring, that \( \widehat{m}_1, \ldots, \widehat{m}_m \) are all its maximal ideals, and that \( \widehat{m} = \widehat{\mathrm{rad}(A)} \).
(22.16) (Completion, units, and localization). — Let $R$ be a ring, $a$ an ideal, and $\kappa: R \to \hat{R}$ the canonical map. Given $t \in R$, for each $n$ denote by $t_n \in R/a^n$ the residue of $t$. Let’s show that $\kappa(t)$ is a unit if and only if each $t_n$ is.

Indeed, by (22.8), we may regard $\hat{R}$ as a submodule of $\prod R/a^n$. Then each $t_n$ is equal to the projection of $\kappa(t)$. Hence $t_n$ is a unit if $\kappa(t)$ is. Conversely, assume $t_n$ is a unit for each $n$. Then there are $u_n \in R$ with $u_n t \equiv 1 \pmod{a^n}$. By the uniqueness of inverses, $u_{n+1} \equiv u_n$ in $R/a^n$ for each $n$. Set $u := (u_n) \in \prod R/a^n$. Then $u \in \hat{R}$, and $u \kappa(t) = 1$. Thus $\kappa(t)$ is a unit.

Set $T := \kappa^{-1}(\hat{R}^\times)$. Then by the above, $T$ consists of the $t \in R$ whose residue $t_n \in R/a^n$ is a unit for each $n$. So (22.11) and (12.2) yield

$$T = \{ t \in R \mid t \text{ lies in no maximal ideal containing } a \}.$$  \hspace{1cm} (22.16.1)

Set $S := 1 + a$. Then $S \subseteq T$ owing to (22.15.1) as no maximal ideal can contain both $x$ and $1 + x$. Hence the UMP of localization (11.23) yields this diagram:

$$\begin{array}{ccc}
S^{-1}R & \xrightarrow{\sigma} & T^{-1}R \\
\varphi & & \varphi \\
\downarrow & & \downarrow \\
R & \xrightarrow{\kappa} & \hat{R}
\end{array}
$$

Further, $S$ and $T$ map into $(R/a^n)^\times$; hence, (11.9), (11.12), and (12.22) yield:

$$R/a^n = S^{-1}R/a^n S^{-1}R = T^{-1}R/a^n T^{-1}R.$$

Therefore, $\hat{R}$ is, by (22.3), equal to the completion of each of $S^{-1}R$ and $T^{-1}R$ in their $a S^{-1}R$-adic and $a T^{-1}R$-adic topologies.

For example, take $a$ to be a maximal ideal $m$. Then $T = R - m$ by (22.15.1). Thus $\hat{R}$ is equal to the completion of the localization $R_m$.

Finally, assume $R$ is Noetherian. Let’s prove that $\sigma$ and $\tau$ are injective. Indeed, say $\tau \sigma(x/s) = 0$. Then $\kappa(x) = 0$ as $\kappa(s)$ is a unit. So $x \in \bigcap a^n$. Hence the Krull Intersection Theorem, (11.24) or (21.14), yields an $s' \in S$ with $s' x = 0$. So $x/s = 0$ in $S^{-1}R$. Thus $\sigma$ is injective. Similarly, $\tau$ is injective.

Theorem (22.17) (Exactness of Completion). — Let $R$ be a Noetherian ring, $a$ an ideal. Then on the finitely generated modules $M$, the functor $M \mapsto \hat{M}$ is exact.

Proof: Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated modules. Set $F^n M' := M' \cap a^n M$. By the Artin–Rees Lemma (21.13), the $F^n M'$ form an $a$-stable filtration. Hence, it yields the same topology, so the same completion, as the $a$-adic filtration by (22.10). Thus $0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0$ is exact by (22.1) and (22.3), as desired. \(\square\)

Exercise (22.18). — Let $A$ be a Noetherian semilocal ring. Prove that an element $x \in A$ is a nonzerodivisor on $A$ if and only if its image $\hat{x} \in \hat{A}$ is one on $\hat{A}$.

Exercise (22.19). — Let $p \in \mathbb{Z}$ be prime. For $n > 0$, define a $\mathbb{Z}$-linear map

$$\alpha_n: \mathbb{Z}/(p) \to \mathbb{Z}/(p^n) \quad \text{by} \quad \alpha_n(1) = p^{n-1}.$$  

Set $A := \bigoplus_{n \geq 1} \mathbb{Z}/(p)$ and $B := \bigoplus_{n \geq 1} \mathbb{Z}/(p^n)$. Set $\alpha := \bigoplus \alpha_n$; so $\alpha: A \to B$.

1. Show that the $p$-adic completion $\hat{A}$ is just $A$.
2. Show that, in the topology on $A$ induced by the $p$-adic topology on $B$, the completion $\hat{A}$ is equal to $\prod_{n=1}^{\infty} \mathbb{Z}/(p)$. 


(3) Show that the natural sequence of $p$-adic completions

$$
\hat{A} \to \hat{B} \to (B/A) \hat{\to}
$$

is not exact at $\hat{B}$. (Thus $p$-adic completion is neither left exact nor right exact.)

**Corollary (22.20).** — Let $R$ be a Noetherian ring, $a$ an ideal, and $M$ a finitely generated module. Then the natural map is an isomorphism:

$$
\hat{R} \otimes M \to \hat{M}.
$$

**Proof:** By (22.17), the functor $M \mapsto \hat{M}$ is exact on the category of finitely generated modules, and so (22.18) yields the conclusion.

**Exercise (22.21).** — Let $R$ be a ring, $a$ an ideal. Show that $M \mapsto \hat{M}$ preserves surjections, and that $\hat{R} \otimes M \to \hat{M}$ is surjective if $M$ is finitely generated.

**Corollary (22.22).** — Let $R$ be a Noetherian ring, $a$ and $b$ ideals, $M$ a finitely generated module. Then, using the $a$-adic topology, we have

1. $(bM)\hat{\to} = b\hat{M} = \hat{b}\hat{M}$ and
2. $(b^n)\hat{\to} = b^n\hat{R} = (b\hat{R})^n = (\hat{b})^n$ for any $n \geq 0$.

**Proof:** In general, the inclusion $bM \to M$ induces a commutative square

$$
\begin{array}{ccc}
R \otimes (bM) & \to & R \otimes M \\
\downarrow & & \downarrow \\
(bM)\hat{\to} & \to & \hat{M}
\end{array}
$$

It is not hard to see that top map’s image is $b(\hat{R} \otimes M)$.

In the present case, the two vertical maps are isomorphisms by (22.20), and the bottom map is injective by (22.21). Thus $(bM)\hat{\to} = b\hat{M}$.

Taking $R$ for $M$ yields $\hat{b} = b\hat{R}$. Hence $b\hat{M} = b\hat{R} \hat{M} = \hat{b}\hat{M}$. Thus (1) holds.

In (1), taking $b^n$ for $b$ and $R$ for $M$ yields $(b^n)\hat{\to} = b^n\hat{R}$. In particular, $\hat{b} = b\hat{R}$; so $(b\hat{R})^n = (\hat{b})^n$. But $b^nR' = (bR')^n$ for any $R$-algebra $R'$. Thus (2) holds.

**Corollary (22.23).** — Let $R$ be a Noetherian ring, $a$ an ideal. Then $\hat{R}$ is flat.

**Proof:** Let $a$ be any ideal. Then $\hat{R} \otimes b = \hat{b}$ by (22.21), and $\hat{b} = b\hat{R}$ by (22.22)(2). Thus $\hat{R}$ is flat by the Ideal Criterion (9.20).

**Exercise (22.24).** — Let $R$ be a Noetherian ring, $a$ an ideal. Prove that $\hat{R}$ is faithfully flat if and only if $a \subset \text{rad}(R)$.

**Exercise (22.25).** — Let $R$ be a Noetherian ring, and $a$ and $b$ ideals. Assume $a \subset \text{rad}(R)$, and use the $a$-adic topology. Prove $b$ is principal if $b\hat{R}$ is.

**Lemma (22.26).** — Let $R$ be a ring, $\alpha: M \to N$ a map of modules, $F^\bullet M$ and $F^\bullet N$ filtrations. Assume $\alpha F^nM \subset F^nN$ for all $n$. Assume $F^nM = M$ and $F^nN = N$ for $n \ll 0$. If the induced map $G^\bullet \alpha$ is injective or surjective, then so is $\alpha$.

**Proof:** For each $n \in \mathbb{Z}$, form the following commutative diagram of $R$-modules:

$$
\begin{array}{ccc}
0 \to F^nM/F^{n+1}M & \to M/F^{n+1}M & \to M/F^nM \to 0 \\
\downarrow G^n\alpha & \downarrow \alpha_{n+1} & \downarrow \alpha_n \\
0 \to F^nN/F^{n+1}N & \to N/F^{n+1}N & \to N/F^nN \to 0
\end{array}
$$
Its rows are exact. So the Snake Lemma (22.22) yields this exact sequence:

\[ \text{Ker } G^n \alpha \rightarrow \text{Ker } \alpha_{n+1} \rightarrow \text{Ker } \alpha_n \rightarrow \text{Coker } G^n \alpha \rightarrow \text{Coker } \alpha_{n+1} \rightarrow \text{Coker } \alpha_n. \]

Assume \( G^* \alpha \) is injective. Then \( \text{Ker } G^n \alpha = 0 \). But \( M/F^n M = 0 \) for \( n \ll 0 \). So by induction \( \text{Ker } \alpha_n = 0 \) for all \( n \). Thus \( \alpha \) is injective by (22.27) and (22.28).

Assume \( G^* \alpha \) is surjective, or \( \text{Coker } G^n \alpha = 0 \). So \( \text{Ker } \alpha_{n+1} \rightarrow \text{Ker } \alpha_n \) is surjective. But \( N/F^n N = 0 \) for \( n \ll 0 \). So by induction, \( \text{Coker } \alpha_n = 0 \) for all \( n \). So

\[
0 \rightarrow \text{Ker } \alpha_n \rightarrow M/F^n M \overset{\phi_n}{\longrightarrow} N/F^n N \rightarrow 0
\]

is exact. Thus \( \alpha \) is surjective by (22.27) and (22.28).

\textbf{Lemma (22.27).} — Let \( R \) be a ring, \( \mathfrak{a} \) an ideal, \( M \) a module, \( F^* M \) an \( \mathfrak{a} \)-filtration. Assume \( R \) is complete, \( M \) is separated, and \( F^n M = M \) for \( n \ll 0 \). Assume \( G^* M \) is module finite over \( G^* R \). Then \( M \) is complete, and is module finite over \( R \).

\textbf{Proof:} Take finitely many generators \( \mu_i \) of \( G^* M \), and replace them by their homogeneous components. Set \( n_i := \deg(\mu_i) \). Lift \( \mu_i \) to \( m_i \in F^{n_i} M \).

Filter \( R \) \( \mathfrak{a} \)-adically. Set \( E := \bigoplus \mathfrak{a} \mathfrak{a} R[-n_i] \). Filter \( E \) with \( F^n E := \bigoplus \mathfrak{a} \mathfrak{a} F^n R[-n_i] \).

Then \( F^n E = E \) for \( n \ll 0 \). Define \( \alpha : E \rightarrow M \) by sending \( 1 \in R[-n_i] \) to \( m_i \in M \). Then \( \alpha F^n E \subset F^n M \) for all \( n \). Also, \( G^* \alpha : G^* E \rightarrow G^* M \) is surjective as the \( \mu_i \) generate. So \( \alpha \) is surjective by (22.27).

Form the following canonical commutative diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{\kappa_E} & \hat{E} \\
\alpha \downarrow & & \hat{\alpha} \\
M & \xrightarrow{\kappa_M} & M
\end{array}
\]

As \( R \) is complete, \( \kappa_R : R \rightarrow \hat{R} \) is surjective by (22.12); hence, \( \kappa_E \) is surjective. Thus \( \kappa_M \) is surjective; that is, \( M \) is complete. As \( M \) is separated, \( \kappa_M \) is injective by (22.27). So \( \kappa_M \) is bijective. So \( \alpha \) is surjective. Thus \( M \) is module finite.

\textbf{Exercise (22.28).} (Nakayama’s Lemma for a complete ring). — Let \( R \) be a ring, \( \mathfrak{a} \) an ideal, and \( M \) a module. Assume \( R \) is complete, and \( M \) separated. Show \( m_1, \ldots, m_n \in M \) generate assuming their images \( m'_1, \ldots, m'_n \) in \( M/\mathfrak{a} M \) generate.

\textbf{Proposition (22.29).} — Let \( R \) be a ring, \( \mathfrak{a} \) an ideal, and \( M \) a module. Assume \( R \) is complete, and \( M \) separated. Assume \( G^* M \) is a Noetherian \( G^* R \)-module. Then \( M \) is a Noetherian \( R \)-module, and every submodule \( N \) is complete.

\textbf{Proof:} Let \( F^* M \) denote the \( \mathfrak{a} \)-adic filtration, and \( F^* N \) the induced filtration: \( F^n N := N \cap F^n M \). Then \( N \) is separated, and \( F^n N = N \) for \( n \ll 0 \). Further, \( G^* N \subset G^* M \). However, \( G^* M \) is Noetherian. So \( G^* N \) is module finite. Thus \( N \) is complete and is module finite over \( R \) by (22.27). Thus \( M \) is Noetherian.

\textbf{Theorem (22.30).} — Let \( R \) be a ring, \( \mathfrak{a} \) an ideal. If \( R \) is Noetherian, so is \( \hat{R} \).

\textbf{Proof:} Assume \( R \) is Noetherian. Then \( G^* R \) is algebra finite over \( R/\mathfrak{a} \) by (21.12), so Noetherian by the Hilbert Basis Theorem, (11.5.2). But \( G^* R = G^* \hat{R} \) by (22.11). Thus \( \hat{R} \) is Noetherian by (22.21) with \( \hat{R} \) for \( R \) and \( \hat{R} \) for \( M \).
Example (22.31). — Let $k$ be a Noetherian ring, $P := k[X_1, \ldots, X_r]$ the polynomial ring, and $A := k[[X_1, \ldots, X_r]]$ the formal power series ring. Then $A$ is the completion of $P$ in the $\langle X_1, \ldots, X_r \rangle$-adic topology by (22.24). Further, $P$ is Noetherian by the Hilbert Basis Theorem, (21.14). Thus $A$ is Noetherian by (22.31).

Assume $k$ is a domain. Then $A$ is a domain. Indeed, $A$ is one if $r = 1$, because
\[(a_m X_1^n + \cdots)(b_n X_1^n + \cdots) = a_m b_n X_1^{m+n} + \cdots.\]

If $r > 1$, then $A = k[[X_1, \ldots, X_r]][[X_{r+1}, \ldots, X_r]]$; so $A$ is a domain by induction.

Set $p_i := \langle X_{i+1}, \ldots, X_r \rangle$. Then $A/p_i = k[[X_1, \ldots, X_i]]$ by (22.10). Hence $p_i$ is prime. So $0 = p_r \subseteq \cdots \subseteq p_0$ is a chain of primes of length $r$. Thus $\dim A \geq r$.

Assume $k$ is a field. Then $A$ is local with maximal ideal $\langle X_1, \ldots, X_r \rangle$ and with residue field $k$ by the above and either by (22.23) or again by (22.11). Therefore, $\dim A \leq r$ by (21.19). Thus $A$ is regular of dimension $r$.

Theorem (22.32) (UMP of Formal Power Series). — Let $R$ be a ring, $R'$ an $R$-algebra, $b$ an ideal of $R'$, and $x_1, \ldots, x_n \in b$. Let $P := R[[X_1, \ldots, X_r]]$ be the formal power series ring. If $R'$ is separated and complete in the $b$-adic topology, then there is a unique $R$-algebra map $\hat{\pi}: P \to R'$ with $\hat{\pi}(X_i) = x_i$ for $1 \leq i \leq n$.

Proof: Form the map $\pi: R[X_1, \ldots, X_n] \to R'$ with $\pi(X_i) = x_i$. By the UMP of completion $\pi$ induces the desired map $\hat{\pi}: P \to R'$.

Alternatively, for each $m$, the map $\pi$ induces a map
\[P/\langle X_1, \ldots, X_r \rangle^m = R[X_1, \ldots, X_n]/\langle X_1, \ldots, X_n \rangle^m \to R'/b^m.\]
Taking inverse limits yields $\hat{\pi}$ owing to (22.28) and (22.29). \qed

Theorem (22.33) (Cohen Structure). — Let $A$ be a complete Noetherian local ring with maximal ideal $m$. Assume $A$ contains a coefficient field $k$; that is, $k \to A/m$. Then $A \simeq k[[X_1, \ldots, X_n]]/a$ for some variables $X_i$ and ideal $a$. Further, if $A$ is regular of dimension $r$, then $A \simeq k[[X_1, \ldots, X_r]]$.

Proof: Take generators $x_1, \ldots, x_n \in m$. Let $\pi: k[[X_1, \ldots, X_n]] \to A$ be the map with $\pi(X_i) = x_i$ of (22.34). Then $G^*\pi$ is surjective. Hence, $\pi$ is surjective by (22.26). Set $a := \operatorname{Ker}(\pi)$. Then $k[[X_1, \ldots, X_n]]/a \to A$.

Assume $A$ is regular of dimension $r$. Take $n := r$. Then $G^*A$ is a polynomial ring in $r$ variables over $k$ by (22.22). And $G^*(k[[X_1, \ldots, X_r]])$ is too by (22.35). Since $G^*\pi$ is surjective, it is bijective by (22.28) with $G^*A$ for both $R$ and $M$. So $\pi$ is bijective by (22.20). Thus $k[[X_1, \ldots, X_r]] \to A$. \qed
23. Discrete Valuation Rings

A discrete valuation is a homomorphism from the multiplicative group of a field to the additive group of integers such that the value of a sum is at least the minimum value of the summands. The corresponding discrete valuation ring consists of the elements whose values are nonnegative, plus 0. We characterize these rings in various ways; notably, we prove they are the normal Noetherian local domains of dimension 1. Then we prove that any normal Noetherian domain is the intersection of all the discrete valuation rings obtained by localizing at its height-1 primes. Finally, we prove Serre’s Criterion for normality of Noetherian domains. Along the way, we study the notions of regular sequence, depth, and Cohen–Macaulayness; these notions are so important that we study them further in an appendix.

(23.1) (Discrete Valuations). — Let $K$ be a field. We define a discrete valuation of $K$ to be a surjective function $v: K^\times \to \mathbb{Z}$ such that, for every $x, y \in K^\times$,

1. $v(x \cdot y) = v(x) + v(y)$,
2. $v(x + y) \geq \min\{v(x), v(y)\}$ if $x \neq -y$. (23.1.1)

Condition (1) just means $v$ is a group homomorphism. Hence, for any $x \in K^\times$,

1. $v(1) = 0$ and $v(x^{-1}) = -v(x)$. (23.1.2)

As a convention, we define $v(0) := \infty$. Consider the sets

$$A := \{x \in K \mid v(x) \geq 0\} \quad \text{and} \quad \mathfrak{m} := \{x \in K \mid v(x) > 0\}.$$  

Clearly, $A$ is a subring, so a domain, and $\mathfrak{m}$ is an ideal. Further, $\mathfrak{m}$ is nonzero as $v$ is surjective. We call $A$ the discrete valuation ring (DVR) of $v$.

Notice that, if $x \in K$, but $x \not\in A$, then $x^{-1} \in \mathfrak{m}$; indeed, $v(x) < 0$, and so $v(x^{-1}) = -v(x) > 0$. Hence, $\text{Frac}(A) = K$. Further,

$$A^\times = \{x \in K \mid v(x) = 0\} = A - \mathfrak{m}.$$ 

Indeed, if $x \in A^\times$, then $v(x) \geq 0$ and $-v(x) = v(x^{-1}) \geq 0$; so $v(x) = 0$. Conversely, if $v(x) = 0$, then $v(x^{-1}) = -v(x) = 0$; so $x^{-1} \in A$, and so $x \in A^\times$. Therefore, by the nonunit criterion, $A$ is a local domain, not a field, and $\mathfrak{m}$ is its maximal ideal.

An element $t \in \mathfrak{m}$ with $v(t) = 1$ is called a (local) uniformizing parameter. Such a $t$ is irreducible, as $t = ab$ with $v(a) \geq 0$ and $v(b) \geq 0$ implies $v(a) = 0$ or $v(b) = 0$ since $1 = v(a) + v(b)$. Further, any $x \in K^\times$ has the unique factorization $x = u t^n$ where $u \in A^\times$ and $n := v(x)$; indeed, $v(u) = 0$ as $u = xt^{-n}$. In particular, $t_1$ is uniformizing parameter if and only if $t_1 = ut$ with $u \in A^\times$; also, $A$ is a UFD.

Moreover, $A$ is a PID; in fact, any nonzero ideal $\mathfrak{a}$ of $A$ has the form

$$\mathfrak{a} = (t^m) \quad \text{where} \quad m := \min\{v(x) \mid x \in \mathfrak{a}\}. \quad \text{(23.1.3)}$$

Indeed, given a nonzero $x \in \mathfrak{a}$, say $x = ut^n$ where $u \in A^\times$. Then $t^n \in \mathfrak{a}$. So $n \geq m$. Set $y := ut^{-m}$. Then $y \in A$ and $x = yt^m$, as desired.

In particular, $\mathfrak{m} = (t)$ and $\dim(A) = 1$. Thus $A$ is regular local of dimension 1.

Example (23.2). — The prototype is this example. Let $k$ be a field, $t$ a variable, and $K := k((t))$ the field of formal Laurent series $x := \sum_{i \geq n} a_i t^i$ with $n \in \mathbb{Z}$ and with $a_i \in k$ and $a_0 \not= 0$. Set $v(x) := n$, the “order of vanishing” of $x$. Clearly, $v$ is a discrete valuation, the formal power series ring $k[[t]]$ is its DVR, and $\mathfrak{m} := (t)$ is its maximal ideal.
The preceding example can be extended to cover any DVR $A$ that contains a field $k$ with $k \twoheadrightarrow A/\langle t \rangle$ where $t$ is a uniformizing power. Indeed, $A$ is a subring of its completion $\hat{A}$ by (22.3), and $\hat{A} = k[[t]]$ by the proof of the Cohen Structure Theorem (22.3.3). Further, clearly, the valuation on $\hat{A}$ restricts to that on $A$.

A second old example is this. Let $p \in \mathbb{Z}$ be prime. Given $x \in \mathbb{Q}$, write $x = ap^n/b$ with $a, b \in \mathbb{Z}$ relatively prime and prime to $p$. Set $v(x) := n$. Clearly, $v$ is a discrete valuation, the localization $\mathbb{Z}(p)$ is its DVR, and $p\mathbb{Z}(p)$ is its maximal ideal. We call $v$ the $p$-adic valuation of $\mathbb{Q}$.

**Lemma (23.3).** — Let $A$ be a local domain, $m$ its maximal ideal. Assume that $m$ is nonzero and principal and that $\bigcap_{n \geq 0} m^n = 0$. Then $A$ is a DVR.

**Proof:** Given a nonzero $x \in A$, there is an $n \geq 0$ such that $x \in m^n - m^{n+1}$. Say $m = \langle t \rangle$. Then $x = ut^n$, and $u \notin m$, so $u \in A^\times$. Set $K := \text{Frac}(A)$. Given $x \in K^\times$, write $x = y/z$ where $y = bt^m$ and $z = ct^k$ with $b, c \in A^\times$. Then $x = ut^n$ with $u := b/c \in A^\times$ and $n := m - k \in \mathbb{Z}$. Define $v: K^\times \to \mathbb{Z}$ by $v(x) := n$. If $ut^n = wt^h$ with $n \geq h$, then $(u/w)t^{n-h} = 1$, and so $n = h$. Thus $v$ is well defined.

Since $v(t) = 1$, clearly $v$ is surjective. To verify (23.3.1), take $x = ut^n$ and $y = wt^h$ with $u, w \in A^\times$. Then $xy = (uw)t^{n+h}$. Thus (1) holds. To verify (2), we may assume $n \geq h$. Then $x + y = t^h(ut^{n-h} + w)$. Hence

$$v(x + y) \geq \min\{n, h\} = \min\{v(x), v(y)\}.$$ 

Thus (2) holds. So $v: K^\times \to \mathbb{Z}$ is a valuation. Clearly, $A$ is the DVR of $v$. □

(23.4) *(Depth).* — Let $R$ be a ring, $M$ a nonzero module, and $x_1, \ldots, x_n \in R$. Set $M_i := M/(x_1, \ldots, x_i)M$. We say the sequence $x_1, \ldots, x_n$ is $M$-regular, or is an $M$-sequence, and we call $n$ its length if $M_n \neq 0$ and $x_i \notin z\text{div}(M_{i-1})$ for all $i$.

Call the supremum of the lengths $n$ of the $M$-sequences found in an ideal $a$ the depth of $a$ on $M$, and denote it by $\text{depth}(a, M)$. By convention, $\text{depth}(a, M) = 0$ means $a$ contains no nonzerodivisor on $M$.

If $M$ is semilocal, call the depth of $\text{rad}(M)$ on $M$ simply the depth of $M$ and denote it by $\text{depth}(M)$. Notice that, in this case, the condition $M_n \neq 0$ is automatic owing to Nakayama’s Lemma (11.2.10).

If $M$ is semilocal and $\text{depth}(M) = \dim(M)$, call $M$ Cohen–Macaulay. When $R$ is semilocal, call $R$ Cohen–Macaulay if $R$ is a Cohen–Macaulay $R$-module.

**Lemma (23.5).** — Let $A$ be a Noetherian local ring, $m$ its maximal ideal, and $M$ a nonzero finitely generated module.

1. Then $\text{depth}(M) = 0$ if and only if $m \in \text{Ass}(M)$.
2. Then $\text{depth}(M) = 1$ if and only if there is an $x \in m$ with $x \notin z\text{div}(M)$ and $m \in \text{Ass}(M/xM)$.
3. Then $\text{depth}(M) \leq \dim(M)$.

**Proof:** Consider (1). If $m \in \text{Ass}(M)$, then it is immediate from the definitions that $m \subset z\text{div}(M)$ and so $\text{depth}(M) = 0$.

Conversely, assume $\text{depth}(M) = 0$. Then $m \subset z\text{div}(M)$. Since $A$ is Noetherian, $z\text{div}(M) = \bigcup_{p \in \text{Ass}(M)} \mathfrak{p}$ by (17.1.23). Since $M$ is also finitely generated, $\text{Ass}(M)$ is finite by (17.2.41). Hence $m = \mathfrak{p}$ for some $p \in \text{Ass}(M)$ by Prime Avoidance, (3.1.14).

Consider (2). Assume $\text{depth}(M) = 1$. Then there is an $M$-sequence of length 1, but none longer. So there is an $x \in m$ with $x \notin z\text{div}(M)$ and $\text{depth}(M/xM) = 0$. Then $m \in \text{Ass}(M/xM)$ by (1).
Conversely, assume there is \( x \in m \) with \( x \not\in z\text{div}(M) \). Then \( \text{depth}(M) \geq 1 \) by definition. Assume \( m \in \text{Ass}(M/\langle xM \rangle) \). Then given any \( y \in m \) with \( y \not\in z\text{div}(M) \), also \( m \in \text{Ass}(M/yM) \) by (17.7.2). So \( \text{depth}(M/yM) = 0 \) by (1). So there is no \( z \in m \) such that \( y, z \) is an \( M \)-sequence. Thus \( \text{depth}(M) \leq 1 \). Thus \( \text{depth}(M) = 1 \).

Consider (3). Given any \( M \)-sequence \( x_1, \ldots, x_n \), set \( M_i := M/(x_1, \ldots, x_i) \). Then \( \text{dim}(M_{i+1}) = \text{dim}(M_i) - 1 \) by (23.7.2). Hence \( \text{dim}(M) - n = \text{dim}(M_n) \geq 0 \). But \( \text{depth}(M) := \sup \{n\} \). Thus (3) holds. \( \square \)

**Exercise (23.8).** — Let \( R \) be a ring, \( M \) a module, and \( x, y \in R \).

1. Assume that \( x, y \) form an \( M \)-sequence. Prove that, given any \( m, n \in M \) with \( xm = yn \), there exists \( p \in M \) with \( m = yp \) and \( n = xp \).

2. Assume that \( x, y \) form an \( M \)-sequence and that \( y \not\in z\text{div}(M) \). Prove that \( y, x \) form an \( M \)-sequence too.

3. Assume that \( R \) is local, that \( x, y \) lie in its maximal ideal \( m \), and that \( M \) is nonzero and Noetherian. Assume that, given any \( m, n \in M \) with \( xm = yn \), there exists \( p \in M \) with \( m = yp \) and \( n = xp \). Prove that \( x, y \) form an \( M \)-sequence.

**Exercise (23.9).** — Let \( A \) be a Noetherian local ring, \( M \) and \( N \) nonzero finitely generated modules, \( F : ((R\text{-mod})) \to ((R\text{-mod})) \) a left-exact functor that preserves the finitely generated modules (such as \( F(\bullet) := \text{Hom}(M, \bullet) \) by (16.2.1)). Show that, for \( d = 1, 2 \), if \( N \) has depth at least \( d \), then so does \( F(N) \).

**Exercise (23.10).** — Let \( R \) be a local ring, \( m \) its maximal ideal, \( M \) a Noetherian module, \( x_1, \ldots, x_n \in m \), and \( \sigma \) a permutation of \( 1, \ldots, n \). Assume \( x_1, \ldots, x_n \) form an \( M \)-sequence, and prove \( x_{\sigma(1)}, \ldots, x_{\sigma(n)} \) do too; first, say \( \sigma \) transposes \( i \) and \( i + 1 \).

**Exercise (23.11).** — Prove that a Noetherian local ring \( A \) of dimension \( r \geq 1 \) is regular if and only if its maximal ideal \( m \) is generated by an \( A \)-sequence. Prove that, if \( A \) is regular, then \( A \) is Cohen–Macaulay.

**Theorem (23.10) (Characterization of DVRs).** — Let \( A \) be a local ring, \( m \) its maximal ideal. Assume \( A \) is Noetherian. Then these five conditions are equivalent:

1. \( A \) is a DVR.
2. \( A \) is a normal domain of dimension 1.
3. \( A \) is a normal domain of depth 1.
4. \( A \) is a regular local ring of dimension 1.
5. \( m \) is principal and of height at least 1.

**Proof:** Assume (1). Then \( A \) is UFD by (23.1); so \( A \) is normal by (11.8.3). Further, \( A \) has just two primes, \( (0) \) and \( m \); so \( \text{dim}(A) = 1 \). Thus (2) holds. Further, (4) holds by (23.1); Clearly, (4) implies (5).

Assume (2). Take a nonzero \( x \in m \). Then \( A/(x) \neq 0 \), so \( \text{Ass}(A/(x)) \neq \emptyset \) by (17.7.1). Now, \( A \) is a local domain of dimension 1. So \( A \) has just two primes: \( (0) \) and \( m \). But \( (0) \notin \text{Ass}(A/(x)) \). So \( m \in \text{Ass}(A/(x)) \). Thus (23.10)(2) yields (3).

Assume (3). By (23.10)(2), there are \( x, y \in m \) such that \( x \) is nonzero and \( y \) has residue \( \bar{y} \in A/(x) \) with \( m = \text{Ann}(\bar{y}) \). So \( ym \subset (x) \). Set \( z := y/x \in \text{Frac}(A) \). Then \( zm = (ym)/x \subset A \). Suppose \( zm \subset m \). Then \( z \) is integral over \( A \) by (11.8.3). But \( A \) is normal, so \( z \in A \). So \( y = zx \in (x) \), a contradiction. Hence, \( 1 \in zm \); so there is \( t \in m \) with \( zt = 1 \). Given \( w \in m \), therefore \( w = (zt)wzz \) in \( A \). Thus \( m \) is principal. Finally, \( \text{ht}(m) \geq 1 \) because \( x \in m \) and \( x \neq 0 \). Thus (5) holds.

Assume (5). Set \( N := \bigcap m^n \). The Krull Intersection Theorem (15.2.9) yields an \( x \in m \) with \( (1 + x)N = 0 \). Then \( 1 + x \in A^\times \). So \( N = 0 \). Further, \( A \) is a domain by
Let \( 2 = 1 \) if and only if \( (S_2) \).

**Exercise (23.11).** — Let \( R \) be a DVR with fraction field \( K \), and \( f \in A \) a nonzero nonunit. Prove \( A \) is a maximal proper subring of \( K \). Prove \( \dim(A) \neq \dim(A_f) \).

**Exercise (23.12).** — Let \( k \) be a field, \( P := k[X,Y] \) the polynomial ring in two variables, \( f \in P \) an irreducible polynomial. Say \( f = \ell(X,Y) + g(X,Y) \) with \( \ell(X,Y) = aX + bY \) for \( a, b \in k \) and with \( g \in \langle X,Y \rangle^2 \). Set \( R := P/(f) \) and \( \mathfrak{p} := \langle X,Y \rangle/(f) \). Prove that \( R_\mathfrak{p} \) is a DVR if and only if \( \ell \neq 0 \). (Thus \( R_\mathfrak{p} \) is a DVR if and only if the plane curve \( C : f = 0 \subset k^2 \) is nonsingular at \((0,0)\); see (23.7)).

**Exercise (23.13).** — Let \( k \) be a field, \( A \) a ring intermediate between the polynomial ring and the formal power series ring in one variable: \( k[X] \subset A \subset k[[X]] \). Suppose that \( A \) is local with maximal ideal \( \langle X \rangle \). Prove that \( A \) is a DVR. (Such local rings arise as rings of power series with curious convergence conditions.)

**Exercise (23.14).** — Let \( L/K \) be an algebraic extension of fields, \( X_1, \ldots, X_n \) variables, \( P \) and \( Q \) the polynomial rings over \( K \) and \( L \) in \( X_1, \ldots, X_n \).

1. Let \( \mathfrak{q} \) be a prime of \( Q \), and \( \mathfrak{p} \) its contraction in \( P \). Prove \( \operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{q}) \).
2. Let \( f, g \in P \) be two polynomials with no common prime factor in \( P \). Prove that \( f \) and \( g \) have no common prime factor \( q \in Q \).

**Exercise (23.15) (Serre’s Conditions).** — Let \( R \) be a Noetherian ring. We say Serre’s Condition \( (R_n) \) holds if, for any prime \( \mathfrak{p} \) of height \( m \leq n \), the localization \( R_\mathfrak{p} \) is regular of dimension \( m \).

For example, \( (R_0) \) holds if and only if \( R_\mathfrak{p} \) is a field for any minimal prime \( \mathfrak{p} \). Also, \( (R_1) \) holds if and only if \( (R_0) \) does and \( R_\mathfrak{p} \) is a DVR for any \( \mathfrak{p} \) of height-1. We say Serre’s Condition \( (S_n) \) holds for an \( R \)-module \( M \) if, for any prime \( \mathfrak{p} \),

\[
\operatorname{depth}(M_\mathfrak{p}) \geq \min\{\dim(M_\mathfrak{p}), n\}.
\]

Note \( \operatorname{depth}(M_\mathfrak{p}) \leq \dim(M_\mathfrak{p}) \) by (23.6) (3). Hence \( (S_n) \) holds if and only if \( M_\mathfrak{p} \) is Cohen–Macaulay when \( \operatorname{depth}(M_\mathfrak{p}) \leq n \). In particular, \( (S_1) \) holds if and only if \( \mathfrak{p} \) is minimal when \( \mathfrak{p} \in \operatorname{Ass}(M) \) by (17.14); that is, \( M \) has no embedded primes.

**Exercise (23.16).** — Let \( R \) be a Noetherian domain, \( M \) a finitely generated module. Show that \( M \) is torsionfree if and only if it satisfies \( (S_1) \).

**Exercise (23.17).** — Let \( R \) be a Noetherian ring. Show that \( R \) is reduced if and only if \( (R_0) \) and \( (S_1) \) hold.

**Lemma (23.18).** — Let \( R \) be a Noetherian domain. Set

\[
\Phi := \{ \mathfrak{p} \text{ prime} \mid \operatorname{ht}(\mathfrak{p}) = 1 \} \quad \text{and} \quad \Sigma := \{ \mathfrak{p} \text{ prime} \mid \operatorname{depth}(R_\mathfrak{p}) = 1 \}.
\]

Then \( \Phi \subseteq \Sigma \), and \( \Phi = \Sigma \) if and only if \( (S_2) \) holds. Further, \( R = \bigcap_{\mathfrak{p} \in \Sigma} R_\mathfrak{p} \).

**Proof:** Given \( \mathfrak{p} \in \Phi \), set \( \mathfrak{q} := \mathfrak{p}R_\mathfrak{p} \). Take \( 0 \neq \mathfrak{q} \subset \mathfrak{q} \). Then \( \mathfrak{q} \) is minimal over \( \langle x \rangle \). So \( \mathfrak{q} \in \operatorname{Ass}(R_\mathfrak{p}/\langle x \rangle) \) by (17.18). Hence \( \operatorname{depth}(R_\mathfrak{p}) = 1 \) by (23.3) (2). Thus \( \Phi \subseteq \Sigma \).

However, \( (S_1) \) holds by (23.17). Hence \( (S_2) \) holds if and only if \( \Phi \subseteq \Sigma \). Thus \( \Phi = \Sigma \) if and only if \( (S_2) \) holds.

Further, \( R \subset R_\mathfrak{p} \) for any prime \( \mathfrak{p} \) by (11.3); so \( R \subset \bigcap_{\mathfrak{p} \in \Sigma} R_\mathfrak{p} \). As to the possible inclusion, take an \( x \in \bigcap_{\mathfrak{p} \in \Sigma} R_\mathfrak{p} \). Say \( x = a/b \) with \( a, b \in R \) and \( b \neq 0 \). Then \( a \in bR_\mathfrak{p} \) for all \( \mathfrak{p} \in \Sigma \). But \( \mathfrak{p} \in \Sigma \) if \( \mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}(R_\mathfrak{p}/bR_\mathfrak{p}) \) by (23.3) (2), so if \( \mathfrak{p} \in \operatorname{Ass}(R/bR) \) by (17.11). Hence \( a \in bR \) by (13.24). Thus \( x \in R \), as desired. □
Theorem (23.19). — Let $R$ be a normal Noetherian domain. Then
\[ R = \bigcap_{p \in \Phi} R_p \text{ where } \Phi := \{ \text{p prime } | \text{ht}(p) = 1 \}. \]

Proof: As $R$ is normal, so is $R_p$ for any prime $p$ by (11.32). So $\text{depth}(R_p) = 1$ if and only if $\text{dim}(R_p) = 1$ by (23.11). Thus (23.18) yields the assertion.

Theorem (23.20) (Serre’s Criterion). — Let $R$ be a Noetherian domain. Then $R$ is normal if and only if $(R_1)$ and $(S_2)$ hold.

Proof: As $R$ is a domain, $(R_0)$ and $(S_1)$ hold by (11.32). If $R$ is normal, then so is $R_p$ for any prime $p$ by (11.32); whence, $(R_1)$ and $(S_2)$ hold by (23.11).

Conversely, assume $R$ satisfies $(R_1)$ and $(S_2)$. Let $x$ be integral over $R$. Then $x$ is integral over $R_p$ for any prime $p$. Now, $R_p$ is a DVR for all $p$ of height 1 as $R$ satisfies $(R_1)$. Hence, $x \in R_p$ for all $p$ of height 1, so for all $p$ of depth 1 as $R$ satisfies $(S_2)$. So $x \in R$ owing to (23.18). Thus $R$ is normal.

Example (23.21). — Let $k$ be an algebraically closed field, $P := k[X, Y]$ the polynomial ring in two variables, $f \in P$ irreducible. Then $\text{dim}(P) = 2$ by (11.32) or by (11.18). Set $R := P/(f)$. Then $R$ is a domain.

Let $p \subset R$ be a nonzero prime. Say $m = m/(f)$. Then $0 \subsetneq \langle f \rangle \subseteq m$ is a chain of primes of length 2, the maximum. Thus $m$ is maximal, and $\text{dim}(R) = 1$.

Hence $m = (X - a, Y - b)$ for some $a, b \in k$ by (11.32). Write
\[ f(X, Y) = \frac{\partial f}{\partial X}(a, b)(X - a) + \frac{\partial f}{\partial Y}(a, b)(Y - b) + g \]
where $g \in m^2$. Then $R_p$ is a DVR if and only if $\frac{\partial f}{\partial X}(a, b)$ and $\frac{\partial f}{\partial Y}(a, b)$ are not both equal to zero owing to (23.22) applied after making the change of variables $X' := X - a$ and $Y' := Y - b$.

Clearly, $R$ satisfies $(S_2)$. Further, $R$ satisfies $(R_1)$ if and only if $R_p$ is a DVR for every nonzero prime $p$. Hence, by Serre’s Criterion, $R$ is normal if and only if $\frac{\partial f}{\partial X}$ and $\frac{\partial f}{\partial Y}$ do not both belong to any maximal ideal $m$ of $P$ containing $f$.

(Put geometrically, $R$ is normal if and only if the plane curve $C : f = 0 \subset k^2$ is nonsingular everywhere.) Thus $R$ is normal if and only if $\langle f, \partial f/\partial X, \partial f/\partial Y \rangle = 1$.

Exercise (23.22). — Prove that a Noetherian domain $R$ is normal if and only if, given any prime $p$ associated to a principal ideal, $pR_p$ is principal.

Exercise (23.23). — Let $R$ be a Noetherian ring, $K$ its total quotient ring,
\[ \Phi := \{ \text{p prime } | \text{ht}(p) = 1 \} \quad \text{and} \quad \Sigma := \{ \text{p prime } | \text{depth}(R_p) = 1 \}. \]
Assuming $(S_1)$ holds for $R$, prove $\Phi \subseteq \Sigma$, and prove $\Phi = \Sigma$ if and only if $(S_2)$ holds.

Further, without assuming $(S_1)$ holds, prove this canonical sequence is exact:
\[ R \to K \to \prod_{p \in \Phi} K_p/R_p. \]

Exercise (23.24). — Let $R$ be a Noetherian ring, and $K$ its total quotient ring. Set $\Phi := \{ \text{p prime } | \text{ht}(p) = 1 \}$. Prove these three conditions are equivalent:
1. $R$ is normal.
2. $(R_1)$ and $(S_2)$ hold.
3. $(R_1)$ and $(S_1)$ hold, and $R \to K \to \prod_{p \in \Phi} K_p/R_p$ is exact.
23. Appendix: Cohen–Macaulayness

Exercise (23.26). — Let \( R \to R' \) be a flat map of Noetherian rings, \( a \subset R \) an ideal, \( M \) a finitely generated \( R \)-module, and \( x_1, \ldots, x_r \) an \( M \)-sequence in \( a \). Set \( M' := M \otimes_R R' \). Assume \( M'/aM' \neq 0 \). Show \( x_1, \ldots, x_r \) is an \( M' \)-sequence in \( aR' \).

Exercise (23.27). — Let \( R \) be a Noetherian ring, \( a \) an ideal, and \( M \) a finitely generated module with \( M/aM \neq 0 \). Let \( x_1, \ldots, x_r \) be an \( M \)-sequence in \( a \) and \( p \in \text{Supp}(M/aM) \). Prove the following statements:

1. \( x_1/1, \ldots, x_r/1 \) is an \( M_p \)-sequence in \( a_p \), and
2. \( \text{depth}(a, M) \leq \text{depth}(a_p, M_p) \).

Proposition (23.28). — Let \( R \) be a Noetherian ring, \( a \) an ideal, and \( M \) a finitely generated module. Assume \( M/aM \neq 0 \). Let \( x_1, \ldots, x_m \) be a finished \( M \)-sequence in \( a \). Then \( m = \text{depth}(a, M) \).

Proof: Let \( y_1, \ldots, y_n \) be a second finished \( M \)-sequence in \( a \). Say \( m \leq n \). Induct on \( m \). Suppose \( m = 0 \). Then \( a \subset z.\text{div}(M) \). Hence \( n = 0 \) too. Now, suppose \( m \geq 1 \).

Set \( M_i := M/\langle x_1, \ldots, x_i \rangle M \) and \( N_j := M/\langle y_1, \ldots, y_j \rangle M \) for all \( i, j \). Set \( U := \bigcup_{i=0}^{m-1} z.\text{div}(M_i) \cup \bigcup_{j=0}^{n-1} z.\text{div}(N_j) \).

Then \( U \) is equal to the union of all associated primes of \( M_i \) for \( i < m \) and of \( N_j \) for \( j < n \) by (17.26.1). And these primes are finite in number by (17.24.13). Suppose \( a \subset U \). Then \( a \) lies in one of the primes, say \( p \in \text{Ass}(M_i) \), by (17.24.13). But \( x_{i+1} \in a - z.\text{div}(M_i) \) and \( a \subset p \subset z.\text{div}(M_i) \), a contradiction. Thus \( a \not\subset U \).

Take \( z \in a - U \). Then \( z \not\in z.\text{div}(M_i) \) for \( i < m \) and \( z \not\in z.\text{div}(N_j) \) for \( j < n \). Now, \( a \subset z.\text{div}(M_m) \) by finiteness. So \( a \subset q \) for some \( q \in \text{Ass}(M_m) \) by (17.24.10). But \( M_m = M_{m-1}/x_mM_{m-1} \). Moreover, \( x_m \) and \( z \) are nonzerodivisors on \( M_{m-1} \).

Also \( x_m, z \in a \subset q \). So \( q \in \text{Ass}(M_{m-1}/zM_{m-1}) \) by (17.24.8). Hence

\[ a \subset z.\text{div}(M/\langle x_1, \ldots, x_{m-1}, z \rangle M). \]

Hence \( x_1, \ldots, x_{m-1}, z \) is finished in \( a \). Similarly, \( y_1, \ldots, y_{n-1}, z \) is finished in \( a \).

Thus we may replace both \( x_m \) and \( y_n \) by \( z \).

By (23.27.2), we may move \( z \) to the front of both sequences. Thus we may assume \( x_1 = y_1 = z \). Then \( M_1 = N_1 \). Further, \( x_2, \ldots, x_m \) and \( y_2, \ldots, y_n \) are finished \( M_1 \)-sequences in \( a \). So by induction, \( m - 1 = n - 1 \). Thus \( m = n \). \( \Box \)

Exercise (23.29). — Let \( R \) be a Noetherian ring, \( a \) an ideal, and \( M \) a finitely generated module with \( M/aM \neq 0 \). Let \( x \in a \) be a nonzerodivisor on \( M \). Show \( \text{depth}(a, M/xM) = \text{depth}(a, M) - 1 \).
EXERCISE (26.3.1). — Let $A$ be a Noetherian local ring, $M$ a finitely generated module, $x \notin \text{z.div}(M)$. Show $M$ is Cohen–Macaulay if and only if $M/xM$ is so.

PROPOSITION (23.31). — Let $R \to R'$ be a map of Noetherian rings, $a \subset R$ an ideal, and $M$ a finitely generated $R$-module with $M/aM \neq 0$. Set $M' := M \otimes_R R'$. Assume $M'/R'$ is faithfully flat. Then $\text{depth}(aR', M') = \text{depth}(a, M)$.

Proof: By (23.22), there is a finished $M$-sequence $x_1, \ldots, x_r$ in $a$. For all $i$, set $M_i := M/(x_1, \ldots, x_i)M$ and $M_i' := M'/(x_1, \ldots, x_i)M'$. By (8.14), we have $M'/aM' = M/aM \otimes_R R'$ and $M_i' = M_i \otimes_R R'$.

So $M'/aM' \neq 0$ by faithful flatness. Hence $x_1, \ldots, x_r$ is an $M'$-sequence by (23.35).

As $x_1, \ldots, x_r$ is finished, $a \subset \text{z.div}(M)$. So $\text{Hom}_R(R/a, M_r) \neq 0$ by (14.24).

However, (8.14) and (8.11) yield $\text{Hom}_R(R/a, M_r) \otimes R R' = \text{Hom}_R(R/a, M_r') = \text{Hom}_R(R'/aR', M_r')$. So $\text{Hom}_R(R'/aR', M_r') \neq 0$ by faithful flatness. So $aR' \subset \text{z.div}(M_r')$ by (14.24). So $x_1, \ldots, x_r$ is a finished $M'$-sequence in $aR'$. Thus (23.23) yields the assertion. □

EXERCISE (23.3.2). — Let $A$ be a Noetherian local ring, and $M$ a nonzero finitely generated module. Prove the following statements:

1. $\text{depth}(M) = \text{depth}(\hat{M})$.

2. $M$ is Cohen–Macaulay if and only if $\hat{M}$ is Cohen–Macaulay.

EXERCISE (23.3.3). — Let $R$ be a Noetherian ring, $a$ an ideal, and $M$ a finitely generated module with $M/aM \neq 0$. Show that there is $p \in \text{Supp}(M/aM)$ with $\text{depth}(a, M) = \text{depth}(a_p, M_p)$.

LEMMA (23.3.4). — Let $A$ be a Noetherian local ring, $m$ its maximal ideal, $a$ another ideal, $M$ a nonzero finitely generated module, and $x \in m - \text{z.div}(M)$. Assume $a \subset \text{z.div}(M)$. Set $M' := M/xM$. Then there is $p \in \text{Ass}(M')$ with $p \supset a$.

Proof: By hypothesis, the sequence $0 \to M \xrightarrow{\mu} M \to M' \to 0$ is exact. Set $H := \text{Hom}(A/a, M)$. Then $H \neq 0$ by (14.22) as $a \subset \text{z.div}(M)$. Further, $H$ is finitely generated by (11.14). So $H/xH \neq 0$ by Nakayama’s Lemma (11.11). Also, $0 \to H \xrightarrow{\mu} H \to \text{Hom}(A/a, M')$ is exact by (7.13); so $H/xH \subset \text{Hom}(A/a, M')$. So $\text{Hom}(A/a, M') \neq 0$. But $\text{Supp}(A/a) = \text{V}(a)$ by (13.3.1). Thus (14.20) yields the desired $p$. □

LEMMA (23.3.5). — Let $R$ be a Noetherian Ring, $M$ a nonzero finitely generated module, $p_0 \in \text{Ass}(M)$, and $p_0 \subsetneq \cdots \subsetneq p_r$ a chain of primes. Assume that there is no prime $p$ with $p_{i-1} \subsetneq p \subsetneq p_i$ for any $i$. Then $\text{depth}(p_r, M) \leq r$.

Proof: If $r = 0$, then $p_0 \subset \text{z.div}(M)$. So $\text{depth}(p_0, M) = 0$, as desired. Induct on $r$. Assume $r \geq 1$. As $p_0 \in \text{Ass}(M)$, we have $p_r \in \text{Supp}(M)$ by (14.1.1); so $M_{p_r} \neq 0$. Further, $\text{depth}(p_r, M) \leq \text{depth}(M_{p_r})$ by (23.20)(2). So localizing at $p_r$, we may assume $R$ is local and $p_r$ is the maximal ideal.

Let $x_1, \ldots, x_s$ be a finished $M$-sequence in $p_{r-1}$. Then as $p_{r-1} \subset p_r$, clearly $M/p_{r-1}M \neq 0$. So $s = \text{depth}(p_{r-1}, M)$ by (23.25). So by induction $s \leq r - 1$. Set $M_s := M/(x_1, \ldots, x_s)M$. Then $p_{r-1} \subset \text{z.div}(M_s)$ by finiteness.

Suppose $p_r \subset \text{z.div}(M_s)$. Then $x_1, \ldots, x_s$ is finished in $p_r$. So $s = \text{depth}(p_r, M)$.
by (23.38), as desired.

Suppose instead $p_i \not\subseteq z.\text{div}(M)$. Then there's $x \in p_i - z.\text{div}(M)$. So $x_1, \ldots, x_s, x$ is an $M$-sequence in $p_i$. By (23.39), there is $p \in \text{Ass}(M_i/xM_i)$ with $p \supseteq p_i - 1$. But $p = \text{Ann}(m)$ for some $m \in M_s/xM_s$, so $x \in p$. Hence $p_i - 1 \subseteq p \subseteq p_i$. Hence, by hypothesis, $p = p_i$. Hence $x_1, \ldots, x_s, x$ is finished in $p_i$. So (23.38) yields $s + 1 = \text{depth}(p_i, M)$. Thus $\text{depth}(p_i, M) \leq r$, as desired. □

**Theorem (23.36) (Unmixedness).** — Let $A$ be a Noetherian local ring, and $M$ a finitely generated module. Assume $M$ is Cohen–Macaulay. Then $M$ has no embedded primes, and all maximal chains of primes in $\text{Supp}(M)$ are of the same length, namely, $\dim(M)$.

**Proof:** Given $p_0 \in \text{Ass}(M)$, take any maximal chain of primes $p_0 \subsetneq \cdots \subsetneq p_r$. Then $p_r$ is the maximal ideal. So $\text{depth}(M) = \text{depth}(p_r, M)$. So $\dim(M) \leq r$ by (23.38). But $\text{depth}(M) = \dim(M)$ as $M$ is Cohen–Macaulay. And $r \leq \dim(M)$ by (21.11). So $r = \dim(M)$. Hence $p_0$ is minimal. Thus $M$ has no embedded primes.

Given any maximal chain of primes $p_0 \subsetneq \cdots \subsetneq p_r$ in $\text{Supp}(M)$, necessarily $p_0$ is minimal. So $p_0 \in \text{Ass}(M)$ by (17.15). Thus, as above, $r = \dim(M)$, as desired. □

**Exercise (23.37).** — Prove that a Cohen–Macaulay local ring $A$ is catenary.

**Proposition (23.38).** — Let $A$ be a Noetherian local ring, $M$ a finitely generated module. Let $x_1, \ldots, x_n$ be nonunits of $A$, and set $M_i := M/(x_1, \ldots, x_i)M$ for all $i$. Assume $M$ is Cohen–Macaulay. Then $x_1, \ldots, x_n$ is an $M$-sequence if and only if it is part of a sop; if so, then $M_n$ is Cohen–Macaulay.

**Proof:** First, assume $x_1, \ldots, x_n$ is part of a sop. Induct on $n$. For $n = 0$, the assertion is trivial. Say $n \geq 1$. By induction $x_1, \ldots, x_{n-1}$ is an $M$-sequence, and $M_{n-1}$ is Cohen–Macaulay. Now, all maximal chains of primes in $\text{Supp}(M_{n-1})$ have the same length by (23.38), and $\dim(M_n) = \dim(M_{n-1}) - 1$ by (21.5). Hence $x_n$ is in no minimal prime of $M_{n-1}$. But $M_{n-1}$ has no embedded primes by (23.39). So $x_n \notin p$ for any $p \in \text{Ass}(M_{n-1})$. So $x_n \notin z.\text{div}(M_{n-1})$ by (17.20). Thus $x_1, \ldots, x_n$ is an $M$-sequence. Finally, as $M_{n-1}$ is Cohen–Macaulay, so is $M_n$ by (23.39).

Conversely, assume $x_1, \ldots, x_n$ is an $M$-sequence. By (23.21), extend it to a finished $M$-sequence $x_1, \ldots, x_r$. Then $\text{depth}(M_r) = 0$, and $M_r$ is Cohen–Macaulay by (23.41) applied recursively. So $\dim(M_r) = 0$. Thus $x_1, \ldots, x_r$ is a sop. □

**Proposition (23.39).** — Let $A$ be a Noetherian local ring, $M$ a finitely generated module, $p \in \text{Supp}(M)$. Set $s := \text{depth}(p, M)$. Assume $M$ is Cohen–Macaulay. Then $M_p$ is a Cohen–Macaulay $A_p$-module of dimension $s$.

**Proof:** Induct on $s$. Assume $s = 0$. Then $p \subset z.\text{div}(M)$. So $p$ lies in some $q \in \text{Ass}(M)$ by (17.20). But $q$ is minimal in $\text{Supp}(M)$ by (23.38). So $q = p$. Hence $\dim(M_p) = 0$. Thus $M_p$ is a Cohen–Macaulay $A_p$-module of dimension 0.

Assume $s \geq 1$. Then there is $x \in p - z.\text{div}(M)$. Set $M' := M/xM$, and set $s' := \text{depth}(p, M')$. Then $M/pM \neq 0$ by (15.31). So $s' = s - 1$ by (23.24), and $M'$ is Cohen–Macaulay by (23.41). Further, $M'_p = M'_p/xM_p$ by (17.20). But $x \in p$. So $M' \neq 0$ by Nakayama’s Lemma (11.11). So $p \in \text{Supp}(M')$. Hence by induction, $M'_p$ is a Cohen–Macaulay $A_p$-module of dimension $s - 1$.

As $x \notin z.\text{div}(M)$, also $x \notin z.\text{div}(M_p)$ by (23.41)(1). Hence $M_p$ is a Cohen–Macaulay $A_p$-module by (23.31). Finally, $\dim(M_p) = s$ by (21.5). □
DEFINITION (23.40). — Let $R$ be a Noetherian ring, and $M$ a finitely generated module. We call $M$ Cohen–Macaulay if $M_m$ is a Cohen–Macaulay $R_m$-module for every maximal ideal $m \in \text{Supp}(M)$. It is equivalent that $M_p$ be a Cohen–Macaulay $R_p$-module for every $p \in \text{Supp}(M)$, because if $p$ lies in the maximal ideal $m$, then $R_p$ is the localization of $R_m$ at the prime ideal $pR_m$ by (11.28), and hence $R_p$ is Cohen–Macaulay if $R_m$ is by (23.39).

We say $R$ is Cohen–Macaulay if $R$ is a Cohen–Macaulay $R$-module.

PROPOSITION (23.41). — Let $R$ be a Noetherian ring. Then $R$ is Cohen–Macaulay if and only if the polynomial ring $R[X]$ is Cohen–Macaulay.

PROOF: First, assume $R[X]$ is Cohen–Macaulay. Given a prime $p$ of $R$, set $\mathfrak{P} := pR[X] + (X)$. Then $\mathfrak{P}$ is prime in $R[X]$ by (4.13). Now, $R[X]/(X) = R$ and $\mathfrak{P}/(X) = p$ owing to (11.30); hence, $R_p = R_p$ by (11.22) (1). Further, (12.22) yields $(R[X]/(X))_p = (R[X]/(X)R[X])_p$. Hence $R[X]_p/(X)R[X]_p = R_p$. But $R[X]_p$ is Cohen–Macaulay by (23.31), and $X$ is plainly a nonzerodivisor; so $R_p$ is Cohen–Macaulay by (23.31). Thus $R$ is Cohen–Macaulay.

Conversely, assume $R$ is Cohen–Macaulay. Given a maximal ideal $\mathfrak{M}$ of $R[X]$, set $m := \mathfrak{M} \cap R$. Then $R[X]_m = (R[X])_m$ by (11.24) (1), and $R[X]_m = R_m[X]$ by (11.30). But $R_m$ is Cohen–Macaulay. Thus, to show $R[X]_m$ is Cohen–Macaulay, replace $R$ by $R_m$, and so assume $R$ is local with maximal ideal $m$.

As $\mathfrak{M}(R/m)[X]$ is maximal, it contains a nonzero polynomial $\overline{f}$. As $R/m$ is a field, we may take $\overline{f}$ monic. Lift $\overline{f}$ to a monic polynomial $f \in \mathfrak{M}$. Set $B := R[X]/(f)$. Then $B$ is a free, module-finite extension of $R$ by (11.26). So $\dim(R) = \dim(B)$ by (12.2). Plainly $\dim(B) \geq \dim(B_m)$. So $\dim(R) \geq \dim(B_m)$.

Further, $B$ is flat over $R$ by (11.7). And $B_m$ is flat over $B$ by (12.24). So $B_m$ is flat over $R$ by (12.12). So any $R$-sequence in $m$ is a $B_m$-sequence by (23.46) as $B_m/mB_m \neq 0$. Hence $\text{depth}(B_m) \geq \text{depth}(R)$.

But $\text{depth}(R) = \dim(R)$ and $\dim(R) \geq \dim(B_m)$. So $\text{depth}(B_m) \geq \dim(B_m)$. But the opposite inequality holds by (23.3). Thus $B_m$ is Cohen–Macaulay. But $B_m = R[X]/(f)$ by (12.22). And $f$ is monic, so a nonzerodivisor. So $R[X]_m$ is Cohen–Macaulay by (23.31). Thus $R[X]$ is Cohen–Macaulay. $\square$

DEFINITION (23.42). — A ring $R$ is called universally catenary if every finitely generated $R$-algebra is catenary.

THEOREM (23.43). — A Cohen–Macaulay ring $R$ is universally catenary.

PROOF: Clearly any quotient of a catenary ring is catenary, as chains of primes can be lifted by (11.3). So it suffices to prove that, for any $n$, the polynomial ring $P$ in $n$ variables over $R$ is catenary.

Notice $P$ is Cohen–Macaulay by induction on $n$, as $P = R$ if $n = 0$, and the induction step holds by (23.31). Now, given nested primes $q \subset p$ in $P$, put $\mathfrak{p}$ in a maximal ideal $m$. Then any chain of primes from $q$ to $p$ corresponds to a chain from $qP_m$ to $pP_m$ by (11.24). But $P_m$ is Cohen–Macaulay, so catenary by (23.37). Thus the assertion holds. $\square$

EXAMPLE (23.44). — Trivially, a field is Cohen–Macaulay. Plainly, a domain of dimension 1 is Cohen–Macaulay. By (23.21), a normal domain of dimension 2 is Cohen–Macaulay. Thus these rings are all universally catenary by (23.31). In particular, we recover (11.19).
Proposition (23.45). — Let $A$ be a regular local ring of dimension $n$, and $M$ a finitely generated module. Assume $M$ is Cohen-Macaulay of dimension $n$. Then $M$ is free.

Proof: Induct on $n$. If $n = 0$, then $A$ is a field by (21.21), and so $M$ is free.

Assume $n \geq 1$. Let $t \in A$ be an element of a regular system of parameters. Then $A/(t)$ is regular of dimension $n - 1$ by (23.20). As $M$ is Cohen-Macaulay of dimension $n$, any associated prime $q$ is minimal in $A$ by (23.36); so $q = (0)$ as $A$ is a domain by (21.23). Hence $t$ is a nonzerodivisor on $M$ by (17.15). So $M/tM$ is Cohen–Macaulay of dimension $n - 1$ by (21.5) and (21.21). Hence by induction, $M/tM$ is free, say of rank $r$.

Let $k$ be the residue field of $A$. Then $M \otimes_A k = (M/tM) \otimes_{A/(t)} k$ by (8.16)(1). So $r = \text{rank}(M \otimes_A k)$.

Set $p := (t)$. Then $A_p$ is a DVR by (23.11). Moreover, $M_p$ is Cohen–Macaulay of dimension 1 by (23.31) as $\text{depth}(\langle t \rangle, M) = 1$. So $M_p$ is torsionfree by (23.11). Therefore $M_p$ is flat by (12.22), so free by (11.21). Set $s := \text{rank}(M_p)$.

Let $k(p)$ be the residue field of $A_p$. Then $M_p \otimes_{A_p} k(p) = M_p/tM_p$ by (5.11)(1). Moreover, $M_p/tM_p = (M/tM)_p$ by (12.22). So $r = s$.

Set $K := \text{Frac}(A)$. Then $M_p \otimes_{A_p} K = M \otimes_A K$ by (11.21)(1). Hence $M \otimes_A K$ has rank $r$. Thus $M$ is free by (13.13). □
24. Dedekind Domains

Dedekind domains are defined as the normal Noetherian domains of dimension 1. We prove they are the Noetherian domains whose localizations at nonzero primes are discrete valuation rings. Next we prove the Main Theorem of Classical Ideal Theory: in a Dedekind domain, every nonzero ideal factors uniquely into primes. Then we prove that a normal domain has a module-finite integral closure in any finite separable extension of its fraction field by means of the trace pairing of the extension. We conclude that a ring of algebraic integers is a Dedekind domain and that, if a domain is algebra finite over a field of characteristic 0, then in the fraction field or in any algebraic extension of it, the integral closure is module finite over the domain and is algebra finite over the field.

Definition (24.1). — A domain $R$ is said to be Dedekind if it is Noetherian, normal, and of dimension 1.

Example (24.2). — Examples of Dedekind domains include the integers $\mathbb{Z}$, the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$, the polynomial ring $k[X]$ in one variable over a field, and any DVR. Indeed, those rings are PIDs, and every PID $R$ is a Dedekind domain: $R$ is Noetherian by definition; $R$ is a UFD, so normal by Gauss’s Theorem, (10.33); and $R$ is of dimension 1 since every nonzero prime is maximal by (2.25).

On the other hand, any local Dedekind domain is a DVR by (23.11).

Example (24.3). — Let $d \in \mathbb{Z}$ be a square-free integer. Set $R := \mathbb{Z} + \mathbb{Z}\eta$ where

$$\eta := \begin{cases} (1 + \sqrt{d})/2 & \text{if } d \equiv 1 \pmod{4}; \\ \sqrt{d} & \text{if not}. \end{cases}$$

Then $R$ is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{d})$ by [2. Prp. (6.14), p. 412]; so $R$ is normal by (11.14). Also, $\dim(R) = \dim(\mathbb{Z})$ by (11.4); so $\dim(R) = 1$. Finally, $R$ is Noetherian by (11.22) as $\mathbb{Z}$ is so and as $R := \mathbb{Z} + \mathbb{Z}\eta$. Thus $R$ is Dedekind.

Example (24.4). — Let $k$ be an algebraically closed field, $P := k[X,Y]$ the polynomial ring in two variables, $f \in P$ irreducible. By (23.11), $R$ is a Noetherian domain of dimension 1, and $R$ is Dedekind if and only if $\langle f, \partial f/\partial X, \partial f/\partial Y \rangle = 1$.

Exercise (24.5). — Let $R$ be a domain, $S$ a multiplicative subset.

1. Assume $\dim(R) = 1$. Prove $\dim(S^{-1}R) = 1$ if and only if there is a nonzero prime $p$ with $p \cap S = \emptyset$.

2. Assume $\dim(R) \geq 1$. Prove $\dim(R) = 1$ if and only if $\dim(R_p) = 1$ for every nonzero prime $p$.

Exercise (24.6). — Let $R$ be a Dedekind domain, $S$ a multiplicative subset. Prove $S^{-1}R$ is a Dedekind domain if and only if there’s a nonzero prime $p$ with $p \cap S = \emptyset$.

Proposition (24.7). — Let $R$ be a Noetherian domain, not a field. Then $R$ is a Dedekind domain if and only if $R_p$ is a DVR for every nonzero prime $p$. 

148
If a Noetherian domain is Dedekind, then $R_p$ is too by $(24.5.6)$; so $R_p$ is a DVR by $(24.5.10)$. Conversely, suppose $R_p$ is a DVR for every nonzero prime $p$. Then, trivially, $R$ satisfies $(R_1)$ and $(S_2)$; so $R$ is normal by Serre’s Criterion. Since $R$ is not a field, $\dim(R) \geq 1$; whence, $\dim(R) = 1$ by $(24.7.8)(2)$. Thus $R$ is Dedekind. \[\square\]

**EXERCISE (24.8).** — Let $R$ be a Dedekind domain, and $a$, $b$, $c$ ideals. By first reducing to the case that $R$ is local, prove that

\[ a \cap (b + c) = (a \cap b) + (a \cap c), \]
\[ a + (b \cap c) = (a + b) \cap (a + c). \]

**PROPOSITION (24.9).** — In a Noetherian domain $R$ of dimension 1, every ideal $a \neq 0$ has a unique factorization $a = q_1 \cdots q_r$ with the $q_i$ primary and their primes $p_i$ distinct; further, $\{p_1, \ldots, p_r\} = \Ass(R/a)$ and $q_i = aR_{p_i} \cap R$ for each $i$.

**Proof:** The Lasker–Noether Theorem, $(18.21)$, yields an irredundant primary decomposition $a = \bigcap q_i$. Say $q_i$ is $p_i$-primary. Then by $(18.19)$ the $p_i$ are distinct and $\{p_i\} = \Ass(R/a)$.

The $q_i$ are pairwise comaximal for the following reason. Suppose $q_i + q_j$ lies in a maximal ideal $m$. Now, $p_i := \sqrt{q_i}$ by $(18.23)$; so $p_i^{n_i} \subseteq q_i$ for some $n_i$ by $(3.32)$. Hence $p_i^{n_i} \subseteq m$. So $p_i \subseteq m$ by $(23.1)$. But $0 \neq a \subseteq p_i$; hence, $p_i$ is maximal since $\dim(R) = 1$. Therefore, $p_i = m$. Similarly, $p_j = m$. Hence $i = j$. Thus the $q_i$ are pairwise comaximal. So the Chinese Remainder Theorem, $(18.30)$, yields $a = \prod q_i$.

As to uniqueness, let $a = \prod q_i$ be any factorization with the $q_i$ primary and their primes $p_i$ distinct. The $p_i$ are minimal containing $a$ as $\dim(R) = 1$; so the $p_i$ lie in $\Ass(R/a)$ by $(17.18)$. By the above reasoning, the $q_i$ are pairwise comaximal and so $\prod q_i = \bigcap q_i$. Hence $a = \bigcap q_i$ is an irredundant primary decomposition by $(18.33)$. So the $p_i$ are unique by the First Uniqueness Theorem, $(18.24)$, and $q_i = aR_{p_i} \cap R$ by the Second Uniqueness Theorem, $(18.25)$, and by $(24.7.8)(3)$. \[\square\]

**THEOREM (24.10)** (Main Theorem of Classical Ideal Theory). — Let $R$ be a domain. Assume $R$ is Dedekind. Then every nonzero ideal $a$ has a unique factorization into primes $p$. In fact, if $v_p$ denotes the valuation of $R_p$, then

\[ a = \prod p_{v_p(a)} \] where $v_p(a) := \min\{ v_p(a) \mid a \in a \}.$

**Proof:** Using $(24.31)$, write $a = \prod q_i$, with the $q_i$ primary, their primes $p_i$ distinct and unique, and $q_i = aR_{p_i} \cap R$. Then $R_{p_i}$ is a DVR by $(24.7.7)$. So $(24.7.16)$ yields $aR_{p_i} = p_{i}^{m_i}R_{p_i}$ with $m_i := \max\{ v_{p_i}(a/s) \mid a \in a$ and $s \in R - p_i \}$. But $v_{p_i}(1/s) = 0$. So $v_{p_i}(a/s) = v_{p_i}(a)$. Hence $m_i := v_{p_i}(a)$. Now, $p_{i}^{m_i}$ is primary by $(18.11)$ as $p_i$ is maximal; so $p_{i}^{m_i}R_{p_i} \cap R = p_{i}^{m_i}$ by $(18.23)$. Thus $q_i = p_{i}^{m_i}$. \[\square\]

**COROLLARY (24.11).** — A Noetherian domain $R$ of dimension 1 is Dedekind if and only if every primary ideal is a power of its radical.

**Proof:** If $R$ is Dedekind, every primary ideal is a power of its radical by $(24.11)$. Conversely, given a nonzero prime $p$, set $m := pR_p$. Then $m \neq 0$. So $m \neq m^2$ by Nakayama’s Lemma. Take $t \in m - m^2$. Then $m$ is the only prime containing $t$, as $\dim(R_p) = 1$ by $(24.3)(2)$. So $tR_p$ is $m$-primary by $(18.11)$. Set $q := tR_p \cap R$. Then $q$ is $p$-primary by $(18.8)$. So $q = p^n$ for some $n$ by hypothesis. But $qR_p = tR_p$ by $(18.11)(3)(b)$. So $tR_p = m^n$. But $t \notin m^2$. So $n = 1$. So $R_p$ is a DVR by $(24.7.11)$. Thus $R$ is Dedekind by $(24.7)$. \[\square\]
EXERCISE (24.17). — Prove that a semilocal Dedekind domain \( A \) is a PID. Begin by proving that each maximal ideal is principal.

EXERCISE (24.13). — Let \( R \) be a Dedekind domain, \( \mathfrak{a} \) and \( \mathfrak{b} \) two nonzero ideals. Prove (1) every ideal in \( R/\mathfrak{a} \) is principal, and (2) \( \mathfrak{b} \) is generated by two elements.

LEMMA (24.14) (Artin Character). — Let \( L \) be a field, \( G \) a group, \( \sigma_i: G \to L^\times \) distinct homomorphisms. Then the \( \sigma_i \) are linearly independent over \( L \) in the vector space of set maps \( \sigma: G \to L \) under valuewise addition and scalar multiplication.

**Proof:** Suppose there’s an equation \( \sum_{i=1}^m a_i \sigma_i = 0 \) with nonzero \( a_i \in L \). Take \( m \geq 1 \) minimal. Now, \( \sigma_i \neq 0 \) as \( \sigma_i: G \to L^\times \); so \( m \geq 2 \). Since \( \sigma_1 \neq \sigma_2 \), there’s an \( x \in G \) with \( \sigma_1(x) \neq \sigma_2(x) \). Then \( \sum_{i=1}^m a_i \sigma_i(x) = \sum_{i=1}^m a_i \sigma_i(xy) = 0 \) for every \( y \in G \) since \( \sigma_i \) is a homomorphism.

Set \( b_i := a_i(1 - \sigma_i(x)/\sigma_1(x)) \). Then

\[
\sum_{i=1}^m b_i \sigma_i = \sum_{i=1}^m a_i \sigma_i - \frac{1}{\sigma_1(x)} \sum_{i=1}^m a_i \sigma_i \sigma_i = 0.
\]

But \( b_1 = 0 \) and \( b_2 \neq 0 \), contradicting the minimality of \( m \).

(24.15) (Trace). — Let \( L/K \) be a finite Galois field extension. Its trace is this:

\[
\text{tr}: L \to K \quad \text{by} \quad \text{tr}(x) := \sum_{\sigma \in \text{Gal}(L/K)} \sigma(x).
\]

Clearly, \( \text{tr} \) is \( K \)-linear. It is nonzero by (24.14) applied with \( G := L^\times \).

Consider the symmetric \( K \)-bilinear **Trace Pairing**:

\[
L \times L \to K \quad \text{by} \quad (x, y) \mapsto \text{tr}(xy).
\]

(24.15.1)

It is nondegenerate for this reason. As \( \text{tr} \) is nonzero, there is \( z \in L \) with \( \text{tr}(z) \neq 0 \). Now, given \( x \in L^\times \), set \( y := z/x \). Then \( \text{tr}(xy) \neq 0 \), as desired.

LEMMA (24.16). — Let \( R \) be a normal domain, \( K \) its fraction field, \( L/K \) a finite Galois field extension, and \( x \in L \) integral over \( R \). Then \( \text{tr}(x) \in R \).

**Proof:** Let \( x^n + a_1 x^{n-1} + \cdots + a_n = 0 \) be an equation of integral dependence for \( x \) over \( R \). Let \( \sigma \in \text{Gal}(L/K) \). Then

\[
(\sigma x)^n + a_1 (\sigma x)^{n-1} + \cdots + a_n = 0;
\]

so \( \sigma x \) is integral over \( R \). Hence \( \text{tr}(x) \) is integral over \( R \), and lies in \( K \). Thus \( \text{tr}(x) \in R \) since \( R \) is normal.

THEOREM (24.17) (Finiteness of integral closure). — Let \( R \) be a normal Noetherian domain, \( K \) its fraction field, \( L/K \) a finite separable field extension, and \( R' \) the integral closure of \( R \) in \( L \). Then \( R' \) is module finite over \( R \), and is Noetherian.

**Proof:** Let \( L_1 \) be the Galois closure of \( L/K \), and \( R'_1 \) the integral closure of \( R \) in \( L_1 \). Let \( z_1, \ldots, z_n \in L_1 \) form a \( K \)-basis. Using (24.15.20), write \( z_i = y_i/a_i \) with \( y_i \in R'_1 \) and \( a_i \in R \). Clearly, \( y_1, \ldots, y_n \) form a basis of \( L_1/K \) contained in \( R'_1 \).

Let \( x_1, \ldots, x_n \) form the dual basis with respect to the Trace Pairing, (24.15.31), so that \( \text{tr}(x_i y_j) = \delta_{ij} \). Given \( b \in R' \), write \( b = \sum c_i x_i \) with \( c_i \in K \). Fix \( j \). Then

\[
\text{tr}(b y_j) = \text{tr} \left( \sum_i c_i x_i y_j \right) = \sum_i c_i \text{tr}(x_i y_j) = c_j \quad \text{for each} \ j.
\]
But \( b_j \in R'_1 \). So \( c_j \in R \) by \((24.17)\). Thus \( R' \subset \sum R x_i \). Since \( R \) is Noetherian, \( R' \) is module finite over \( R \)-module and Noetherian owing to \((16.12)\). \(\square\)

**Corollary (24.18).** — Let \( R \) be a Dedekind domain, \( K \) its fraction field, \( L/K \) a finite separable field extension. Then the integral closure \( R' \) of \( R \) in \( L \) is Dedekind.

**Proof:** First, \( R' \) is module finite over \( R \) by \((24.17)\); so \( R' \) is Noetherian by \((16.12)\). Second, \( R' \) is normal by \((10.32)\). Finally, \( \dim(R') = \dim(R) \) by \((15.12)\), and \( \dim(R) = 1 \) as \( R \) is Dedekind. Thus \( R \) is Dedekind. \(\square\)

**Theorem (24.19).** — A ring of algebraic integers is a Dedekind domain.

**Proof:** By \((24.17)\), \( \mathbb{Z} \) is a Dedekind domain; whence, so is its integral closure in any field that is a finite extension of \( \mathbb{Q} \) by \((24.18)\). \(\square\)

**Theorem (24.20)** (Noether on Finiteness of Integral Closure). — Let \( k \) be a field of characteristic 0, and \( R \) an algebra-finite domain over \( k \). Set \( K := \text{Frac}(R) \). Let \( L/K \) be a finite field extension (possibly \( L = K \)), and \( R' \) the integral closure of \( R \) in \( L \). Then \( R' \) is module finite over \( R \) and is algebra finite over \( k \).

**Proof:** By the Noether Normalization Lemma, \((15.1)\), \( R \) is module finite over a polynomial subring \( P \). Then \( P \) is normal by Gauss’s Theorem, \((10.32)\), and Noetherian by the Hilbert Basis Theorem, \((16.12)\); also, \( L/\text{Frac}(P) \) is a finite field extension, which is separable as \( k \) is of characteristic 0. Thus \( R' \) is module finite over \( P \) by \((24.17)\), and so \( R' \) is plainly algebra finite over \( k \). \(\square\)

**\((24.21)\) (Other cases).** — In \((24.18)\), even if \( L/K \) is inseparable, the integral closure \( R' \) of \( R \) in \( L \) is still Dedekind; see \((24.18)\).

However, Akizuki constructed an example of a DVR \( R \) and a finite inseparable extension \( L/\text{Frac}(R) \) such that the integral closure of \( R \) is a DVR, but is not module finite over \( R \). The construction is nicely explained in \([11]\) Secs. 9.4(1) and 9.5. Thus separability is a necessary hypothesis in \((24.17)\).

Noether’s Theorem, \((24.20)\), remains valid in positive characteristic, but the proof is more involved. See \([4]\) (13.13), p. 297.
25. Fractional Ideals

A fractional ideal is defined to be a submodule of the fraction field of a domain. A fractional ideal is called invertible if its product with another fractional ideal is equal to the given domain. We characterize the invertible fractional ideals as those that are nonzero, finitely generated, and principal locally at every maximal ideal. We prove that, in a Dedekind domain, any two nonzero ordinary ideals have an invertible fractional ideal as their quotient. We characterize Dedekind domains as those domains whose ordinary ideals are, equivalently, all invertible, all projective, or all finitely generated and flat. Further, we prove a Noetherian domain is Dedekind if and only if every torsionfree module is flat. Finally, we prove the ideal class group is equal to the Picard group; the former is the group of invertible fractional ideals modulo those that are principal, and the latter is the group, under tensor product, of isomorphism classes of modules local free of rank 1.

**Definition (25.1).** — Let $R$ be a domain, and set $K := \text{Frac}(R)$. We call an $R$-submodule $M$ of $K$ a fractional ideal. We call $M$ principal if there is an $x \in K$ with $M = Rx$.

Given another fractional ideal $N$, form these two new fractional ideals:

$$MN := \{ \sum x_i y_i \mid x_i \in M \text{ and } y_i \in N \} \quad \text{and} \quad (M : N) := \{ z \in K \mid zN \subset M \}.$$

We call them the product of $M$ and $N$ and the quotient of $M$ by $N$.

**Exercise (25.2).** — Let $R$ be a domain, $M$ and $N$ nonzero fractional ideals. Prove that $M$ is principal if and only if there exists some isomorphism $M \simeq R$.

Construct the following canonical surjection and canonical isomorphism:

$$\pi : M \otimes N \to MN \quad \text{and} \quad \varphi : (M : N) \to \text{Hom}(N, M).$$

**Proposition (25.3).** — Let $R$ be a domain, and $K := \text{Frac}(R)$. Consider these finiteness conditions on a fractional ideal $M$:

1. There exist ordinary ideals $a$ and $b$ with $b \neq 0$ and $(a : b) = M$.
2. There exists an $x \in K^\times$ with $xM \subset R$.
3. There exists a nonzero $x \in R$ with $xM \subset R$.
4. $M$ is finitely generated.

Then (1), (2), and (3) are equivalent, and they are implied by (4). Further, all four conditions are equivalent for every $M$ if and only if $R$ is Noetherian.

**Proof:** Assume (1) holds. Take any nonzero $x \in b$. Given $m \in M$, clearly $xm \in a \subset R$; so $xM \subset R$. Thus (2) holds.

Assume (2) holds. Write $x = a/b$ with $a, b \in R$ and $a, b \neq 0$. Then $aM \subset bR \subset R$. Thus (3) holds.

If (3) holds, then $xM$ and $xR$ are ordinary, and $M = (xM : xR)$; thus (1) holds.

Assume (4) holds. Say $y_1/x_1, \ldots, y_n/x_n \in K^\times$ generate $M$ with $x_i, y_i \in R$. Set $x := \prod x_i$. Then $x \neq 0$ and $xM \subset R$. Thus (3) holds.

Assume (3) holds and $R$ is Noetherian. Then $xM \subset R$. So $xM$ is finitely generated, say by $y_1, \ldots, y_n$. Then $y_1/x, \ldots, y_n/x$ generate $M$. Thus (4) holds.

Finally, assume all four conditions are equivalent for every $M$. If $M$ is ordinary, then (3) holds with $x := 1$, and so (4) holds. Thus $R$ is Noetherian. 

152
**Lemma (25.4).** — Let $R$ be a domain, $M$ and $N$ fractional ideals. Let $S$ be a multiplicative subset. Then

$$S^{-1}(MN) = (S^{-1}M)(S^{-1}N) \quad \text{and} \quad S^{-1}(M : N) \subset (S^{-1}M : S^{-1}N),$$

with equality if $N$ is finitely generated.

**Proof:** Given $x \in S^{-1}(MN)$, write $x = (\sum m_i n_i)/s$ with $m_i \in M$, with $n_i \in N$, and with $s \in S$. Then $x = \sum (m_i/s)(n_i/1)$, and so $x \in (S^{-1}M)(S^{-1}N)$. Thus $S^{-1}(MN) \subset (S^{-1}M)(S^{-1}N)$.

Conversely, given $x \in (S^{-1}M)(S^{-1}N)$, say $x = \sum (m_i/s_i)(n_i/t_i)$ with $m_i \in M$ and $n_i \in N$ and $s_i, t_i \in S$. Set $s := \prod s_i$ and $t := \prod t_i$. Then

$$x = \sum (m_i n_i/s_i t_i) = \sum m'_i n'_i / st \in S^{-1}(MN)$$

with $m'_i \in M$ and $n'_i \in N$. Thus $S^{-1}(MN) \supset (S^{-1}M)(S^{-1}N)$, so equality holds.

Given $z \in S^{-1}(M : N)$, write $z = x/s$ with $x \in (M : N)$ and $s \in S$. Given $y \in S^{-1}N$, write $y = n/t$ with $n \in N$ and $t \in S$. Then $z \cdot n = xn/st$ and $xn \in M$ and $st \in S$. So $z \in (S^{-1}M : S^{-1}N)$. Thus $S^{-1}(M : N) \subset (S^{-1}M : S^{-1}N)$.

Conversely, say $N$ is generated by $n_1, \ldots, n_r$. Given $z \in (S^{-1}M : S^{-1}N)$, write $zn_i/1 = m_i/s_i$ with $m_i \in M$ and $s_i \in S$. Set $s := \prod s_i$. Then $sz \cdot n_i \in M$. So $sz \in (M : N)$. Hence $z \in S^{-1}(M : N)$, as desired. \qed

**Definition (25.5).** — Let $R$ be a domain. We call a fractional ideal $M$ locally principal if, for every maximal ideal $m$, the localization $M_m$ is principal over $R_m$.

**Exercise (25.6).** — Let $R$ be a domain, $M$ and $N$ fractional ideals. Prove that the map $\pi: M \otimes N \to MN$ is an isomorphism if $M$ is locally principal.

**Proposition (25.7).** (Invertible fractional ideals). — Let $R$ be a domain. A fractional ideal $M$ is said to be invertible if there is some fractional ideal $M^{-1}$ with $MM^{-1} = R$.

For example, a nonzero principal ideal $Rx$ is invertible, as $(Rx)(R \cdot 1/x) = R$.

**Proposition (25.8).** — Let $R$ be a domain, $M$ an invertible fractional ideal. Then $M^{-1}$ is unique; in fact, $M^{-1} = (R : M)$.

**Proof:** Clearly $M^{-1} \subset (R : M)$ as $MM^{-1} = R$. But, if $x \in (R : M)$, then $x \cdot 1 \in (R : M)MM^{-1} \subset M^{-1}$, so $x \in M^{-1}$. Thus $(R : M) \subset M^{-1}$, as desired. \qed

**Exercise (25.9).** — Let $R$ be a domain, $M$ and $N$ fractional ideals.

1. Assume $N$ is invertible, and show that $(M : N) = M \cdot N^{-1}$.

2. Show that both $M$ and $N$ are invertible if and only if their product $MN$ is, and that if so, then $(MN)^{-1} = N^{-1}M^{-1}$.

**Lemma (25.10).** — An invertible ideal is finitely generated and nonzero.

**Proof:** Let $R$ be the domain, $M$ the ideal. Say $1 = \sum m_i n_i$ with $m_i \in M$ and $n_i \in M^{-1}$. Let $m \in M$. Then $m = \sum m_i n_i$. But $m n_i \in R$ as $m \in M$ and $n_i \in M^{-1}$. So the $m_i$ generate $M$. Trivially, $M \neq 0$. \qed

**Lemma (25.11).** — Let $A$ be a local domain, $M$ a fractional ideal. Then $M$ is invertible if and only if $M$ is principal and nonzero.

**Proof:** Assume $M$ is invertible. Say $1 = \sum m_i n_i$ with $m_i \in M$ and $n_i \in M^{-1}$. As $A$ is local, $A - A^\times$ is an ideal. So there's a $j$ with $m_j n_j \in A^\times$. Let $m \in M$. Then $mn_j \in A$. Set $\alpha := (mn_j)(m_jn_j)^{-1} \in A$. Then $\alpha m = mn_j$. Thus $M = Am_j$.

Conversely, if $M$ is principal and nonzero, then it’s invertible by (25.7). \qed
Let \( M \) be a Dedekind domain, \( M := (a : b) \). Then \( M \) is invertible, and has a unique factorization into powers of primes \( p \): if \( v_p \) denotes the valuation of \( R_p \) and if \( p^v := (p^{-1})^{-v} \) when \( v < 0 \), then

\[
M = \prod v_p(M) \quad \text{where} \quad v_p(M) := \min\{ v_p(x) \mid x \in M \}.
\]

Further, \( v_p(M) = \min\{ v_p(x_i) \} \) if the \( x_i \) generate \( M \).

Proof: First, \( R \) is Noetherian. So (24.72) yields that \( M \) is finitely generated and that there is a nonzero \( x \in R \) with \( xM \subset R \). Also, each \( R_p \) is a DVR by (24.17). So \( xM_p \) is principal by (24.71). Thus \( M \) is invertible by (25.13).

The Main Theorem of Classical Ideal Theory, (25.11), yields \( xM = \prod p^{v_p(x)} \) and \( xR = \prod p^{v_p(x)} \). But \( v_p(xM) = v_p(x) + v_p(M) \). Thus (25.14) yields

\[
M = (xM : xR) = \prod p^{v_p(x)+v_p(M)} \cdot \prod p^{-v_p(x)} = \prod p^{v_p(M)}.
\]

Further, given \( x \in M \), say \( x = \sum_{i=1}^na_ix_i \) with \( a_i \in R \). Then (24.71) yields

\[
v_p(x) \geq \min\{ v_p(a_i) \} \geq \min\{ v_p(x_i) \}
\]

by induction on \( n \). Thus \( v_p(M) = \min\{ v_p(x_i) \} \).

Exercise (25.16). — Show that a ring is a PID if and only if it’s a Dedekind domain and a UFD.

(25.16) \( \text{Invertible modules} \). — Let \( R \) be an arbitrary ring. We call a module \( M \) invertible if there is another module \( N \) with \( M \otimes N \simeq R \).

Up to (noncanonical) isomorphism, \( N \) is unique if it exists: if \( N' \otimes M \simeq R \), then

\[
N = R \otimes N \simeq (N' \otimes M) \otimes N = N' \otimes (M \otimes N) \simeq N' \otimes R = N'.
\]

Exercise (25.14). — Let \( R \) be an arbitrary ring, \( M \) an invertible module. Prove that \( M \) is finitely generated, and that, if \( R \) is local, then \( M \) is free of rank 1.

Exercise (25.15). — Show these conditions on an \( R \)-module \( M \) are equivalent:

1. \( M \) is invertible.
2. \( M \) is finitely generated, and \( M_m \simeq R_m \) at each maximal ideal \( m \).
3. \( M \) is locally free of rank 1.

Assuming these conditions hold, show that \( M \otimes \text{Hom}(M, R) = R \).

Proposition (25.19). — Let \( R \) be a domain, \( M \) a fractional ideal. Then the following conditions are equivalent:
Every nonzero ordinary ideal is a set. But (1) holds. □

The Picard Group

Let $R$ be a ring. We denote the collection of isomorphism classes of invertible modules by $\text{Pic}(R)$. By (25.21), every invertible module is finitely generated, so isomorphic to a quotient of $R^n$ for some integer $n$. Hence, $\text{Pic}(R)$ is a set. Further, $\text{Pic}(R)$ is, clearly, a group under tensor product.
with the class of \( R \) as identity. We call \( \text{Pic}(R) \) the **Picard Group** of \( R \).

Assume \( R \) is a domain, not a field. Set \( K := \text{Frac}(R) \). Given an invertible abstract module \( M \), we can embed \( M \) into \( K \) as follows. Set \( S := R - 0 \), and form the canonical map \( M \to S^{-1}M \). It is injective owing to \((12.17)\) if the multiplication map \( \mu_x : M \to M \) is injective for any \( x \in S \). Fix \( x \), and let’s prove \( \mu_x \) is injective.

Let \( m \) be a maximal ideal. Clearly, \( M_m \) is an invertible \( R_m \)-module. So \( M_m \cong R_m \) by \((25.17)\). Hence \( \mu_x : M_m \to M_m \) is injective. Therefore, \( \mu_x : M \to M \) is injective by \((13.43)\). Thus \( M \) embeds canonically into \( S^{-1}M \). Now, \( S^{-1}M \) is a localization of \( M_m \), so is a 1-dimensional \( K \)-vector space, again as \( M_m \cong R_m \). Choose an isomorphism \( S^{-1}M \cong K \). It yields the desired embedding of \( M \) into \( K \).

Hence, \((25.19)\) implies \( M \) is also invertible as a fractional ideal.

The invertible fractional ideals \( N \), clearly, form a group \( \mathcal{I}(R) \). Sending an \( N \) to its isomorphism class yields a map \( \kappa : \mathcal{I}(R) \to \text{Pic}(R) \) by \((45.18)\). By the above, \( \kappa \) is a group homomorphism by \((76.1)\). It’s not hard to check that its kernel is the group \( \mathcal{P}(R) \) of principal ideals and that \( \mathcal{P}(R) = K^\times / R^\times \).

We call \( \mathcal{I}(R)/\mathcal{P}(R) \) the **Ideal Class Group** of \( R \). Thus \( \mathcal{I}(R)/\mathcal{P}(R) = \text{Pic}(R) \); in other words, the Ideal Class Group is canonically isomorphic to the Picard Group.

Every invertible fractional ideal is, by \((25.25)\), finitely generated and nonzero, so of the form \((a : b)\) where \( a \) and \( b \) are nonzero ordinary ideals by \((25.3)\). Conversely, by \((25.13)\) and \((25.24)\), every fractional ideal of this form is invertible if and only if \( R \) is Dedekind. In fact, then \( \mathcal{I}(R) \) is the free abelian group on the prime ideals. Further, then \( \text{Pic}(R) = 0 \) if and only if \( R \) is UFD, or equivalently by \((25.13)\), a PID. See \[\text{Ch. 11, Sects. 10–11, pp. 424–437}\] for a discussion of the case in which \( R \) is a ring of quadratic integers, including many examples where \( \text{Pic}(R) \neq 0 \).
26. Arbitrary Valuation Rings

A valuation ring is a subring of a field such that the reciprocal of any element outside the subring lies in it. Valuation rings are normal local domains. They are maximal under domination of local rings: that is, one contains the other, and the inclusion map is a local homomorphism. Given any domain, its normalization is equal to the intersection of all the valuation rings containing it. Given a 1-dimensional Noetherian domain and a finite extension of its fraction field with a proper subring containing the domain, that subring too is 1-dimensional and Noetherian, this is the Krull–Akizuki Theorem. So normalizing a Dedekind domain a proper subring containing the domain, that subring too is 1-dimensional and Noetherian.

**Definition (26.1).** — A subring $V$ of a field $K$ is said to be a valuation ring of $K$ if, whenever $z \in K - V$, then $1/z \in V$.

**Proposition (26.2).** — Let $V$ be a valuation ring of a field $K$, and set 
\[ m := \{1/z \mid z \in K - V\} \cup \{0\}. \]
Then $V$ is local, $m$ is its maximal ideal, and $K$ is its fraction field.

**Proof:** Clearly $m = V - V^\times$. Let’s show $m$ is an ideal. Take a nonzero $a \in V$ and nonzero $x, y \in m$. Suppose $ax \notin m$. Then $ax \in V^\times$. So $a(1/ax) \in V$. So $1/x \in V$. So $x \in V^\times$, a contradiction. Thus $ax \in m$. Now, by hypothesis, either $x/y \in V$ or $y/x \in V$. Say $y/x \in V$. Then $1 + (y/x) \in V$. So $x = (1 + (y/x))x \in m$. Thus $m$ is an ideal. Hence $V$ is local and $m$ is its maximal ideal by (5.4.2). Finally, $K$ is its fraction field, because whenever $z \in K - V$, then $1/z \in V$. \[ \square \]

**Exercise (26.3).** — Let $V$ be a domain. Show that $V$ is a valuation ring if and only if, given any two ideals $a$ and $b$, either $a$ lies in $b$ or $b$ lies in $a$.

**Exercise (26.4).** — Let $V$ be a valuation ring of $K$, and $V \subset W \subset K$ a subring. Prove that $W$ is also a valuation ring of $K$, that its maximal ideal $p$ lies in $V$, that $V/p$ is a valuation ring of the field $W/p$, and that $W = V_p$.

**Exercise (26.5).** — Prove that a valuation ring $V$ is normal.

**Lemma (26.6).** — Let $R$ be a domain, $a$ an ideal, $K := \text{Frac}(R)$, and $x \in K^\times$. Then either $1 \notin aR[x]$ or $1 \notin aR[1/x]$.

**Proof:** Assume $1 \in aR[x]$ and $1 \in aR[1/x]$. Then there are equations 
\[ 1 = a_0 + \cdots + a_nx^n \quad \text{and} \quad 1 = b_0 + \cdots + b_mx^m \quad \text{with all} \quad a_i, b_j \in a. \]
Assume $n, m$ minimal and $m \leq n$. Multiply through by $1 - b_0$ and $a_nx^n$, getting 
\[ 1 - b_0 = (1 - b_0)a_0 + \cdots + (1 - b_0)a_nx^n \quad \text{and} \]
\[ (1 - b_0)a_nx^n = a_nb_1x^{n-1} + \cdots + a_nb_mx^{n-m}. \]
Combine the latter equations, getting 
\[ 1 - b_0 = (1 - b_0)a_0 + \cdots + (1 - b_0)a_{n-1}x^{n-1} + a_nb_1x^{n-1} + \cdots + a_nb_mx^{n-m}. \]
Simplify, getting an equation of the form $1 = c_0 + \cdots + c_{n-1}x^{n-1}$ with $c_i \in a$, which contradicts the minimality of $n$. \[ \square \]
Let \( A, B \) be local rings, and \( m, n \) their maximal ideals. We say \( B \) dominates \( A \) if \( B \supset A \) and \( n \cap A = m \); in other words, the inclusion map \( \varphi : A \to B \) is a local homomorphism.

**Proposition (26.8).** — Let \( K \) be a field, \( A \) any local subring. Then \( A \) is dominated by a valuation ring \( V \) of \( K \) with algebraic residue field extension.

**Proof:** Let \( m \) be the maximal ideal of \( A \). Let \( S \) be the set of pairs \((R, n)\) where \( R \subset K \) is a subring containing \( A \) and where \( n \subset R \) is a maximal ideal with \( n \cap A = m \) and with \( R/n \) an algebraic extension of \( A/m \). Then \((A, m) \in S \). Order \( S \) as follows: \((R, n) \leq (R', n')\) if \( R \subset R' \) and \( n = n' \cap R \). Let \((R_\lambda, n_\lambda)\) form a totally ordered subset. Set \( B := \bigcup R_\lambda \) and \( M = \bigcap n_\lambda \). Plainly \( M \cap R_\lambda = n_\lambda \) and \( B/M = \bigcap R_\lambda/n_\lambda \) for all \( \lambda \). So any \( y \in B/M \) is in \( R_\lambda/n_\lambda \) for some \( \lambda \). Hence \( B/M \) is a field and is algebraic over \( A/m \). Thus by Zorn’s Lemma, \( B/M \) has a maximal element, say \((V, \mathfrak{m})\).

For any nonzero \( x \in K \), set \( V' := V[x] \) and \( V'' := V[1/x] \). By (26.8), either \( 1 \notin \frak{m} V' \) or \( 1 \notin \frak{m} V'' \). Say \( 1 \notin \frak{m} V' \). Then \( \frak{m} V'' \) is proper, so it is contained in a maximal ideal \( \mathfrak{m}' \) of \( V' \). Since \( \frak{m}' \cap V = \frak{m} \) and \( V \cap \mathfrak{m}' \) is proper, \( \frak{m}' \cap V = \frak{m} \).

Further \( V'/\frak{m}' \) is generated as a ring over \( V/\frak{m} \) by the residue \( x' \) of \( x \). Hence \( x' \) is algebraic over \( V/\frak{m} \); otherwise, \( V'/\frak{m}' \) would be a polynomial ring, so not a field.

Hence \((V', \frak{m}') \in S \), and \((V', \frak{m}') \supset (V, \frak{m}) \). By maximality, \( V = V' \); so \( x \in V \).

Thus \( V \) is a valuation ring of \( K \). So \( V \) is local, and \( \frak{m} \) is its unique maximal ideal.

Finally, \((V, \frak{m}) \in S \); so \( V \) dominates \( A \) with algebraic residue field extension. \( \square \)

**Exercise (26.9).** — Let \( K \) be a field, \( S \) the set of local subrings ordered by domination. Show that the valuation rings of \( K \) are the maximal elements of \( S \).

**Theorem (26.10).** — Let \( R \) be any subring of a field \( K \). Then the integral closure \( \overline{R} \) of \( R \) in \( K \) is the intersection of all valuation rings \( V \) of \( K \) containing \( R \). Further, if \( R \) is local, then the \( V \) dominating \( R \) with algebraic residue field extension suffice.

**Proof:** Every valuation ring \( V \) is normal by (26.8). So if \( V \supset R \), then \( V \supset \overline{R} \).

Thus \((\bigcap_{V \supset R} V) \supset \overline{R} \).

To prove the opposite inclusion, take any \( x \in K - \overline{R} \). To find a valuation ring \( V \) with \( V \supset R \) and \( x \notin V \), set \( y := 1/x \). If \( 1/y \in R[y] \), then for some \( n \),

\[
\frac{1}{y} = a_0 y^n + a_1 y^{n-1} + \cdots + a_n \quad \text{with} \quad a_\lambda \in R.
\]

Multiplying by \( x^n \) yields \( x^{n+1} - a_n x^n - \cdots - a_0 = 0 \). So \( x \in \overline{R} \), a contradiction.

Thus \( 1 \notin y R[y] \). So there is a maximal ideal \( m \) of \( R[y] \) containing \( y \). Then the composition \( R \to R[y] \to R[y]/m \) is surjective as \( y \in m \). Its kernel is \( m \cap R \), so \( m \cap R \) is a maximal ideal of \( R \). By (26.8), there is a valuation ring \( V \) that dominates \( R[y]/m \) with algebraic residue field extension; whence, if \( R \) is local, then \( V \) also dominates \( R \), and the residue field of \( R[y]/m \) is equal to that of \( R \). But \( y \in m \); so \( x = 1/y \notin V \), as desired. \( \square \)

**Valuations.** — We call an additive abelian group \( \Gamma \) totally ordered if \( \Gamma \) has a subset \( \Gamma_+ \) that is closed under addition and satisfies \( \Gamma_+ \cup \{0\} \cup -\Gamma_+ = \Gamma \).

Given \( x, y \in \Gamma \), write \( x > y \) if \( x - y \in \Gamma_+ \). Note that either \( x > y \) or \( x = y \) or \( y > x \).

Consider, if \( x > y \), then \( x + z > y + z \) for any \( z \in \Gamma \).

Let \( V \) be a domain, and set \( K := \text{Frac}(V) \) and \( \Gamma := K^\times/V^\times \). Write the group \( \Gamma \) additively, and let \( v : K^\times \to \Gamma \) be the quotient map. It is a homomorphism:

\[
v(xy) = v(x) + v(y).
\]
Set $\Gamma_+ := v(V - 0) - 0$. Then $\Gamma_+$ is closed under addition. Clearly, $V$ is a valuation ring if and only if $-\Gamma_+ \cup \{0\} \sqcup \Gamma_+ = \Gamma$, so if and only if $\Gamma$ is totally ordered.

Assume $V$ is a valuation ring. Let’s prove that, for all $x, y \in K^x$,

$$v(x + y) \geq \min\{v(x), v(y)\} \quad \text{if} \quad x \neq -y. \quad (26.11.2)$$

Indeed, say $v(x) \geq v(y)$. Then $z := x/y \in V$. So $v(z + 1) \geq 0$. Hence

$$v(x + y) = v(z + 1) + v(y) \geq v(y) = \min\{v(x), v(y)\},$$

Note that (26.11.1) and (26.11.2) are the same as (1) and (2) of (26.11.1).

Conversely, start with a field $K$, with a totally ordered additive abelian group $\Gamma$, and with a surjective homomorphism $v : K^x \to \Gamma$ satisfying (26.11.2). Set

$$V := \{x \in K^x \mid v(x) \geq 0\} \cup \{0\}.$$

Then $V$ is a valuation ring, and $\Gamma = K^x/V^x$. We call such a $v$ a valuation of $K$, and $\Gamma$ the value group of $v$ or of $V$.

For example, a DVR $V$ of $K$ is just a valuation ring with value group $\mathbb{Z}$, since any $x \in K^x$ has the form $x = ut^n$ with $u \in V^x$ and $n \in \mathbb{Z}$.

**Example (26.12).** — Fix totally ordered additive abelian group $\Gamma$, and a field $K$. Form the $k$-vector space $R$ with basis the symbols $X^a$ for $a \in \Gamma$. Define $X^a X^b := X^{a+b}$, and extend this product to $R$ by linearity. Then $R$ is a $k$-algebra with $X^0 = 1$. We call $R$ the group algebra of $\Gamma$. Define $v : (R - 0) \to \Gamma$ by

$$v(\sum r_a X^a) := \min\{a \mid r_a \neq 0\}.$$

Then for $x, y \in (R - 0)$, clearly $v(xy) = v(x) + v(y)$ because $k$ is a domain and $\Gamma$ is ordered. Hence $R$ is a domain. Moreover, if $v(x + y) = a$, then either $v(x) \leq a$ or $v(y) \leq a$. Thus $v(x + y) \geq \min\{v(x), v(y)\}$.

Set $K := \text{Frac}(R)$, and extend $v$ to a map $v: K^x \to \Gamma$ by $v(x/y) := v(x) - v(y)$ if $y \neq 0$. Clearly $v$ is well defined, surjective, and a homomorphism. Further, for $x, y \in K^x$, clearly $v(x + y) \geq \min\{v(x), v(y)\}$. Thus $v$ is a valuation with group $\Gamma$.

Set $R' := \{x \in R \mid v(x) \geq 0\}$ and $p := \{x \in R \mid v(x) > 0\}$. Clearly, $R'$ is a ring, and $p$ is a prime of $R'$. Further, $p^\circ$ is the valuation ring of $v$.

There are many choices for $\Gamma$ other than $\mathbb{Z}$. Examples include the additive rationals, the additive reals, its subgroup generated by two incommensurate reals, and the lexicographically ordered product of any two totally ordered abelian groups.

**Proposition (26.13).** — Let $v$ be a valuation of a field $K$, and $x_1, \ldots, x_n \in K^x$ with $n \geq 2$. Set $m := \min\{v(x_i)\}$.

1. If $n = 2$ and if $v(x_1) \neq v(x_2)$, then $v(x_1 + x_2) = m$.
2. If $x_1 + \cdots + x_n = 0$, then $m = v(x_i) = v(x_j)$ for some $i \neq j$.

**Proof:** For (1), say $v(x_1) > v(x_2)$; so $v(x_2) = m$. Set $z := x_1/x_2$. Then

$$v(z) > 0.$$ 

Also

$$v(-z) = v(z) + v(-1) > 0.$$ 

Now,

$$0 = v(1) = v(z + 1 - z) \geq \min\{v(z + 1), v(-z)\} \geq 0.$$ 

Hence $v(z + 1) = 0$. Now, $x_1 + x_2 = (z + 1)x_2$. Therefore, $v(x_1 + x_2) = v(x_2) = m$.

Thus (1) holds.

For (2), reorder the $x_i$ so $v(x_i) = m$ for $i \leq k$ and $v(x_i) > m$ for $i > k$.

By induction, (26.11.2) yields $v(x_{k+1} + \cdots + x_n) \geq \min_{>k}\{v(x_i)\}$. Therefore,$v(x_{k+1} + \cdots + x_n) > m$. If $k = 1$, then (1) yields $v(0) = v(x_1 + (x_2 + \cdots + x_n)) = m$, a contradiction. So $k > 1$, and $v(x_1) = v(x_2) = m$, as desired. \(\square\)
EXERCISE (26.16.1). — Let $V$ be a valuation ring, such as a DVR, whose value group $\Gamma$ is Archimedean; that is, given any nonzero $\alpha, \beta \in \Gamma$, there’s $n \in \mathbb{Z}$ such that $n\alpha > \beta$. Show that $V$ is a maximal proper subring of its fraction field $K$.

EXERCISE (26.16.15). — Let $V$ be a valuation ring. Show that
(1) every finitely generated ideal $a$ is principal, and
(2) $V$ is Noetherian if and only if $V$ is a DVR.

LEMMA (26.16). — Let $R$ be a 1-dimensional Noetherian domain, $K$ its fraction field, $M$ a torsionfree module, and $x \in R$ nonzero. Then $\ell(R/xR) < \infty$. Further,
\[ \ell(M/xM) \leq \dim_K (M \otimes_R K) \ell(R/xR), \tag{26.16.1} \]
with equality if $M$ is finitely generated.

**Proof:** Set $r := \dim_K(M \otimes_R K)$. If $r = \infty$, then (26.16.1) is trivial; so we may assume $r < \infty$.


First, assume $M$ is finitely generated. Choose any $K$-basis $m_1/s_1, \ldots, m_r/s_r$ of $M_K$ with $m_i \in M$ and $s_i \in S$. Then $m_1/1, \ldots, m_r/1$ is also a basis. Define an $R$-map $\alpha: R^r \to M$ by sending the standard basis elements to the $m_i$. Then its localization $\alpha_K$ is an $R$-isomorphism. But $\ker(\alpha)$ is a submodule of $R^r$, so torsionfree. Further, $S^{-1}\ker(\alpha) = \ker(\alpha_K) = 0$. Hence $\ker(\alpha) = 0$. Thus $\alpha$ is injective.

Set $N := \ker(\alpha)$. Then $N_K = 0$, and $N$ is finitely generated. Hence, $\text{Supp}(N)$ is a proper closed subset of $\text{Spec}(R)$. But $\dim(R) = 1$ by hypothesis. Hence, $\text{Supp}(N)$ consists entirely of maximal ideals. So $\ell(N) < \infty$ by (15.21).

Similarly, $\text{Supp}(R/xR)$ is closed and proper in $\text{Spec}(R)$. So $\ell(R/xR) < \infty$.

Consider the standard exact sequence:
\[ 0 \to N' \to N \to N/xN \to 0 \quad \text{where} \quad N' := \ker(\mu_x). \]

Apply Additivity of Length, (19.43); it yields $\ell(N') = \ell(N/xN)$.

Since $M$ is torsionfree, $\mu_x: M \to M$ is injective. Consider this commutative diagram with exact rows:
\[ \begin{array}{c}
0 \to R^r \xrightarrow{\alpha_x} M \to N \to 0 \\
\mu_x \downarrow \quad \mu_x \downarrow \\
0 \to R^r \xrightarrow{\alpha_x} M \to N \to 0
\end{array} \]

Apply the snake lemma (15.14). It yields this exact sequence:
\[ 0 \to N' \to (R/xR)^r \to M/xM \to N/xN \to 0. \]

Hence $\ell(M/xM) = \ell((R/xR)^r)$ by additivity. But $\ell((R/xR)^r) = r \ell(R/xR)$ also by additivity. Thus equality holds in (26.16.1) when $M$ is finitely generated.

Second, assume $M$ is arbitrary, but (26.16.1) fails. Then $M$ possesses a finitely generated submodule $M'$ whose image $H$ in $M/xM$ satisfies $\ell(H) > r\ell(R/xR)$. Now, $M_K \supset M'_K$; so $r \geq \dim_K(M'_K)$. Therefore,
\[ \ell(M'/xM') \geq \ell(H) > r\ell(R/xR) \geq \dim_K(M'_K) \ell(R/xR). \]

However, together these inequalities contradict the first case with $M'$ for $M$. \qed
Given a nonzero ideal $\alpha'$ of $R'$, take any nonzero $x \in \alpha'$. Since $K'/K$ is finite, there is an equation $a_nx^n + \cdots + a_0 = 0$ with $a_i \in R$ and $a_0 \neq 0$. Then $a_0 \in \alpha' \cap R$. Further, $(26.16)$ yields $\ell(R/a_0R) < \infty$.

Clearly, $R'$ is a domain, so a torsionfree $R$-module. Further, $R' \otimes_R K \subset K'$; hence, $\dim_K(R' \otimes_R K) < \infty$. Therefore, $(26.10)$ yields $\ell_R(R'/a_0R') < \infty$.

But $\alpha'/a_0R' \subset R'/a_0R'$. So $\ell_R(\alpha'/a_0R') < \infty$. So $\alpha'/a_0R'$ is finitely generated over $R$ by $(19.21)(3)$. Hence $\alpha'$ is finitely generated over $R'$. Thus $R'$ is Noetherian.

Set $R'' := R'/a_0R'$. Clearly, $\ell_{R''}R'' \leq \ell_{R'}R''$. So $\ell_{R'}R'' < \infty$. So, in $R''$, every prime is maximal by $(19.4)$. So if $\alpha'$ is prime, then $\alpha'/a_0R'$ is maximal, whence $\alpha'$ maximal. So in $R$, every nonzero prime is maximal. Thus $R'$ is 1-dimensional. $\square$

**Corollary (26.18).** — Let $R$ be a 1-dimensional Noetherian domain, such as a Dedekind domain. Let $K$ be its fraction field, $K'$ a finite extension field, and $R'$ the normalization of $R$ in $K'$. Then $R'$ is Dedekind.

**Proof:** Since $R$ is 1-dimensional, it’s not a field. But $R'$ is the normalization of $R$. So $R'$ is not a field by $(19.4)$. Hence, $R'$ is Noetherian and 1-dimensional by $(26.17)$. Thus $R'$ is Dedekind by $(24.10)$.

**Corollary (26.19).** — Let $K'/K$ be a field extension, $V'$ a valuation ring of $K'$ not containing $K$. Set $V := V' \cap K$. Then $V$ is a DVR if $V'$ is, and the converse holds if $K'/K$ is finite.

**Proof:** It follows easily from $(26.1)$ that $V$ is a valuation ring, and from $(26.10)$ that its value group is a subgroup of that of $V'$. Now, a nonzero subgroup of $\mathbb{Z}$ is a copy of $\mathbb{Z}$. Thus $V$ is a DVR if $V'$ is.

Conversely, assume $V$ is a DVR, so Noetherian and 1-dimensional. Now, $V' \supseteq K$, so $V' \subseteq K'$. Hence, $V'$ is Noetherian by $(26.17)$, so a DVR by $(26.15)(2)$. $\square$

**Exercise (26.21).** — Let $R$ be a Noetherian domain, $K := \text{Frac}(R)$, and $L$ a finite extension field (possibly $L = K$). Prove the integral closure $\overline{R}$ of $R$ in $L$ is the intersection of all DVRs $V'$ of $L$ containing $R$ by modifying the proof of $(26.10)$: show $y$ is contained in a height-1 prime $p$ of $R[y]$ and apply $(26.18)$ to $R[y]_p$. 

**Theorem (26.17) (Krull–Akizuki).** — Let $R$ be a 1-dimensional Noetherian domain, $K$ its fraction field, $K'$ a finite extension field, and $R'$ a proper subring of $K'$ containing $R$. Then $R'$ is, like $R$, a 1-dimensional Noetherian domain.

**Proof:** Given a nonzero ideal $\alpha'$ of $R'$, take any nonzero $x \in \alpha'$. Since $K'/K$ is finite, there is an equation $a_nx^n + \cdots + a_0 = 0$ with $a_i \in R$ and $a_0 \neq 0$. Then $a_0 \in \alpha' \cap R$. Further, $(26.16)$ yields $\ell(R/a_0R) < \infty$.

Clearly, $R'$ is a domain, so a torsionfree $R$-module. Further, $R' \otimes_R K \subset K'$; hence, $\dim_K(R' \otimes_R K) < \infty$. Therefore, $(26.10)$ yields $\ell_R(R'/a_0R') < \infty$.

But $\alpha'/a_0R' \subset R'/a_0R'$. So $\ell_R(\alpha'/a_0R') < \infty$. So $\alpha'/a_0R'$ is finitely generated over $R$ by $(19.21)(3)$. Hence $\alpha'$ is finitely generated over $R'$. Thus $R'$ is Noetherian.

Set $R'' := R'/a_0R'$. Clearly, $\ell_{R''}R'' \leq \ell_{R'}R''$. So $\ell_{R'}R'' < \infty$. So, in $R''$, every prime is maximal by $(19.4)$. So if $\alpha'$ is prime, then $\alpha'/a_0R'$ is maximal, whence $\alpha'$ maximal. So in $R$, every nonzero prime is maximal. Thus $R'$ is 1-dimensional. $\square$
 Solutions

1. Rings and Ideals

Exercise (1.5). — Let \( \varphi : R \to R' \) be a map of rings, \( a \) an ideal of \( R \), and \( b \) an ideal of \( R' \). Set \( a^e := \varphi(a)R' \) and \( b^e := \varphi^{-1}(b) \). Prove these statements:

1. Then \( a^e \subseteq a \) and \( b^e \subseteq b \).
2. Then \( a^e + b^e = a^e \).
3. If \( b \) is an extension, then \( b^e \) is the largest ideal of \( R \) with extension \( b \).
4. If two extensions have the same contraction, then they are equal.

Solution: For (1), given \( x \in a \), note \( \varphi(x) = x \cdot 1 \in aR' \). So \( x \in \varphi^{-1}(aR') \), or \( x \in a^e \). Thus \( a \subseteq a^e \). Next, \( \varphi(\varphi^{-1}(b)) \subseteq b \). But \( b \) is an ideal of \( R' \). So \( \varphi(\varphi^{-1}(b)) \subseteq b \) or \( a^e \subseteq b \). Thus (1) holds.

For (2), note \( a^e \subseteq a^e \) by (1) applied with \( b := a^e \). Thus \( a \subseteq a^e \). Similarly, \( b^e \subseteq b^e \) by (1) applied with \( a := b^e \). Thus (2) holds.

For (3), say \( b = a^e \). Then \( b^e = a^e \). But \( a^e = a^e \) by (2). Hence \( b^e \) has extension \( b \). Further, it’s the largest such ideal, as \( a^e \subseteq a \) by (1). Thus (3) holds.

For (4), say \( b^e_i = b_i^e \) for extensions \( b_i \). Then \( b_i^e = b_i \) by (3). Thus (4) holds.

Exercise (1.7). — Let \( R \) be a ring, \( a \) an ideal, and \( P := R[X_1, \ldots, X_n] \) the polynomial ring. Prove \( P/aP = (R/a)[X_1, \ldots, X_n] \).

Solution: The two \( R \)-algebras are equal, as they have the same UMP: each is universal among \( R \)-algebras \( R' \) with distinguished elements \( x_1, \ldots, x_n \) and with \( aR' = 0 \). Namely, the structure map \( \varphi : R \to R' \) factors through a unique map \( \pi : P \to R' \) such that \( \pi(x_i) = x_i \) for all \( i \) by (1.3); then \( \pi \) factors through a unique map \( P/aP \to R' \) as \( aR' = 0 \) by (1.4). On the other hand, \( \varphi \) factors through a unique map \( \psi : R/a \to R' \) as \( aR' = 0 \) by (1.4); then \( \psi \) factors through a unique map \( (R/a)[X_1, \ldots, X_n] \to R' \) such that \( \pi(x_i) = x_i \) for all \( i \) by (1.3).

Exercise (1.10). — Let \( R \) be ring, and \( P := R[X_1, \ldots, X_n] \) the polynomial ring. Let \( m \leq n \) and \( a_1, \ldots, a_m \in R \). Set \( p := \langle X_1 - a_1, \ldots, X_m - a_m \rangle \). Prove that \( P/p = R[X_{m+1}, \ldots, X_n] \).

Solution: First, assume \( m = n \). Set \( P' := R[X_1, \ldots, X_{n-1}] \) and \( p' := \langle X_1 - a_1, \ldots, X_{n-1} - a_{n-1} \rangle \subseteq P' \). By induction on \( n \), we may assume \( P'/p' = R \). However, \( P = P'[X_n] \). Hence \( P/pP' = (P'/p')[X_n] \) by (1.2). Thus \( P/pP = R[X_n] \).

We have \( P/p = (P/pP)/p(P/pP) \) by (1.3). But \( p = p'P + \langle X_n - a_n \rangle P \). Hence \( p(P/pP) = \langle X_n - a_n \rangle (P/pP) \). So \( P/p = R[X_n]/(X_n - a_n) \). So \( P/p = R \) by (1.3).

In general, \( P = R[X_{m+1}, \ldots, X_n] \). Thus \( P/p = R[X_{m+1}, \ldots, X_n] \) by (1.3).

Exercise (1.14) (Chinese Remainder Theorem). — Let \( R \) be a ring.

1. Let \( a \) and \( b \) be comaximal ideals; that is, \( a + b = R \). Prove

   \( a \cap b = a \cap b \) and \( R/ab = (R/a) \times (R/b) \).
(2) Let \(a\) be comaximal to both \(b\) and \(b'\). Prove \(a\) is also comaximal to \(bb'\).

(3) Let \(a, b\) be comaximal, and \(m, n \geq 1\). Prove \(a^m\) and \(b^n\) are comaximal.

(4) Let \(a_1, \ldots, a_n\) be pairwise comaximal. Prove

(a) \(a_1\) and \(a_2 \cdots a_n\) are comaximal;
(b) \(a_1 \cap \cdots \cap a_n = a_1 \cdots a_n\);
(c) \(R/(a_1 \cdots a_n) \rightarrow \prod(R/a_i)\).

SOLUTION: To prove (1)(a), note that always \(ab \subseteq a \cap b\). Conversely, \(a + b = R\) implies \(x + y = 1\) with \(x \in a\) and \(y \in b\). So given \(z \in a \cap b\), we have \(z = xz + yz \in ab\).

To prove (1)(b), form the map \(R \rightarrow R/a \times R/b\) that carries an element to its pair of residues. The kernel is \(a \cap b\), which is \(ab\) by (1). So we have an injection

\[\varphi: R/ab \rightarrow R/a \times R/b.\]

To show that \(\varphi\) is surjective, take any element \((\bar{x}, \bar{y})\) in \(R/a \times R/b\). Say \(\bar{x}\) and \(\bar{y}\) are the residues of \(x\) and \(y\). Since \(a + b = R\), we can find \(a \in a\) and \(b \in b\) such that \(a + b = y - x\). Then \(\varphi(x + a) = (\bar{x}, \bar{y})\), as desired. Thus (1) holds.

To prove (2), note that

\[R = (a + b)(a + b') = (a^2 + ba + ab') + bb' \subseteq a + bb' \subseteq R.\]

To prove (3), note that (2) implies \(a\) and \(b^n\) are comaximal for any \(n \geq 1\) by induction on \(n\). Hence, \(b^n\) and \(a^n\) are comaximal for any \(m \geq 1\).

To prove (4)(a), assume \(a_1\) and \(a_2 \cdots a_{n-1}\) are comaximal by induction on \(n\). By hypothesis, \(a_1\) and \(a_n\) are comaximal. Thus (2) yields (a).

To prove (4)(b) and (4)(c), again proceed by induction on \(n\). Thus (1) yields

\[a_1 \cap (a_2 \cap \cdots \cap a_n) = a_1 \cap (a_2 \cdots a_n) = a_1 a_2 \cdots a_n;\]

\[R/(a_1 \cdots a_n) \rightarrow R/a_1 \times R/(a_2 \cdots a_n) \rightarrow \prod(R/a_i).\n\]

Exercise (1.15). — First, given a prime number \(p\) and a \(k \geq 1\), find the idempotents in \(\mathbb{Z}/(p^k)\). Second, find the idempotents in \(\mathbb{Z}/(12)\). Third, find the number of idempotents in \(\mathbb{Z}/(n)\) where \(n = \prod_{i=1}^{N} p_i^{n_i}\) with \(p_i\) distinct prime numbers.

SOLUTION: First, let \(m \in \mathbb{Z}\) be idempotent modulo \(p^k\). Then \(m(m-1)\) is divisible by \(p^k\). So either \(m\) or \(m - 1\) is divisible by \(p^k\), as \(m\) and \(m - 1\) have no common prime divisor. Hence \(0\) and \(1\) are the only idempotents in \(\mathbb{Z}/(p^k)\).

Second, since \(-3 + 4 = 1\), the Chinese Remainder Theorem \((\text{IRT})\) yields

\[\mathbb{Z}/(12) = \mathbb{Z}/(3) \times \mathbb{Z}/(4).\]

Hence \(m\) is idempotent modulo 12 if and only if \(m\) is idempotent modulo 3 and modulo 4. By the previous case, we have the following possibilities:

\[m \equiv 0 \pmod{3}\text{ and } m \equiv 0 \pmod{4};\]
\[m \equiv 1 \pmod{3}\text{ and } m \equiv 1 \pmod{4};\]
\[m \equiv 1 \pmod{3}\text{ and } m \equiv 0 \pmod{4};\]
\[m \equiv 0 \pmod{3}\text{ and } m \equiv 1 \pmod{4}.\]

Therefore, \(m \equiv 0, 1, 4, 9 \pmod{12}\).

Third, for each \(i\), the two numbers \(p_1^{n_1} \cdots p_{i-1}^{n_{i-1}}\) and \(p_i^{n_i}\) have no common prime divisor. Hence some linear combination is equal to 1 by the Euclidean Algorithm. So the principal ideals they generate are comaximal. Hence by induction on \(N\), the
Chinese Remainder Theorem yields

\[ \mathbb{Z}/\langle n \rangle = \prod_{i=1}^{N} \mathbb{Z}/\langle p_i^{n_i} \rangle. \]

So \( m \) is idempotent modulo \( n \) if and only if \( m \) is idempotent modulo \( p_i^{n_i} \) for all \( i \); hence, if and only if \( m \) is 0 or 1 modulo \( p_i^{n_i} \) for all \( i \) by the first case. Thus there are \( 2^N \) idempotents in \( \mathbb{Z}/\langle n \rangle \).

**Exercise (1.16).** — Let \( R := R' \times R'' \) be a product of rings, \( a \subset R \) an ideal. Show \( a = a' \times a'' \) with \( a' \subset R' \) and \( a'' \subset R'' \) ideals. Show \( R/a = (R'/a') \times (R''/a'') \).

**Solution:** Set \( a' := \{ x' \mid (x', 0) \in a \} \) and \( a'' := \{ x'' \mid (0, x'') \in a \} \). Clearly \( a' \subset R' \) and \( a'' \subset R'' \) are ideals. Clearly,

\[ a \supset a' \times 0 + 0 \times a'' = a' \times a''. \]

The opposite inclusion holds, because if \( a \ni (x', x'') \), then

\[ a \ni (x', x'') \cdot (1, 0) = (x', 0) \quad \text{and} \quad a \ni (x', x'') \cdot (0, 1) = (0, x''). \]

Finally, the equation \( R/a = (R/a') \times (R/a'') \) is now clear from the construction of the residue class ring. ∎

**Exercise (1.17).** — Let \( R \) be a ring, and \( e, e' \) idempotents. (See (III.7) also.)

1. Set \( a := \langle e \rangle \). Show \( a \) is idempotent; that is, \( a^2 = a \).
2. Let \( a \) be a principal idempotent ideal. Show \( a(f) \) with \( f \) idempotent.
3. Set \( e'' := e + e' - ee' \). Show \( \langle e, e' \rangle = \langle e'' \rangle \) and \( e'' \) is idempotent.
4. Let \( e_1, \ldots, e_r \) be idempotents. Show \( \langle e_1, \ldots, e_r \rangle = \langle f \rangle \) with \( f \) idempotent.
5. Assume \( R \) is Boolean. Show every finitely generated ideal is principal.

**Solution:** For (1), note \( a^2 = \langle e^2 \rangle \) since \( a = \langle e \rangle \). But \( e^2 = e \). Thus (1) holds.

For (2), say \( a = \langle g \rangle \). Then \( a^2 = \langle g^2 \rangle \). But \( a^2 = a \). So \( g = xy \) for some \( x \). Set \( f := xy \). Then \( f \in a \); so \( \langle f \rangle \subset a \). And \( g = fg \). So \( a \subset \langle f \rangle \). Thus (2) holds.

For (3), note \( \langle e'' \rangle \subset \langle e, e' \rangle \). Conversely, \( ee'' = e + e' - e^2 = e + ee' - e' = e \).

By symmetry, \( e'e'' = e' \). So \( \langle e, e' \rangle \subset \langle e'' \rangle \) and \( e''^2 = ee'' + e'e'' - ee'e'' = e'' \). Thus (4) holds.

For (4), induct on \( r \). Thus (3) yields (4).

For (5), recall that every element of \( R \) is idempotent. Thus (4) yields (5). ∎

### 2. Prime Ideals

**Exercise (2.2).** — Let \( a \) and \( b \) be ideals, and \( p \) a prime ideal. Prove that these conditions are equivalent: (1) \( a \subset p \) or \( b \subset p \); and (2) \( a \cap b \subset p \); and (3) \( ab \subset p \).

**Solution:** Trivially, (1) implies (2). If (2) holds, then (3) follows as \( ab \subset a \cap b \).

Finally, assume \( a \not\subset p \) and \( b \not\subset p \). Then there are \( x \in a \) and \( y \in b \) with \( x, y \not\in p \).

Hence, since \( p \) is prime, \( xy \not\in p \). However, \( xy \in ab \). Thus (3) implies (1). ∎

**Exercise (2.4).** — Given a prime number \( p \) and an integer \( n \geq 2 \), prove that the residue ring \( \mathbb{Z}/\langle p^n \rangle \) does not contain a domain as a subring.

**Solution:** Any subring of \( \mathbb{Z}/\langle p^n \rangle \) must contain 1, and 1 generates \( \mathbb{Z}/\langle p^n \rangle \) as an abelian group. So \( \mathbb{Z}/\langle p^n \rangle \) contains no proper subrings. However, \( \mathbb{Z}/\langle p^n \rangle \) is not a domain, because in it, \( p \cdot p^{n-1} = 0 \) but neither \( p \) nor \( p^{n-1} \) is 0.
Exercise (2.5). — Let $R := R' \times R''$ be a product of two rings. Show that $R$ is a domain if and only if either $R'$ or $R''$ is a domain and the other is 0.

Solution: Assume $R$ is a domain. As $(1, 0) \cdot (0, 1) = (0, 0)$, either $(1, 0) = (0, 0)$ or $(0, 1) = (0, 0)$. Correspondingly, either $R' = 0$ and $R = R''$, or $R'' = 0$ and $R = R''$. The assertion is now obvious.

Exercise (2.18). — Let $R$ be a ring, $p$ a prime ideal, $R[X]$ the polynomial ring. Show that $pR[X]$ and $pR[X] + (X)$ are prime ideals of $R[X]$, and that if $p$ is maximal, then so is $pR[X] + (X)$.

Solution: Note $R[X]/pR[X] = (R/p)[X]$ by $(2.21)$. But $R/p$ is a domain by $(4.14)$. So $R[X]/pR[X]$ is a domain by $(4.3)$. Thus $pR[X]$ is prime by $(4.3)$.

Note $(pR[X] + (X))/pR[X]$ is equal to $\langle X \rangle \subset (R/p)[X]$. But $(R/p)[X]/\langle X \rangle$ is equal to $R/p$ by $(4.8)$. So $R[X]/(pR[X] + (X))$ is equal to $R/p$ by $(4.14)$. But $R/p$ is a domain by $(4.14)$. Thus $pR[X] + (X)$ is prime again by $(4.4)$.

Assume $p$ is maximal. Then $R/p$ is a field by $(4.14)$. But, as just noted, $R/p$ is equal to $R[X]/(pR[X] + (X))$. Thus $pR[X] + (X)$ is maximal again by $(4.17)$.

Exercise (2.11). — Let $R := R' \times R''$ be a product of rings, $p \subset R$ an ideal. Show $p$ is prime if and only if either $p = p' \times R''$ with $p'$ a prime of $R'$ prime or $p = R' \times p''$ with $p'' \subset R''$ prime.

Solution: Simply combine $(1.16)$, $(2.4)$, and $(4.3)$.

Exercise (2.16). — Let $k$ be a field, $R$ a nonzero ring, $\varphi: k \rightarrow R$ a ring map. Prove $\varphi$ is injective.

Solution: By $(1.71)$, $1 \neq 0$ in $R$. So $\ker(\varphi) \neq k$. So $\ker(\varphi) = 0$ by $(4.17)$. Thus $\varphi$ is injective.

Exercise (2.10). — Let $R$ be a domain, and $R[X_1, \ldots, X_n]$ the polynomial ring in $n$ variables. Let $m \leq n$, and set $p := \langle X_1, \ldots, X_m \rangle$. Prove $p$ is a prime ideal.

Solution: Simply combine $(2.4)$, $(4.3)$, and $(1.11)$.

Exercise (2.12). — Let $R$ be a domain, and $x, y \in R$. Assume $\langle x \rangle = \langle y \rangle$. Show $x = uy$ for some unit $u$.

Solution: By hypothesis, $x = uy$ and $y = vx$ for some $u, v \in R$. So $x = 0$ if and only if $y = 0$; if so, take $u := 1$. Assume $x \neq 0$. Now, $x = uvx$, or $x(1 - uv) = 0$. But $R$ is a domain. So $1 - uv = 0$. Thus $u$ is a unit.

Exercise (2.19). — Let $B$ be a Boolean ring. Show that every prime $p$ is maximal, and $B/p = \mathbb{F}_2$.

Solution: Given $x \in B/p$, plainly $x(x - 1) = 0$. But $B/p$ is a domain by $(4.16)$. So $z = 0, 1$. Thus $B/p = \mathbb{F}_2$. Plainly, $\mathbb{F}_2$ is a field. So $p$ is maximal by $(4.17)$.

Exercise (2.20). — Let $R$ be a ring. Assume that, given $x \in R$, there is $n \geq 2$ with $x^n = x$. Show that every prime $p$ is maximal.

Solution: Given $y \in R/p$, say $y(y^{n-1} - 1) = 0$ with $n \geq 2$. But $R/p$ is a domain by $(2.5)$. So $y = 0$ or $yy^{n-2} = 1$. So $R/p$ is a field. Thus $p$ is maximal by $(2.14)$.

Exercise (2.22). — Prove the following statements, or give a counterexample.
(1) The complement of a multiplicative subset is a prime ideal.
(2) Given two prime ideals, their intersection is prime.
(3) Given two prime ideals, their sum is prime.
(4) Given a ring map \( \varphi : R \to R' \), the operation \( \varphi^{-1} \) carries maximal ideals of \( R' \) to maximal ideals of \( R \).
(5) In \( (\mathbb{Z}, a) \), an ideal \( n' \subset R/a \) is maximal if and only if \( \kappa^{-1}n' \subset R \) is maximal.

**Solution:** (1) False. In the ring \( \mathbb{Z} \), consider the set \( S \) of powers of 2. The complement \( T \) of \( S \) contains 3 and 5, but not 8; so \( T \) is not an ideal.

(2) False. In the ring \( \mathbb{Z} \), consider the prime ideals \( (2) \) and \( (3) \); their intersection \( \langle 2 \rangle \cap \langle 3 \rangle \) is equal to \( \langle 6 \rangle \), which is not prime.

(3) False. Since \( 2 \cdot 3 - 5 = 1 \), we have \( \langle 3 \rangle + \langle 5 \rangle = \mathbb{Z} \).

(4) False. Let \( \varphi : \mathbb{Z} \to \mathbb{Q} \) be the inclusion map. Then \( \varphi^{-1}(0) = \langle 0 \rangle \).

(5) True. By \( (\mathbb{Z}, a) \), the operation \( b' \mapsto \kappa^{-1}b' \) sets up an inclusion-preserving bijective correspondence between the ideals \( b' \supset n' \) and the ideals \( b \supset \kappa^{-1}n' \).

**Exercise (2.23).** — Let \( k \) be a field, \( P := k[X_1, \ldots, X_n] \) the polynomial ring, \( f \in P \) nonzero. Let \( d \) be the highest power of any variable appearing in \( f \).

1. Let \( S \subset k \) have at least \( d + 1 \) elements. Proceeding by induction on \( n \), find \( a_1, \ldots, a_n \in S \) with \( f(a_1, \ldots, a_n) \neq 0 \).

2. Using the algebraic closure \( K \) of \( k \), find a maximal ideal \( \mathfrak{m} \) of \( P \) with \( f \notin \mathfrak{m} \).

**Solution:** Consider (1). Assume \( n = 1 \). Then \( f \) has at most \( d \) roots by \( (2.18) \), p. 392. So \( f(a_1) \neq 0 \) for some \( a_1 \in S \).

Assume \( n \geq 1 \). Say \( f = \sum g_iX_i^d \) with \( g_i \in k[X_2, \ldots, X_n] \). But \( f \neq 0 \). So \( g_i \neq 0 \) for some \( i \). By induction, \( g_i(a_2, \ldots, a_n) \neq 0 \) for some \( a_2, \ldots, a_n \in S \). By \( n = 1 \), find \( a_1 \in S \) such that \( f(a_1, \ldots, a_n) = \sum g_i(a_2, \ldots, a_n)a_i^d \neq 0 \). Thus (1) holds.

Consider (2). As \( K \) is infinite, (1) yields \( a_1, \ldots, a_n \in K \) with \( f(a_1, \ldots, a_n) \neq 0 \). Define \( \varphi : P \to K \) by \( \varphi(X_i) = a_i \). Then \( \text{Im}(\varphi) \subset K \) is the \( k \)-subalgebra generated by the \( a_i \). It is a field by \( (2.6) \), p. 495. Set \( \mathfrak{m} := \ker(\varphi) \). Then \( \mathfrak{m} \) is maximal by \( (\mathbb{Z}, a) \) and \( (2.2) \), and \( f_i \notin \mathfrak{m} \) as \( \varphi(f_i) = f(a_1, \ldots, a_n) \neq 0 \). Thus (2) holds.

**Exercise (2.26).** — Prove that, in a PID, elements \( x \) and \( y \) are relatively prime (share no prime factor) if and only if the ideals \( \langle x \rangle \) and \( \langle y \rangle \) are comaximal.

**Solution:** Say \( \langle x \rangle + \langle y \rangle = \langle d \rangle \). Then \( d = \text{gcd}(x, y) \), as is easy to check. The assertion is now obvious.

**Exercise (2.29).** — Preserve the setup of \( (2.28) \). Let \( f := a_0X^n + \cdots + a_n \) be a polynomial of positive degree \( n \). Assume that \( R \) has infinitely many prime elements \( p \), or simply that there is a \( p \) such that \( p \nmid a_0 \). Show that \( \langle f \rangle \) is not maximal.

**Solution:** Set \( a := (p, f) \). Then \( a \nmid \langle f \rangle \), because \( p \) is not a multiple of \( f \). Set \( k := \mathbb{Z}/(p) \). Since \( p \) is irreducible, \( k \) is a domain by \( (2.21) \) and \( (2.25) \). Let \( f' \in k[X] \) denote the image of \( f \). By hypothesis, \( \deg(f') = n \geq 1 \). Hence \( f' \) is not a unit by \( (2.22) \) since \( k \) is a domain. Therefore, \( \langle f' \rangle \) is proper. But \( P/\mathfrak{a} \overset{\sim}{\rightarrow} k[X]/\langle f' \rangle \) by \( (1.2) \) and \( (2.10) \). So \( a \) is proper. Thus \( \langle f \rangle \) is not maximal.

### 3. Radicals

**Exercise (3.3).** — Let \( R \) be a ring, \( a \subset \text{rad}(R) \) an ideal, \( w \in R \), and \( w' \in R/a \) its residue. Prove that \( w \in R^x \) if and only if \( w' \in (R/a)^x \). What if \( a \nsubseteq \text{rad}(R) \)?
SOLUTION: Plainly, \( w \in R^\times \) implies \( w' \in (R/a)^\times \), whether \( a \subset \text{rad}(R) \) or not.

Assume \( a \subset \text{rad}(R) \). As every maximal ideal of \( R \) contains \( \text{rad}(R) \), the operation \( m \mapsto m/a \) establishes a bijective correspondence between the maximal ideals of \( R \) and those of \( R/a \) owing to (1.35). So \( w \) belongs to a maximal ideal of \( R \) if and only if \( w' \) belongs to one of \( R/a \). Thus \( w \in R^\times \) if and only if \( w' \in (R/a)^\times \) by (2.33).

Assume \( a \not\subset \text{rad}(R) \). Then there is a maximal ideal \( m \subset R \) with \( a \not\subset m \). So \( a + m = R \). So there are \( a \in a \) and \( v \in m \) with \( a + v = w \). Thus \( v \not\in R^\times \), but the residue of \( v \) is \( w' \), even if \( w' \in (R/a)^\times \). For example, take \( R := \mathbb{Z} \) and \( a := \langle 2 \rangle \) and \( w := 3 \). Then \( w \not\in R^\times \), but the residue of \( w \) is \( 1 \in (R/a)^\times \).

Exercise (3.8). — Let \( A \) be a local ring. Find its idempotents \( e \).

Solution: Let \( m \) be the maximal ideal. Then \( 1 \not\in m \), so either \( e \not\in m \) or \( 1 - e \not\in m \). Say \( e \notin m \). Then \( e \) is a unit by (3.6). But \( e(1 - e) = 0 \). Thus \( e = 1 \). Similarly, if \( 1 - e \notin m \), then \( e = 0 \).

Alternatively, (3.7) implies that \( A \) is not the product of two nonzero rings. So (3.8) implies that either \( e = 0 \) or \( e = 1 \).

Exercise (3.9). — Let \( A \) be a ring, \( m \) a maximal ideal such that \( 1 + m \) is a unit for every \( m \in m \). Prove \( A \) is local. Is this assertion still true if \( m \) is not maximal?

Solution: Take \( y \in A - m \). Since \( m \) is maximal, \( \langle y \rangle + m = A \). Hence there exist \( x \in R \) and \( m \in m \) such that \( xy + m = 1 \), or in other words, \( xy = 1 - m \). So \( xy \) is a unit by hypothesis; whence, \( y \) is a unit. Thus \( A \) is local by (3.6).

No, the assertion is not true if \( m \) is not maximal. Indeed, take any ring that is not local, for example \( \mathbb{Z} \), and take \( m := \langle 0 \rangle \).

Exercise (3.13). — Let \( \varphi: R \to R' \) be a map of rings, \( p \) an ideal of \( R \). Prove

1. there is an ideal \( q \) of \( R' \) with \( \varphi^{-1}(q) = p \); if and only if \( \varphi^{-1}(pR') = p \);
2. if \( p \) is prime with \( \varphi^{-1}(pR') = p \), then there’s a prime \( q \) of \( R' \) with \( \varphi^{-1}(q) = p \).

Solution: In (1), given \( q \), note \( \varphi(p) \subset q \), as always \( \varphi(\varphi^{-1}(q)) \subset q \). So \( pR' \subset q \). Hence \( \varphi^{-1}(pR') \subset \varphi^{-1}(q) = p \). But, always \( p \subset \varphi^{-1}(pR') \). Thus \( \varphi^{-1}(pR') = p \). The converse is trivial: take \( q := pR' \).

In (2), set \( S := \varphi(R - p) \). Then \( S \cap pR' = \emptyset \), as \( \varphi(x) \in pR' \) implies \( x \in \varphi^{-1}(pR') \) and \( \varphi^{-1}(pR') = p \). So there’s a prime \( q \) of \( R' \) containing \( pR' \) and disjoint from \( S \) by (3.9). So \( \varphi^{-1}(q) \cap \varphi^{-1}(pR') = p \) and \( \varphi^{-1}(q) \cap (R - p) = \emptyset \). Thus \( \varphi^{-1}(q) = p \).

Exercise (3.14). — Use Zorn’s lemma to prove that any prime ideal \( p \) contains a prime ideal \( q \) that is minimal containing any given subset \( s \subset p \).

Solution: Let \( S \) be the set of all prime ideals \( q \) such that \( s \subset q \subset p \). Then \( p \in S \), so \( S \neq \emptyset \). Order \( S \) by reverse inclusion. To apply Zorn’s Lemma, we must show that, for any decreasing chain \( \{q_\lambda \} \) of prime ideals, the intersection \( q := \bigcap q_\lambda \) is a prime ideal. Plainly \( q \) is always an ideal. So take \( x, y \notin q \). Then there exists \( \lambda \) such that \( x, y \notin q_\lambda \). Since \( q_\lambda \) is prime, \( xy \notin q_\lambda \). So \( xy \notin q \). Thus \( q \) is prime.

Exercise (3.16). — Let \( R \) be a ring, \( S \) a subset. Show that \( S \) is saturated multiplicative if and only if \( R - S \) is a union of primes.
SOLUTION: First, assume $S$ is saturated multiplicative. Take $x \in R - S$. Then $xy \notin S$ for all $y \in R$; in other words, $(x) \cap S = \emptyset$. Then (3.16) gives a prime $p \supset (x)$ with $p \cap S = \emptyset$. Thus $R - S$ is a union of primes.

Conversely, assume $R - S$ is a union of primes $p$. Then $1 \in S$ as 1 lies in no $p$. Take $x, y \in R$. Then $x, y \in S$ if and only if $x, y$ lie in no $p$; if and only if $xy$ lies in no $p$, as every $p$ is prime; if and only if $xy \in S$. Thus $S$ is saturated multiplicative.

Exercise (3.17). Let $R$ be a ring, and $S$ a multiplicative subset. Define its \textbf{saturation} to be the subset

$$
\overline{S} := \{ x \in R \mid \text{there is } y \in R \text{ with } xy \in S \}.
$$

(1) Show (a) that $\overline{S} \supset S$, and (b) that $\overline{S}$ is saturated multiplicative, and (c) that any saturated multiplicative subset $T$ containing $S$ also contains $\overline{S}$.

(2) Show that $R - \overline{S}$ is the union $U$ of all the primes $p$ with $p \cap S = \emptyset$.

(3) Let $a$ be an ideal; assume $S = 1 + a$; set $W := \bigcup_{p \supset a} p$. Show $R - \overline{S} = W$.

(4) Given $f \in R$, let $\overline{S}_f$ denote the saturation of the multiplicative subset of all powers of $f$. Given $f, g \in R$, show $\overline{S}_f \subset \overline{S}_g$ if and only if $\sqrt{f} \supset \sqrt{g}$.

Solution: Consider (1). Trivially, if $x \in S$, then $x \cdot 1 \in S$. Thus (a) holds.

Hence $1 \in \overline{S}$ as $1 \in S$. Now, take $x, x' \in \overline{S}$. Then there are $y, y' \in R$ with $xy, x'y' \in S$. But $S$ is multiplicative. So $(xx')(yy') \in S$. Hence $xx' \in \overline{S}$. Thus $\overline{S}$ is multiplicative. Further, take $x, x' \in R$ with $xx' \in \overline{S}$. Then there is $y \in R$ with $xy \in S$. So $x, x' \in \overline{S}$. Thus $S$ is saturated. Thus (b) holds.

Finally, consider (c). Given $x \in \overline{S}$, there is $y \in R$ with $xy \in S$. So $xy \in T$. But $T$ is saturated multiplicative. So $x \in T$. Thus $T \supset \overline{S}$. Thus (c) holds.

Consider (2). Plainly, $R-U$ contains $S$. Further, $R-U$ is saturated multiplicative by (3.16). So $R-U \supset \overline{S}$ by (1)(c). Thus $U \subset R - \overline{S}$. Conversely, $R - \overline{S}$ is a union of primes $p$ by (3.16). Plainly, $p \cap S = \emptyset$ for all $p$. So $U \supset R - \overline{S}$. Thus (2) holds.

For (3), first take a prime $p$ with $p \cap S = \emptyset$. Then $1 \notin p + a$; else, $1 = p + a$ with $p \in p$ and $a \in a$, and so $1 - p = a \in p \cap S$. So $p + a$ lies in a maximal ideal $m$ by (3.12). Then $a \subset m$; so $m \subset W$. But also $p \subset m$. Thus $U \subset W$.

Conversely, take $p \supset a$. Then $1 + p \subset 1 + a = S$. But $p \cap (1+p) = \emptyset$. So $p \cap S = \emptyset$. Thus $U \subset W$. Thus $U = W$. Thus (2) yields (3).

Consider (4). By (1), $\overline{S}_f \subset \overline{S}_g$ if and only if $f \in \overline{S}_g$. By definition of saturation, $f \in \overline{S}_g$ if and only if $hf = g^n$ for some $h$ and $n$. By definition of radical, $hf = g^n$ for some $h$ and $n$ if and only if $g \in \sqrt{\overline{S}_f}$. Plainly, $g \in \sqrt{\overline{S}_f}$ if and only if $\sqrt{g} \supset \sqrt{\overline{S}_f}$. Thus (4) holds.

Exercise (3.18). Let $R$ be a nonzero ring, $S$ a subset. Show $S$ is maximal in the set $\mathcal{S}$ of multiplicative subsets $T$ of $R$ with $0 \notin T$ if and only if $R - S$ is a \textbf{minimal} prime—that is, it is a prime containing no smaller prime.

Solution: First, assume $S$ is maximal in $\mathcal{S}$. Then $S$ is equal to its saturation $\overline{S}$, as $S \supset \overline{S}$ and $\overline{S}$ is multiplicative by (3.17) (1)(a), (b) and as $0 \in \overline{S}$ would imply $0 = 0 \cdot y \in S$ for some $y$. So $R - S$ is a union of primes $p$ by (3.16). Fix a $p$. Then (3.13) yields in $p$ a minimal prime $q$. Then $S \subset R - q$. But $R - q \in \mathcal{S}$ by (2.11). As $S$ is maximal, $S = R - q$, or $R - S = q$. Thus $R - S$ is a minimal prime.

Conversely, assume $R - S$ is a minimal prime $q$. Then $S \in \mathcal{S}$ by (2.11). Given $T \in \mathcal{S}$ with $S \subset T$, note $R - T = \bigcup p$ with $p$ prime by (3.16). Fix a $p$. Now, $S \subset T \subset \overline{T}$. So $q \supset p$. But $q$ is minimal. So $q = p$. But $p$ is arbitrary,
Let \( k \) be a field, \( S \subset k \) a subset of cardinality \( d \) at least 2.

(1) Let \( P := k[X_1, \ldots, X_n] \) be the polynomial ring, \( f \in P \) nonzero. Assume the highest power of any \( X_i \) in \( f \) is less than \( d \). Proceeding by induction on \( n \), show there are \( a_1, \ldots, a_n \in S \) with \( f(a_1, \ldots, a_n) \neq 0 \).

(2) Let \( V \) be a \( k \)-vector space, and \( W_1, \ldots, W_r \) proper subspaces. Assume \( r < d \). Show \( \bigcup_i W_i \neq V \).

(3) In (2), let \( W \subset \bigcup_i W_i \) be a subspace. Show \( W \subset W_i \) for some \( i \).

(4) Let \( R \) a \( k \)-algebra, \( a, a_1, \ldots, a_r \) ideals with \( a \supset \bigcup_i a_i \). Show \( a \subset a_i \) for some \( i \).

**Solution:** For (1), first assume \( n = 1 \). Then \( f \) has degree at most \( d \), so at most \( d \) roots by [2, (1.8), p. 392]. So there’s \( a_1 \in S \) with \( f(a_1) \neq 0 \).

Assume \( n > 1 \). Say \( f = \sum_j g_j X_j^i \) with \( g_j \in k[X_2, \ldots, X_n] \). But \( f \neq 0 \). So \( g_i \neq 0 \) for some \( i \). By induction, there are \( a_2, \ldots, a_n \in S \) with \( g_i(a_2, \ldots, a_n) \neq 0 \).

So there’s \( a_1 \in S \) with \( f(a_1, \ldots, a_n) = \sum_j g_j(a_2, \ldots, a_n) a_1^i \neq 0 \). Thus (1) holds.

For (2), for all \( i \), take \( v_i \in V - W_i \). Form their span \( V' \subset V \). Set \( n := \dim V' \) and \( W_i' := W_i \cap V' \). Then \( n < \infty \), and it suffices to show \( \bigcup_i W_i' \neq V' \).

Identify \( V' \) with \( k^n \). Form the polynomial ring \( P := k[X_1, \ldots, X_n] \). For each \( i \), take a linear form \( f_i \in P \) that vanishes on \( W_i' \). Set \( f := f_1 \cdots f_r \). Then \( r \) is the highest power of any variable in \( f \). But \( r < d \). So (1) yields \( a_1, \ldots, a_n \in S \) with \( f(a_1, \ldots, a_n) \neq 0 \). Then \( (a_1, \ldots, a_n) \in V' - \bigcup_i W_i' \).

For (3), for all \( i \), set \( U_i := W \cap W_i \). Then \( \bigcup_i U_i = W \). So (2) implies \( U_i = W \) for some \( i \). Thus \( W \subset W_i \).

Finally, (4) is a special case of (3), as every ideal is a \( k \)-vector space. \( \square \)

**Exercise (3.21).** Let \( k \) be a field, \( R := k[X, Y] \) the polynomial ring in two variables, \( m := (X, Y) \). Show \( m \) is a union of strictly smaller primes.

**Solution:** Since \( R \) is a UFD, and \( m \) is maximal, so prime, any nonzero \( f \in m \) has a prime factor \( p \in \mathbb{P} \). Thus \( m = \bigcup_{p \in \mathbb{P}} \langle p \rangle \), as \( m \) is not principal. \( \square \)

**Exercise (3.23).** Find the nilpotents in \( \mathbb{Z}/(n) \). In particular, take \( n = 12 \).

**Solution:** An integer \( m \) is nilpotent modulo \( n \) if and only if some power \( m^k \) is divisible by \( n \). The latter holds if and only if every prime factor of \( n \) occurs in \( m \).

In particular, in \( \mathbb{Z}/(12) \), the nilpotents are 0 and 6. \( \square \)

**Exercise (3.24).** Let \( R \) be a ring. (1) Assume every ideal not contained in \( \text{nil}(R) \) contains a nonzero idempotent. Prove that \( \text{nil}(R) = \text{rad}(R) \). (2) Assume \( R \) is Boolean. Prove that \( \text{nil}(R) = \text{rad}(R) = \{0\} \).

**Solution:** or (1), recall \((3.22)\), that \( \text{nil}(R) \subset \text{rad}(R) \). To prove the opposite inclusion, set \( R' := R/\text{nil}(R) \). Assume \( \text{rad}(R') \neq \{0\} \). Then there is a nonzero idempotent \( e \in \text{rad}(R') \). Then \( e(1 - e) = 0 \). But \( 1 - e \) is a unit by \((3.22)\). So \( e = 0 \), a contradiction. Hence \( \text{rad}(R') = \{0\} \). Thus \((3.23)\) yields (1).

For (2), recall from \((1.7)\) that every element of \( R \) is idempotent. So \( \text{nil}(R) = \{0\} \), and every nonzero ideal contains a nonzero idempotent. \( \square \)

**Exercise (3.25).** Let \( \varphi: R \to R' \) be a ring map, \( b \subset R' \) a subset. Prove \( \varphi^{-1} \sqrt{b} = \sqrt{\varphi^{-1} b} \).
SOLUTION: Below, (1) is clearly equivalent to (2); and (2), to (3); and so forth:

(1) \( x \in \varphi^{-1} \sqrt{b} \);  \quad (2) \( \varphi x \in \sqrt{b} \);
(3) \( (\varphi x)^n \in b \) for some \( n \);  \quad (4) \( \varphi(x^n) \in b \) for some \( n \);
(5) \( x^n \in \varphi^{-1}b \) for some \( n \);  \quad (6) \( x \in \sqrt{\varphi^{-1}b} \).

\[ \square \]

EXERCISE (3.38). — Let \( R \) be a ring, \( X \) a variable. Show that
\[ \text{rad}(R[X]) = \text{nil}(R[X]) = \text{nil}(R)R[X]. \]

SOLUTION: First, recall that \( \text{rad}(R[X]) \supset \text{nil}(R[X]) \) by (3.37). Next, recall that \( \text{nil}(R[X]) \supset \text{nil}(R)R[X] \) by (3.31). Finally, given \( f := a_0 + \cdots + a_n X^n \) in \( \text{rad}(R[X]) \), note that \( 1 + Xf \) is a unit by (3.27). So \( a_0, \ldots, a_n \) are nilpotent by (3.36)(2). So \( f \in \text{nil}(R)R[X] \). Thus \( \text{nil}(R)R[X] \supset \text{rad}(R[X]) \), as desired. \[ \square \]

EXERCISE (3.26). — Let \( e, e' \in \text{Idem}(R) \). Assume \( \sqrt{\langle e \rangle} = \sqrt{\langle e' \rangle} \). Show \( e = e' \).

SOLUTION: By hypothesis, \( e^n \in \langle e' \rangle \) for some \( n \geq 1 \). But \( e^2 = e \), so \( e^n = e \). So \( e = xe' \) for some \( x \). So \( e = xe' = ee' \). By symmetry, \( e' = ee' \). Thus \( e = e' \). \[ \square \]

EXERCISE (3.27). — Let \( R \) be a ring, \( \mathfrak{a}_1, \mathfrak{a}_2 \) comaximal ideals with \( \mathfrak{a}_1 \mathfrak{a}_2 \subset \text{nil}(R) \).
Show there are complementary idempotents \( e_1 \) and \( e_2 \) with \( e_i \in \mathfrak{a}_i \).

SOLUTION: Since \( \mathfrak{a}_1 \) and \( \mathfrak{a}_2 \) are comaximal, there are \( x_i \in \mathfrak{a}_i \) with \( x_1 + x_2 = 1 \).
Given \( n \geq 1 \), expanding \( (x_1 + x_2)^{2n-1} \) and collecting terms yields \( a_1 x_1^n + a_2 x_2^n = 1 \) for suitable \( a_i \in R \). Now, \( x_1 x_2 \in \text{nil}(R) \); take \( n \geq 1 \) so that \( (x_1 x_2)^n = 0 \). Set \( e_1 := a_1 x_1^n \in \mathfrak{a}_1 \). Then \( e_1 + e_2 = 1 \) and \( e_1 e_2 = 0 \). Thus \( e_1 \) and \( e_2 \) are complementary idempotents by (3.11). \[ \square \]

EXERCISE (3.28). — Let \( R \) be a ring, \( \mathfrak{a} \) an ideal, \( \kappa: R \to R/\mathfrak{a} \) the quotient map. Assume \( \mathfrak{a} \subset \text{nil}(R) \). Prove that \( \text{Idem}(\kappa) \) is bijective.

SOLUTION: Note that \( \text{Idem}(\kappa) \) is injective by (3.22) and (3.3).
As to surjectivity, given \( e' \in \text{Idem}(R/\mathfrak{a}) \), take \( z \in R \) with residue \( e' \). Then \( \langle z \rangle \) and \( (1 - z) \) are trivially comaximal. And \( \langle z \rangle (1 - z) \subset \mathfrak{a} \subset \text{nil}(R) \) as \( \kappa(z - z^2) = 0 \).
So (3.27) yields complementary idempotents \( e_1 \in \langle z \rangle \) and \( e_2 \in \langle 1 - z \rangle \).
Say \( e_1 = xz \) with \( x \in R \). Then \( \kappa(e_1) = xe' \). So \( \kappa(e_1) = x e'^2 = \kappa(e_1)e' \).
Similarly, \( \kappa(e_2) = \kappa(e_2)(1 - e') \). So \( \kappa(e_2) = 1 - \kappa(e_1) \). So \( (1 - \kappa(e_1))e' = 0 \),
or \( e' = \kappa(e_1)e' \). But \( \kappa(e_1) = \kappa(e_1)e' \). So \( \kappa(e_1) = e' \). Thus \( \text{Idem}(\kappa) \) is surjective. \[ \square \]

EXERCISE (3.30). — Let \( R \) be a ring. Prove the following statement equivalent:

(1) \( R \) has exactly one prime \( p \);
(2) every element of \( R \) is either nilpotent or a unit;
(3) \( R/\text{nil}(R) \) is a field.

SOLUTION: Assume (1). Let \( x \in R \) be a nonunit. Then \( x \in p \). So \( x \) is nilpotent by the Scheinnullstellsensatz (3.23). Thus (2) holds.
Assume (2). Then every \( x \notin \text{nil}(R) \) has an inverse. Thus (3) holds.
Assume (3). Then \( \text{nil}(R) \) is maximal by (3.12). But any prime of \( R \) contains \( \text{nil}(R) \) by (3.24). Thus (1) holds. \[ \square \]

EXERCISE (3.32). — Let \( R \) be a ring, and \( \mathfrak{a} \) an ideal. Assume \( \sqrt{\mathfrak{a}} \) is finitely generated. Show \( (\sqrt{\mathfrak{a}})^n \subset \mathfrak{a} \) for all large \( n \).
SOLUTION: Let $x_1, \ldots, x_m$ be generators of $\sqrt{a}$. For each $i$, there is $n_i$ such that $x_n^i \in a$. Let $n > \sum (n_i - 1)$. Given $a \in \sqrt{a}$, write $a = \sum_{i=1}^m y_i x_i$ with $y_i \in R$. Then $a^n$ is a linear combination of terms of the form $x_1^{n_1} \cdots x_m^{n_m}$ with $\sum_{i=1}^m n_i = n$. Hence $j_i \geq n_i$ for some $i$, because if $j_i \leq n_i - 1$ for all $i$, then $\sum j_i \leq \sum (n_i - 1)$. Thus $a^n \in a$, as desired. \hfill \Box

EXERCISE (3.33). — Let $R$ be a ring, $q$ an ideal, $p$ a finitely generated prime. Prove that $p = \sqrt{q}$ if and only if there is $n \geq 1$ such that $p \supset q \supset p^n$.

SOLUTION: If $p = \sqrt{q}$, then $p \supset q \supset p^n$ by (3.32). Conversely, if $q \supset p^n$, then clearly $\sqrt{q} \supset p$. Further, since $p$ is prime, if $p \supset q$, then $p \supset \sqrt{q}$. \hfill \Box

EXERCISE (3.35). — Let $R$ be a ring. Assume $R$ is reduced and has finitely many minimal prime ideals $p_1, \ldots, p_n$. Prove that $\phi: R \to \prod (R/p_i)$ is injective, and for each $i$, there is some $(x_1, \ldots, x_n) \in \text{Im}(\phi)$ with $x_i \neq 0$ but $x_j = 0$ for $j \neq i$.

SOLUTION: Clearly $\text{Ker}(\phi) = \bigcap p_i$. Now, $R$ is reduced and the $p_i$ are its minimal primes; hence, (3.34) and (3.35) yield

\[ \langle 0 \rangle = \sqrt{\langle 0 \rangle} = \bigcap p_i. \]

Thus $\text{Ker}(\phi) = \langle 0 \rangle$, and so $\phi$ is injective.

Finally, fix $i$. Since $p_i$ is minimal, $p_i \not\supset p_j$ for $j \neq i$; say $a_j \in p_j - p_i$. Set $a := \prod_{j \neq i} a_j$. Then $a \in p_j - p_i$ for all $j \neq i$. So take $(x_1, \ldots, x_n) := \phi(a)$. \hfill \Box

EXERCISE (3.36). — Let $R$ be a ring, $X$ a variable, $f := a_0 + a_1 X + \cdots + a_n X^n$ and $g := b_0 + b_1 X + \cdots + b_m X^m$ polynomials with $a_0 \neq 0$ and $b_m \neq 0$. Call $f$ primitive if $\langle a_0, \ldots, a_n \rangle = R$. Prove the following statements:

1. Then $f$ is nilpotent if and only if $a_0, \ldots, a_n$ are nilpotent.
2. Then $f$ is a unit if and only if $a_0$ is a unit and $a_1, \ldots, a_n$ are nilpotent.
3. If $f$ is a zerodivisor, then there is a nonzero $b \in R$ with $b f = 0$; in fact, if $f g = 0$ with $m$ minimal, then $f b_m = 0$ (or $m = 0$).
4. Then $f g$ is primitive if and only if $f$ and $g$ are primitive.

SOLUTION: In (1), if $a_0, \ldots, a_n$ are nilpotent, so is $f$ owing to (3.31). Conversely, say $a_i \notin \text{nil}(R)$. Then the Schinzel-Mullins-Stellensatz (3.24) yields a prime $p \subset R$ with $a_i \notin p$. So $f \notin pR[X]$. But $pR[X]$ is prime by (2.18). So plainly $f \notin \text{nil}(R[X])$.

Alternatively, say $f^k = 0$. Then $(a_n X^n)^k = 0$. So $f - a_n X^n$ is nilpotent owing to (3.31). So $a_0, \ldots, a_{n-1}$ are nilpotent by induction on $n$. Thus (1) holds.

For (2), suppose $a_0$ is a unit and $a_1, \ldots, a_n$ are nilpotent. Then $a_1 X + \cdots + a_n X^n$ is nilpotent by (1), so belongs to $\text{rad}(R)$ by (3.31). Thus $f$ is a unit by (3.2).

Conversely, suppose $f g = 1$. Then $a_0 b_0 = 1$. Thus $a_0$ and $b_0$ are units.

Further, given a prime $p \subset R$, let $\kappa_p: R[X] \to (R/p)[X]$ be the canonical map. Then $\kappa_p(f) \kappa_p(g) = 1$. But $R/p$ is a domain by (2.24). So $\deg \kappa_p(f) = 0$ owing to (3.34). So $a_1, \ldots, a_n \in p$. But $p$ is arbitrary. Thus $a_1, \ldots, a_n \in \text{nil}(R)$ by (3.24).

Alternatively, let’s prove $a_n^{-1} b_{m-n} = 0$ by induction on $r$. Set $c_i := \sum_{j+k=i} a_j b_k$. Then $\sum c_i X^i = f g$. But $f g = 1$. So $c_i = 0$ for $i > 0$. Taking $i := m + n$ yields $a_n b_m = 0$. Then $c_{m+n-r} = 0$ yields $a_n b_{m-n} + a_{n-1} b_{m-(r-1)} + \cdots = 0$. Multiplying by $a_n^r$ yields $a_n^{-1} b_{m-n} = 0$ by induction. So $a_n^{-1} b_0 = 0$. But $b_0$ is a unit. So $a_n^{-1} = 0$. So $a_n X^n \in \text{rad}(R[X])$ by (3.24). But $f$ is a unit. So $f - a_n X^n$ is a unit by (3.2). So $a_1, \ldots, a_{n-1}$ are nilpotent by induction on $n$. Thus (2) holds.

For (3), suppose $f b_m \neq 0$. Say $a_r b_m \neq 0$, but $a_{r+i} b_m = 0$ for all $i > 0$. Fix $i > 0$.
and set $h := a_r + g$. Then $fh = 0$ if $fg = 0$. Also $h = 0$ or $\deg(h) < m$. So $h = 0$ if $m$ is minimal. In particular, $a_r + b_m = 0$. But $i > 0$ is arbitrary. Also $fg = 0$ yields $a_r b_n + a_{r+1} b_{m-1} + \cdots = 0$. So $a_r b_m = 0$, a contradiction. Thus (3) holds.

For (4), given $m \subset R$ maximal, let $\kappa_m : R[X] \to (R/m)[X]$ be the canonical map. Then $h \in R[X]$ is primitive if and only if $\kappa_m(h) \neq 0$ for all $m$, owing to (4.31). But $R/m$ is a field by (2.17). So $(R/m)[X]$ is a domain by (4.3). Hence $\kappa_p(fg) = 0$ if and only if $\kappa_p(f) = 0$ or $\kappa_p(g) = 0$. Thus (4) holds.

**Exercise (3.37).** — Generalize (4.30) to the polynomial ring $P := R[X_1, \ldots, X_r]$.

For (3), reduce to the case of one variable $Y$ via this standard device: take $d$ suitably large, and define $\varphi : P \to R[Y]$ by $\varphi(X_i) := Y^d$.

**Solution:** Let $f, g \in P$. Write $f = \sum a_{(i)} X^{(i)}$ where $(i) := (i_1, \ldots, i_r)$ and $X^{(i)} := X_1^{i_1} \cdots X_r^{i_r}$. Call $f$ **primitive** if the $a_{(i)}$ generate $R$. Set $(0) := (0, \ldots, 0)$. Then (1)–(4) generalize as follows:

1. (1') Then $f$ is nilpotent if and only if all $a_{(i)}$ are nilpotent for all $(i)$.
2. (2') Then $f$ is a unit if and only if $a_{(0)}$ is a unit and all $a_{(i)}$ are nilpotent for $(i) \neq (0)$.
3. (3') Assume $f$ is a zerodivisor. Then there is a nonzero $c \in R$ with $cf = 0$.
4. (4') Then $fg$ is primitive if and only if $f$ and $g$ are primitive.

To prove (1')–(2'), set $R' := R[X_2, \ldots, X_r]$, and say $f = \sum f_i X_1^{i_r}$ with $f_i \in R'$.

In (1'), if $f$ is nilpotent, so are all $f_i$ by (3.36)(1); hence by induction on $r$, so are all $a_{(i)}$. Conversely, if all $a_{(i)}$ are nilpotent, so is $f$ by (3.31). Thus (1') holds.

In (2'), if $a_{(0)}$ is a unit and all $a_{(i)}$ are nilpotent for $(i) \neq (0)$, then $\sum_{i \neq (0)} a_{(i)} X^{(i)}$ is nilpotent by (1), so belongs to $\text{rad}(R)$ by (3.24.10). Then $f$ is a unit by (3.2).

Conversely, suppose $f$ is a unit. Then $f_0$ is a unit, and $f_i$ is nilpotent for $i > 0$ by (3.30)(2). So $a_{(0)}$ is a unit, and $a_{(i)}$ is nilpotent if $i_1 = 0$ and $(i) \neq (0)$, by induction on $r$. Also, $a_{(i)}$ is nilpotent if $i_1 > 0$ by (1'). Thus (2') holds.

In (3'), there’s a nonzero $g \in P$ with $fg = 0$. Take $d$ larger than any exponent of any $X_i$ found in $f$ or $g$. Form the $R$-algebra map $\varphi : P \to R[Y]$ with $\varphi(X_i) = Y^{d - 1}$. Then $\varphi(f)\varphi(g) = 0$. But $\varphi(X^{(i)}) = Y^{i_1 + \cdots + i_r - d + 1}$. So $\varphi$ carries distinct monomials in $f$ to distinct monomials in $\varphi(f)$, and the same for $g$. So $\varphi(f)$ has the same coefficients as $f$, and $\varphi(g)$ the same as $g$. So $\varphi(g) \neq 0$. Hence $\varphi(f)$ is a zerodivisor. So (3.30)(3) yields a nonzero $c \in R$ with $c\varphi(f) = 0$. Hence $ca_{(i)} = 0$ for all $a_{(i)}$. So $cf = 0$. Thus (3') holds.

For (4'), use the solution of (3.33)(4) with $X$ replaced by $X_1, \ldots, X_r$. □

**Exercise (3.39).** — Let $R$ be a ring, $a$ an ideal, $X$ a variable, the formal power series ring, $\mathfrak{M} \subset R[[X]]$ an ideal, and $f := \sum a_n X^n \in R[[X]]$. Set $m := \mathfrak{M} \cap R$ and $\mathfrak{A} := \{ \sum b_n X^n \mid b_n \in a \}$. Prove the following statements:

1. If $f$ is nilpotent, then $a_n$ is nilpotent for all $n$. The converse is false.
2. Then $f \in \text{rad}(R[[X]])$ if and only if $a_0 \in \text{rad}(R)$.
3. Assume $X \in \mathfrak{M}$. Then $X$ and $m$ generate $\mathfrak{M}$.
4. Assume $\mathfrak{M}$ is maximal. Then $X \in \mathfrak{M}$ and $m$ is maximal.
5. If $a$ is finitely generated, then $a R[[X]] = \mathfrak{A}$. The converse may fail.

**Solution:** For (1), assume $f$ and $a_i$ for $i < n$ are nilpotent. Set $g := \sum_{i \geq n} a_i X^i$. Then $g = f - \sum_{i < n} a_i X^i$. So $g$ is nilpotent by (3.33); say $g^m = 0$ with $m > 1$. Then $a_n^m = 0$. Thus by induction $a_n$ is nilpotent for all $n$.

The converse is false. For example, set $P := \mathbb{Z}[X_2, X_3, \ldots]$ for variables $X_n$. Set
R := P/(X_2^n, X_3^n, \ldots). Let a_n be the residue of X_n. Then a_n = 0, but \( \sum a_n X^n \) is not nilpotent. Thus (1) holds.

For (2), given \( g = \sum b_n X^n \in \text{rad}(R[[X]]) \), note that \( 1 + fg \) is a unit if and only if \( 1 + a_n b_0 \) is a unit by (3.14). Thus (3.14) yields (2) holds.

For (3), note \( \mathfrak{M} \) contains \( X \) and \( m \), so the ideal they generate. But \( f = a_0 + Xg \) for some \( g \in R[[X]] \). So if \( f \in \mathfrak{M} \), then \( a_0 \in \mathfrak{M} \cap R = m \). Thus (3) holds.

For (4), note that \( X \in \text{rad}(R[[X]]) \) by (2). So \( X \) and \( m \) generate \( \mathfrak{M} \) by (3). So \( P/n = R/m \) by (3.14). Thus (3.14) yields (4).

In (5), plainly \( aR[[X]] \subseteq \mathfrak{A} \). Now, assume \( f := \sum a_n X^n \in \mathfrak{A} \), or all \( a_n \in a \). Say \( b_1, \ldots, b_m \in a \) generate. Then \( a_n = \sum_{i=1}^m c_i b_i \) for some \( c_i \in R \). Thus, as desired, \( f = \sum_{n \geq 0} \left( \sum_{i=1}^m c_i b_i \right) X^n = \sum_{i=1}^m b_i \left( \sum_{n \geq 0} c_i X^n \right) \in aR[[X]] \).

For a counterexample, take \( a_0, a_1, \ldots \) to be variables. Take \( R := \mathbb{Z}[a_1, a_2, \ldots] \) and \( a := (a_1, a_2, \ldots) \). Given \( g \in aR[[X]] \), say \( g = \sum_{i=1}^m b_i g_i \) with \( b_i \in a \) and \( g_i = \sum_{n \geq 0} b_n X^n \). Choose \( p \) greater than the maximum \( n \) such that \( a_n \) occurs in any \( b_i \). Then \( \sum_{i=1}^m b_i g_i \in \langle a_1, \ldots, a_{p-1} \rangle \), but \( a_p \notin \langle a_1, \ldots, a_{p-1} \rangle \). Therefore, \( g \neq f := \sum a_n X^n \). Thus \( f \notin aR[[X]] \), but \( f \in \mathfrak{A} \).

4. Modules

Exercise (4.3). Let \( R \) be a ring, \( M \) a module. Consider the set map \( \rho: \text{Hom}(R,M) \rightarrow M \) defined by \( \rho(\theta) := \theta(1) \). Show that \( \rho \) is an isomorphism, and describe its inverse.

Solution: First off, \( \rho \) is \( R \)-linear, because \( \rho(x\theta + x'\theta') = (x\theta + x'\theta')(1) = x\theta(1) + x'\theta'(1) = x\rho(\theta) + x'\rho(\theta') \).

Set \( H := \text{Hom}(R,M) \). Define \( \alpha: M \rightarrow H \) by \( \alpha(m)(x) := xm \). It is easy to check that \( \alpha \rho = 1_H \) and \( \rho \alpha = 1_M \). Thus \( \rho \) and \( \alpha \) are inverse isomorphisms by (3.12).

Exercise (4.12). Let \( R \) be a domain, and \( x \in R \) nonzero. Let \( M \) be the submodule of \( \text{Frac}(R) \) generated by \( 1, x^{-1}, x^{-2}, \ldots \). Suppose that \( M \) is finitely generated. Prove that \( x^{-1} \in R \), and conclude that \( M = R \).

Solution: Suppose \( M \) is generated by \( m_1, \ldots, m_k \). Say \( m_i = \sum_{j=0}^{n_i} a_{ij} x^{-j} \) for some \( n_i \) and \( a_{ij} \in \mathbb{Z} \). Set \( n := \max\{n_i\} \). Then \( 1, x^{-1}, \ldots, x^{-n} \) generate \( M \). So \( x^{-n+1} = a_n x^{-n} + \cdots + a_1 x^{-1} + a_0 \) for some \( a_i \in R \). Thus \( x^{-1} = a_n + \cdots + a_1 x^{-1} + a_0 x^n \in R \).

Finally, as \( x^{-1} \in R \) and \( R \) is a ring, also \( x^{-1}, x^{-2}, \ldots \in R \); so \( M \subseteq R \). Conversely, \( M \supset R \) as \( 1 \in M \). Thus \( M = R \).

Exercise (4.13). A finitely generated free module has finite rank.

Solution: Say \( e_\lambda \) for \( \lambda \in \Lambda \) form a free basis, and \( m_1, \ldots, m_r \) generate. Then \( m_i = \sum x_{ij} e_\lambda \) for some \( x_{ij} \). Consider the \( e_\lambda \) that occur. Plainly, they are finite in number, and generate. So they form a finite free basis, as desired.
Exercise (4.16). — Let $\Lambda$ be an infinite set, $R_\lambda$ a nonzero ring for $\lambda \in \Lambda$. Endow $\prod R_\lambda$ and $\bigoplus R_\lambda$ with componentwise addition and multiplication. Show that $\prod R_\lambda$ has a multiplicative identity (so is a ring), but that $\bigoplus R_\lambda$ does not (so is not a ring).

Solution: Consider the vector $(1)$ whose every component is 1. Obviously, $(1)$ is a multiplicative identity of $\prod R_\lambda$. On the other hand, no restricted vector $(x_\lambda)$ can be a multiplicative identity in $\bigoplus R_\lambda$: indeed, because $\Lambda$ is infinite, $x_\mu$ must be zero for some $\mu$. So $(x_\lambda) \cdot (y_\lambda) \neq (y_\lambda)$ if $y_\mu \neq 0$.

Exercise (4.17). — Let $R$ be a ring, $M$ a module, and $M'$, $M''$ submodules. Show that $M = M' \oplus M''$ if and only if $M = M' + M''$ and $M' \cap M'' = 0$.

Solution: Assume $M = M' \oplus M''$. Then $M$ is the set of pairs $(m', m'')$ with $m' \in M'$ and $m'' \in M''$ by (1); further, $M'$ is the set of $(m', 0)$, and $M'$ is that of $(0, m'')$. So plainly $M = M' + M''$ and $M' \cap M'' = 0$.

Conversely, consider the map $M' \oplus M'' \to M$ given by $(m', m'') \mapsto m' + m''$. It is surjective if $M = M' + M''$. It is injective if $M' \cap M'' = 0$; indeed, if $m' + m'' = 0$, then $m' = -m'' \in M' \cap M'' = 0$, and so $(m', m'') = 0$ as desired.

Exercise (4.18). — Let $L$, $M$, and $N$ be modules. Consider a diagram

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N$$

where $\alpha$, $\beta$, $\rho$, and $\sigma$ are homomorphisms. Prove that

$$M = L \oplus N \quad \text{and} \quad \alpha = \iota_L, \ \beta = \pi_N, \ \sigma = \iota_N, \ \rho = \pi_L$$

if and only if the following relations hold:

$$\beta \alpha = 0, \ \beta \sigma = 1, \ \rho \sigma = 0, \ \rho \alpha = 1,$$

$$\alpha \rho + \sigma \beta = 1.$$

Solution: If $M = L \oplus N$ and $\alpha = \iota_L, \ \beta = \pi_N, \ \sigma = \iota_N, \ \rho = \pi_L$, then the definitions immediately yield $\alpha \rho + \sigma \beta = 1$ and $\beta \alpha = 0, \ \beta \sigma = 1, \ \rho \sigma = 0, \ \rho \alpha = 1$.

Conversely, assume $\alpha \rho + \sigma \beta = 1$ and $\beta \alpha = 0, \ \beta \sigma = 1, \ \rho \sigma = 0, \ \rho \alpha = 1$. Consider the maps $\varphi : M \to L \oplus N$ and $\theta : L \oplus N \to M$ given by $\varphi m := (\rho m, \beta m)$ and $\theta(l, n) := \alpha l + \sigma n$. They are inverse isomorphisms, because

$$\varphi \theta(l, n) = (\rho \alpha l + \rho \sigma n, \ \beta \alpha l + \beta \sigma n) = (l, n) \quad \text{and} \quad \theta \varphi m = \alpha \rho m + \sigma \beta m = m.$$  

Lastly, $\beta = \pi_N \varphi$ and $\rho = \pi_L \varphi$ by definition of $\varphi$, and $\alpha = \theta L$ and $\sigma = \theta N$ by definition of $\theta$.

Exercise (4.19). — Let $L$ be a module, $\Lambda$ a nonempty set, $M_\lambda$ a module for $\lambda \in \Lambda$. Prove that the injections $\iota_\lambda : M_\kappa \to \bigoplus M_\lambda$ induce an injection

$$\bigoplus \text{Hom}(L, M_\lambda) \hookrightarrow \text{Hom}(L, \bigoplus M_\lambda),$$

and that it is an isomorphism if $L$ is finitely generated.

Solution: For $\lambda \in \Lambda$, let $\alpha_\lambda : L \to M_\lambda$ be maps, almost all 0. Then

$$(\sum \iota_\lambda \alpha_\lambda)(l) = (\alpha_\lambda(l)) \in \bigoplus M_\lambda.$$  

So if $\sum \iota_\lambda \alpha_\lambda = 0$, then $\alpha_\lambda = 0$ for all $\lambda$. Thus the $\iota_\lambda$ induce an injection.

Assume $L$ is finitely generated, say by $l_1, \ldots, l_k$. Let $\alpha : L \to \bigoplus M_\lambda$ be a map. Then each $\alpha(l_i)$ lies in a finite direct subsum of $\bigoplus M_\lambda$. So $\alpha(L)$ lies in one too. Set $\alpha_\kappa := \pi_\kappa \alpha$ for all $\kappa \in \Lambda$. Then almost all $\alpha_\kappa$ vanish. So $(\alpha_\kappa)$ lies in $\bigoplus \text{Hom}(L, M_\lambda)$, and $\sum \iota_\kappa \alpha_\kappa = \alpha$. Thus the $\iota_\kappa$ induce a surjection, so an isomorphism.
EXERCISE (4.20). — Let $a$ be an ideal, $\Lambda$ a nonempty set, $M_\Lambda$ a module for $\lambda \in \Lambda$. Prove $a(\bigoplus M_\lambda) = \bigoplus aM_\lambda$. Prove $a(\prod M_\lambda) = \prod aM_\lambda$ if $a$ is finitely generated.

**Solution:** First, $a(\bigoplus M_\lambda) \subseteq \bigoplus aM_\lambda$ because $a \cdot (m_\lambda) = (am_\lambda)$. Conversely, $a(\bigoplus M_\lambda) \supseteq \bigoplus aM_\lambda$ because $(a\lambda m_\lambda) = \sum a\lambda m_\lambda$ since the sum is finite.

Second, $a(\prod M_\lambda) \subset \prod aM_\lambda$ as $a(m_\lambda) = (am_\lambda)$. Conversely, say $a$ is generated by $f_1, \ldots, f_n$. Then $a(\prod M_\lambda) \supseteq \prod aM_\lambda$. Indeed, take $(m_\lambda) \in \prod aM_\lambda$. Then for each $\lambda$, there is $n_\lambda$ such that $m_\lambda = \sum a\lambda m_\lambda$ with $a\lambda \in a$ and $m_\lambda \in M_\lambda$. Write $a\lambda = \sum_{i=1}^n x_{\lambda ij} f_i$ with the $x_{\lambda ij}$ scalars. Then

$$(m_\lambda) = \left( \sum_{j=1}^n \sum_{i=1}^m f_i x_{\lambda ij} m_\lambda \right) = \sum_{i=1}^n f_i \left( \sum_{j=1}^n x_{\lambda ij} m_\lambda \right) \in a(\prod M_\lambda). \quad \Box$$

5. Exact Sequences

EXERCISE (5.5). — Let $M'$ and $M''$ be modules, $N \subset M'$ a submodule. Set $M := M' \oplus M''$. Using (5.12)(1) and (5.8) and (5.9), prove $M/N = M'/N \oplus M''$.

**Solution:** By (5.12)(1) and (5.9), the two sequences $0 \to M'' \to M' \to 0$ and $0 \to N \to M' \to M'/N \to 0$ are exact. So by (5.9), the sequence

$$0 \to N \to M' \oplus M'' \to (M'/N) \oplus M'' \to 0$$

is exact. Thus (6.3) yields the assertion. \quad \Box

EXERCISE (5.6). — Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence. Prove that, if $M'$ and $M''$ are finitely generated, then so is $M$.

**Solution:** Let $m_1', \ldots, m_n' \in M$ map to elements generating $M''$. Let $m \in M$, and write its image in $M''$ as a linear combination of the images of the $m_1'$. Let $m'' \in M$ be the same combination of the $m_1'$. Set $m' := m - m''$. Then $m'$ maps to $0$ in $M''$; so $m'$ is the image of an element of $M'$.

Let $m_1', \ldots, m_{i'}' \in M$ be the images of elements generating $M'$. Then $m'$ is a linear combination of the $m_{i'}'$. So $m$ is a linear combination of the $m_{i'}'$ and $m_i'$. Thus the $m_{i'}'$ and $m_i'$ together generate $M$. \quad \Box

EXERCISE (5.11). — Let $M'$, $M''$ be modules, and set $M := M' \oplus M''$. Let $N$ be a submodule of $M$ containing $M'$, and set $N'' := N \cap M''$. Prove $N = M' \oplus N''$.

**Solution:** Form the sequence $0 \to M' \to N \to \pi_{M''} N \to 0$. It splits by (5.11) as $(\pi_{M''} N) \circ \iota_M = 1_{M'}$. Finally, if $(m', m'') \in N$, then $(0, m'') \in N$ as $M' \subset N$; hence, $\pi_{M''} N = N''$. \quad \Box

EXERCISE (5.12). — Criticize the following misstatement of (5.14): given a 3-term exact sequence $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$, there is an isomorphism $M \cong M' \oplus M''$ if and only if there is a section $\sigma: M'' \to M$ of $\beta$ and $\alpha$ is injective.

**Solution:** We have $\alpha: M' \to M$, and $\iota_{M'}: M' \to M' \oplus M''$, but (5.12) requires that they be compatible with the isomorphism $M \cong M' \oplus M''$ and, similarly for $\beta: M \to M''$ and $\pi_{M''}: M' \oplus M'' \to M''$.

Let’s construct a counterexample (due to B. Noohi). For each integer $n \geq 2$, let $M_n$ be the direct sum of countably many copies of $\mathbb{Z}/(n)$. Set $M := \bigoplus M_n$.

First, let us check these two statements:
(1) For any finite abelian group $G$, we have $G \oplus M \simeq M$.

(2) For any finite subgroup $G \subset M$, we have $M/G \simeq M$.

Statement (1) holds since $G$ is isomorphic to a direct sum of copies of $\mathbb{Z}/\langle n \rangle$ for various $n$ by the structure theorem for finite abelian groups [24 (6.4), p. 472], [24 Thm. 13.3, p. 200].

To prove (2), write $M = B \bigoplus M'$, where $B$ contains $G$ and involves only finitely many components of $M$. Then $M' \simeq M$. Therefore, (6.11) and (1) yield

$$M/G \simeq (B/G) \oplus M' \simeq M.$$  

To construct the counterexample, let $p$ be a prime number. Take one of the $\mathbb{Z}/\langle p^2 \rangle$ components of $M$, and let $M' \subset \mathbb{Z}/\langle p^2 \rangle$ be the cyclic subgroup of order $p$. There is no retraction $\mathbb{Z}/\langle p^2 \rangle \to M'$, so there is no retraction $M \to M'$ either, since the latter would induce the former. Finally, take $M'' := M/M'$. Then (1) and (2) yield $M \simeq M' \oplus M''$.

\[\square\]

**Exercise (5.14).** — Referring to (4.8), give an alternative proof that $\beta$ is an isomorphism by applying the Snake Lemma to the diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & M \\
& \downarrow & \kappa \\
& \beta & \\
0 & \longrightarrow & M/L \\
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & N/M \\
& \downarrow & \gamma \\
& & 0 \\
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & (N/L)/(M/L) \\
& & 0 \\
\end{array}
\]

**Solution:** The Snake Lemma yields an exact sequence,

$$L \xrightarrow{\lambda} L \rightarrow \text{Ker}(\beta) \rightarrow 0;$$

hence, Ker$(\beta)$ = 0. Moreover, $\beta$ is surjective because $\kappa$ and $\lambda$ are.

\[\square\]

**Exercise (5.15) (Five Lemma).** — Consider this commutative diagram:

$$
\begin{array}{ccc}
M_4 & \xrightarrow{\alpha_4} & M_3 \\
\gamma_4 & \downarrow & \gamma_3 \\
N_4 & \xrightarrow{\beta_3} & N_3 \\
\end{array} \quad \begin{array}{ccc}
M_3 & \xrightarrow{\alpha_3} & M_2 \\
\gamma_2 & \downarrow & \gamma_1 \\
N_3 & \xrightarrow{\beta_2} & N_2 \\
\end{array} \quad \begin{array}{ccc}
M_2 & \xrightarrow{\alpha_2} & M_1 \\
\gamma_1 & \downarrow & \gamma_0 \\
N_2 & \xrightarrow{\beta_1} & N_1 \\
\end{array} \quad \begin{array}{ccc}
M_1 & \xrightarrow{\alpha_1} & M_0 \\
\gamma_0 & \downarrow & \gamma_0 \\
N_1 & \xrightarrow{\beta_0} & N_0 \\
\end{array}
$$

Assume it has exact rows. Via a chase, prove these two statements:

(1) If $\gamma_3$ and $\gamma_1$ are surjective and if $\gamma_0$ is injective, then $\gamma_2$ is surjective.

(2) If $\gamma_3$ and $\gamma_1$ are injective and if $\gamma_4$ is surjective, then $\gamma_2$ is injective.

**Solution:** Let’s prove (1). Take $n_2 \in N_2$. Since $\gamma_1$ is surjective, there is $m_1 \in M_1$ such that $\gamma_1(m_1) = \beta_2(n_2)$. Then $\gamma_0\alpha_1(m_1) = \beta_1\gamma_1(m_1) = \beta_1\beta_2(n_2) = 0$ by commutativity and exactness. Since $\gamma_0$ is injective, $\alpha_1(m_1) = 0$. Hence exactness yields $m_2 \in M_2$ with $\alpha_2(m_2) = m_1$. So $\beta_2(\gamma_2(m_2) - n_2) = \gamma_1\alpha_2(m_2) - \beta_2(n_2) = 0$.

Hence exactness yields $n_3 \in N_3$ with $\beta_3(n_3) = \gamma_2(m_2) - n_2$. Since $\gamma_3$ is surjective, there is $m_3 \in M_3$ with $\gamma_3(m_3) = n_3$. Then $\gamma_2\alpha_3(m_3) = \beta_3\gamma_3(m_3) = \gamma_2(m_2) - n_2$. Hence $\gamma_2(m_2 - \alpha_3(m_3)) = n_2$. Thus $\gamma_2$ is surjective.

The proof of (2) is similar. 

\[\square\]
**Exercise (5.16) (Nine Lemma).** — Consider this commutative diagram:

\[
\begin{array}{ccc}
0 & \to & L' \\
\downarrow & & \downarrow \\
0 & \to & L \\
\downarrow & & \downarrow \\
0 & \to & L'' \\
\downarrow & & \downarrow \\
0 & \to & M' \\
\downarrow & & \downarrow \\
0 & \to & M \\
\downarrow & & \downarrow \\
0 & \to & M'' \\
\downarrow & & \downarrow \\
0 & \to & N' \\
\downarrow & & \downarrow \\
0 & \to & N \\
\downarrow & & \downarrow \\
0 & \to & N'' \\
\end{array}
\]

Assume all the columns are exact and the middle row is exact. Prove that the first row is exact if and only if the third is.

**Solution:** The first row is exact if the third is owing to the Snake Lemma \((5.13)\) applied to the bottom two rows. The converse is proved similarly. □

**Exercise (5.17).** — Consider this commutative diagram with exact rows:

\[
\begin{array}{ccc}
M' & \xrightarrow{\beta'} & M \\
\downarrow & \alpha & \downarrow \\
N' & \xrightarrow{\gamma'} & N \\
\end{array}
\]

Assume \(\alpha'\) and \(\gamma\) are surjective. Given \(n \in N\) and \(m'' \in M''\) with \(\alpha''(m'') = \gamma'(n)\), show that there is \(m \in M\) such that \(\alpha(m) = n\) and \(\gamma(m) = m''\).

**Solution:** Since \(\gamma\) is surjective, there is \(m_1 \in M\) with \(\gamma(m_1) = m''\). Then \(\gamma'(n - \alpha(m_1)) = 0\) as \(\alpha''(m'') = \gamma'(n)\) and as the right-hand square is commutative. So by exactness of the bottom row, there is \(n' \in N'\) with \(\beta'(n') = n - \alpha(m_1)\). Since \(\alpha'\) is surjective, there is \(m' \in M'\) with \(\alpha'(m') = n'\). Set \(m := m_1 + \beta(m')\). Then \(\gamma(m) = m''\) as \(\alpha = 0\). Further, \(\alpha(m) = \alpha(m_1) + \beta'(n') = n\) as the left-hand square is commutative. Thus \(m\) works. □

**Exercise (5.22).** — Show that a free module \(R^{\oplus \Lambda}\) is projective.

**Solution:** Given \(\beta: M \to N\) and \(\alpha: R^{\oplus \Lambda} \to N\), use the UMP of \((4.11)\) to define \(\gamma: R^{\oplus \Lambda} \to M\) by sending the standard basis vector \(e_{\lambda}\) to any lift of \(\alpha(e_{\lambda})\), that is, any \(m_{\lambda} \in M\) with \(\beta(m_{\lambda}) = \alpha(e_{\lambda})\). (The Axiom of Choice permits a simultaneous choice of all \(m_{\lambda}\) if \(\Lambda\) is infinite.) Clearly \(\alpha = \beta \gamma\). Thus \(R^{\oplus \Lambda}\) is projective. □

**Exercise (5.24).** — Let \(R\) be a ring, \(P\) and \(N\) finitely generated modules with \(P\) projective. Prove Hom\((P, N)\) is finitely generated, and is finitely presented if \(N\) is.

**Solution:** Since \(P\) is finitely generated, there is a surjection \(R^{\oplus m} \xrightarrow{\alpha} P\) for some \(m\) by \((3.14)\). Set \(K := \text{Ker}(\alpha)\). Since \(P\) is projective, the sequence

\[
0 \to K \to R^{\oplus m} \to P \to 0
\]

splits by \((5.20)\). Hence Hom\((P, N) \oplus \text{Hom}(K, N) = \text{Hom}(R^{\oplus m}, N)\) by \((4.15.2)\). But Hom\((R^{\oplus m}, N) = \text{Hom}(R, N)^{\oplus m} = N^{\oplus m}\) by \((4.15.2)\) and \((1.8)\). So since \(N\) is finitely generated, Hom\((R^{\oplus m}, N)\) is finitely generated too. Now, Hom\((P, N)\) is a
Let \(\text{Hom}(R^{\oplus m}, N)\) by (5.34). So \(\text{Hom}(P, N)\) is finitely generated too.

Suppose now there is a finite presentation \(F_2 \to F_1 \to N \to 0\). Then (5.24) and (5.28) yield the exact sequence

\[
\text{Hom}(R^{\oplus m}, F_2) \to \text{Hom}(R^{\oplus m}, F_1) \to \text{Hom}(R^{\oplus m}, N) \to 0.
\]

But the \(\text{Hom}(R^{\oplus m}, F_i)\) are free of finite rank by (14.1.3) and (14.1.9). Thus \(\text{Hom}(R^{\oplus m}, N)\) is finitely presented.

As above, \(\text{Hom}(K, N)\) is finitely generated. Consider the (split) exact sequence

\[
0 \to \text{Hom}(K, N) \to \text{Hom}(R^{\oplus m}, N) \to \text{Hom}(P, N) \to 0.
\]

Thus (5.28) implies \(\text{Hom}(P, N)\) is finitely presented.

\(\square\)

**Exercise (5.26).** Let \(R\) be a ring, and \(0 \to L \to R^n \to M \to 0\) an exact sequence. Prove \(M\) is finitely presented if and only if \(L\) is finitely presented.

**Solution:** Assume \(M\) is finitely presented; say \(R^\ell \to R^m \to M \to 0\) is a finite presentation. Let \(L'\) be the image of \(R^\ell\). Then \(L' \oplus R^n \cong L \oplus R^m\) by Schanuel’s Lemma (5.26). Hence \(L\) is a quotient of \(R^\ell \oplus R^n\). Thus \(L\) is finitely generated.

Conversely, assume \(L\) is generated by \(\ell\) elements. They yield a surjection \(R^\ell \twoheadrightarrow L\) by (4.10)(1). It yields a sequence \(R^\ell \to R^n \to M \to 0\). The latter is, plainly, exact. Thus \(M\) is finitely presented.

\(\square\)

**Exercise (5.27).** Let \(R\) be a ring, \(X_1, X_2, \ldots\) infinitely many variables. Set \(P := R[X_1, X_2, \ldots]\) and \(M := P/\langle X_1, X_2, \ldots \rangle\). Is \(M\) finitely presented? Explain.

**Solution:** No, otherwise by (5.28), the ideal \(\langle X_1, X_2, \ldots \rangle\) would be generated by some \(f_1, \ldots, f_n \in P\), so also by \(X_1, \ldots, X_m\) for some \(m\), but plainly it isn’t.

\(\square\)

**Exercise (5.29).** Let \(0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0\) be a short exact sequence with \(M\) finitely generated and \(N\) finitely presented. Prove \(L\) is finitely generated.

**Solution:** Let \(R\) be the ground ring. Say \(M\) is generated by \(m\) elements. They yield a surjection \(\mu: R^m \twoheadrightarrow M\) by (11.1)(1). As in (5.28), \(\mu\) induces the following commutative diagram, with \(\lambda\) surjective:

\[
\begin{array}{ccc}
0 & \to & K \\
\downarrow{\lambda} & & \downarrow{\mu} \\
0 & \to & L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0
\end{array}
\]

By (5.28), \(K\) is finitely generated. Thus \(L\) is too, as \(\lambda\) is surjective.

\(\square\)

**Exercise (5.36).** Let \(R\) be a ring, and \(a_1, \ldots, a_m \in R\) with \(\langle a_1 \rangle \supset \cdots \supset \langle a_m \rangle\). Set \(M := (R/\langle a_1 \rangle) \oplus \cdots \oplus (R/\langle a_m \rangle)\). Show that \(F_r(M) = \langle a_1 \cdots a_{m-r} \rangle\).

**Solution:** Form the presentation \(R^m \xrightarrow{\alpha} R^m \to M \to 0\) where \(\alpha\) has matrix

\[
A = \begin{pmatrix}
a_1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & a_m
\end{pmatrix}
\]

Set \(s := m - r\). Now, \(a_i \in \langle a_{i-1} \rangle\) for all \(i > 1\). Hence \(a_i \cdots a_s \in \langle a_1 \cdots a_s \rangle\) for all \(1 \leq i_1 < \cdots < i_s \leq m\). Thus \(I_s(A) = \langle a_1 \cdots a_s \rangle\), as desired.

\(\square\)
Exercise (5.37). — In the setup of (5.36), assume \( a_1 \) is a nonunit.

(1) Show that \( m \) is the smallest integer such that \( F_m(M) = R \).

(2) Let \( n \) be the largest integer such that \( F_n(M) = (0) \); set \( k := m - n \). Assume \( R \) is a domain. Show (a) that \( a_i \neq 0 \) for \( i < k \) and \( a_i = 0 \) for \( i \geq k \), and (b) that \( M \) determines each \( a_i \) up to unit multiple.

Solution: For (1), note there’s a presentation \( R^m \rightarrow R^m \rightarrow M \rightarrow 0 \); see the solution to (5.36). So \( F_m(M) = R \) by (5.36). On the other hand, \( F_{m-1}(M) = \langle a_1 \rangle \) by (5.36). So \( F_{m-1}(M) \neq R \) as \( a_1 \) is a nonunit. Thus (1) holds.

For (2)(a), note \( F_{n+1}(M) \neq (0) \) and \( F_n(M) = (0) \). Hence \( a_1 \cdots a_{k-1} \neq 0 \) and \( a_1 \cdots a_k = 0 \) by (5.36). But \( R \) is a domain. Hence \( a_1, \ldots, a_i \neq 0 \) for \( i < k \) and \( a_k = 0 \). But \( \langle a_k \rangle \supset \cdots \supset \langle a_m \rangle \). Hence \( a_i = 0 \) for \( i \geq k \). Thus (2)(a) holds.

For (2)(b), given \( b_1, \ldots, b_p \in R \) with \( b_1 \) a nonunit, with \( \langle b_1 \rangle \supset \cdots \supset \langle b_p \rangle \) and \( M = \langle R/b_1 \rangle \oplus \cdots \oplus \langle R/b_p \rangle \), note that (1) yields \( p = m \) and that (2)(a) yields \( b_i \neq 0 \) for \( i < k \) and \( b_i = 0 \) for \( i \geq k \).

Given \( i \), (5.37) yields \( \langle a_1 \cdots a_i \rangle = \langle b_1 \cdots b_i \rangle \). But \( R \) is a domain. So (4.12) yields a unit \( u_i \) such that \( a_1 \cdots a_i = u_i b_1 \cdots b_i \). So

\[
u_{i-1} b_1 \cdots b_{i-1} a_i = u_i b_1 \cdots b_i.
\]

If \( i < k \), then \( b_1 \cdots b_{i-1} \neq 0 \); whence, \( u_{i-1} a_i = u_i b_i \). Thus (2)(b) holds. \( \square \)

6. Direct Limits

Exercise (6.3). — (1) Show that the condition (6.2)(1) is equivalent to the commutativity of the corresponding diagram:

\[
\begin{array}{ccc}
\text{Hom}_C(B, C) & \rightarrow & \text{Hom}_{C'}(F(B), F(C)) \\
\downarrow & & \downarrow \\
\text{Hom}_C(A, C) & \rightarrow & \text{Hom}_{C'}(F(A), F(C))
\end{array}
\]  

(6.3.1)

(2) Given \( \gamma : C \rightarrow D \), show (6.2)(1) yields the commutativity of this diagram:

\[
\begin{array}{ccc}
\text{Hom}_C(B, C) & \rightarrow & \text{Hom}_{C'}(F(B), F(C)) \\
\downarrow & & \downarrow \\
\text{Hom}_C(A, D) & \rightarrow & \text{Hom}_{C'}(F(A), F(D))
\end{array}
\]

Solution: In (6.3.1), the left-hand vertical map is given by composition with \( \alpha \), and the right-hand vertical map is given by composition with \( F(\alpha) \). So the composition of the top map and the right-hand map sends \( \beta \) to \( F(\beta)F(\alpha) \), whereas the composition of the left-hand map with the bottom map sends \( \beta \) to \( F(\beta \alpha) \). These two images are always equal if and only if (6.3.1) commutes. Thus (1) holds if and only if (6.3.1).

As to (2), the argument is similar. \( \square \)

Exercise (6.5). — Let \( \mathcal{C} \) and \( \mathcal{C}' \) be categories, \( F : \mathcal{C} \rightarrow \mathcal{C}' \) and \( F' : \mathcal{C}' \rightarrow \mathcal{C} \) an adjoint pair. Let \( \varphi_{A,A'} : \text{Hom}_\mathcal{C}(FA, A') \cong \text{Hom}_\mathcal{C}(A, F'A') \) denote the natural bijection, and set \( \eta_A := \varphi_{A,F(A)}(1_{FA}) \). Do the following:
(1) Prove $\eta_A$ is natural in $A$; that is, given $g: A \to B$, the induced square

$$
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & F'FA \\
g \downarrow & & \downarrow F'g \\
B & \xrightarrow{\eta_B} & F'FB
\end{array}
$$

is commutative. We call the natural transformation $A \mapsto \eta_A$ the **unit** of $(F, F')$.

(2) Given $f': FA \to A'$, prove $\varphi_{A,A'}(f') = F'f' \circ \eta_A$.

(3) Prove the natural map $\eta_A: A \to F'FA$ is **universal** from $A$ to $F'$; that is, given $f: A \to F'A'$, there is a unique map $f': FA \to A'$ with $F'f' \circ \eta_A = f$.

(4) Conversely, instead of assuming $(F, F')$ is an adjoint pair, assume given a natural transformation $\eta: 1_C \to F'$ satisfying (1) and (3). Prove the equation in (2) defines a natural bijection making $(F, F')$ an adjoint pair, whose unit is $\eta$.

(5) Identify the units in the two examples in (6.3): the “free module” functor and the “polynomial ring” functor.

(Dually, we can define a **counit** $\varepsilon: FF' \to 1_C$, and prove similar statements.)

**Solution:** For (1), form this canonical diagram, with horizontal induced maps:

$$
\begin{array}{ccc}
\text{Hom}_C(FA, FA) & \xrightarrow{(Fg)_*} & \text{Hom}_C(FA, FB) & \xleftarrow{(Fg)^*} & \text{Hom}_C(FB, FB) \\
\varphi_{A,FA} & & \varphi_{A,FB} & & \varphi_{B,FB} \\
\text{Hom}_C(A, F'FA) & \xrightarrow{(F'g)_*} & \text{Hom}_C(A, F'FB) & \xleftarrow{\varphi_{B,FA}} & \text{Hom}_C(B, F'FB)
\end{array}
$$

It commutes since $\varphi$ is natural. Follow $1_{FA}$ out of the upper left corner to find $F'Fg \circ \eta_A = \varphi_{A,FB}(Fg)$ in $\text{Hom}_C(A, F'FB)$. Follow $1_{FB}$ out of the upper right corner to find $\varphi_{A,FB}(Fg) = \eta_B \circ g$ in $\text{Hom}_C(A, F'FB)$. Thus $(F'Fg) \circ \eta_A = \eta_B \circ g$.

For (2), form this canonical commutative diagram:

$$
\begin{array}{ccc}
\text{Hom}_C(FA, FA) & \xrightarrow{f'} & \text{Hom}_C(FA, A') \\
\varphi_{A,FA} & & \varphi_{A,A'} \\
\text{Hom}_C(A, F'FA) & \xrightarrow{(F'f')_*} & \text{Hom}_C(A, F'A')
\end{array}
$$

Follow $1_{FA}$ out of the upper left-hand corner to find $\varphi_{A,A'}(f') = F'f' \circ \eta_A$.

For (3), given an $f$, note that (2) yields $\varphi_{A,A'}(f') = f$; whence, $f' = \varphi_{A,A'}^{-1}(f)$. Thus $f'$ is unique. Further, an $f'$ exists: just set $f' := \varphi_{A,A'}^{-1}(f)$.

For (4), set $\psi_{A,A'}(f') := F'f' \circ \eta_A$. As $\eta_A$ is universal, given $f: A \to F'A'$, there is a unique $f': FA \to A'$ with $F'f' \circ \eta_A = f$. Thus $\psi_{A,A'}$ is a bijection:

$$
\psi_{A,A'} : \text{Hom}_C(FA, A') \xrightarrow{\sim} \text{Hom}_C(A, F'A').
$$

Also, $\psi_{A,A'}$ is natural in $A$, as $\eta_A$ is natural in $A$ and $F'$ is a functor. And, $\psi_{A,A'}$ is natural in $A'$, as $F'$ is a functor. Clearly, $\psi_{A,FA}(1_{FA}) = \eta_A$. Thus (4) holds.

For (5), use the notation of (6.3). Clearly, if $F$ is the “free module” functor, then $\eta_A: \Lambda \to R^\Lambda$ carries an element of $\Lambda$ to the corresponding standard basis vector. Further, if $F$ is the “polynomial ring” functor and if $A$ is the set of variables $X_1, \ldots, X_n$, then $\eta_A(X_i)$ is just $X_i$ viewed in $R[X_1, \ldots, X_n]$. □

**Exercise (6.9).** — Let $\alpha: L \to M$ and $\beta: L \to N$ be two maps in a category $\mathcal{C}$. Their **pushout** is defined as the object of $\mathcal{C}$ universal among objects $P$ equipped with a pair of maps $\gamma: M \to P$ and $\delta: N \to P$ such that $\gamma \alpha = \delta \beta$. Express the
pushout as a direct limit. Show that, in \((\text{Sets})\), the pushout is the disjoint union \(M \sqcup N\) modulo the smallest equivalence relation \(\sim\) with \(m \sim n\) if there is \(\ell \in L\) with \(\alpha(\ell) = m\) and \(\beta(\ell) = n\). Show that, in \((\text{R-mod})\), the pushout is equal to the direct sum \(M \oplus N\) modulo the image of \(L\) under the map \((\alpha, -\beta)\).

**Solution:** Let \(\Lambda\) be the category with three objects \(\lambda\), \(\mu\), and \(\nu\) and two non-identity maps \(\lambda \to \mu\) and \(\lambda \to \nu\). Define a functor \(\lambda \mapsto M_\lambda\) by \(M_\lambda := L\), \(M_\mu := M\), \(M_\nu := N\), \(\alpha_\mu := \alpha\), and \(\alpha_\nu := \beta\). Set \(Q := \varinjlim M_\lambda\). Then writing

\[
\begin{array}{ccc}
N & \xleftarrow{\beta} & L \\
\eta_\nu & \downarrow & \eta_\lambda & \downarrow \\
Q & \xrightarrow{1_R} & Q & \xrightarrow{1_R} \\
\end{array}
\quad \text{as} \quad
\begin{array}{ccc}
L & \xrightarrow{\alpha} & M \\
\beta & \downarrow & \eta_\mu \\
N & \xrightarrow{\eta_\nu} & Q
\end{array}
\]

we see that \(Q\) is equal to the pushout of \(\alpha\) and \(\beta\); here \(\gamma = \eta_\mu\) and \(\delta = \eta_\nu\).

In \((\text{Sets})\), take \(\gamma\) and \(\delta\) to be the inclusions followed by the quotient map. Clearly \(\gamma\alpha = \delta\beta\). Further, given \(P\) and maps \(\gamma' : M \to P\) and \(\delta' : N \to P\), they define a unique map \(M \sqcup N \to P\), and it factors through the quotient if and only if \(\gamma'\alpha = \delta'\beta\). Thus \((M \sqcup N)\//\sim\) is the pushout.

In \((\text{R-mod})\), take \(\gamma\) and \(\delta\) to be the inclusions followed by the quotient map. Then for all \(\ell \in L\), clearly \(\iota_M \alpha(\ell) - \iota_N \beta(\ell) = (\alpha(\ell), -\beta(\ell))\). Hence \(\iota_M \alpha(\ell) - \iota_N \beta(\ell)\) is in \(\text{Im}(L)\). Hence, \(\iota_M \alpha(\ell)\) and \(\iota_N \beta(\ell)\) have the same image in the quotient. Thus \(\gamma\alpha = \delta\beta\). Given \(\gamma' : M \to P\) and \(\delta' : N \to P\), they define a unique map \(M \oplus N \to P\), and it factors through the quotient if and only if \(\gamma'\alpha = \delta'\beta\). Thus \((M \oplus N)\//\text{Im}(L)\) is the pushout.

**Exercise (6.16).** — Let \(\mathcal{C}\) be a category, \(\Sigma\) and \(\Lambda\) small categories.

1. Prove \(\mathcal{C}^{\Sigma \times \Lambda} = (\mathcal{C}^\Lambda)^\Sigma\) with \((\sigma, \lambda) \mapsto M_{\sigma, \lambda}\) corresponding to \(\sigma \mapsto (\lambda \mapsto M_{\sigma, \lambda})\).

2. Assume \(\mathcal{C}\) has direct limits indexed by \(\Sigma\) and by \(\Lambda\). Prove that \(\mathcal{C}\) has direct limits indexed by \(\Sigma \times \Lambda\) and that \(\varinjlim_{\lambda \in \Lambda} \varinjlim_{\sigma \in \Sigma} \mathcal{C}^{\Sigma \times \Lambda} = \varinjlim_{(\sigma, \lambda) \in \Sigma \times \Lambda} \mathcal{C}^{\Sigma \times \Lambda}\).

**Solution:** Consider (1). In \(\Sigma \times \Lambda\), a map \((\sigma, \lambda) \to (\tau, \mu)\) factors in two ways:

\[
(\sigma, \lambda) \to (\tau, \lambda) \to (\tau, \mu) \quad \text{and} \quad (\sigma, \lambda) \to (\sigma, \mu) \to (\tau, \mu).
\]

So, given a functor \((\sigma, \lambda) \mapsto M_{\sigma, \lambda}\), there is a commutative diagram like \((6.13.1)\). It shows that the map \(\sigma \to \tau\) in \(\Sigma\) induces a natural transformation from \(\lambda \mapsto M_{\sigma, \lambda}\) to \(\lambda \mapsto M_{\tau, \lambda}\). Thus the rule \(\sigma \mapsto (\lambda \mapsto M_{\sigma, \lambda})\) is a functor from \(\Sigma\) to \(\mathcal{C}^\Lambda\).

A map from \((\sigma, \lambda) \mapsto M_{\sigma, \lambda}\) to a second functor \((\sigma, \lambda) \mapsto N_{\sigma, \lambda}\) is a collection of maps \(\theta_{\sigma, \lambda} : M_{\sigma, \lambda} \to N_{\sigma, \lambda}\) such that, for every map \((\sigma, \lambda) \to (\tau, \mu)\), the square

\[
\begin{array}{ccc}
M_{\sigma, \lambda} & \xrightarrow{\theta_{\sigma, \lambda}} & M_{\tau, \mu} \\
\downarrow & & \downarrow \\
N_{\sigma, \lambda} & \xrightarrow{\theta_{\sigma, \mu}} & N_{\tau, \mu}
\end{array}
\]

is commutative. Factoring \((\sigma, \lambda) \to (\tau, \mu)\) in two ways as above, we get a commutative cube. It shows that the \(\theta_{\sigma, \lambda}\) define a map in \((\mathcal{C}^\Lambda)^\Sigma\).

This passage from \(\mathcal{C}^{\Sigma \times \Lambda}\) to \((\mathcal{C}^\Lambda)^\Sigma\) is reversible. Thus (1) holds.

As to (2), assume \(\mathcal{C}\) has direct limits indexed by \(\Sigma\) and \(\Lambda\). Then \(\mathcal{C}^\Lambda\) has direct limits indexed by \(\Sigma\) by \((6.13)\). So the functors \(\varinjlim_{\lambda \in \Lambda} : \mathcal{C}^\Lambda \to \mathcal{C}\) and \(\varinjlim_{\sigma \in \Sigma} : (\mathcal{C}^\Lambda)^\Sigma \to \mathcal{C}^\Lambda\) exist, and they are the left adjoints of the diagonal functors \(\mathcal{C} \to \mathcal{C}^\Lambda\) and \(\mathcal{C}^\Lambda \to (\mathcal{C}^\Lambda)^\Sigma\) by \((6.8)\). Hence the composition \(\varinjlim_{\lambda \in \Lambda} \varinjlim_{\sigma \in \Sigma}\) is
the left adjoint of the composition of the two diagonal functors. But the latter is just the diagonal \( C \to C^{\Delta} \) owing to (1). So this diagonal has a left adjoint, which is necessarily \( \lim_{\Delta} \) owing to the uniqueness of adjoints. Thus (2) holds. □

**Exercise (6.17).** Let \( \lambda \mapsto M_\lambda \) and \( \lambda \mapsto N_\lambda \) be two functors from a small category \( \Lambda \) to \( ((R \text{-mod})) \), and \( \{ \theta_\lambda : M_\lambda \to N_\lambda \} \) a natural transformation. Show

\[
\lim \text{Coker}(\theta_\lambda) = \text{Coker}(\lim M_\lambda \to \lim N_\lambda).
\]

Show that the analogous statement for kernels can be false by constructing a counterexample using the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
Z & \xrightarrow{\mu_2} & Z \\
\mu_2 & & \mu_2 \\
Z & \xrightarrow{\mu_2} & Z / \langle 2 \rangle
\end{array}
\]

**Solution:** By (6.8), the cokernel is a direct limit, and by (6.14), direct limits commute; thus, the asserted equation holds.

To construct the desired counterexample using the given diagram, view its rows as expressing the cokernel \( Z / \langle 2 \rangle \) as a direct limit over the category \( \Lambda \) of (6.8). View the left two columns as expressing a natural transformation \( \theta_\lambda \), and view the third column as expressing the induced map between the two limits. The latter map is 0; so its kernel is \( Z / \langle 2 \rangle \). However, \( \text{Ker}(\theta_\lambda) = 0 \) for \( \lambda \in \Lambda \); so \( \lim \text{Ker}(\theta_\lambda) = 0 \). □

**7. Filtered direct limits**

**Exercise (7.2).** Let \( R \) be a ring, \( M \) a module, \( \Lambda \) a set, \( M_\lambda \) a submodule for each \( \lambda \in \Lambda \). Assume \( \bigcup M_\lambda = M \). Assume, given \( \lambda, \mu \in \Lambda \), there is \( \nu \in \Lambda \) such that \( M_\lambda, M_\mu \subset M_\nu \). Order \( \Lambda \) by inclusion: \( \lambda \leq \mu \) if \( M_\lambda \subset M_\mu \). Prove \( M = \lim M_\lambda \).

**Solution:** Let us prove that \( M \) has the UMP characterizing \( \lim M_\lambda \). Given homomorphisms \( \beta_\lambda : M_\lambda \to P \) with \( \beta_\lambda = \beta_\nu |_{M_\lambda} \) when \( \lambda \leq \nu \), define \( \beta : M \to P \) by \( \beta(m) := \beta_\lambda(m) \) if \( m \in M_\lambda \). Such a \( \lambda \) exists as \( \bigcup M_\lambda = M \). If also \( m \in M_\mu \) and \( M_\lambda, M_\mu \subset M_\nu \), then \( \beta_\lambda(m) = \beta_\mu(m) = \beta_\nu(m) \); so \( \beta \) is well defined. Clearly, \( \beta : M \to P \) is the unique set map such that \( \beta |_{M_\lambda} = \beta_\lambda \). Further, given \( m, n \in M \) and \( x \in R \), there is \( \nu \) such that \( m, n \in M_\nu \). So \( \beta(m+n) = \beta_\nu(m+n) = \beta_\nu(m) + \beta_\nu(n) \) and \( \beta(xm) = \beta_\nu(xm) = x \beta_\nu(m) \). Thus \( \beta \) is \( R \)-linear. Thus \( M = \lim M_\lambda \). □

**Exercise (7.3).** Show that every module \( M \) is the filtered direct limit of its finitely generated submodules.

**Solution:** Every element \( m \in M \) belongs to the submodule generated by \( m \); hence, \( M \) is the union of all its finitely generated submodules. Any two finitely generated submodules are contained in a third, for example, their sum. So the assertion results from (7.2) with \( \Lambda \) the set of all finite subsets of \( M \). □

**Exercise (7.4).** Show that every direct sum of modules is the filtered direct limit of its finite direct summands.
Consider an element of the direct sum. It has only finitely many nonzero components. So it lies in the corresponding finite direct sum. Thus the union of the subsums is the whole direct sum. Now, given any two finite direct subsums, their sum is a third. Thus the finite subsets of indices form a directed partially ordered set \( \Lambda \). So the assertion results from (7.2).

**Exercise (7.9).** — Let \( R := \lim_{n} R_{\lambda} \) be a filtered direct limit of rings.

1. Prove that \( R = 0 \) if and only if \( R_{\lambda} = 0 \) for some \( \lambda \).
2. Assume that each \( R_{\lambda} \) is a domain. Prove that \( R \) is a domain.
3. Assume that each \( R_{\lambda} \) is a field. Prove that \( R \) is a field.

**Solution:** For (1), first assume \( R = 0 \). Fix any \( \kappa \). Then \( 1 \in R_{\kappa} \) maps to \( 0 \in R \). So (7.8) with \( Z \) for \( R \) yields some transition map \( \alpha_{\kappa} : R_{\kappa} \to R_{\lambda} \) with \( \alpha_{\kappa} 1 = 0 \). But \( \alpha_{\kappa} 1 = 1 \). Thus 1 = 0 in \( R_{\lambda} \). So \( R_{\lambda} = 0 \) by (1.1).

Conversely, assume \( R_{\lambda} = 0 \). Then 1 = 0 in \( R_{\lambda} \). So 1 = 0 in \( R \), as the transition map \( \alpha_{\lambda} : R_{\lambda} \to R \) carries 1 to 1 and 0 to 0. Thus \( R = 0 \) by (1.1). Thus (1) holds.

In (2), given \( x, y \in R \) with \( xy = 0 \), we can lift \( x, y \) back to some \( x_{\lambda}, y_{\lambda} \in R_{\lambda} \) for some \( \lambda \) by (7.8)(1) and (7.9)(1). Then \( x_{\lambda}y_{\lambda} \) maps to 0 in \( R \). So (7.8)(3) yields a transition map \( \alpha_{\mu} : R_{\mu} \to R_{\lambda} \) with \( \alpha_{\mu}(x_{\lambda}y_{\lambda}) = 0 \). But \( \alpha_{\mu}(x_{\lambda}y_{\lambda}) = \alpha_{\lambda}(x_{\lambda})\alpha_{\mu}(y_{\lambda}) \), and \( R_{\mu} \) is a domain. So either \( \alpha_{\mu}(x_{\lambda}) = 0 \) or \( \alpha_{\mu}(y_{\lambda}) = 0 \). Hence, either \( x = 0 \) or \( y = 0 \). Thus \( R \) is a domain.

For (3), given \( x \in R - 0 \), we can lift \( x \) back to some \( x_{\lambda} \in R_{\lambda} \) for some \( \lambda \) by (7.8)(1). Then \( x_{\lambda} \neq 0 \) as \( x \neq 0 \). But \( R_{\lambda} \) is a field. So there is \( y_{\lambda} \in R_{\lambda} \) with \( x_{\lambda}y_{\lambda} = 1 \). Say \( y_{\lambda} \) maps to \( y \in R \). Then \( xy = 1 \). So \( R \) is a field. Thus (3) holds.

**Exercise (7.10).** — Let \( M := \lim_{n} M_{\lambda} \) be a filtered direct limit of modules, with transition maps \( \alpha_{\lambda} : M_{\lambda} \to M_{\mu} \) and insertions \( \alpha_{\lambda} : M_{\lambda} \to M_{\lambda} \). For each \( \lambda \), let \( N_{\lambda} \subset M_{\lambda} \) be a submodule, and let \( N \subset M \) be a submodule. Prove that \( N_{\lambda} = \alpha_{\lambda}^{-1}N \) for all \( \lambda \) if and only if (a) \( N_{\lambda} = (\alpha_{\lambda})^{-1}N_{\mu} \) for all \( \alpha_{\lambda} \) and (b) \( \bigcup \alpha_{\lambda}N_{\lambda} = N \).

**Solution:** First, assume \( N_{\lambda} = \alpha_{\lambda}^{-1}N \) for all \( \lambda \). Recall \( \alpha_{\lambda} = \alpha_{\mu}^{-1} \alpha_{\lambda} \) for all \( \alpha_{\lambda} \). So \( \alpha_{\lambda}^{-1}N = (\alpha_{\mu})^{-1} \alpha_{\lambda}^{-1}N \). Thus (a) holds.

Further, \( N_{\lambda} = \alpha_{\lambda}^{-1}N \) implies \( \alpha_{\lambda}N_{\lambda} \subset N \). So \( \bigcup \alpha_{\lambda}N_{\lambda} \subset N \). Finally, for any \( m \in M \), there is \( \lambda \) and \( m_{\lambda} \in M_{\lambda} \) with \( m = \alpha_{\lambda}m_{\lambda} \) by (7.8)(1). But \( N_{\lambda} := \alpha_{\lambda}^{-1}N \); hence, if \( m \in N \), then \( m_{\lambda} \in N_{\lambda} \), so \( m \in \alpha_{\lambda}N_{\lambda} \). Thus (b) holds too.

Conversely, assume (b). Then \( \alpha_{\lambda}N_{\lambda} \subset N \), or \( N_{\lambda} \subset \alpha_{\lambda}^{-1}N \), for all \( \lambda \).

Assume (a) too. Given \( \lambda \) and \( m_{\lambda} \in \alpha_{\lambda}^{-1}N \), note \( \alpha_{\lambda}m_{\lambda} \in N = \bigcup \alpha_{\mu}N_{\mu} \). So there is \( \mu \) and \( n_{\mu} \in N_{\mu} \) with \( \alpha_{\mu}n_{\mu} = \alpha_{\lambda}m_{\lambda} \). So (7.8)(2) yields \( \nu \) and \( \alpha_{\mu} \) and \( \alpha_{\lambda} \) with
The functor \( \lambda \mapsto \alpha_{\lambda} \) given \( \mathbb{Set} \). Let \( \alpha_{\lambda} \) be a submodule for all \( \lambda \). Assume \( \alpha_{\lambda} \alpha_{\lambda} \subseteq \alpha_{\mu} \) for each transition map \( \alpha_{\mu} \). Set \( \alpha := \lim \alpha_{\lambda} \). If each \( \alpha_{\lambda} \) is prime, show \( \alpha \) is prime. If each \( \alpha_{\lambda} \) is maximal, show \( \alpha \) is maximal.

**Exercise (7.15).** Let \( R := \lim \ R_{\lambda} \) be a filtered direct limit of rings, \( \alpha_{\lambda} \subseteq R_{\lambda} \) an ideal for each \( \lambda \). Assume \( \alpha_{\lambda} \alpha_{\lambda} \subseteq \alpha_{\mu} \) for each transition map \( \alpha_{\mu} \). Set \( \alpha := \lim \alpha_{\lambda} \). Hence \( \lim \alpha_{\lambda} \subseteq \alpha_{\mu} \). Set \( \alpha := \lim \alpha_{\lambda} \). Thus \( \lim \alpha_{\lambda} \subseteq \alpha_{\mu} \). Assume \( \alpha_{\lambda} \alpha_{\lambda} \subseteq \alpha_{\mu} \) for each transition map \( \alpha_{\mu} \). Set \( \alpha := \lim \alpha_{\lambda} \). Thus \( \lim \alpha_{\lambda} \subseteq \alpha_{\mu} \). The assertions.

**Exercise (7.16).** Let \( M := \lim \ M_{\lambda} \) be a filtered direct limit of modules, with transition maps \( \alpha_{\mu} : M_{\lambda} \rightarrow M_{\mu} \) and insertions \( \alpha_{\lambda} : M_{\lambda} \rightarrow M \). Prove \( \lim \alpha_{\lambda} N_{\mu} \subseteq N_{\mu} \) for all \( \lambda \). Assume \( \alpha_{\lambda} \alpha_{\lambda} \subseteq \alpha_{\mu} \) for all \( \alpha_{\mu} \). Prove \( \lim \alpha_{\lambda} N_{\mu} = \bigcup \alpha_{\lambda} N_{\mu} \).

**Exercise (7.17).** Let \( R := \lim \ R_{\lambda} \) be a filtered direct limit of rings. Prove that \( \lim \text{nil}(R_{\lambda}) = \text{nil}(R) \).

**Exercise (7.18).** Let \( R := \lim \ R_{\lambda} \) be a filtered direct limit of rings. Assume each ring \( R_{\lambda} \) is local, say with maximal ideal \( m_{\lambda} \), and assume each transition map \( \alpha_{\mu} : R_{\lambda} \rightarrow R_{\mu} \) is local. Set \( m := \lim m_{\lambda} \). Prove that \( R \) is local with maximal ideal \( m \). Assume \( \alpha_{\lambda} \) is local. Then \( \lim \alpha_{\lambda} m_{\mu} \subseteq m_{\mu} \) and that each insertion \( \alpha_{\lambda} : R_{\lambda} \rightarrow R_{\mu} \) is local.

**Exercise (7.19).** Let \( \Lambda \) and \( \Lambda' \) be small categories, \( C : \Lambda' \rightarrow \Lambda \) a functor. Assume \( \Lambda' \) is filtered. Assume \( C \) is cofinal; that is,

1. given \( \lambda \in \Lambda \), there is a map \( \lambda \rightarrow C \lambda' \) for some \( \lambda' \in \Lambda' \), and
2. given \( \psi, \varphi : \lambda \rightarrow C \lambda' \), there is \( \chi : \lambda' \rightarrow \lambda' \) with \( (C \chi) \psi = (C \chi) \varphi \).

Let \( \lambda \mapsto M_{\lambda} \) be a functor from \( \Lambda \) to \( \mathcal{C} \) whose direct limit exists. Show that

\[
\lim_{\lambda' \in \Lambda'} M_{\lambda'} = \lim_{\lambda \in \Lambda} M_{\lambda};
\]

more precisely, show that the right side has the UMP characterizing the left.
SOLUTION: Let $P$ be an object of $\mathcal{C}$. For $\lambda' \in \Lambda'$, take maps $\gamma_{\lambda'}: MC_{\lambda'} \to P$ compatible with the transition maps $MC_{\lambda'} \to MC_{\lambda''}$. Given $\lambda \in \Lambda$, choose a map $\lambda \to C\lambda'$, and define $\beta_{\lambda}: M_{\lambda} \to P$ to be the composition

$$\beta_{\lambda}: M_{\lambda} \longrightarrow MC_{\lambda'} \xrightarrow{\gamma_{\lambda'}} P.$$ 

Let’s check that $\beta_{\lambda}$ is independent of the choice of $\lambda \to C\lambda'$.

Given a second choice $\lambda \to C\lambda''$, there are maps $\lambda'' \to \mu'$ and $\lambda \to \mu'$ for some $\mu' \in \Lambda'$ such that the compositions $\lambda \to C\lambda' \to C\mu' \to C\mu'_1$ and $\lambda \to C\lambda'' \to C\mu' \to C\mu'_1$ are equal since $C$ is cofinal. Therefore, $\lambda \to C\lambda''$ gives rise to the same $\beta_{\lambda}$, as desired.

Clearly, the $\beta_{\lambda}$ are compatible with the transition maps $M_{\kappa} \to M_{\lambda}$. So the $\beta_{\lambda}$ induce a map $\beta: \lim_{\kappa} M_{\kappa} \to P$ with $\beta_{\alpha} = \beta_{\lambda}$ for every inclusion $\alpha: M_{\lambda} \to \lim_{\kappa} M_{\kappa}$. In particular, this equation holds when $\lambda = C\lambda'$ for any $\lambda \in \Lambda'$, as required. □

Exercise (7.21). — Show that every $R$-module $M$ is the filtered direct limit over a directed set of finitely presented modules.

SOLUTION: By (6.20), there is a presentation $R^{\oplus \Psi_1} \xrightarrow{\alpha} R^{\oplus \Psi_2} \to M \to 0$. For $i = 1, 2$, let $\Lambda_i$ be the set of finite subsets $\Psi_i$ of $\Phi_i$, and order $\Lambda_i$ by inclusion. Clearly, an inclusion $\Psi_i \hookrightarrow \Phi_i$ yields an injection $R^{\oplus \Psi_i} \to R^{\oplus \Phi_i}$, which is given by extending vectors by 0. Hence (6.26) yields $\lim_{\gamma_{\lambda} \in \Lambda} R^{\oplus C_{\lambda}} = \lim_{\gamma_{\lambda} \in \Lambda} R^{\oplus \Phi_i}$. Let $\Lambda \subset \Lambda_1 \times \Lambda_2$ be the set of pairs $\lambda := (\Psi_1, \Psi_2)$ such that $\alpha$ induces a map $\alpha_{\lambda}: R^{\oplus \Psi_1} \to R^{\oplus \Psi_2}$. Order $\Lambda$ by componentwise inclusion. Clearly, $\Lambda$ is directed. For $\lambda \in \Lambda$, set $M_{\lambda} := \text{Coker}(\alpha_{\lambda})$. Then $M_{\lambda}$ is finitely presented.

For $i = 1, 2$, the projection $C_i: \Lambda \to \Lambda_i$ is surjective, so cofinal. Hence, (7.21) yields $\lim_{\gamma_{\lambda} \in \Lambda} R^{\oplus C_{\lambda}} = \lim_{\gamma_{\lambda} \in \Lambda} R^{\oplus \Psi_i}$. Thus (6.17) yields $\lim_{\gamma_{\lambda} \in \Lambda} M_{\lambda} = M$. □

8. Tensor Products

Exercise (8.4). — Let $R$ be a ring, $R'$ an $R$-algebra, and $M$ an $R'$-module. Set $M' := R' \otimes_R M$. Define $\alpha: M \to M'$ by $\alpha m := 1 \otimes m$. Prove $M$ is a direct summand of $M'$ with $\alpha = \iota_M$, and find the retraction (projection) $\pi_M: M' \to M$.

Solution: As the canonical map $R' \times M \to M'$ is bilinear, $\alpha$ is linear. Define $\mu: M \times R' \to M'$ by $\mu(x, m) := xm$. Plainly $\mu$ is $R$-bilinear. So $\mu$ induces an $R$-linear map $\rho: M' \to M$. Then $\rho$ is a retraction of $\alpha$, as $\rho(\alpha(m)) = 1 \cdot m$. Let $\beta: M \to \text{Coker}(\alpha)$ be the quotient map. Then (6.39) implies that $M$ is a direct summand of $M'$ with $\alpha = \iota_M$ and $\rho = \pi_M$. □

Exercise (8.7). — Let $R$ be a domain, $a$ a nonzero ideal. Set $K := \text{Frac}(R)$. Show that $a \otimes_R K = K$.

Solution: Define a map $\beta: a \times K \to K$ by $\beta(x, y) := xy$. It is clearly $R$-bilinear. Given any $R$-bilinear map $\alpha: a \times K \to P$, fix a nonzero $z \in a$, and define an $R$-linear map $\gamma: K \to P$ by $\gamma(y) := \alpha(z, y/z)$. Then $\alpha = \gamma \beta$ as

$$\alpha(x, y) = \alpha(xz, y/z) = \alpha(z, xy/z) = \gamma(xy) = \gamma \beta(x, y).$$

Clearly, $\beta$ is surjective. So $\gamma$ is unique with this property. Thus the UMP implies that $K = a \otimes_R K$. (Also, as $\gamma$ is unique, $\gamma$ is independent of the choice of $z$.) Alternatively, form the linear map $\varphi: a \otimes K \to K$ induced by the bilinear map $\beta$. Since $\beta$ is surjective, so is $\varphi$. Now, given any $w \in a \otimes K$, say $w = \sum a_i \otimes x_i / x_i$ with
all $x_i$ and $x$ in $R$. Set $a := \sum a_i x_i \in a$. Then $w = a \otimes (1/x)$. Hence, if $\varphi(w) = 0$, then $a/x = 0$; so $a = 0$ and so $w = 0$. Thus $\varphi$ is injective, so bijective.

**Exercise (8.9).** — Let $R$ be a ring, $R'$ an $R$-algebra, $M$, $N$ two $R'$-modules. Show there is a canonical $R$-linear map $\tau: M \otimes_R N \to M \otimes_{R'} N$.

Let $K \subset M \otimes_R N$ denote the $R$-submodule generated by all the differences $(x'm) \otimes n - m \otimes (x'n)$ for $x' \in R'$ and $m \in M$ and $n \in N$. Show $K$ is equal to $\ker(\tau)$, and $\tau$ is surjective. Show $\tau$ is an isomorphism if $R'$ is a quotient of $R$.

**Solution:** The canonical map $\beta': M \times N \to M \otimes_{R'} N$ is $R'$-bilinear, so $R$-bilinear. Hence, by (8.3), it factors: $\beta' = \tau \beta$ where $\beta: M \times N \to M \otimes_R N$ is the canonical map and $\tau$ is the desired map.

Set $Q := (M \otimes_R N)/K$. Then $\tau$ factors through a map $\tau': Q \to M \otimes_{R'} N$ since each generator $(x'm) \otimes n - m \otimes (x'n)$ of $K$ maps to 0 in $M \otimes_{R'} N$.

By (8.2), there is an $R'$-structure on $M \otimes_R N$ with $y'(m \otimes n) = m \otimes (y'n)$, and so by (8.2)'(1), another one with $y'(m \otimes n) = (y'm) \otimes n$. Clearly, $K$ is a submodule for each structure, so $Q$ is too. But on $Q$ the two structures coincide. Further, the canonical map $M \times N \to Q$ is $R'$-bilinear. Hence the latter factors through $M \otimes_R N$, furnishing an inverse to $\tau'$. So $\tau': Q \to M \otimes_{R'} N$. Hence $\ker(\tau)$ is equal to $K$, and $\tau$ is surjective.

Finally, suppose $R'$ is a quotient of $R$. Then every $x' \in R'$ is the residue of some $x \in R$. So each $(x'm) \otimes n - m \otimes (x'n)$ is equal to 0 in $M \otimes_{R'} N$ as $x'm = x(m)$ and $x'n = xn$. Hence $\ker(\tau)$ vanishes. Thus $\tau$ is an isomorphism.

**Exercise (8.12).** — In the setup of (8.11), find the unit $\eta_M$ of each adjunction.

**Solution:** Consider the left adjoint $FM := M \otimes_R R'$ of restriction of scalars. A map $\theta: FM \to P$ corresponds to the map $M \to P$ carrying $m$ to $\theta(m \otimes 1_{R'})$. Take $P := FM$ and $\theta := 1_{FM}$. Thus $\eta_M: M \to FM$ is given by $\eta_M m = m \otimes 1_{R'}$.

Consider the right adjoint $F'P := \text{Hom}_R(R', P)$ of restriction of scalars. A map $\mu: M \to P$ corresponds to the map $M \to F'P$ carrying $m$ to the map $\nu: R' \to P$ defined by $\nu x := x(\mu m)$. Take $P := M$ and $\mu := 1_M$. Thus $\eta_M: M \to F'M$ is given by $(\eta_M m)(x) = x\mu m$.

**Exercise (8.15).** — Let $M$ and $N$ be nonzero $k$-vector spaces. Prove $M \otimes N \neq 0$.

**Solution:** Vector spaces are free modules; say $M = k^\oplus \Phi$ and $N = k^\oplus \Psi$. Then (8.13) yields $M \otimes N = k^\oplus(\Phi \times \Psi)$ as $k \otimes k = k$ by (5.0) (2). Thus $M \otimes N \neq 0$.

**Exercise (8.16).** — Let $R$ be a ring, $a$ and $b$ ideals, and $M$ a module.

1. Use (8.13) to show that $(R/a) \otimes M = M/aM$.
2. Use (1) to show that $(R/a) \otimes (R/b) = R/(a + b)$.

**Solution:** To prove (1), view $R/a$ as the cokernel of the inclusion $a \to R$. Then (8.13) implies that $(R/a) \otimes M$ is the cokernel of $a \otimes M \to R \otimes M$. Now, $R \otimes M = M$ and $x \otimes m = xm$ by (5.0) (2). Correspondingly, $a \otimes M \to M$ has $aM$ as image. The assertion follows. (Caution: $a \otimes M \to M$ needn’t be injective; if it’s not, then $a \otimes M \neq aM$. For example, take $R := \mathbb{Z}$, take $a := (2)$, and take $M := \mathbb{Z}/(2)$; then $a \otimes M \to M$ is just multiplication by $2$ on $\mathbb{Z}/(2)$, and so $aM = 0$.)

To prove (2), apply (1) with $M := R/b$. Note $a(R/b) = (a + b)/b$ by (8.5.1). So $R/a \otimes R/b = (R/b)/((a + b)/b)$.

The latter is equal to $R/(a + b)$ by (8.5.2).
Exercise (8.17). — Show $\mathbb{Z}/\langle m \rangle \otimes \mathbb{Z}/\langle n \rangle = 0$ if $m$ and $n$ are relatively prime.

Solution: The hypothesis yields $\langle m \rangle + \langle n \rangle = \mathbb{Z}$. Thus (8.14)(2) yields

$$
\mathbb{Z}/\langle m \rangle \otimes \mathbb{Z}/\langle n \rangle = \mathbb{Z}/\langle \langle m \rangle + \langle n \rangle \rangle = 0.
$$

Exercise (8.19). — Let $F : ((R\text{-mod}) \to (R\text{-mod}))$ be a linear functor. Show that $F$ always preserves finite direct sums. Show that $\theta(M) : M \otimes F(R) \to F(M)$ is surjective if $F$ preserves surjections and $M$ is finitely generated, and that $\theta(M)$ is an isomorphism if $F$ preserves cokernels and $M$ is finitely presented.

Solution: The first assertion follows from the characterization of the direct sum of two modules in terms of maps (4.13), since $F$ preserves the relations there.

The second assertion follows from the first via the second part of the proof of Watt’s Theorem (5.18), but with $\Sigma$ and $A$ finite.

Exercise (8.24). — Let $R$ be a ring, $M$ a module, $X$ a variable. Let $M[X]$ be the set of polynomials in $X$ with coefficients in $M$, that is, expressions of the form $\sum_{i=0}^{\infty} m_i X^i$ with $m_i \in M$. Prove $M \otimes_R R[X] = M[X]$ as $R[X]$-modules.

Solution: Plainly, $M[X]$ is an $R[X]$-module. Define $b : M \times R[X] \to M[X]$ by $b(m, \sum a_i X^i) := \sum a_i m X^i$. Then $b$ is $R$-bilinear, so induces an $R$-linear map $\beta : M \otimes_R R[X] \to M[X]$. Plainly, $\beta$ is $R[X]$-linear. By (4.14), any $t \in M \otimes_R R[X]$ can be written as $t = \sum m_i \otimes X^i$ for some $m_i \in M$. Then $\beta t = \sum m_i X^i$. If $\beta t = 0$, then $m_i = 0$ for all $i$, and so $t = 0$. Given $u := \sum m_i X^i \in M[X]$, set $t := \sum m_i \otimes X^i$. Then $\beta t = u$. Thus $\beta$ is bijective, as desired.

Alternatively, for any $R[X]$-module $P$, define an $R$-linear map $\varphi_M : \text{Hom}_{R[X]}(M[X], P) \to \text{Hom}_R(M, P)$ by $\varphi_M \alpha := \alpha|_M$.

If $\varphi_M \alpha = 0$, then $\alpha(\sum m_i X^i) = \sum (\alpha m_i) X^i = 0 = \alpha(\sum m_i X^i)$, because $\alpha$ is $R[X]$-linear and $\alpha|_M = 0$; thus $\varphi_M \alpha$ is injective. Given $\gamma : M \to P$, define $\alpha : M[X] \to P$ by $\alpha(\sum m_i X^i) = \sum (\gamma m_i) X^i$. Then $\alpha$ is $R[X]$-linear, and $\varphi_M \alpha = \gamma$. Thus $\varphi_M \alpha$ is surjective, so bijective. Thus $M \to M[X]$ is a left adjoint of restriction of scalars. But $M \to M \otimes_R R[X]$ is too by (8.11). Thus $M[X] = M \otimes_R R[X]$.

Exercise (8.25). — Let $R$ be a ring, $(R'_\sigma)_{\sigma \in \Sigma}$ a family of algebras. For each finite subset $J$ of $\Sigma$, let $R'_J$ be the tensor product of the $R'_\sigma$ for $\sigma \in J$. Prove that the assignment $J \to R'_J$ extends to a filtered direct system and that $\varprojlim R'_J$ exists and is the coproduct of the family $(R'_\sigma)_{\sigma \in \Sigma}$.

Solution: Let $\Lambda$ be the set of subsets of $\Sigma$, partially ordered by inclusion. Then $\Lambda$ is a filtered small category by (7.11). Further, the assignment $J \to R'_J$ extends to a functor from $\Lambda$ to $((R\text{-alg}))$ as follows: by induction, (5.22) implies that $R'_J$ is the coproduct of the family $(R'_\sigma)_{\sigma \in J}$, so that, first, for each $\sigma \in J$, there is a canonical algebra map $\iota_\sigma : R'_\sigma \to R'_J$, and second, given $J \subset K$, the $\iota_\sigma$ for $\sigma \in K$ induce an algebra map $\alpha_K' : R'_J \to R'_K$. So $\varprojlim R'_J$ exists in $((R\text{-alg}))$ by (7.11).

Given a family of algebra maps $\varphi_\sigma : R'_\sigma \to R''$, for each $J$, there is a compatible map $\varphi_J : R'_J \to R''$, since $R'_J$ is the coproduct of the $R'_\sigma$. Further, the various $\varphi_J$ are compatible, so they induce a compatible map $\varphi : \varprojlim R'_J \to R''$. Thus $\varprojlim R'_J$ is the coproduct of the $R'_\sigma$. \qed
Exercise (8.26). — Let $X$ be a variable, $\omega$ a complex cubic root of 1, and $\sqrt[3]{2}$ the real cube root of 2. Set $k := \mathbb{Q}(\omega)$ and $K := k[\sqrt[3]{2}]$. Show $K = k[X]/(X^3 - 2)$ and then $K \otimes_k K \times K \times K$.

Solution: Note $\omega$ is a root of $X^2 + X + 1$, which is irreducible over $\mathbb{Q}$; hence, $[k : \mathbb{Q}] = 2$. But the three roots of $X^3 - 2 \in \mathbb{Q}[X]$ are $\sqrt[3]{2}$ and $\omega \sqrt[3]{2}$ and $\omega^2 \sqrt[3]{2}$. Therefore, $X^3 - 2$ has no root in $k$. So $X^3 - 2$ is irreducible over $k$. Thus $k[X]/(X^3 - 2) \cong K$.

Note $K[X] = K \otimes_k k[X]$ as $k$-algebras by (S.29). So (S.42)(2) and (S.43) and (S.10)(1) yield

\[
\begin{align*}
K[X]/(X^3 - 2) \otimes_k K &= k[X]/(X^3 - 2) \otimes_{k[X]} (k[X] \otimes_k K) \\
&= k[X]/(X^3 - 2) \otimes_{k[X]} K[X] = K[X]/(X^3 - 2).
\end{align*}
\]

However, $X^3 - 2$ factors in $K$ as follows:

\[
X^3 - 2 = (X - \sqrt[3]{2})(X - \omega \sqrt[3]{2})(X - \omega^2 \sqrt[3]{2}).
\]

So the Chinese Remainder Theorem, (I.13), yields

\[
K[X]/(X^3 - 2) = K \times K \times K,
\]

because $K[X]/(X - \omega^i \sqrt[3]{2}) \cong K$ for any $i$ by (I.3).

9. Flatness

Exercise (9.4). — Let $R$ be a ring, $R'$ an algebra, $F$ an $R$-linear functor from $((R\text{-mod}))$ to $((R'\text{-mod}))$. Assume $F$ is exact. Prove the following equivalent:

1. $F$ is faithful.
3. $F(R/m) \neq 0$ for every maximal ideal $m$ of $R$.
4. A sequence $M' \to M \to M''$ is exact if $FM' \xrightarrow{F\alpha} FM \xrightarrow{F\beta} FM''$ is.

Solution: To prove (1) implies (2), suppose $FM = 0$. Then $1_{FM} = 0$. But always $1_{FM} = F(1_M)$. Hence (1) yields $1_M = 0$. So $M = 0$. Thus (2) holds.

Conversely, assume (2). Given $\alpha : M \to N$ with $F\alpha = 0$, set $I := \text{Im}(\alpha)$. As $F$ is exact, (E.2) yields $FI = \text{Im}(F\alpha)$. Hence $FI = 0$. So (2) yields $I = 0$. Thus $\alpha = 0$. Thus (1) holds. Thus (1) and (2) are equivalent.

To prove (2) implies (3), take $M := R/m$.

Conversely, assume (3). Given $0 \neq m \in M$, form $\alpha : R \to M$ by $\alpha(x) := xm$. Set $\mathfrak{a} := \text{Ker}(\alpha)$. Let $m \supseteq \mathfrak{a}$ be a maximal ideal. We get a surjection $R/\mathfrak{a} \to R/m$ and an injection $R/\mathfrak{a} \hookrightarrow M$. They induce a surjection $F(R/\mathfrak{a}) \twoheadrightarrow F(R/m)$ and an injection $F(R/\mathfrak{a}) \hookrightarrow FM$ as $F$ is exact. But $F(R/m) \neq 0$ by (3). So $F(R/\mathfrak{a}) \neq 0$. Thus (2) holds. Thus (1) and (2) and (3) are equivalent.

To prove (3) implies (4), set $I := \text{Im}(\alpha)$ and $K := \text{Ker}(\beta)$. Now, $F(\beta\alpha) = 0$. So (3) yields $\beta\alpha = 0$. Hence $I \subseteq K$. But $F$ is exact; so $F(K/I) = FK/FI$, and (E.2) yields $FI = \text{Im}(F\alpha)$ and $FK = \text{Ker}(F\beta)$. Hence $F(K/I) = 0$. But (1) implies (2). So $K/I = 0$. Thus (4) holds.

Conversely, assume (4). Given $\alpha : M \to N$ with $F\alpha = 0$, set $K := \text{Ker}(\alpha)$. As $F$ is exact, (E.2) yields $FK = \text{Ker}(F\alpha)$. Hence $FK \to FM$ is exact. So (4) implies $K \to M \to 0$ is exact. So $\alpha = 0$. Thus (1) holds, as desired.

Exercise (9.8). — Show that a ring of polynomials $P$ is faithfully flat.
SOLUTION: The monomials form a free basis, so $P$ is faithfully flat by (9.7). □

EXERCISE (9.10). — Let $R$ be a ring, $M$ and $N$ flat modules. Show that $M \otimes N$ is flat. What if “flat” is replaced everywhere by “faithfully flat”? □

SOLUTION: Associativity (5.10) yields $(M \otimes N) \otimes \bullet = M \otimes (N \otimes \bullet)$; in other words, $(M \otimes N) \otimes \bullet = (M \otimes \bullet) \circ (N \otimes \bullet)$. So $(M \otimes N) \otimes \bullet$ is the composition of two exact functors. Hence it is exact. Thus $M \otimes N$ is flat.

Similarly if $M$ and $N$ are faithfully flat, then $M \otimes N \otimes \bullet$ is faithful and exact. So $M \otimes N$ is faithfully flat. □

EXERCISE (9.11). — Let $R$ be a ring, $M$ a flat module, $R'$ an algebra. Show that $M \otimes_R R'$ is flat over $R'$. What if “flat” is replaced everywhere by “faithfully flat”? □

SOLUTION: Cancellation (5.11) yields $(M \otimes_R R') \otimes_{R'} \bullet = M \otimes_R \bullet$. But $M \otimes_R \bullet$ is exact, as $M$ is flat over $R$. Thus $M \otimes_R R'$ is flat over $R'$.

Similarly, if $M$ is faithfully flat over $R$, then $M \otimes_R \bullet$ is faithful too. Thus $M \otimes_R R'$ is faithfully flat over $R'$. □

EXERCISE (9.12). — Let $R$ be a ring, $R'$ a flat algebra, $M$ a flat $R'$-module. Show that $M$ is flat over $R$. What if “flat” is replaced everywhere by “faithfully flat”? □

SOLUTION: Cancellation (5.11) yields $M \otimes_R \bullet = M \otimes_R R' \otimes_{R'} \bullet$. But $R' \otimes_R \bullet$ and $M \otimes_R \bullet$ are exact; so their composition $M \otimes_R \bullet$ is too. Thus $M$ is flat over $R$.

Similarly, as the composition of two faithful functors is, plainly, faithful, the assertion remains true if “flat” is replaced everywhere by “faithfully flat.” □

EXERCISE (9.13). — Let $R$ be a ring, $R'$ an algebra, $R''$ an $R'$-algebra, and $M$ an $R''$-module. Assume that $M$ is flat over $R$ and faithfully flat over $R'$. Prove that $R'$ is flat over $R$.

SOLUTION: Let $N' \rightarrow N$ be an injective map of $R$-modules. Then the map $N' \otimes_R M \rightarrow N \otimes_R M$ is injective as $M$ is flat over $R$. But by Cancellation (5.11), that map is equal to this one:

$$(N' \otimes_R R') \otimes_{R'} M \rightarrow (N \otimes_R R') \otimes_{R'} M.$$

And $M$ is faithfully flat over $R'$. Hence the map $N' \otimes_R R' \rightarrow N \otimes_R R'$ is injective by (9.7). Thus $R'$ is flat over $R$. □

EXERCISE (9.14). — Let $R$ be a ring, $a$ an ideal. Assume $R/a$ is flat. Show $a = a^2$.

SOLUTION: Since $R/a$ is flat, tensoring it with the inclusion $a \hookrightarrow R$ yields an injection $a \otimes_R (R/a) \hookrightarrow R \otimes_R (R/a)$. But the image vanishes: $a \otimes r = 1 \otimes ar = 0$. Further, $a \otimes_R (R/a) = a/a^2$ by (5.14). Hence $a/a^2 = 0$. Thus $a = a^2$. □

EXERCISE (9.15). — Let $R$ be a ring, $R'$ a flat algebra. Prove equivalent:

1. $R'$ is faithfully flat over $R$.
2. For every $R$-module $M$, the map $M \rightarrow M \otimes_R R'$ by $m \rightarrow m \otimes 1$ is injective.
3. Every ideal $a$ of $R$ is the contraction of its extension, or $a = \varphi^{-1}(aR')$.
4. Every prime $p$ of $R$ is the contraction of some prime $q$ of $R'$, or $p = \varphi^{-1} q$.
5. Every maximal ideal $m$ of $R$ extends to a proper ideal, or $mR' \neq R'$.
6. Every nonzero $R$-module $M$ extends to a nonzero module, or $M \otimes_R R' \neq 0$. 

Solutions: (9.15) 189
Assume (1). In (2), set $K := \ker \alpha$. Then the canonical sequence
\[ 0 \to K \otimes R' \to M \otimes R' \xrightarrow{\alpha \otimes 1} M \otimes R' \otimes R' \]
is exact. But $\alpha \otimes 1$ has a retraction, namely $m \otimes x \otimes y \mapsto m \otimes xy$. So $\alpha \otimes 1$ is injective. Hence $K \otimes_R R' = 0$. So (1) implies $K = 0$ by (8.13). Thus (2) holds.

Assume (2). Then $R/a \to (R/a) \otimes R'$ is injective. But $(R/a) \otimes R' \to R' / aR'$ by (8.16)(1). So $\varphi^{-1}(aR') = a$. Thus (3) holds.

Assume (3). Then (8.16)(2) yields (4).

Assume (4). Then every maximal ideal $m$ of $R$ is the contraction of some prime $q$ of $R'$. So $mR' \subseteq q$. Thus (5) holds.

Assume (5). Consider (6). Take a nonzero $m \in M$, and set $M' := Rm$. As $R'$ is flat, the inclusion $M' \to M$ yields an injection $M' \otimes R' \to M \otimes R'$.

Note $M' = R/a$ for some $a$ by (9.17). So $M' \otimes_R R' = R'/aR'$ by (8.16)(1). Take a maximal ideal $m \supset a$. Then $aR' \subseteq mR'$. But $mR' \not\subseteq R'$ by (5). Hence $R'/aR' \neq 0$. So $M' \otimes_R R' \neq 0$. Hence $M \otimes R' \neq 0$. Thus (6) holds.

Finally, (6) and (1) are equivalent by (9.20).

**Exercise (9.17).** — Let $R$ be a ring, $0 \to M' \xrightarrow{\alpha} M \to M'' \to 0$ an exact sequence with $M$ flat. Assume $N \otimes M' \xrightarrow{\otimes \alpha} N \otimes M$ is injective for all $N$. Prove $M''$ is flat.

**Solution:** Let $\beta: N \to P$ be an injection. It yields the following commutative diagram with exact rows by hypothesis and by (5.13):
\[
\begin{array}{ccc}
0 & \to & N \otimes M' \\
\downarrow{\beta \otimes M'} & & \downarrow{\beta \otimes M} \\
0 & \to & P \otimes M'
\end{array}
\]
\[
\begin{array}{ccc}
& & \beta \otimes M'' \\
\downarrow & & \downarrow \\
0 & \to & P/N \otimes M' \to P/N \otimes M
\end{array}
\]
Since $M$ is flat, $\ker(\beta \otimes M) = 0$. So the Snake Lemma (5.13) applied to the top two rows yields $\ker(\beta \otimes M'') = 0$. Thus $M''$ is flat.

**Exercise (9.18).** — Prove that an $R$-algebra $R'$ is faithfully flat if and only if the structure map $\varphi: R \to R'$ is injective and the quotient $R' / \varphi R$ is flat over $R$.

**Solution:** Assume $R'$ is faithfully flat. Then for every $R$-module $M$, the map $M \xrightarrow{\varphi} M \otimes_R R'$ is injective by (9.16). Taking $M := R$ shows $\varphi$ is injective. And, since $R'$ is flat, $R'/\varphi R$ is flat by (9.17).

Conversely, assume $\varphi$ is injective and $R'/\varphi(R)$ is flat. Then $M \to M \otimes_R R'$ is injective for every module $M$ by (9.16)(1), and $R'$ is flat by (9.16)(2). Thus $R'$ is faithfully flat by (9.16).

**Exercise (9.21).** — Let $R$ be a ring, $R'$ an algebra, $M$ and $N$ modules. Show that there is a canonical map
\[ \sigma: \text{Hom}_R(M, N) \otimes_R R' \to \text{Hom}_R(M \otimes_R R', N \otimes_R R'). \]
Assume $R'$ is flat over $R$. Show that if $M$ is finitely generated, then $\sigma$ is injective, and that if $M$ is finitely presented, then $\sigma$ is an isomorphism.

**Solution:** Simply put $R' := R$ and $P := R'$ in (9.21), put $P := N \otimes_R R'$ in the second equation in (8.11), and combine the two results.
Exercise (9.25) (Equational Criterion for Flatness). — Prove that Condition (9.24) (4) can be reformulated as follows: Given any relation \( \sum x_i y_i = 0 \) with \( x_i \in R \) and \( y_i \in M \), there are \( x_{ij} \in R \) and \( y'_j \in M \) such that
\[
\sum_j x_{ij} y'_j = y_i \quad \text{for all } i \quad \text{and} \quad \sum_i x_{ij} x_i = 0 \quad \text{for all } j.
\]
\[(9.25.1)\]

Solution: Assume (9.24) (4) holds. Let \( e_1, \ldots, e_m \) be the standard basis of \( R^m \).

Given a relation \( \sum x_i y_i = 0 \), define \( \alpha: R^m \to M \) by \( \alpha(e_i) := y_i \) for each \( i \). Set \( k := \sum x_i e_i \). Then \( \alpha(k) = 0 \). So (9.24) (4) yields a factorization \( \alpha: R^m \xrightarrow{\varphi} R^n \xrightarrow{x} M \) with \( \varphi(k) = 0 \). Let \( e'_1, \ldots, e'_n \) be the standard basis of \( R^n \), and set \( y'_j := \beta(e'_j) \) for each \( j \).

Let \( (x_{ij}) \) be the \( n \times m \) matrix of \( \varphi \); that is, \( \varphi(e_i) = \sum x_{ji} e'_j \). Then \( y_i = \sum x_{ji} y'_j \). Now, \( \sum_i x_{ji} x_i = 0 \); hence, \( \sum_{i,j} x_{ij} x_i e'_j = 0 \). Thus (9.25) (4) holds.

Conversely, given \( \alpha: R^m \to M \) and \( k \in \text{Ker}(\alpha) \), write \( k = \sum x_i e_i \). Assume (9.25) (4). Let \( \varphi: R^m \to R^n \) be the map with matrix \((x_{ij})\); that is, \( \varphi(e_i) = \sum x_{ji} e'_j \). Then \( \varphi(k) = \sum x_i x_{ji} e'_j = 0 \). Define \( \beta: R^n \to M \) by \( \beta(e'_j) := y'_j \). Then \( \beta \varphi(e_i) = y_i \); hence, \( \beta \varphi = \alpha \). Thus (9.25) (4) holds.

Exercise (9.28). — Let \( R \) be a ring, \( M \) a module. Prove (1) if \( M \) is flat, then for \( x \in R \) and \( m \in M \) with \( xm = 0 \), necessarily \( m \in \text{Ann}(x)M \), and (2) the converse holds if \( R \) is a Principal Ideal Ring (PIR); that is, every ideal \( a \) is principal.

Solution: For (1), assume \( M \) is flat and \( xm = 0 \). Then (9.24) yields \( x \in R \) and \( m_j \in M \) with \( \sum x_i m_j = m \) and \( x_i x = 0 \) for all \( j \). Thus \( m \in \text{Ann}(x)M \).

Alternatively, \( 0 \to \text{Ann}(x) \to R \xrightarrow{\mu_x} R \) is always exact. Tensoring with \( M \) gives \( 0 \to \text{Ann}(x) \otimes M \to M \xrightarrow{\mu_x} M \), which is exact as \( M \) is flat. So \( \text{Im}(\text{Ann}(x) \otimes M) = \text{Ker}(\mu_x) \). But always \( \text{Im}(\text{Ann}(x) \otimes M) = \text{Ann}(x)M \). Thus (1) holds.

For (2), it suffices, by (9.24), to show \( \alpha: a \otimes M \to aM \) is injective. Since \( R \) is a PIR, \( a = \langle x \rangle \) for some \( x \in R \). So, given \( z \in a \otimes M \), there are \( y_i \in R \) and \( m_i \in M \) such that \( z = \sum_i y_i x \otimes m_i \). Set \( m := \sum_i y_i m_i \).

Then
\[ z = \sum_i x \otimes y_i m_i = x \otimes \sum_i y_i m_i = x \otimes m. \]

Suppose \( z \in \text{Ker}(\alpha) \). Then \( xm = 0 \). Hence \( m \in \text{Ann}(x)M \) by hypothesis. So \( m = \sum_j z_j n_j \) for some \( z_j \in \text{Ann}(x) \) and \( n_j \in M \). Hence
\[ z = x \otimes \sum_j z_j n_j = \sum_j z_j x \otimes n_j = 0. \]

Thus \( \alpha \) is injective. Thus (2) holds.

10. Cayley–Hamilton Theorem

Exercise (10.6). — Let \( R \) be a nonzero ring, \( \alpha: R^m \to R^n \) a map of free modules. Assume \( \alpha \) is surjective. Show that \( m \geq n \).

Solution: Let \( \mathfrak{m} \) be a maximal ideal. Then \( \alpha \) induces a \( R^m/\mathfrak{m} R^m \to R^n/\mathfrak{m} R^n \), which is surjective. Plainly, that map can be rewritten as \( (R/\mathfrak{m})^m \to (R/\mathfrak{m})^n \). But \( R/\mathfrak{m} \) is a field. Thus \( m \geq n \).

Exercise (10.7). — Let \( R \) be a ring, \( a \) an ideal. Assume \( a \) is finitely generated and idempotent (or \( a = a^2 \)). Prove there is a unique idempotent \( e \) with \( \langle e \rangle = a \).

Solution: By (10.353), there is \( a \subseteq M \) such that \( (1 - e)a = 0 \). So for all \( x \in a \), we have \((1 - e)x = 0\), or \( x = ex \). Thus \( a = \langle e \rangle \) and \( e = e^2 \).

Finally, \( e \) is unique by (10.248).
EXERCISE (10.8). — Let \( R \) be a ring, \( \mathfrak{a} \) an ideal. Prove the following conditions are equivalent:

1. \( R/\mathfrak{a} \) is projective over \( R \).
2. \( R/\mathfrak{a} \) is flat over \( R \), and \( \mathfrak{a} \) is finitely generated.
3. \( \mathfrak{a} \) is finitely generated and idempotent.
4. \( \mathfrak{a} \) is generated by an idempotent.
5. \( \mathfrak{a} \) is a direct summand of \( R \).

**Solution:** Suppose (1) holds. Then \( R/\mathfrak{a} \) is flat by (11.7). Further, the sequence \( 0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0 \) splits by (11.4). So (11.4) yields a surjection \( \rho: R \to \mathfrak{a} \). Hence \( \mathfrak{a} \) is principal. Thus (2) holds.

If (2) holds, then (3) holds by (11.14). If (3) holds, then (4) holds by (11.7). If (4) holds, then (5) holds by (11.5). If (5) holds, then \( R \cong \mathfrak{a} \oplus R/\mathfrak{a} \), and so (1) holds by (11.4). \( \square \)

EXERCISE (10.9). — Prove the following conditions on a ring \( R \) are equivalent:

1. \( R \) is **absolutely flat**: that is, every module is flat.
2. Every finitely generated ideal is a direct summand of \( R \).
3. Every finitely generated ideal is idempotent.
4. Every principal ideal is idempotent.

**Solution:** Assume (1). Let \( \mathfrak{a} \) be a finitely generated ideal. Then \( R/\mathfrak{a} \) is flat by hypotheses. So \( \mathfrak{a} \) is a direct summand of \( R \) by (11.4). Thus (2) holds.

Conditions (2) and (3) are equivalent by (11.4).

Trivially, if (3) holds, then (4) does. Conversely, assume (4). Given a finitely generated ideal \( \mathfrak{a} \), say \( \mathfrak{a} = \langle x_1, \ldots, x_n \rangle \). Then each \( \langle x_i \rangle \) is idempotent by hypothesis. So \( \langle x_i \rangle = \langle f_i \rangle \) for some idempotent \( f_i \) by (11.7) (2). Then \( \mathfrak{a} = \langle f_1, \ldots, f_n \rangle \). Hence \( \mathfrak{a} \) is idempotent by (11.7) (4), (1). Thus (3) holds.

Assume (2). Let \( M \) be a module, and \( \mathfrak{a} \) a finitely generated ideal. Then \( \mathfrak{a} \) is a direct summand of \( R \) by hypothesis. So \( R/\mathfrak{a} \) is flat by (11.4). Hence \( \mathfrak{a} \otimes M \to \mathfrak{a} M \) by (11.4) (1); cf. the proof of (8.12) (1). So \( M \) is flat by (9.20). Thus (1) holds. \( \square \)

EXERCISE (10.10). — Let \( R \) be a ring.

1. Assume \( R \) is Boolean. Prove \( R \) is absolutely flat.
2. Assume \( R \) is absolutely flat. Prove any quotient ring \( R' \) is absolutely flat.
3. Assume \( R \) is absolutely flat. Prove every nonunit \( x \) is a zerodivisor.
4. Assume \( R \) is absolutely flat and local. Prove \( R \) is a field.

**Solution:** In (1), as \( R \) is Boolean, every element is idempotent. Hence every principal ideal is idempotent by (11.7) (1). Thus (10.4) yields (1).

For (2), let \( \mathfrak{b} \subset R' \) be principal, say \( \mathfrak{b} = \langle \overline{x} \rangle \). Let \( x \in R \) lift \( \overline{x} \). Then \( \langle x \rangle \) is idempotent by (10.3). Hence \( \mathfrak{b} \) is also idempotent. Thus (10.4) yields (2).

For (3) and (4), take a nonunit \( x \). Then \( \langle x \rangle \) is idempotent by (10.3). So \( x = ax^2 \) for some \( a \). Then \( x(ax - 1) = 0 \). But \( x \) is a nonunit. So \( ax - 1 \neq 0 \). Thus (3) holds.

Suppose \( R \) is local, say with maximal ideal \( \mathfrak{m} \). Since \( x \) is a nonunit, \( x \in \mathfrak{m} \). So \( ax \in \mathfrak{m} \). So \( ax - 1 \notin \mathfrak{m} \). So \( ax - 1 \) is a unit. But \( x(ax - 1) = 0 \). So \( x = 0 \). Thus \( x \) is the only nonunit. Thus (4) holds. \( \square \)

EXERCISE (10.14). — Let \( R \) be a ring, \( \mathfrak{a} \) an ideal, and \( \alpha: M \to N \) a map of modules. Assume that \( \mathfrak{a} \subset \text{rad}(R) \), that \( N \) is finitely generated, and that the induced map \( \overline{\alpha}: M/\mathfrak{a}M \to N/\mathfrak{a}N \) is surjective. Show that \( \alpha \) is surjective.
SOLUTION: Since $\pi$ is surjective, $\alpha(M) + aN = N$. Since $N$ is finitely generated, so is $N/\alpha(N)$. Hence $\alpha(M) = N$ by (10.14). Thus $\alpha$ is surjective. \hfill $\square$

EXERCISE (10.15). — Let $R$ be a ring, $m \subseteq \text{rad}(R)$ an ideal. Let $\alpha, \beta : M \rightarrow N$ be two maps of finitely generated modules. Assume that $\alpha$ is an isomorphism and that $\beta(M) \subseteq mN$. Set $\gamma := \alpha + \beta$. Show that $\gamma$ is an isomorphism.

SOLUTION: As $\alpha$ is surjective, given $n \in N$, there is $m \in M$ with $\alpha(m) = n$. So

$$n = \alpha(m) + \beta(m) - \beta(m) \in \gamma(M) + mN.$$  

But $M/N$ is finitely generated as $M$ is. Hence $\gamma(M) = N$ by (10.14). So $\alpha^{-1}\gamma$ is surjective, and therefore an isomorphism by (10.14). Thus $\gamma$ is an isomorphism. \hfill $\square$

EXERCISE (10.16). — Let $A$ be a local ring, $m$ the maximal ideal, $M$ a finitely generated $A$-module, and $m_1, \ldots, m_n \in M$. Set $k := A/m$ and $M' := M/mM$, and write $m_i'$ for the image of $m_i$ in $M'$. Prove that $m_1', \ldots, m_n' \in M'$ form a basis of the $k$-vector space $M'$ if and only if $m_1, \ldots, m_n$ form a minimal generating set of $M$ (that is, no proper subset generates $M$), and prove that every minimal generating set of $M$ has the same number of elements.

SOLUTION: By (10.13), reduction mod $m$ gives a bijective correspondence between generating sets of $M$ as an $A$-module, and generating sets of $M'$ as an $A$-module, or equivalently by (10.13), as an $k$-vector space. This correspondence preserves inclusion. Hence, a minimal generating set of $M$ corresponds to a minimal generating set of $M'$, that is, to a basis. But any two bases have the same number of elements. \hfill $\square$

EXERCISE (10.17). — Let $A$ be a local ring, $k$ its residue field, $M$ and $N$ finitely generated modules. (1) Show that $M = 0$ if and only if $M \otimes_A k = 0$. (2) Show that $M \otimes_A N \neq 0$ if $M \neq 0$ and $N \neq 0$.

SOLUTION: Let $m$ be the maximal ideal. Then $M \otimes k = M/mM$ by (8.10). So (1) is nothing but a form of Nakayama’s Lemma (10.11).

In (2), $M \otimes k \neq 0$ and $N \otimes k \neq 0$ by (1). So $(M \otimes k) \otimes (N \otimes k) \neq 0$ by (8.13) and (8.14). But $(M \otimes k) \otimes (N \otimes k) = (M \otimes N) \otimes (k \otimes k)$ by the associative and commutative laws, (8.14) and (8.10). Finally, $k \otimes k = k$ by (8.10). \hfill $\square$

EXERCISE (10.19). — Let $A \rightarrow B$ be a local homomorphism, $M$ a finitely generated $B$-module. Prove that $M$ is faithfully flat over $A$ if and only if $M$ is flat over $A$ and nonzero. Conclude that, if $B$ is flat over $A$, then $B$ is faithfully flat over $A$.

SOLUTION: Plainly, to prove the first assertion, it suffices to show that $M \otimes_A \bullet$ is faithful if and only if $M \neq 0$. Now, if $M \otimes_A \bullet$ is faithful, then $M \otimes N \neq 0$ whenever $N \neq 0$ by (8.3). But $M \otimes A = M$ by the Unitary Law, and $A \neq 0$. Thus $M \neq 0$.

Conversely, suppose $M \neq 0$. Denote the maximal ideals of $A$ and $B$ by $m$ and $n$. Then $nM \neq M$ by Nakayama’s Lemma (10.10). But $mB \subset n$ as $A \rightarrow B$ is a local homomorphism. So $M/mM \neq 0$. But $M/mM = M \otimes (A/m)$ by (8.14). Thus (8.3) implies $M \otimes_A \bullet$ is faithful. Finally, the second assertion is the special case with $M := B$. \hfill $\square$

EXERCISE (10.22). — Let $G$ be a finite group of automorphisms of a ring $R$. Form the subring $R^G$ of invariants. Show that every $x \in R$ is integral over $R^G$, in fact, over the subring $R'$ generated by the elementary symmetric functions in the conjugates $gx$ for $g \in G$. 

Solutions: (10.22) 193
**Solution:** Given an \( x \in R \), form \( F(X) := \prod_{g \in G} (X - gx) \). Then the coefficients of \( F(X) \) are the elementary symmetric functions in the conjugates \( gx \) for \( g \in G \); hence, they are invariant under the action of \( G \). So \( F(x) = 0 \) is a relation of integral dependence for \( x \) over \( R[G] \), in fact, over its subring \( R' \).

**Exercise (10.24).** — Let \( k \) be a field, \( P := k[X] \) the polynomial ring in one variable, \( f \in P \). Set \( R := k[X^2] \subset P \). Using the free basis 1, \( X \) of \( P \) over \( R \), find an explicit equation of integral dependence of degree 2 on \( f \) for \( f \).

**Solution:** Write \( f = f_e + f_o \), where \( f_e \) and \( f_o \) are the polynomials formed by the terms of \( f \) of even and odd degrees. Say \( f_o = gX \). Then the matrix of \( \mu_f \) is \((f_e gX^2)\). Its characteristic polynomial is \( T^2 - 2f_e T + f_e^2 - f_o^2 \). So the Cayley–Hamilton Theorem yields \( f^2 - 2f_e f + f_e^2 - f_o^2 = 0 \). □

**Exercise (10.29).** — Let \( R_1, \ldots, R_n \) be \( R \)-algebras that are integral over \( R \). Show that their product \( \prod R_i \) is a integral over \( R \).

**Solution:** Let \( y = (y_1, \ldots, y_n) \in \prod_{i=1}^n R_i \). Since \( R_i/R \) is integral, \( R_i[y_i] \) is a module-finite \( R \)-subalgebra of \( R_i \) by (10.28). Hence \( \prod_{i=1}^n R_i[y_i] \) is a module-finite \( R \)-subalgebra of \( \prod_{i=1}^n R_i \) by (10.13) and induction on \( n \). Now, \( y \in \prod_{i=1}^n R_i[y_i] \). Therefore, \( y \) is integral over \( R \) by (10.28). Thus \( \prod_{i=1}^n R_i \) is integral over \( R \). □

**Exercise (10.31).** — For \( 1 \leq i \leq r \), let \( R_i \) be a ring, \( R_i' \) an extension of \( R_i \), and \( x_i \in R_i' \). Set \( R := \prod R_i \), set \( R' := \prod R_i' \), and set \( x := (x_1, \ldots, x_r) \). Prove

1. \( x \) is integral over \( R \) if and only if each \( x_i \) is integral over \( R_i \) for each \( i \);
2. \( R \) is integrally closed in \( R' \) if and only if each \( R_i \) is integrally closed in \( R_i' \).

**Solution:** Assume \( x \) is integral over \( R \). Say \( x^n + a_1 x^{n-1} + \cdots + a_n = 0 \) with \( a_j \in R \). Say \( a_j := (a_{ij}, \ldots, a_{ij}) \). Fix \( i \). Then \( x_i^n + a_{i1} x_i^{n-1} + \cdots + a_{in} = 0 \). So \( x_i \) is integral over \( R_i \).

Conversely, suppose each \( x_i \) is integral over \( R_i \). Say \( x_i^n + a_{i1} x_i^{n-1} + \cdots + a_{in} = 0 \). Set \( n := \max n_i \), set \( a_{ij} := 0 \) for \( j > n_i \), and set \( a_j := (a_{ij}, \ldots, a_{ij}) \in R \) for each \( j \). Then \( x^n + a_{i1} x^{n-1} + \cdots + a_{in} = 0 \). Thus \( x \) is integral over \( R \). Thus (1) holds.

Assertion (2) is an immediate consequence of (1). □

**Exercise (10.35).** — Let \( K \) be a field, \( X \) and \( Y \) variables. Set

\[ R := k[X,Y] / \langle Y^2 - X^2 - X^3 \rangle, \]

and let \( x, y \in R \) be the residues of \( X, Y \). Prove that \( R \) is a domain, but not a field. Set \( t := y/x \in \frac{R}{x} \). Prove that \( k[t] \) is the integral closure of \( R \) in \( \frac{R}{x} \).

**Solution:** As \( k[X,Y] \) is a UFD and \( Y^2 - X^2 - X^3 \) is irreducible, \( \langle Y^2 - X^2 - X^3 \rangle \) is prime by (24.1); however, it is not maximal by (24.2). Hence \( R \) is a domain by (24.3), but not a field by (24.7).

Note \( Y^2 - X^2 - X^3 = 0 \). Hence \( x = t^2 - 1 \) and \( y = t^3 - t \). So \( k[t] \supset k[x,y] = R \). Further, \( t \) is integral over \( R \); so \( k[t] \) is integral over \( R \) by (2) \( \Rightarrow \) (1) of (10.28).

Finally, \( k[t] \) has \( \frac{R}{x} \) as fraction field. Further, \( \frac{R}{x} \neq R \), so \( x \) and \( y \) cannot be algebraic over \( k \); hence, \( t \) must be transcendental. So \( k[t] \) is normal by (10.28)(1). Thus \( k[t] \) is the integral closure of \( R \) in \( \frac{R}{x} \). □

11. Localization of Rings

Solutions: (11.2)
Exercise (11.2). — Let $R$ be a ring, $S$ a multiplicative subset. Prove $S^{-1}R = 0$ if and only if $S$ contains a nilpotent element.

Solution: By (11.4), $S^{-1}R = 0$ if and only if $1/1 = 0/1$. But by construction, $1/1 = 0/1$ if and only if $0 \in S$. Finally, since $S$ is multiplicative, $0 \in S$ if and only if $S$ contains a nilpotent element.

Exercise (11.4). — Find all intermediate rings $Z \subset R \subset Q$, and describe each $R$ as a localization of $Z$. As a starter, prove $Z[2/3] = S^{-1}Z$ where $S = \{3^i \mid i \geq 0\}$.

Solution: Clearly $Z[2/3] \subset Z[1/3]$ as $2/3 = 2 \cdot (1/3)$. But the opposite inclusion holds because, by the very definition of lowest terms if $r$ and $s$ have no common prime divisor. Let $S$ be the multiplicative subset generated by $P$, that is, the smallest multiplicative subset containing $P$. Clearly, $S$ is equal to the set of all products of elements of $P$.

First note that, if $p \in P$, then $1/p \in R$. Indeed, take an element $x = r/ps \in R$ in lowest terms. Then $sx = r/p \in R$. Also the Euclidean algorithm yields $m, n \in Z$ such that $mp + nr = 1$. Then $1/p = m + nsx \in R$, as desired. Hence $S^{-1}Z \subset R$. But the opposite inclusion holds because, by the very definition of $S$, every element of $R$ is of the form $r/s$ for some $s \in S$. Thus $S^{-1}Z = R$.

Exercise (11.7). — Let $R'$ and $R''$ be rings. Consider $R := R' \times R''$ and set $S := \{(1,1), (1,0)\}$. Prove $R' = S^{-1}R$.

Solution: Let’s show that the projection map $\pi: R' \times R'' \to R'$ has the UMP of (11.6). First, note that $\pi S = \{1\} \subset R'^\times$. Let $\psi: R' \times R'' \to B$ be a ring map such that $\psi(1,0) \in B^\times$. Then in $B$,

$$\psi(1,0) \cdot \psi(0,x) = \psi((1,0) \cdot (0,x)) = \psi(0,0) = 0 \text{ in } B.$$  

Hence $\psi(0,x) = 0$ for all $x \in R''$. So $\psi$ factors uniquely through $\pi$ by (11.6).

Exercise (11.8). — Take $R$ and $S$ as in (11.7). On $R \times S$, impose this relation:

$$(x,s) \sim (y,t) \quad \text{if} \quad xt = ys.$$  

Prove that it is not an equivalence relation.

Solution: Observe that, for any $z \in R''$, we have

$$((1,z), (1,1)) \sim ((1,0), (1,0)).$$  

However, if $z \neq 0$, then

$$((1,z), (1,1)) \not\sim ((1,0), (1,1)).$$  

Thus although $\sim$ is reflexive and symmetric, it is not transitive if $R'' \neq 0$.

Exercise (11.9). — Let $R$ be a ring, $S \subset T$ a multiplicative subsets, $\overline{S}$ and $\overline{T}$ their saturations; see (11.2). Set $U := (S^{-1}R)^\times$. Show the following:

1. $U = \{ x/s \mid x \in \overline{S} \text{ and } s \in S \}$.
2. $\varphi^{-1}U = \overline{S}$.
3. $S^{-1}R = T^{-1}R$ if and only if $\overline{S} = \overline{T}$.
4. $\overline{S^{-1}R} = S^{-1}R$. 


SOLUTION: In (1), given $x \in S$ and $s \in S$, take $y \in R$ such that $xy \in S$. Then $x/s \cdot sy/xy = 1$ in $S^{-1}R$. Thus $x/s \in U$. Conversely, say $x/s \cdot y/t = 1$ in $S^{-1}R$ with $x, y \in R$ and $s, t \in S$. Then there's $u \in S$ with $xyu = stu$ in $R$. But $stu \in S$. Thus $x \in S$. Thus (1) holds.

For (1), set $V := \varphi_S^{-1}T$. Then $V$ is saturated multiplicatively by (11.14). Further, $V \supset S$ by (11.14). Thus (1)(c) of (6.17) yields $V \supset S$. Conversely, take $x \in V$. Then $x/1 \in T$. So (1) yields $x/1 = y/s$ with $y \in S$ and $s \in S$. So there's $t \in S$ with $xst = yt$ in $R$. But $\overline{S} \supset S$ by (1)(a) of (6.17), and $S$ is multiplicative by (1)(b) of (6.17); so $yt \in S$. But $\overline{S}$ is saturated again by (1)(b). Thus $x \in \overline{S}$. Thus $V = \overline{S}$.

In (3), if $S^{-1}R = T^{-1}R$, then (2) implies $\overline{S} = T$. Conversely, if $\overline{S} = T$, then (4) implies $S^{-1}R = T^{-1}R$.

As to (4), note that, in any product, a unit is a unit if and only if each factor is. So a ring map $\varphi: R \to R'$ carries $S$ into $R'^x$ if and only if $\varphi$ carries $S$ into $R'^x$. Thus $S^{-1}R$ and $S^{-1}R$ are characterized by equivalent UMPs. Thus (4) holds. □

**Exercise (11.10).** — Let $R$ be a ring, $S \subset T \subset U$ and $W$ multiplicative subsets.

1. Show there's a unique $R$-algebra map $\varphi^S_T: S^{-1}R \to T^{-1}R$ and $\varphi^U_V \varphi^S_T = \varphi^S_U$.

2. Given a map $\varphi: S^{-1}R \to W^{-1}R$, show $S \subset \overline{S} \subset W$ and $\varphi = \varphi_{W_1}$.

3. Let $\Lambda$ be a set, $S_\lambda \subset S$ a multiplicative subset for all $\lambda \in \Lambda$. Assume $\bigcup S_\lambda = S$. Assume given $\lambda, \mu \in \Lambda$, there is $\nu$ such that $S_\lambda, S_\mu \subset S_\nu$. Order $\Lambda$ by inclusion: $\lambda \leq \mu$ if $S_\lambda \subset S_\mu$. Using (1), show $\lim_{\lambda} S_\lambda^{-1}R = S^{-1}R$.

**Solution:** For (1), note $\varphi_T S \subset \varphi_T T \subset (T^{-1}R)^x$. So (11.2) yields a unique $R$-algebra map $\varphi^S_T: S^{-1}R \to T^{-1}R$. By uniqueness, $\varphi^T_U \varphi^S_T = \varphi^S_U$. Thus (1) holds.

For (2), note $\varphi(S^{-1}R)^x \subset (W^{-1}R)^x$. So $\varphi^{-1}(S^{-1}R)^x \subset \varphi^{-1}(W^{-1}R)^x$. But $\varphi^{-1}(S^{-1}R)^x = \overline{S}$ and $\varphi^{-1}(W^{-1}R)^x = W$ by (11.3). Also $S \subset \overline{S}$ by (5.17)(a). Thus (2) holds.

For (3), notice $\Lambda$ is directed. Given $\lambda \leq \mu$, set $\alpha^{\lambda}_\mu := \varphi^S_{\nu_{\lambda}}$. Then $\alpha^{\lambda}_\mu \alpha^{\lambda}_\nu = \alpha^{\lambda}_\nu$ if $\lambda \leq \mu \leq \nu$. Thus $\lim_{\lambda} S_\lambda^{-1}R$ exists as a filtered direct limit of $R$-algebras by (4.2).

Given $\lambda$, set $\beta_\lambda := \varphi^S_\lambda$. Then $\beta_\lambda \alpha^{\lambda}_\mu = \beta_\lambda$. So the $\beta_\lambda$ induce an $R$-algebra map $\beta: \lim_{\lambda} S_\lambda^{-1}R \to S^{-1}R$ with $\beta_\lambda = \beta \alpha_\lambda$ where $\alpha_\lambda$ is the insertion of $S_\lambda^{-1}R$.

Take $x \in \text{Ker}(\beta)$. There are $\lambda$ and $x_\lambda/s_\lambda \in S_\lambda^{-1}R$ such that $\alpha_\lambda(x_\lambda/s_\lambda) = x$ by (7.2)(1). Then $\beta_\lambda(x_\lambda/s_\lambda) = 0$. So there is $s \in S$ with $sx_\lambda = 0$. But $x \in S_\mu$ for some $\mu \geq \lambda$. Hence $\alpha^{\lambda}_\mu(x_\lambda/s_\lambda) = 0$. So $x = \alpha_\mu x_\lambda/s_\lambda = 0$. Thus $\beta$ is injective.

As to surjectivity, take $x/s \in S^{-1}R$. Then $s \in S_\lambda$ for some $\lambda$, so $x/s \in S_\lambda^{-1}R$. Hence $\beta_\lambda(x/s) = x/s$. Thus $\beta$ is surjective, so an isomorphism. Thus (3) holds. □

**Exercise (11.11).** — Let $R$ be a ring, $S_0$ the set of nonzerodivisors.

1. Show $S_0$ is the largest multiplicative subset $S$ with $\varphi_S: R \to S^{-1}R$ injective.

2. Show every element $x/s$ of $S_0^{-1}R$ is either a zerodivisor or a unit.

3. Suppose every element of $R$ is either a zerodivisor or a unit. Show $R = S_0^{-1}R$.

**Solution:** For (1), let $s \in S$ and $x \in R$ with $sx = 0$. Then $\varphi_S(sx) = 0$. So $\varphi_S(s) \varphi_S(x) = 0$. But $\varphi_S(s)$ is a unit. So $\varphi_S(x) = 0$. But $\varphi_S$ is injective. So $x = 0$. Thus $S \subset S_0$; that is, (1) holds.

For (2), take $x/s \in S_0^{-1}R$, and suppose it is a nonzerodivisor. Then $x/1$ is also a nonzerodivisor. Hence $x \in S_0$, for if $x/1 = 0$, then $x/1 \cdot y/1 = 0$, so $\varphi_{S_0}(y) = y/1 = 0$, so $y = 0$ as $\varphi_{S_0}$ is injective. Therefore, $x/s$ is a unit. Thus (2) holds.

In (3), by hypothesis, $S_0 \subset R^x$. So $R \cong S_0^{-1}R$ by (11.8); that is, (3) holds. □
Solutions: (11.29) 197

Exercise (11.17). — Let \( R \) be a ring, \( S \) a multiplicative subset, \( a \) and \( b \) ideals. Show (1) if \( a \subseteq b \), then \( a^S \subseteq b^S \); (2) \( (a^S)^S = a^S \); and (3) \( (ab^S)^S = (ab)^S \).

Solution: For (1), take \( x \in a^S \). Then there is \( s \in S \) with \( sx \in a \). If \( a \subseteq b \), then \( sx \in b \), and so \( x \in b^S \). Thus (1) holds.

To show (2), proceed by double inclusion. First, note \( a^S \supseteq a \) by (11.14)(2). So \( (a^S)^S \supseteq a^S \) again by (11.14)(2). Conversely, given \( x \in (a^S)^S \), there is \( s \in S \) with \( sx \in a^S \). So there is \( t \in S \) with \( tsx \in a \). But \( ts \in S \). So \( x \in a^S \). Thus (2) holds.

To show (3), proceed by double inclusion. First, note \( a \subseteq a^S \) and \( b \subseteq b^S \) by (11.14)(2). So \( ab \subseteq a^Sb^S \). Thus (1) yields \( (ab)^S \subseteq (a^Sb^S)^S \).

Conversely, given \( x \in a^Sb^S \), say \( x = \sum y_iz_i \), with \( y_i \in a^S \) and \( z_i \in b^S \). Then there are \( s_i,t_i \in S \) such that \( s_iy_i \in a \) and \( t_iz_i \in b \). Set \( u := \prod s_it_i \). Then \( u \in S \) and \( ux \in ab \). So \( x \in (ab)^S \). Thus \( a^Sb^S \subseteq (ab)^S \). So \( (a^Sb^S)^S \subseteq ((ab)^S)^S \) by (1). But \( ((ab)^S)^S = (ab)^S \) by (2). Thus (3) holds.

Exercise (11.18). — Let \( R \) be a ring, \( S \) a multiplicative subset. Prove that

\[
\text{nil}(R)(S^{-1}R) = \text{nil}(S^{-1}R).
\]

Solution: Proceed by double inclusion. Given an element of \( \text{nil}(R)(S^{-1}R) \), put it in the form \( x/s \) with \( x \in \text{nil}(R) \) and \( s \in S \) using (11.14)(1). Then \( x^n = 0 \) for some \( n \geq 1 \).

Conversely, take \( x/s \in \text{nil}(S^{-1}R) \). Then \( (x/s)^m = 0 \) with \( m \geq 1 \). So there’s \( t \in S \) with \( tx^m = 0 \) by (11.14)(1). Then \( (tx)^m = 0 \). So \( tx \in \text{nil}(R) \). But \( tx/ts = x/s \). So \( x/s \in \text{nil}(R)(S^{-1}R) \) by (11.14)(1). Thus \( \text{nil}(R)(S^{-1}R) \subseteq \text{nil}(S^{-1}R) \).

Exercise (11.24). — Let \( R'/R \) be an integral extension of rings, and \( S \) a multiplicative subset of \( R \). Show that \( S^{-1}R' \) is integral over \( S^{-1}R \).

Solution: Given \( x/s \in S^{-1}R' \), let \( x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \) be an equation of integral dependence of \( x \) on \( R \). Then

\[
(x/s)^n + (a_{n-1}/1)(1/s)(x/s)^{n-1} + \cdots + a_0(1/s)^n = 0
\]

is an equation of integral dependence of \( x/s \) on \( S^{-1}R \), as required.

Exercise (11.25). — Let \( R \) be a domain, \( K \) its fraction field, \( L \) a finite extension field, and \( \overline{R} \) the integral closure of \( R \) in \( L \). Show that \( L \) is the fraction field of \( \overline{R} \). Show that, in fact, every element of \( L \) can be expressed as a fraction \( b/a \) where \( b \) is in \( \overline{R} \) and \( a \) is in \( R \).

Solution: Let \( x \in L \). Then \( x \) is algebraic (integral) over \( K \), say

\[
x^n + y_1x^{n-1} + \cdots + y_n = 0
\]

with \( y_i \in K \). Write \( y_i = a_i/a \) with \( a_1, \ldots, a_n, a \in R \). Then

\[
(ax)^n + a_1(ax)^{n-1} + \cdots + a^n = 0
\]

Set \( b := ax \). Then \( b \in \overline{R} \) and \( x = b/a \).

Exercise (11.26). — Let \( R \subset R' \) be domains, \( K \) and \( L \) their fraction fields. Assume that \( R' \) is a finitely generated \( R \)-algebra, and that \( L \) is a finite dimensional \( K \)-vector space. Find an \( f \in R \) such that \( R'_f \) is module finite over \( R_f \).

Solution: Let \( x_1, \ldots, x_n \) generate \( R' \) over \( R \). Using (11.24), write \( x_i = b_i/a_i \) with \( b_i \) integral over \( R \) and \( a_i \) in \( R \). Set \( f := \prod a_i \). The \( x_i \) generate \( R'_f \) as an \( R_f \)-algebra; so the \( b_i \) do too. Thus \( R'_f \) is module finite over \( R_f \) by (11.28).
EXERCISE (11.29). — Let $R$ be a ring, $S$ and $T$ multiplicative subsets.

1. Set $T' := \varphi_S(T)$ and assume $S \subset T$. Prove

$$T^{-1}R = T'^{-1}(S^{-1}R) = T^{-1}(S^{-1}R).$$

2. Set $U := \{st \in R \mid s \in S$ and $t \in T\}$. Prove

$$T^{-1}(S^{-1}R) = S^{-1}(T^{-1}R) = U^{-1}R.$$

SOLUTION: A proof similar to that of (11.27) shows $T^{-1}R = T'^{-1}(S^{-1}R)$. By (11.27), $T'^{-1}(S^{-1}R) = T^{-1}(S^{-1}R)$. Thus (1) holds.

As $1 \in T$, obviously $S \subset U$. So (1) yields $U^{-1}R = U^{-1}(S^{-1}R)$. Now, clearly $U^{-1}(S^{-1}R) = T^{-1}(S^{-1}R)$. Similarly, $U^{-1}RS^{-1}(T^{-1}R)$. Thus (2) holds. \hfill \Box

EXERCISE (11.32) (Localization and normalization commute). — Given a domain $R$ and a multiplicative subset $S$ with $0 \notin S$. Show that the localization of the normalization $S^{-1}R$ is equal to the normalization of the localization $S^{-1}R$.

SOLUTION: Since $0 \notin S$, clearly $\text{Frac}(S^{-1}R) = \text{Frac}(R)$ owing to (11.23). Now, $S^{-1}R$ is integral over $S^{-1}R$ by (11.21). Thus $S^{-1}R \subset S^{-1}R$.

Conversely, given $x \in S^{-1}R$, consider an equation of integral dependence:

$$x^n + a_1x^{n-1} + \cdots + a_n = 0.$$

Say $a_i = b_i/s_i$ with $b_i \in R$ and $s_i \in S$; set $s := \prod s_i$. Multiplying by $s^n$ yields

$$(sx)^n + sa_1(sx)^{n-1} + \cdots + s^n a_n = 0.$$

Hence $sx \in R$. So $x \in S^{-1}R$. Thus $S^{-1}R \supset S^{-1}R$, as desired. \hfill \Box

12. Localization of Modules

EXERCISE (12.4). — Let $R$ be a ring, $S$ a multiplicative subset, and $M$ a module. Show that $M = S^{-1}M$ if and only if $M$ is an $S^{-1}R$-module.

SOLUTION: If $M = S^{-1}M$, then $M$ is an $S^{-1}R$-module since $S^{-1}M$ is by (12.23). Conversely, if $M$ is an $S^{-1}R$-module, then $M$ equipped with the identity map has the UMP that characterizes $S^{-1}M$; whence, $M = S^{-1}M$. \hfill \Box

EXERCISE (12.5). — Let $R$ be a ring, $S \subset T$ multiplicative subsets, $M$ a module. Set $T_1 := \varphi_S(T) \subset S^{-1}R$. Show $T^{-1}M = T_1^{-1}(S^{-1}M) = T_1^{-1}(S^{-1}M)$.

SOLUTION: Let’s check that both $T^{-1}(S^{-1}M)$ and $T_1^{-1}(S^{-1}M)$ have the UMP characterizing $T^{-1}M$. Let $\psi: M \to N$ be an $R$-linear map into an $T^{-1}R$-module. Then the multiplication map $\mu_s: N \to N$ is bijective for all $s \in T$ by (12.21), so for all $s \in S$ since $S \subset T$. Hence $\psi$ factors via a unique $S^{-1}R$-linear map $\rho: S^{-1}M \to N$ by (12.23) and by (12.21) again.

Similarly, $\rho$ factors through a unique $T^{-1}R$-linear map $\rho': T^{-1}(S^{-1}M) \to N$. Hence $\psi = \rho' \varphi_T \varphi_S$, and $\rho'$ is clearly unique, as required. Also, $\rho$ factors through a unique $T_1^{-1}(S^{-1}R)$-linear map $\rho'_1: T_1^{-1}(S^{-1}M) \to N$. Hence $\psi = \rho'_1 \varphi_{T_1} \varphi_S$, and $\rho'_1$ is clearly unique, as required. \hfill \Box
EXERCISE (12.6). — Let $R$ be a ring, $S$ a multiplicative subset. Show that $S$ becomes a filtered category when equipped as follows: given $s, t \in S$, set

$$\text{Hom}(s, t) := \{ x \in R \mid xs = t \}.$$  

Given a module $M$, define a functor $S \to ((R-\text{mod}))$ as follows: for $s \in S$, set $M_s := M$; to each $x \in \text{Hom}(s, t)$, associate $\mu_x : M_s \to M_t$. Define $\beta_s : M_s \to S^{-1}M$ by $\beta_s(m) := m/s$. Show the $\beta_s$ induce an isomorphism $\varprojlim M_s \cong S^{-1}M$.

**Solution:** Clearly, $S$ is a category. Now, given $s, t \in S$, set $u := st$. Then $u \in S$; also $t \in \text{Hom}(s, u)$ and $s \in \text{Hom}(t, u)$. Given $x, y \in \text{Hom}(s, t)$, we have $xs = t$ and $ys = t$. So $s \in \text{Hom}(t, u)$ and $xs = ys$ in $\text{Hom}(s, u)$. Thus $S$ is filtered.

Further, given $x \in \text{Hom}(s, t)$, we have $\beta_x \mu_x = \beta_s$ since $m/s = xm/t$ as $xs = t$. So the $\beta_s$ induce a homomorphism $\beta : \varprojlim M_s \to S^{-1}M$. Now, every element of $S^{-1}M$ is of the form $m/s$, and $m/s := \beta_s(m)$; hence, $\beta$ is surjective.

Each $m \in \varprojlim M_s$ lifts to an $m' \in M_s$ for some $s \in S$ by (12.1). Assume $\beta m = 0$. Then $\beta m' = 0$ as the $\beta_s$ induce $\beta$. But $\beta m' = m'/s$. So there is $t \in S$ with $tm' = 0$. So $\mu_t m' = 0$ in $M_{st}$, and $\mu_t m' \mapsto m$. So $m = 0$. Thus $\beta$ is injective, so an isomorphism. \hfill \square

EXERCISE (12.7). — Let $R$ be a ring, $S$ a multiplicative subset, $M$ a module. Prove $S^{-1}M = 0$ if $\text{Ann}(M) \cap S \neq \emptyset$. Prove the converse if $M$ is finitely generated. 

**Solution:** Say $f \in \text{Ann}(M) \cap S$. Let $m/t \in S^{-1}M$. Then $f/1 \cdot m/t = fm/t = 0$. Hence $m/t = 0$.

Conversely, assume $S^{-1}M = 0$, and say $m_1, \ldots, m_n$ generate $M$. Then for each $i$, there is $f_i \in S$ with $f_im_i = 0$. Then $\prod f_i \in \text{Ann}(M) \cap S$, as desired. \hfill \square

EXERCISE (12.8). — Let $R$ be a ring, $M$ a finitely generated module, $a$ an ideal. 

(1) Set $S := 1 + a$. Show that $S^{-1}a$ lies in the radical of $S^{-1}R$.

(2) Use (1), Nakayama’s Lemma (11.11), and (12.7), but not the determinant trick (11.2), to prove this part of (11.3): if $M = aM$, then $sM = 0$ for an $s \in S$.

**Solution:** For (1), use (11.7) as follows. Take $a/(1 + b) \in S^{-1}a$ with $a, b \in a$. Then for $x \in R$ and $c \in a$, we have

$$1 + (a/(1 + b))(x/(1 + c)) = (1 + (b + c + bc + ax))/(1 + b)(1 + c).$$

The latter is a unit in $S^{-1}R$, as $b + c + bc + ax \in a$. So $a/(1 + b) \in \text{rad}(S^{-1}R)$ by (11.2). Thus (1) holds.

For (2), assume $M = aM$. Then $S^{-1}M = S^{-1}aS^{-1}M$ by (12.7). So $S^{-1}M = 0$ by (1) and (11.11). So (12.7) yields an $s \in S$ with $sM = 0$. Thus (2) holds. \hfill \square

EXERCISE (12.12). — Let $R$ be a ring, $S$ a multiplicative subset, $P$ a projective module. Then $S^{-1}P$ is a projective $S^{-1}R$-module.

**Solution:** By (11.2), there is a module $K$ such that $F := K \oplus P$ is free. So (11.11) yields that $S^{-1}F = S^{-1}P \oplus S^{-1}K$ and that $S^{-1}F$ is free over $S^{-1}R$.

Hence $S^{-1}P$ is a projective $S^{-1}R$-module again by (11.2). \hfill \square

EXERCISE (12.14). — Let $R$ be a ring, $S$ a multiplicative subset, $M$ and $N$ modules. Show $S^{-1}(M \otimes_R N) = S^{-1}M \otimes_R S^{-1}N = S^{-1}M \otimes_{S^{-1}R} S^{-1}N = S^{-1}M \otimes_{S^{-1}R} S^{-1}N$. 

Solutions: (12.14)
SOLUTION: By 12.13, \( S^{-1}(M \otimes_R N) = S^{-1}R \otimes_R (M \otimes_R N) \). The latter is equal to \( (S^{-1}R \otimes_R M) \otimes_R N \) by associativity (12.11). Again by (12.13), the latter is equal to \( S^{-1}M \otimes_R N \). Thus the first equality holds.

By cancellation (12.14), \( S^{-1}M \otimes_R N = S^{-1}M \otimes_{S^{-1}R} (S^{-1}R \otimes_R N) \), and the latter is equal to \( S^{-1}M \otimes_{S^{-1}R} S^{-1}N \) by (12.13). Thus the second equality holds.

Finally by (12.3), the kernel of the map \( S^{-1}M \otimes_R S^{-1}N \to S^{-1}M \otimes_{S^{-1}R} S^{-1}N \) is generated by elements \((xm/s) \otimes (n/1) - (m/1) \otimes (xn/s)\) with \( m \in M, n \in N, x \in R, \) and \( s \in S \). Those elements are zero because \( \mu_s \) is an isomorphism on the \( S^{-1}R \)-module \( S^{-1}M \otimes_R S^{-1}N \). Thus the third equality holds.

Exercise (12.15). — Let \( R \) be a ring, \( R' \) an algebra, \( S \) a multiplicative subset, \( M \) a finitely presented module, and \( r \) an integer. Show

\[ F_r(M \otimes_R R') = F_r(M)R' \quad \text{and} \quad F_r(S^{-1}M) = F_r(M)S^{-1}R = S^{-1}F_r(M). \]

SOLUTION: Let \( R^n \xrightarrow{\alpha} R^m \to M \to 0 \) be a presentation. Then, by (12.9),

\[ (R')^n \otimes_{R'} R' \to (R')^m \otimes_{R'} R' \to 0 \]

is a presentation. Further, the matrix \( A \) of \( \alpha \) induces the matrix of \( \alpha \otimes 1 \). Thus

\[ F_r(M \otimes_R R') = F_r(M)R' = F_r(M)R'. \]

For the last equalities, take \( R' := S^{-1}R \). Then \( F_r(S^{-1}M) = F_r(M)S^{-1}R \) by (12.10). Finally, \( F_r(M)S^{-1}R = S^{-1}F_r(M) \) by (12.7).

Exercise (12.18). — Let \( R \) be a ring, \( S \) a multiplicative subset.

1. Let \( M_1 \xrightarrow{\alpha} M_2 \) be a map of modules, which restricts to a map \( N_1 \to N_2 \) of submodules. Show \( \alpha(N_1^S) \subseteq N_2^S \); that is, there is an induced map \( N_1^S \to N_2^S \).

2. Let \( 0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \) be a left exact sequence, which restricts to a left exact sequence \( 0 \to N_1 \to N_2 \to N_3 \) of submodules. Show there is an induced left exact sequence of saturations: \( 0 \to N_1^S \to N_2^S \to N_3^S \).

SOLUTION: For (1), take \( m \in N_1^S \). Then there is \( s \in S \) with \( sm \in N_1 \). So \( \alpha(sm) = sa(m) \). Thus (1) holds.

In (2), \( \alpha(N_1^S) \subseteq N_2^S \) and \( \beta(N_2^S) \subseteq N_3^S \) by (1). Trivially, \( \alpha|N_1^S \) is injective, and \( \beta|N_2^S = 0 \). Finally, given \( m_2 \in \text{Ker}(\beta|N_2^S) \), there is \( s \in S \) with \( sm_2 \in N_2 \), and exactness yields \( m_1 \in M_1 \) with \( \alpha(m_1) = m_2 \). Then \( \beta(sm_2) = s\beta(m_2) = 0 \). So exactness yields \( n_1 \in N_1 \) with \( \alpha(n_1) = sm_2 \). Also \( \alpha(sm_1) = sa(m_1) = sm_2 \). But \( \alpha \) is injective. Hence \( sm_1 = n_1 \). So \( m_1 \in N_1^S \), and \( \alpha(m_1) = m_2 \). Thus (2) holds.

Exercise (12.19). — Let \( R \) be a ring, \( M \) a module, and \( S \) a multiplicative subset. Set \( T^S M := \langle 0 \rangle^S \). We call it the \( S \)-torsion submodule of \( M \). Prove the following:

1. \( T^S(M/T^S M) = 0 \).
2. \( T^S M = \text{Ker}(\varphi_S) \).
3. Let \( \alpha \colon M \to N \) be a map. Then \( \alpha(T^S M) \subseteq T^S N \).
4. Let \( 0 \to M' \to M \to M'' \) be exact. Then so is \( 0 \to T^S M' \to T^S M \to T^S M'' \).
5. Let \( S_1 \subseteq S \) be a multiplicative subset. Then \( T^S(S_1^{-1} M) = S_1^{-1} T^S M \).

SOLUTION: For (1), given an element of \( T^S(M/T^S M) \), let \( m \in M \) represent it. Then there is \( s \in S \) with \( sm \in T^S M \). So there is \( t \in S \) with \( tsm = 0 \). So \( m \in T^S M \). Thus (1) holds. Assertion (2) holds by (12.7) (3).

Assertions (3) and (4) follow from (12.13) (1) and (2).

For (5), given \( m/s_1 \in S_1^{-1} T^S(M) \) with \( s_1 \in S_1 \) and \( m \in T^S M \), take \( s \in S \) with \( sm = 0 \). Then \( sm/s_1 = 0 \). So \( m/s_1 \in T^S(S_1^{-1} M) \). Thus \( S_1^{-1} T^S(M) \subseteq T^S(S_1^{-1} M) \).
For the opposite inclusion, given \( m/s_1 \in TS(S^{-1}M) \) with \( m \in M \) and \( s_1 \in S_1 \), take \( t/t_1 \) with \( t \in S \) and \( t_1 \in S_1 \) and \( t/t_1 \cdot m/s_1 = 0 \). Then \( tm/1 = 0 \). So there is \( s' \in S_1 \) with \( s'tm = 0 \) by (12.27);(3). But \( s't \in S \) as \( S_1 \subseteq S \). So \( m \in TS(M) \). Thus \( m/s_1 \in S^{-1}TS(M) \). Thus (5) holds. □

Exercise (12.28). — Set \( R := \mathbb{Z} \) and \( S = \mathbb{Z} - \langle n \rangle \). Then \( M := \bigoplus_{n \geq 2} \mathbb{Z}/\langle n \rangle \) and \( N := M \). Show that the map \( \sigma \) of (12.28) is not injective.

Solution: Given \( m > 0 \), let \( e_n \) be the \( n \)th standard basis element for some \( n > m \). Then \( m \cdot e_n \neq 0 \). Hence \( \mu_R: R \to \text{Hom}_R(M, M) \) is injective. But \( S^{-1}M = 0 \), as any \( x \in M \) has only finitely many nonzero components; so \( kx = 0 \) for some nonzero integer \( k \). So \( \text{Hom}(S^{-1}M, S^{-1}M) = 0 \). Thus \( \sigma \) is not injective. □

13. Support

Exercise (13.2). — Let \( R \) be a ring, \( p \in \text{Spec}(R) \). Show that \( p \) is a closed point — that is, \( \{p\} \) is a closed set — if and only if \( p \) is a maximal ideal.

Solution: If \( p \) is maximal, then \( V(p) = \{p\} \); so \( p \) is closed.

Conversely, suppose \( p \) is not maximal. Then \( p \not= \bigcap m \) for some maximal ideal \( m \). If \( p \in V(a) \), then \( m \in V(a) \) too. So \( \{p\} \neq V(a) \). Thus \( \{p\} \) is not closed. □

Exercise (13.3). — Let \( R \) be a ring, and set \( X := \text{Spec}(R) \). Let \( X_1, X_2 \subseteq X \) be closed subsets. Show that the following four conditions are equivalent:

(1) \( X_1 \cup X_2 = X \) and \( X_1 \cap X_2 = \emptyset \).
(2) There are complementary idempotents \( e_1, e_2 \in R \) with \( V(e_i) = X_i \).
(3) There are comaximal ideals \( a_1, a_2 \subseteq R \) with \( a_1a_2 = 0 \) and \( V(a_i) = X_i \).
(4) There are ideals \( a_1, a_2 \subseteq R \) with \( a_1 + a_2 = R \) and \( V(a_i) = X_i \).

Finally, given any \( e_i \) and \( a_i \) satisfying (2) and either (3) or (4), necessarily \( e_i \in a_i \).

Solution: Assume (1). Take ideals \( a_1, a_2 \) with \( V(a_i) = X_i \). Then (13.11) yields

\[
V(a_1a_2) = V(a_1) \cup V(a_2) = X = V(0) \quad \text{and} \quad V(a_1 + a_2) = V(a_1) \cap V(a_2) = \emptyset = V(R).
\]

So \( \sqrt{a_1a_2} = \sqrt{(0)} \) and \( \sqrt{a_1} + \sqrt{a_2} = \sqrt{R} \) again by (13.11). Hence (13.27) yields (2).

Assume (2). Set \( a_i := \langle e_i \rangle \). As \( e_1 + e_2 = 1 \) and \( e_1e_2 = 0 \), plainly (3) holds.

Assume (3). As the \( a_i \) are comaximal, the Chinese Remainder Theorem (13.13) yields \( a_1 \cap a_2 = a_1a_2 \). But \( a_1a_2 = 0 \). So \( a_1 + a_2 = R \) by (13.17). Thus (4) holds.

Assume (4). Then (13.14) yields (1) as follows:

\[
X_1 \cup X_2 = V(a_1) \cup V(a_2) = V(a_1a_2) = V(0) = X \quad \text{and} \quad X_1 \cap X_2 = V(a_1) \cap V(a_2) = V(a_1 + a_2) = V(R) = \emptyset.
\]

Finally, say \( e_i \) and \( a_i \) satisfy (2) and either (3) or (4). Then \( \sqrt{e_i} = \sqrt{a_i} \) by (13.11). So \( e_i^n \in a_i \) for some \( n \geq 1 \). But \( e_i^2 = e_i \), so \( e_i^n = e_i \). Thus \( e_i \in a_i \). □

Exercise (13.4). — Let \( \varphi: R \to R' \) be a map of rings, \( a \) an ideal of \( R \), and \( b \) an ideal of \( R' \). Set \( \varphi^*: \text{Spec}(\varphi) \). Prove these two statements:

(1) Every prime of \( R \) is a contraction of a prime if and only if \( \varphi^* \) is surjective.
(2) If every prime of \( R' \) is an extension of a prime, then \( \varphi^* \) is injective.

Is the converse of (2) true?
SOLUTION: Note \( \varphi^*(q) := \varphi^{-1}(q) \) by (13.1.2). Hence (1) holds.

Given two primes \( q_1 \) and \( q_2 \) that are extensions, if \( q_1' = q_2' \), then \( q_1 = q_2 \) by (13.3). Thus (2) holds.

The converse of (2) is false. Take \( R \) to be a domain. Set \( R' := R[X]/(X^2) \). Then \( R'/\langle X \rangle = R \) by (13.3). So \( \langle X \rangle \) is prime by (13.4). But \( \langle X \rangle \) is not an extension, as \( X \notin \mathfrak{a}R' \) for any proper ideal \( \mathfrak{a} \) of \( R \). Finally, every prime \( q \) of \( R' \) contains the residue \( x \) of \( X \), as \( x^2 = 0 \). Hence \( q \) is generated by \( q \cap R \) and \( x \). But \( q \cap R = \varphi^*(q) \). Thus \( \varphi^* \) is injective. \( \square \)

EXERCISE (13.5). Let \( R \) be a ring, \( S \) a multiplicative subset. Set \( X := \text{Spec}(R) \) and \( Y := \text{Spec}(S^{-1}R) \). Set \( \varphi^*_S := \text{Spec}(\varphi_S) \) and \( S^{-1}X := \text{Im} \varphi^*_S \subset X \). Show (1) that \( S^{-1}X \) consists of the primes \( p \) of \( R \) with \( p \cap S = \emptyset \) and (2) that \( \varphi^*_S \) is a homeomorphism of \( Y \) onto \( S^{-1}X \).

SOLUTION: Note \( \varphi^*_S(q) := \varphi^{-1}_S(q) \) by (13.1.2). Hence (13.1.2)(2) yields (1) and also the bijectivity of \( \varphi^*_S \). But \( \varphi^*_S \) is continuous by (13.4). So it remains to show that \( \varphi^*_S \) is closed. Given an ideal \( b \subset S^{-1}R \), set \( a := \varphi^{-1}_S(b) \). It suffices to show

\[
\varphi^*_S(V(b)) = S^{-1}X \cap V(a). \tag{13.5.1}
\]

Given \( p \in \varphi^*_S(V(b)) \), say \( p = \varphi^*_S(q) \) and \( q \in V(b) \). Then \( p = \varphi^{-1}_S(q) \) and \( q \supset b \) by (13.1.1). So \( p = \varphi^{-1}_S(q) \supset \varphi^{-1}_S(b) =: a \). So \( p \in V(a) \). But \( p \in \varphi^*_S(V(b)) \subset S^{-1}X \). Thus \( \varphi^*_S(V(b)) \subset S^{-1}X \cap V(a) \).

Conversely, given \( p \in S^{-1}X \cap V(a) \), say \( p = \varphi^*_S(q) \). Then \( p = \varphi^{-1}_S(q) \) and \( p \supset a := \varphi^{-1}_S(b) \). So \( \varphi^{-1}_S(q) \supset \varphi^{-1}_S(b) \). So \( \varphi^{1}_S(q)R \supset \varphi^{1}_S(b)R \). So \( q \supset b \) by (13.1.2)(1)(b). So \( q \in V(b) \). So \( p = \varphi^*_S(q) \in \varphi^*_S(V(b)) \). Thus (13.5.1) holds, as desired. Thus (2) holds. \( \square \)

EXERCISE (13.6). Let \( \theta : R \to R' \) be a ring map, \( S \subset R \) a multiplicative subset. Set \( X := \text{Spec}(R) \) and \( Y := \text{Spec}(R') \) and \( \theta^* := \text{Spec}(\theta) \). Via (13.5.2), identify \( \text{Spec}(S^{-1}R) \) and \( \text{Spec}(S^{-1}R') \) with their images \( S^{-1}X \subset X \) and \( S^{-1}Y \subset Y \). Show (1) \( S^{-1}Y = \theta^{-1}(S^{-1}X) \) and (2) \( \text{Spec}((S^{-1}X) \cup \text{Spec}(S^{-1}Y) \cup \text{Spec}(S^{-1}Y)) \) holds as desired. Thus (2) holds.

SOLUTION: Given \( q \in Y \), elementary set-theory shows that \( q \cap \theta(S) = \emptyset \) if and only if \( q \cap S = \emptyset \) if and only if \( \theta^{-1}(q) \cap S = \emptyset \). So \( q \in S^{-1}Y \) if and only if \( \theta^{-1}(q) \in S^{-1}X \) by (13.5.1)(1) and (13.5.2). But \( \varphi^*_S(q) := \varphi^*_S(q) \) by (13.5.2). Thus (1) holds.

Finally, \( (S^{-1}X \cup \text{Spec}(S^{-1}Y) \cup \text{Spec}(S^{-1}Y)) \) holds as desired. Thus (2) holds. \( \square \)

EXERCISE (13.7). Let \( \theta : R \to R' \) be a ring map, \( a \subset R \) an ideal. Set \( b := aR' \). Let \( \overline{\theta} : R/a \to R'/b \) be the induced map. Set \( X := \text{Spec}(R) \) and \( Y := \text{Spec}(R') \). Set \( \theta^* := \text{Spec}(\theta) \) and \( \overline{\theta}^* := \text{Spec}(\overline{\theta}) \). Via (13.4), identify \( \text{Spec}(R/a) \) and \( \text{Spec}(R'/b) \) with \( V(a) \subset X \) and \( V(b) \subset Y \). Show (1) \( \overline{V}(b) = \theta^{-1}(V(a)) \) and (2) \( \overline{\theta}^* = \theta^*|_{\overline{V}(b)} \). Show (2) holds.

SOLUTION: Given \( q \in Y \), observe that \( q \supset b \) if and only if \( \theta^{-1}(q) \supset a \). As follows. By (13.4)(1) in its notation, \( q \supset b \) yields \( q^e \supset a^e \supset a \), and \( q^e \supset a \) yields \( q \supset q^e \supset a^e \). Thus (1) holds.

Plainly, \( \overline{\theta}(q/b) = (\theta^{-1}/a) \). Thus (13.1.2) yields (2). \( \square \)

EXERCISE (13.8). Let \( \theta : R \to R' \) be a ring map, \( p \subset R \) a prime, \( k \) the residue field of \( R_p \). Set \( \theta^* := \text{Spec}(\theta) \). Show (1) that \( \theta^{-1}(p) \) is canonically homeomorphic to \( \text{Spec}(R' \otimes_R k) \) and (2) that \( p \in \text{Im} \theta^* \) if and only if \( R' \otimes_R k \neq 0 \).
SOLUTION: First, take $S := R - p$ and apply (13.10) to obtain
\[ \text{Spec}(R'_p) = \theta^*^{-1}(\text{Spec}(R_p)) \quad \text{and} \quad \text{Spec}(\theta_p) = \theta^*|\text{Spec}(R'_p). \]
Next, take $\alpha := pR_p$ and apply (13.21) to $\theta_p : R_p \to R'_p$ to obtain
\[ \text{Spec}(R'/pR') = \text{Spec}(\theta_p)^{-1} \cdot V(pR_p). \]
But $\theta^{-1}(pR_p) = p$ by (13.20)(2); so $V(pR_p) = p$. Therefore,
\[ \text{Spec}(R'_p/pR'_p) = (\theta^*|\text{Spec}(R'_p))^{-1}(p) = \theta^*^{-1}(p). \]
But $k := R_p/pR_p$. So $R'/pR' = k \otimes_R R'$. Thus (1) holds.
Finally, (1) implies $p \in \text{Im} \theta^*$ if and only if $\text{Spec}(R' \otimes_R k) \neq \emptyset$. Thus (2) holds. □

EXERCISE (13.9). — Let $R$ be a ring, $p$ a prime ideal. Show that the image of $\text{Spec}(R_p)$ in $\text{Spec}(R)$ is the intersection of all open neighborhoods of $p$ in $\text{Spec}(R)$.

SOLUTION: By (13.5), the image $X_p$ consists of the primes contained in $p$. Given $f \in R - p$, note that $D(f)$ contains every prime contained in $p$, or $X_p \subset D(f)$. But the principal open sets form a basis of the topology of $X$ by (13.4). Hence $X_p$ is contained in the intersection, $Z$ say, of all open neighborhoods of $p$. Conversely, given a prime $q \notin p$, there is $g \in q - p$. So $D(g)$ is an open neighborhood of $p$, and $q \notin D(g)$. Thus $X_p = Z$, as desired. □

EXERCISE (13.10). — Let $\varphi : R \to R'$ and $\psi : R \to R''$ be ring maps, and define $\theta : R \to R' \otimes_R R''$ by $\theta(x) := \varphi(x) \otimes \psi(x)$. Show
\[ \text{Im} \text{Spec}(\theta) = \text{Im} \text{Spec}(\varphi) \cap \text{Im} \text{Spec}(\psi). \]

SOLUTION: Given $p \in X$, set $k := R_p/pR_p$. Then (8.11) and (8.14) yield
\[ (R' \otimes_R R'') \otimes_R k = (R' \otimes_R (R'' \otimes_R k) = (R' \otimes_R k) \otimes_R (R'' \otimes_R k). \]
So $(R' \otimes_R R'') \otimes_R k \neq 0$ if and only if $R' \otimes_R k \neq 0$ and $R'' \otimes_R k \neq 0$ by (8.15). Hence (13.5)(2) implies that $p \in \text{Im} \text{Spec}(\theta)$ if and only if $p \in \text{Im} \text{Spec}(\varphi)$ and $p \in \text{Im} \text{Spec}(\psi)$, as desired. □

EXERCISE (13.11). — Let $R$ be a filtered direct limit of rings $R_\lambda$ with transition maps $\alpha^\lambda_\mu$ and insertions $\alpha_\lambda$. For each $\lambda$, let $\varphi_\lambda : R' \to R_\lambda$ be a ring map with $\varphi_\mu = \alpha^\lambda_\mu \varphi_\lambda$ for all $\alpha^\lambda_\mu$, so that $\varphi := \alpha_\lambda \varphi_\lambda$ is independent of $\lambda$. Show
\[ \text{Im} \text{Spec}(\varphi) = \bigcap_\lambda \text{Im} \text{Spec}(\varphi_\lambda). \]

SOLUTION: Given $q \in \text{Spec}(R')$, set $k := R'_p/qR'_q$. Then (8.13) yields
\[ R \otimes_{R'} k = \text{lim}_\lambda \text{Spec}(R_\lambda \otimes_{R'} k). \]
So $R \otimes_{R'} k \neq 0$ if and only if $R_\lambda \otimes_{R'} k \neq 0$ for all $\lambda$ by (13.11)(1). Hence (13.5)(2) implies that $p \in \text{Im} \text{Spec}(\varphi)$ if and only if $p \in \text{Im} \text{Spec}(\varphi_\lambda)$ for all $\lambda$, as desired. □

EXERCISE (13.12). — Let $A$ be a domain with just one nonzero prime $p$. Set $K := \text{Frac}(A)$ and $R := (A/p) \times K$. Define $\varphi : A \to R$ by $\varphi(x) := (x', x)$ with $x'$ the residue of $x$. Set $\varphi^* := \text{Spec}(\varphi)$. Show $\varphi^*$ is bijective, but not a homeomorphism.

SOLUTION: Note $p$ is maximal; so $A/p$ is a field. The primes of $R$ are $(0, K)$ and $(A/p, 0)$ by (13.10). Plainly, $\varphi^{-1}(0, K) = p$ and $\varphi^{-1}(A/p, 0) = 0$. So $\varphi^*$ is bijective. Finally, $\text{Spec}(R)$ is discrete, but $\text{Spec}(A)$ has $p \in V(0)$; so $\varphi^*$ is not a homeomorphism. □
EXERCISE (13.13). — Let $\varphi: R \to R'$ be a ring map, and $b$ an ideal of $R'$. Set $\varphi^* := \text{Spec}(\varphi)$. Show that the closure $\overline{\varphi^*(V(b))}$ in $\text{Spec}(R)$ is equal to $V(\varphi^{-1}b)$ and (2) that $\varphi^*(\text{Spec}(R'))$ is dense in $\text{Spec}(R)$ if and only if $\text{Ker}(\varphi) \subseteq \text{nil}(R)$.

Solution: For (1), given $p \in \varphi^*(V(b))$, say $p = \varphi^{-1}(\mathfrak{P})$ where $\mathfrak{P}$ is a prime of $R'$ with $\mathfrak{P} \ni b$. Then $\varphi^{-1}\mathfrak{P} \ni \varphi^{-1}b$. So $p \ni \varphi^{-1}b$, or $p \in V(\varphi^{-1}b)$. Thus $\varphi^*(V(b)) \subset V(\varphi^{-1}b)$. But $V(\varphi^{-1}b)$ is closed. So $\overline{\varphi^*(V(b))} \subset V(\varphi^{-1}b)$.

Conversely, given $p \in V(\varphi^{-1}b)$, note $p \ni \sqrt{\varphi^{-1}b}$. Take a neighborhood $D(f)$ of $p$; then $f \notin p$. Hence $f \notin \sqrt{\varphi^{-1}b}$. But $\sqrt{\varphi^{-1}b} = \varphi^{-1}(\sqrt{b})$ by (13.10). Hence $\varphi(f) \notin \sqrt{b}$. So there's a prime $\mathfrak{P} \ni b$ with $\varphi(f) \notin \mathfrak{P}$ by the Scheinnullstellensatz (13.25). So $\sqrt{\varphi^{-1}b} \in \varphi^*(V(b))$. Further, $f \notin \sqrt{\varphi^{-1}b}$, or $\varphi^{-1}\mathfrak{P} \ni D(f)$. Therefore, $\sqrt{\varphi^{-1}b} \in \varphi^*(V(b)) \cap D(f)$. So $\overline{\varphi^*(V(b))} \cap D(f) \neq \emptyset$. So $p \in \varphi^*(V(b))$. Thus (1) holds.

For (2), take $b := (0)$. Then (1) yields $\overline{\varphi^*(V(b))} = V(\text{Ker}(\varphi))$. But by (13.14), $V(b) = \text{Spec}(R')$ and $\text{Spec}(R) = V((0))$. So $\overline{\varphi^*(\text{Spec}(R'))} = \text{Spec}(R)$ if and only if $V((0)) = V(\text{Ker}(\varphi))$. The latter holds if and only if nil($R$) = $\sqrt{\text{Ker}(\varphi)}$ by (13.14), so plainly if and only if nil($R$) $\supset$ Ker($\varphi$). Thus (2) holds.

EXERCISE (13.14). — Let $R$ be a ring, $R'$ a flat algebra with structure map $\varphi$. Show that $R'$ is faithfully flat if and only if Spec($\varphi$) is surjective.

Solution: Owing to the definition of Spec($\varphi$) in (13.14), the assertion amounts to the equivalence of (1) and (3) of (13.14).

EXERCISE (13.15). — Let $\varphi: R \to R'$ be a flat map of rings, $q$ a prime of $R'$, and $p = \varphi^{-1}(q)$. Show that the induced map Spec($R'_q$) $\to$ Spec($R_p$) is surjective.

Solution: Since $p = \varphi^{-1}(q)$, clearly $\varphi(R - p) \subset (R' - q)$. Thus $\varphi$ induces a local homomorphism $R_p \to R'_q$. Moreover, $R'_q$ is flat over $R_p$ as $R'_q = R_p \otimes_R R'$ by (13.14), and $R_p \otimes_R R'$ is flat over $R_p$ by (13.14). Also $R'_q$ is flat over $R'_p$ by (13.14). Hence $R'_q$ is flat over $R_p$ by (13.14). So $R'_q$ is faithfully flat over $R_p$ by (13.14). Hence Spec($R'_q$) $\to$ Spec($R_p$) is surjective by (13.14).

EXERCISE (13.16). — Let $R$ be a ring. Given $f \in R$, set $S_f := \{f^n \mid n \geq 0\}$, and let $\overline{S_f}$ denote its saturation; see (13.17). Given $f, g \in R$, show that the following conditions are equivalent:

1. $D(g) \subset D(f)$.
2. $V(\langle g \rangle) \subset V(\langle f \rangle)$.
3. $\sqrt{\langle g \rangle} \subset \sqrt{\langle f \rangle}$.
4. $\overline{S_f} \subset \overline{S_g}$.
5. $g \in \sqrt{\langle f \rangle}$.
6. $f \in \overline{S_g}$.
7. There is a unique $R$-algebra map $\varphi_f: \overline{S_f}^{-1}R \to \overline{S_g}^{-1}R$.
8. There is an $R$-algebra map $R_f \to R_g$.

Show that, if these conditions hold, then the map in (8) is equal to $\varphi_g^f$.

Solution: First, (1) and (2) are equivalent by (13.14), and (2) and (3) are too. Plainly, (3) and (5) are equivalent. Further, (3) and (4) are equivalent by (13.17)(4). Always $f \in \overline{S_f}$; so (4) implies (6). Conversely, (6) implies $S_f \subset \overline{S_g}$; whence, (13.17)(1)(c) yields (4). Finally, (8) implies (4) by (13.14)(2). And (4) implies (7) by (13.14)(1). But $\overline{S_f}^{-1}R = \overline{S_f}^{-1}R$ and $\overline{S_g}^{-1}R = \overline{S_g}^{-1}R$ by (13.14); whence, (7) implies both (8) and the last statement.
Exercise (13.17). — Let $R$ be a ring. (1) Show that $D(f) \to R_f$ is a well-defined contravariant functor from the category of principal open sets and inclusions to ($R$-alg)). (2) Given $p \in \text{Spec}(R)$, show $\lim_{\to_{D(f) \ni p}} R_f = R_p$.

Solution: Consider (1). By (13.11), if $D(g) \subset D(f)$, then there is a unique $R$-algebra map $\varphi'_g : S_f^{-1} R \to S_g^{-1} R$. By uniqueness, if $D(h) \subset D(g) \subset D(f)$, then $\varphi''_h \circ \varphi'_g = \varphi'_h \circ \varphi'_g$; also $\varphi'_f = 1$. Further, if $D(g) = D(f)$, then $S_f \subset S_g$ and $S_g \subset S_f$, so $S_f = S_g$ and $\varphi'_f = 1$. Finally, $R_f = S_f^{-1} R$ by (11.11). Thus (1) holds.

For (2), notice (13.11) yields an inclusion-reversing bijective correspondence between the principal open sets $D(f)$ and the saturated multiplicative subsets $S_f$. Further, $D(f) \ni p$ if and only if $f \notin p$ by (13.11).

Set $S := R - p$. By (13.10), $S$ is saturated multiplicative. So $S \ni S_f$ if and only if $f \notin p$ by (13.11)(1)(c). Also, $S = \bigcup_{f \notin p} S_f$. But $R_f = S_f^{-1} R_f$ by (11.11). Thus

$$\lim_{\to_{D(f) \ni p}} R_f = \lim_{\to_{S_f \subset S}} S_f^{-1} R.$$

By the definition of saturation in (13.12), $S_{fg} \ni f, g$. By (13.12)(1)(b), $S_{fg}$ is saturated multiplicative. So $S_{fg} \supset S_f, S_g$ by (13.12)(1)(c). So $\lim S_f^{-1} R = S^{-1} R$ by (11.11)(2). But $S^{-1} R = R_p$ by definition. Thus (2) holds.

Exercise (13.18). — A topological space is called irreducible if it’s nonempty and if every pair of nonempty open subsets meet. Let $R$ be a ring. Set $X := \text{Spec}(R)$ and $\mathfrak{n} := \text{nil}(R)$. Show that $X$ is irreducible if and only if $\mathfrak{n}$ is prime.

Solution: Given $g \in R$, take $f := 0$. Plainly, $D(f) = \emptyset$; see (13.11). So in (13.11) the equivalence of (1) and (5) means $D(g) = \emptyset$ if and only if $g \in \mathfrak{n}$.

Suppose $\mathfrak{n}$ is not prime. Then there are $f, g \in R$ with $f, g \notin \mathfrak{n}$ but $fg \in \mathfrak{n}$. The above yields $D(f) \neq \emptyset$ and $D(g) \neq \emptyset$ but $D(fg) = \emptyset$. Further, $D(f) \cap D(g) = D(fg)$ by (13.11)(1). Thus $X$ is not irreducible.

Suppose $X$ is not irreducible, say $U$ and $V$ are nonempty open sets with $U \cap V = \emptyset$. By (13.11), the $D(f)$ form a basis of the topology: fix $f, g \in R$ with $0 \notin D(f) \subset U$ and $0 \notin D(g) \subset V$. Then $D(f) \cap D(g) = \emptyset$. But $D(f) \cap D(g) = D(fg)$ by (13.11)(1). Hence, the first paragraph implies $f, g \notin \mathfrak{n}$ but $fg \in \mathfrak{n}$. Thus $\mathfrak{n}$ is not prime. □

Exercise (13.19). — Let $X$ be a topological space, $Y$ an irreducible subspace.

1. Show that the closure $\overline{Y}$ of $Y$ is also irreducible.
2. Show that $Y$ is contained in a maximal irreducible subspace.
3. Show that the maximal irreducible subspaces of $X$ are closed, and cover $X$. They are called its irreducible components. What are they if $X$ is Hausdorff?
4. Let $R$ be a ring, and take $X := \text{Spec}(R)$. Show that its irreducible components are the closed sets $V(p)$ where $p$ is a minimal prime.

Solution: For (1), let $U, V$ be nonempty open sets of $\overline{Y}$. Then $U \cap Y$ and $V \cap Y$ are open in $Y$, and nonempty. But $Y$ is irreducible. So $(U \cap Y) \cap (V \cap Y) \neq \emptyset$. So $U \cap V \neq \emptyset$. Thus (1) holds.

For (2), let $S$ be the set of irreducible subspaces containing $Y$. Then $Y \in S$, and $S$ is partially ordered by inclusion. Given a totally ordered subset $\{Y_\lambda\}$ of $S$, set $Y' := \bigcup Y_\lambda$. Then $Y'$ is irreducible: given nonempty open sets $U, V$ of $Y'$, there is $Y_\lambda$ with $U \cap Y_\lambda \neq \emptyset$ and $V \cap Y_\lambda \neq \emptyset$; so $(U \cap Y_\lambda) \cap (V \cap Y_\lambda) \neq \emptyset$ as $Y_\lambda$ is irreducible. Thus Zorn’s Lemma yields (2). □
For (3), note that (1) implies the maximal irreducible subspaces are closed, and that (2) implies they cover, as every point is irreducible. Finally, if \( X \) is Hausdorff, then any two points have disjoint open neighborhoods; hence, every irreducible subspace consists of a single point.

For (4), take \( Y \) to be an irreducible component. Then \( Y \) is closed by (1); so \( Y = \text{Spec}(\mathcal{R}/\mathfrak{a}) \) for some ideal \( \mathfrak{a} \) by \((\text{13.18})\). But \( Y \) is irreducible. So \( \text{nil}(\mathcal{R}/\mathfrak{a}) \) is prime by \((\text{13.18})\). Hence \( \sqrt{\mathfrak{a}} \) is prime. So \( \sqrt{\mathfrak{a}} \) contains a minimal prime \( \mathfrak{p} \) of \( \mathcal{R} \) by \((\text{5.22})\). Set \( Z := \text{Spec}(\mathcal{R}/\mathfrak{p}) \). Then \( Z = \mathcal{V}(\mathfrak{p}) \supset \mathcal{V}(\sqrt{\mathfrak{a}}) = \mathcal{V}(\mathfrak{a}) = Y \) by \((\text{13.18})\). Further, \( Z \) is irreducible by \((\text{13.18})\). So \( Z = Y \) by maximality. Thus \( Y = \mathcal{V}(\mathfrak{p}) \).

Conversely, given a minimal prime \( \mathfrak{q} \), set \( Z := \text{Spec}(\mathcal{R}/\mathfrak{q}) \). Then \( Z \) is irreducible by \((\text{13.18})\). So \( Z \) is contained, by (2), in a maximal irreducible subset, say \( Y \). By the above, \( Y = \mathcal{V}(\mathfrak{p}) \) for some prime \( \mathfrak{p} \). Then \( \mathfrak{p} \subseteq \mathfrak{q} \) by \((\text{13.18})\). Hence \( \mathfrak{p} = \mathfrak{q} \) by minimality. Thus (4) holds.

**Exercise (13.21).** — Let \( \mathcal{R} \) be a ring, \( X := \text{Spec}(\mathcal{R}) \), and \( U \) an open subset. Show \( U \) is quasi-compact if and only if \( X - U = \mathcal{V}(\mathfrak{a}) \) where \( \mathfrak{a} \) is finitely generated.

**Solution:** Assume \( U \) is quasi-compact. By \((\text{13.11})\), \( U = \bigcup D(f_\lambda) \) for some \( f_\lambda \).

So \( U = \bigcup_{\lambda} D(f_\lambda) \) for some \( f_\lambda \). Thus \( X - U = \bigcap \mathcal{V}(f_\lambda) = \mathcal{V}(\langle f_\lambda, \ldots, f_\lambda \rangle) \).

Conversely, assume \( X - U = \mathcal{V}(\langle f_1, \ldots, f_n \rangle) \). Then \( U = \bigcup_{\lambda} D(f_\lambda) \).

**Exercise (13.22).** — Let \( \mathcal{R} \) be a ring, \( M \) a module, \( m \in M \). Set \( X := \text{Spec}(\mathcal{R}) \).

Assume \( X = \bigcup D(f_\lambda) \) for some \( f_\lambda \), and \( m/1 = 0 \) in \( \mathcal{M}_f \) for all \( \lambda \). Show \( m = 0 \).

**Solution:** Since \( m/1 = 0 \) in \( \mathcal{R}_{f_\lambda} \), there is \( n_\lambda > 0 \) such that \( f_\lambda^{n_\lambda} m = 0 \). But \( X = \bigcup D(f_\lambda) \). Hence every prime excludes some \( f_\lambda \), so also \( f_\lambda^{n_\lambda} \). So there are \( \lambda_1, \ldots, \lambda_n \) and \( x_1, \ldots, x_n \) with \( 1 = \sum x_i f_\lambda^{n_\lambda} \). Thus \( m = \sum x_i f_\lambda^{n_\lambda} m = 0 \).

**Exercise (13.23).** — Let \( \mathcal{R} \) be a ring; set \( X := \text{Spec}(\mathcal{R}) \). Prove that the four following conditions are equivalent:

1. \( \mathcal{R}/\text{nil}(\mathcal{R}) \) is absolutely flat.
2. \( X \) is Hausdorff.
3. \( X \) is \( T_1 \); that is, every point is closed.
4. Every prime \( \mathfrak{p} \) of \( \mathcal{R} \) is maximal.

Assume (1) holds. Prove that \( X \) is **totally disconnected**; namely, no two distinct points lie in the same connected component.

**Solution:** Note \( X = \text{Spec}(\mathcal{R}/\text{nil}(\mathcal{R})) \) as \( X = \mathcal{V}(0) = \mathcal{V}(\sqrt{0}) = \text{Spec}(\mathcal{R}/\sqrt{0}) \) by \((\text{13.11})\).

Hence we may replace \( R \) by \( \mathcal{R}/\text{nil}(\mathcal{R}) \), and thus assume \( \text{nil}(\mathcal{R}) = 0 \).

Assume (1). Given distinct primes \( \mathfrak{p}, \mathfrak{q} \subseteq X \), take \( x \in \mathfrak{p} - \mathfrak{q} \). Then \( x \in \langle x \rangle \) by \((\text{13.14})\) (4). So there is \( y \in \mathcal{R} \) with \( x = x^2 y \). Set \( a_1 := \langle x \rangle \) and \( a_2 := \langle 1 - xy \rangle \).

Set \( X_1 := \mathcal{V}(a_1) \). Then \( \mathfrak{p} \subseteq X_1 \) as \( x \in \mathfrak{p} \). Further, \( \mathfrak{q} \subseteq X_2 \) as \( 1 - xy \in \mathfrak{q} \) since \( x(1 - xy) = 0 \in \mathfrak{q} \), but \( x \notin \mathfrak{q} \).

The \( a_i \) are comaximal as \( xy + (1 - xy) = 1 \). Further \( a_1 a_2 = 0 \) as \( (1 - xy) = 0 \).

So \( X_1 \cup X_2 = X \) and \( X_1 \cap X_2 = \emptyset \) by \((\text{13.20})\).

Hence the \( X_i \) are disjoint open and closed sets. Thus (2) holds, and \( X \) is totally disconnected.

In general, a Hausdorff space is \( T_1 \). Thus (2) implies (3).

Conditions (3) and (4) are equivalent by \((\text{13.22})\).

Assume (4). Then every prime \( \mathfrak{m} \) is both maximal and minimal. So \( \mathcal{R}_m \) is a
local ring with $mR_m$ as its only prime by (13.24). Hence $mR_m = \text{nil}(R_m)$ by the Scheinnullstellensatz (13.24). But $\text{nil}(R_m) = \text{nil}(R)_m$ by (13.15). And $\text{nil}(R) = 0$. Thus $R_m/mR_m = R_m$. So $R_m$ is a field. Hence $R$ is absolutely flat by (13.15)(2). Thus (1) holds.

Exercise (13.24). — Let $B$ be a Boolean ring, and set $X := \text{Spec}(B)$. Show a subset $U \subset X$ is both open and closed if and only if $U = \text{D}(f)$ for some $f \in B$. Further, show $X$ is a compact Hausdorff space. (Following Bourbaki, we shorten “quasi-compact” to “compact” when the space is Hausdorff.)

Solution: Let $f \in B$. Then $\text{D}(f) \cup \text{D}(1-f) = X$ whether $B$ is Boolean or not; indeed, if $p \in X - \text{D}(f)$, then $f \in p$, so $1 - f \notin p$, so $p \in \text{D}(1-f)$. However, $\text{D}(f) \cap \text{D}(1-f) = \emptyset$; indeed, if $p \in \text{D}(f)$, then $f \notin p$, but $f(1-f) = 0$ as $B$ is Boolean, so $1 - f \notin p$, so $p \notin \text{D}(1-f)$. Thus $X - \text{D}(f) = \text{D}(1-f)$. Thus $\text{D}(f)$ is closed as well as open.

Conversely, let $U \subset X$ be open and closed. Then $U$ is quasi-compact, as $U$ is closed and $X$ is quasi-compact by (13.20). So $X - U = \text{V}(a)$ where $a$ is finitely generated by (13.21). Since $B$ is Boolean, $a = \langle f \rangle$ for some $f \in B$ by (13.12)(5). Thus $U = \text{D}(f)$.

Finally, let $p, q$ be prime ideals with $p \neq q$. Then there is $f \in p - q$. So $p \notin \text{D}(f)$, but $q \in \text{D}(f)$. By the above, $\text{D}(f)$ is both open and closed. Thus $X$ is Hausdorff.

Exercise (13.25) (Stone’s Theorem). — Show every Boolean ring $B$ is isomorphic to the ring of continuous functions from a compact Hausdorff space $X$ to $\mathbb{F}_2$ with the discrete topology. Equivalently, show $B$ is isomorphic to the ring $R$ of open and closed subsets of $X$; in fact, $X := \text{Spec}(B)$, and $B \rightarrow R$ is given by $f \mapsto \text{D}(f)$.

Solution: The two statements are equivalent by (13.24). Further, $X := \text{Spec}(B)$ is compact Hausdorff, and its open and closed subsets are precisely the $\text{D}(f)$ by (13.23). Thus $f \mapsto \text{D}(f)$ is a well defined function, and is surjective.

This function preserves multiplication owing to (13.21). To show it preserves addition, we must show that, for any $f, g \in B$,

$$\text{D}(f + g) = (\text{D}(f) - \text{D}(g)) \cup (\text{D}(g) - \text{D}(f)).$$

(13.25.1)

Fix a prime $p$. There are four cases. First, if $f \notin p$ and $g \in p$, then $f + g \notin p$. Second, if $g \notin p$ but $f \in p$, then again $f + g \notin p$. In both cases, $p$ lies in the (open) sets on both sides of (13.25.1).

Third, if $f \in p$ and $g \in p$, then $f + g \in p$. The first three cases do not use the hypothesis that $B$ is Boolean. The fourth does. Suppose $f \notin p$ and $g \notin p$. Now, $B/p = \mathbb{F}_2$ by (2.13). So the residues of $f$ and $g$ are both equal to 1. But $1 + 1 = 0 \in \mathbb{F}_2$. So again $f + g \in p$. Thus in both the third and fourth cases, $p$ lies in neither side of (13.25.1). Thus (13.25.1) holds.

Finally, to show that $f \mapsto \text{D}(f)$ is injective, suppose that $\text{D}(f)$ is empty. Then $f \notin \text{nil}(B)$. But $\text{nil}(B) = \{0\}$ by (13.23). Thus $f = 0$.

Alternatively, if $\text{D}(f) = \text{D}(g)$, then $\text{V}(\langle f \rangle) = \text{V}(\langle g \rangle)$, so $\sqrt{\langle f \rangle} = \sqrt{\langle g \rangle}$ by (13.7). But $f, g \in \text{Idem}(B)$ as $B$ is Boolean. Thus $f = g$ by (13.23).

Exercise (13.31). — Let $R$ be a ring, $a$ an ideal, $M$ a module. Prove that

$$\text{Supp}(M/aM) \subset \text{Supp}(M) \cap \text{V}(a),$$

with equality if $M$ is finitely generated.
**Solution:** First, (13.32.1) yields \( M/aM = M \otimes R/a \). But \( \text{Ann}(R/a) = a \); hence (13.32.2) yields \( \text{Supp}(R/a) = V(a) \). Thus (13.32.3) yields the assertion. \( \square \)

**Exercise (13.32).** — Let \( \varphi : R \to R' \) be a map of rings, \( M \) an \( R \)-module. Prove
\[
\text{Supp}(M \otimes R R') \subset \text{Spec}(\varphi)^{-1}(\text{Supp}(M)), \tag{13.32.1}
\]
with equality if \( M \) is finitely generated.

**Solution:** Fix a prime \( q \subset R' \). Set \( p := \varphi^{-1}q \), so \( \text{Spec}(\varphi)(p) = q \). Apply, in order, (13.32.3), twice Cancellation (8.14), and again (13.32.3) to obtain
\[
(M \otimes R R')_q = (M \otimes R R') \otimes_{R'} R'_q = M \otimes R R'_q
= (M \otimes R R'_p) \otimes_{R_p} R'_q = M_p \otimes_{R_p} R'_q. \tag{13.32.2}
\]

First, assume \( q \in \text{Supp}(M \otimes R R') \); that is, \((M \otimes R R')_q \neq 0\). Then (13.32.2) implies \( M_p \neq 0 \); that is, \( p \in \text{Supp}(M) \). Thus (13.32.4) holds.

Conversely, assume \( q \in \text{Spec}(\varphi)^{-1}(\text{Supp}(M)) \). Then \( p \in \text{Supp}(M) \), or \( M_p \neq 0 \). Set \( k := R_q/pR_p \). Then \( M_p/pM_p = M_q \otimes_{R_q} k \) and \( R'_q/pR'_q = R'_q \otimes_{R_q} k \) by (8.14)(1).

Hence Cancellation (8.14), the Associative Law (8.14), and (13.32.2) yield
\[
(M_p/pM_p) \otimes_{k} (R'_q/pR'_q) = (M_q \otimes_{R_q} k) \otimes_{k} (R'_q/pR'_q)
= M_p \otimes_{R_p} (R'_q \otimes_{R_q} k) = (M_p \otimes_{R_p} R'_q) \otimes_{R_q} k
= (M \otimes R R')_q \otimes_{R_q} k. \tag{13.32.3}
\]

Assume \( M \) is finitely generated. Then \( M_p/pM_p \neq 0 \) by Nakayama’s Lemma (13.32.4) over \( R_p \). And \( R'_q/pR'_q \neq 0 \) by Nakayama’s Lemma (13.32.4) over \( R'_q \) as \( pR' \subset q \). So \( (M_p/pM_p) \otimes_{k} (R'_q/pR'_q) \neq 0 \) by (8.14). So (13.32.2) implies \((M \otimes R R')_q \neq 0\), or \( q \in \text{Supp}(M \otimes R R') \). Thus equality holds in (13.32.4). \( \square \)

**Exercise (13.33).** — Let \( R \) be a ring, \( M \) a module, \( p \in \text{Supp}(M) \). Prove
\[
V(p) \subset \text{Supp}(M).
\]

**Solution:** Let \( q \in V(p) \). Then \( q \supset p \). So \( M_p = (M_q)_p \) by (13.32.4)(1). Now, \( p \in \text{Supp}(M) \). So \( M_p \neq 0 \). Hence \( M_q \neq 0 \). Thus \( q \in \text{Supp}(M) \). \( \square \)

**Exercise (13.34).** — Let \( Z \) be the integers, \( Q \) the rational numbers, and set \( M := Q/Z \). Find \( \text{Supp}(M) \), and show that it is not Zariski closed.

**Solution:** Let \( p \in \text{Spec}(R) \). Then \( M_p = Q_p/Z_p \) since localization is exact by (13.32.4). Now, \( Q_p = Q \) by (13.32.4) and (13.32.4) since \( Q \) is a field. If \( p \neq (0) \), then \( Z_p \neq Q_p \) since \( pZ_p \cap Z = \{ p \} \) by (13.32.4). If \( p = (0) \), then \( Z_p = Q_p \). Thus \( \text{Supp}(M) \) consists of all the nonzero primes of \( Z \).

Finally, suppose \( \text{Supp}(M) = V(a) \). Then \( a \) lies in every nonzero prime; so \( a = (0) \). But \( (0) \) is prime. Hence \( (0) \in V(a) = \text{Supp}(M) \), contradicting the above. Thus \( \text{Supp}(M) \) is not closed. \( \square \)

**Exercise (13.36).** — Let \( R \) be a domain, \( M \) a module, set \( S := R - 0 \), and set \( T(M) := T^S(M) \). We call \( T(M) \) the **torsion submodule** of \( M \), and we say \( M \) is **torsionfree** if \( T(M) = 0 \).

Prove \( M \) is torsionfree if and only if \( M_m \) is torsionfree for all maximal ideals \( m \).
SOLUTION: Given an \( m \), note that \( R - m \subset S \). So (13.36.1) yields

\[
T(M_m) = T(M)_m. \tag{13.36.1}
\]

Assume \( M \) is torsionfree. Then \( M_m \) is torsionfree for all \( m \) by (13.36.1). Conversely, if \( M_m \) is torsionfree for all \( m \), then \( T(M)_m = 0 \) for all \( m \) by (13.36.1). Hence \( T(M) = 0 \) by (13.36.1). Thus \( M \) is torsionfree.

Exercise (13.37). — Let \( R \) be a ring, \( P \) a module, \( M, N \) submodules. Assume \( M_m = N_m \) for every maximal ideal \( m \). Show \( M = N \). First assume \( M \subset N \).

Solution: If \( M \subset N \), then (13.37) yields \( (N/M)_m = N_m/M_m = 0 \) for each \( m \); so \( N/M = 0 \) by (13.35). The general case follows by replacing \( N \) by \( M + N \) owing to (13.37) (4), (5).

Exercise (13.38). — Let \( R \) be a ring, \( M \) a module, and \( a \) an ideal. Suppose \( M_m = 0 \) for all maximal ideals \( m \) containing \( a \). Show that \( M = aM \).

Solution: Given any maximal ideal \( m \), note that \( (aM)_m = a_m M_m \) by (13.24). But \( M_m = 0 \) if \( m \supset a \) by hypothesis. And \( a_m = R_m \) if \( m \not
supset a \) by (13.23) (2). Hence \( M_m = (aM)_m \) in any case. Thus (13.37) yields \( M = aM \).

Alternatively, form the ring \( R/a \) and its module \( M/aM \). Given any maximal ideal \( m' \) of \( R/a \), say \( m' = m/a \). By hypothesis, \( M_m = 0 \). But \( M_m/(aM)_m = (M/aM)_m \) by (13.22). Thus \( (M/aM)_m = 0 \). So \( M/aM = 0 \) by (13.35). Thus \( M = aM \).

Exercise (13.39). — Let \( R \) be a ring, \( P \) a module, \( M \) a submodule, and \( p \in P \) an element. Assume \( p/1 \in M_m \) for every maximal ideal \( m \). Show \( p \in M \).

Solution: Set \( N := M + Rp \). Then \( N_m = M_m + R_m \cdot p/1 \) for every \( m \). But \( p/1 \in M_m \). Hence \( N_m = M_m \). So \( N = M \) by (13.37). Thus \( p \in M \).

Exercise (13.40). — Let \( R \) be a domain, \( a \) an ideal. Show \( a = \bigcap_m aR_m \) where \( m \) runs through the maximal ideals and the intersection takes place in \( \text{Frac}(R) \).

Solution: Plainly, \( a \subset \bigcap R_m \). Conversely, take \( x \in \bigcap aR_m \). Then \( x \in aR_m \) for every \( m \). But \( aR_m = a_m \) by (13.24). So (13.35) yields \( x \in a \) as desired.

Exercise (13.41). — Prove these three conditions on a ring \( R \) are equivalent:

1. \( R \) is reduced.
2. \( S^{-1}R \) is reduced for all multiplicatively closed sets \( S \).
3. \( R_m \) is reduced for all maximal ideals \( m \).

If \( R_m \) is a domain for all maximal ideals \( m \), is \( R \) necessarily a domain?

Solution: Assume (1) holds. Then \( \text{nil}(R) = 0 \). But \( \text{nil}(R)(S^{-1}R) = \text{nil}(S^{-1}R) \) by (1.8). Thus (2) holds. Trivially (2) implies (3).

Assume (3) holds. Then \( \text{nil}(R_m) = 0 \). Hence \( \text{nil}(R)_m = 0 \) by (13.18) and (13.26). So \( \text{nil}(R) = 0 \) by (13.35). Thus (1) holds. Thus (1)–(3) are equivalent.

Finally, the answer is no. For example, take \( R := k_1 \times k_2 \) with \( k_i := \mathbb{Z}/(2) \). The primes of \( R \) are \( p := \langle (1,0) \rangle \) and \( q := \langle (0,1) \rangle \) by (2.32). Further, \( R_q = k_1 \) by (11.13), as \( R - q = \{ (1,1), (1,0) \} \). Similarly \( R_p = k_2 \). But \( R \) is not a domain, as \( (1,0) \cdot (0,1) = (0,0) \), although \( R_m \) is a domain for all maximal ideals \( m \).

In fact, take \( R := R_1 \times R_2 \) for any domains \( R_i \). Then again \( R \) is not a domain, but \( R_p \) is a domain for all primes \( p \) by (13.32) (2) below.

Exercise (13.42). — Let \( R \) be a ring, \( \Sigma \) the set of minimal primes. Prove this:
(1) If \( R_p \) is a domain for any prime \( p \), then the \( p \in \Sigma \) are pairwise comaximal.

(2) \( R_p \) is a domain for any prime \( p \) and \( \Sigma \) is finite if and only if \( R = \prod_{i=1}^{n} R_i \) where \( R_i \) is a domain. If so, then \( R_i = R/p_i \) with \( \{p_1, \ldots, p_n\} = \Sigma \).

**Solution:** Consider (1). Suppose \( p, q \in \Sigma \) are not comaximal. Then \( p + q \) lies in some maximal ideal \( m \). Hence \( R_m \) contains two minimal primes, \( pR_m \) and \( qR_m \), by (13.41). However, \( R_m \) is a domain by hypothesis, and so \( \langle 0 \rangle \) is its only minimal prime. Hence \( pR_m = qR_m \). So \( p = q \). Thus (1) holds.

Consider (2). Assume \( R_p \) is a domain for any \( p \). Then \( R \) is reduced by (13.44). Assume, also, \( \Sigma \) is finite. Form the canonical map \( \varphi: R \to \prod_{p \in \Sigma} R/p; \) it is injective by (4.34), and surjective by (1) and the Chinese Remainder Theorem (11.43). Thus \( R \) is a finite product of domains.

Conversely, assume \( R = \prod_{i=1}^{n} R_i \) where \( R_i \) is a domain. Let \( p \) be a prime of \( R \). Then \( R_p = \prod (R_i)_p \) by (13.43). Each \( (R_i)_p \) is a domain by (13.43). But \( R_p \) is local. So \( R_p = (R_i)_p \) for some \( i \) by (13.10). Thus \( R_p \) is a domain. Further, owing to (13.10), each \( p_i \in \Sigma \) has the form \( p_i = \prod a_j \) where, after renumbering, \( a_i(0) \) and \( a_j = R_j \) for \( j \neq i \). Thus the \( i \)th projection gives \( R/p_i \to R_i \). Thus (2) holds. □

**Exercise (13.44).** — Let \( R \) be a ring, \( M \) a module. Prove elements \( m_\lambda \in M \) generate \( M \) if and only if, at every maximal ideal \( m \), their images \( m_\lambda \) generate \( M_m \).

**Solution:** The \( m_\lambda \) define a map \( \alpha: R^\oplus(\lambda) \to M \). By (13.48), it is surjective if and only if \( \alpha_m: (R^\oplus(\lambda))_m \to M_m \) is surjective for all \( m \). But \( (R^\oplus(\lambda))_m = R_m^\oplus(\lambda) \) by (13.44). Hence (13.40)(1) yields the assertion. □

**Exercise (13.47).** — Let \( R \) be a ring, \( R' \) a flat algebra, \( p' \) a prime in \( R' \), and \( p \) its contraction in \( R \). Prove that \( R'_p \) is a faithfully flat \( R_p \)-algebra.

**Solution:** First, \( R'_p \) is flat over \( R_p \) by (13.46). Next, \( R'_p \) is flat over \( R'_p \) by (13.44) and (13.45) as \( R - p \subset R' - p' \). Hence \( R'_p \) is flat over \( R_p \) by (13.42). But a flat local homomorphism is faithfully flat by (11.13). □

**Exercise (13.48).** — Let \( R \) be a ring, \( S \) a multiplicative subset.

(1) Assume \( R \) is absolutely flat. Show \( S^{-1}R \) is absolutely flat.

(2) Show \( R \) is absolutely flat if and only if each \( R_m \) is a field for each maximal \( m \).

**Solution:** In (1), given \( x \in R \), note that \( \langle x \rangle \) is idempotent by (10.4). Hence \( \langle x \rangle = \langle x^2 \rangle = \langle (x^2) \rangle \). So there is \( y \in R \) with \( x = x^2y \).

Given \( a/s \in S^{-1}R \), there are, therefore, \( b, t \in R \) with \( a = a^2b \) and \( s = s^2t \). So \( s(st - 1) = 0 \). So \( (st - 1)1 \cdot s/1 = 0 \). But \( s/1 \) is a unit. Hence \( s/1 \cdot t/1 - 1 = 0 \).

So \( a/s = (a/s)^2 \cdot b/t \). So \( a/s \in \langle a/s \rangle^2 \). Thus \( \langle a/s \rangle \) is idempotent. Hence \( S^{-1}R \) is absolutely flat by (10.9). Thus (1) holds.

Alternatively, given an \( S^{-1}R \)-module \( M \), note \( M \) is also an \( R \)-module, so \( R \)-flat by (1). Hence \( M \otimes S^{-1}R \) is \( S^{-1}R \)-flat by (13.11). But \( M \otimes S^{-1}R = S^{-1}M \) by (14.42), and \( S^{-1}M = M \) by (14.43). Thus \( M \) is \( S^{-1}R \)-flat. Thus again (1) holds.

For (2), first assume \( R \) is absolutely flat. By (1), each \( R_m \) is absolutely flat. So by (10.11)(4), each \( R_m \) is a field.

Conversely, assume each \( R_m \) is a field. Then, given an \( R \)-module \( M \), each \( M_m \) is \( R_m \)-flat. So \( M \) is \( R \)-flat by (13.47). Thus (2) holds. □

**Exercise (13.52).** — Given \( n \), prove an \( R \)-module \( P \) is locally free of rank \( n \) if and only if \( P \) is finitely generated and \( P_m \cong R_m^n \) holds at each maximal ideal \( m \).
SOLUTION: If $P$ is locally free of rank $n$, then $P$ is finitely generated by $(\text{13.35})$. Also, for any $p \in \text{Spec}(R)$, there’s $f \in R - p$ with $P_f \simeq R^n_f$; so $P_p \simeq R^n_p$ by $(\text{14.42})$.

As to the converse, given any prime $p$, take a maximal ideal $m$ containing it. Assume $P_m \simeq R^n_m$. Take a free basis $p_1/f_1^{k_1}, \ldots, p_n/f_n^{k_n}$ of $P_m$ over $R_m$. The $p_i$ define a map $\alpha: R^n \to P$, and $\alpha_m: R^n_m \to P_m$ is bijective, so surjective.

Assume $P$ is finitely generated. Then $(\text{14.42})$ provides $f \in R - m$ such that $\alpha_f: R^n_f \to P_f$ is surjective. Hence $\alpha_q: R^n_q \to P_q$ is surjective for every $q \in \text{D}(f)$ by $(\text{14.38})$ and $(\text{11.24})$. Assume $P_q \simeq R^n_q$ if also $q$ is maximal. So $\alpha_q$ is bijective by $(\text{11.24})$. Clearly, $\alpha_q = (\alpha_f)(qR_f)$. Hence $\alpha_f: R^n_f \to P_f$ is bijective owing to $(\text{11.24})$ with $R_f$ for $R$, as desired.

EXERCISE (13.53). — Let $A$ be a semilocal ring, $P$ a locally free module of rank $n$. Show that $P$ is free of rank $n$.

SOLUTION: As $P$ is locally free, $P$ is finitely presented by $(\text{13.51})$, and $P_m \simeq A^n_m$ at each maximal $m$ by $(\text{13.52})$. But $A$ is semilocal. So $P \simeq A^n$ by $(\text{13.39})$.

EXERCISE (13.54). — Let $R$ be a ring, $M$ a finitely presented module, $n \geq 0$. Show that $M$ is locally free of rank $n$ if and only if $F_{n-1}(M) = \{0\}$ and $F_n(M) = R$.

SOLUTION: Assume $M$ is locally free of rank $n$. Then so is $M_m$ for any maximal ideal $m$ by $(\text{13.52})$. So $F_{n-1}(M_m) = \{0\}$ and $F_n(M_m) = R_m$ by $(\text{13.45})$. But $F_r(M_m) = F_r(M)$ for all $r$ by $(\text{14.16})$. So $F_{n-1}(M_m) = \{0\}$ and $F_n(M_m) = R_m$ by $(\text{13.57})$. The converse follows via reversing the above steps.


EXERCISE (14.4). — Let $R \subset R'$ be an integral extension of rings, and $p$ a prime of $R$. Suppose $R'$ has just one prime $p'$ over $p$. Show (a) that $p'R'_p$ is the only maximal ideal of $R'_p$, (b) that $R'_p = R'_p$, and (c) that $R'_p$ is integral over $R_p$.

SOLUTION: Since $R'$ is integral over $R$, the localization $R'_p$ is integral over $R_p$ by $(\text{14.24})$. Moreover, $R'_p$ is a local ring with unique maximal ideal $pR'_p$ by $(\text{14.22})$. Hence, every maximal ideal of $R'_p$ lies over $pR'_p$ by $(\text{14.33})$. But every maximal ideal of $R'_p$ is the extension of some prime $q' \subset R'$ by $(\text{14.22})$, and therefore $q'$ lies over $p$ in $R$. So, by hypothesis, $q' = p'$. Thus $p'R'_p$ is the only maximal ideal of $R'_p$ that has (a) holds. So $R'_p - p'R'_p$ consists of units. Hence $(\text{14.25})$ and $(\text{14.6})$ yield (b). But $R'_p$ is integral over $R_p$; so (c) holds too.

EXERCISE (14.5). — Let $R \subset R'$ be an integral extension of domains, and $p$ a prime of $R$. Suppose $R'$ has at least two distinct primes $p'$ and $q'$ lying over $p$. Show that $R'_p$ is not integral over $R_p$. Show that, in fact, if $y$ lies in $q'$, but not in $p'$, then $1/y \in R'_p$ is not integral over $R_p$.

SOLUTION: Suppose $1/y$ is integral over $R_p$. Say

$$(1/y)^n + a_1(1/y)^{n-1} + \cdots + a_n = 0$$

with $n \geq 1$ and $a_i \in R_p$. Multiplying by $y^{n-1}$, we obtain

$$1/y = -(a_1 + \cdots + a_n y^{n-1}) \in R'_p.$$  

However, $y \in q'$, so $y \in q'R'_p$. Hence $1 \in q'R'_p$. So $q' \cap (R - p) \neq \emptyset$ by $(\text{14.14})$. But $q' \cap R = p$, a contradiction. So $1/y$ is not integral over $R_p$. 

□
**Exercise (14.6).** — Let \( k \) be a field, and \( X \) an indeterminate. Set \( R' := k[X] \), and \( Y := X^2 \), and \( R := k[Y] \). Set \( p := (Y - 1)R \) and \( p' := (X - 1)R' \). Is \( R'_p \) integral over \( R_p \)? Explain.

**Solution:** Note that \( K = \Box \) as \( R' \) is generated by 1 and \( X \) as an \( R \)-module.

Suppose the characteristic is not 2. Set \( q' := (X + 1)R' \). Then both \( p' \) and \( q' \) contain \( Y - 1 \), so lie over the maximal ideal \( p \) of \( R \). Further \( X + 1 \) lies in \( q' \), but not in \( p' \). Hence \( R'_p \) is not integral over \( R_p \) by (14.15).

Suppose the characteristic is 2. Then \( (X - 1)^2 = Y - 1 \). Let \( q' \subset R' \) be a prime over \( p \). Then \( (X - 1)^2 \in q' \). So \( p' \subset q' \). But \( p' \) is maximal. So \( q' = p' \). Thus \( R' \) has just one prime \( p' \) over \( p \). Hence \( R'_p \) is integral over \( R_p \) by (14.15).

**Exercise (14.12).** — Let \( R \) be a reduced ring, \( \Sigma \) the set of minimal primes. Prove that \( z \cdot \text{div}(R) = \bigcup_{p \in \Sigma} p \) and that \( R_p = \text{Frac}(R/p) \) for any \( p \in \Sigma \).

**Solution:** If \( p \in \Sigma \), then \( p \subset z \cdot \text{div}(R) \) by (14.11). Thus \( z \cdot \text{div}(R) \supset \bigcup_{p \in \Sigma} p \).

Conversely, say \( xy = 0 \). If \( x \notin p \) for some \( p \in \Sigma \), then \( y \in \Sigma \). So if \( x \notin \bigcup_{p \in \Sigma} p \), then \( y \in \bigcap_{p \in \Sigma} p \). But \( \bigcap_{p \in \Sigma} p = (0) \) by the Schönfliesstellsatz (5.19) and (5.33). So \( y = 0 \). Thus, if \( x \notin \bigcup_{p \in \Sigma} p \), then \( x \notin z \cdot \text{div}(R) \). Thus \( z \cdot \text{div}(R) \subset \bigcup_{p \in \Sigma} p \). Thus \( z \cdot \text{div}(R) = \bigcup_{p \in \Sigma} p \).

Fix \( p \in \Sigma \). Then \( R_p \) is reduced by (14.27). Further, \( R_p \) has only one prime, namely \( pR_p \), by (14.20). Hence \( R_p \) is a field, and \( pR_p = (0) \). But by (14.28), \( R_p/pR_p = \text{Frac}(R/p) \). Thus \( R_p = \text{Frac}(R/p) \).

**Exercise (14.13).** — Let \( R \) be a ring, \( \Sigma \) the set of minimal primes, and \( K \) the total quotient ring. Assume \( \Sigma \) is finite. Prove these three conditions are equivalent:

1. \( R \) is reduced.
2. \( z \cdot \text{div}(R) = \bigcup_{p \in \Sigma} p \), and \( R_p \text{Frac}(R/p) \) for each \( p \in \Sigma \).
3. \( K/pK = \text{Frac}(R/p) \) for each \( p \in \Sigma \), and \( K = \prod_{p \in \Sigma} K/pK \).

**Solution:** Assume (1) holds. Then (14.24) yields (2). Assume (2) holds. Set \( S := R - z \cdot \text{div}(R) \). Let \( q \) be a prime of \( R \) with \( q \cap S = \emptyset \). Then \( q \subset z \cdot \text{div}(R) \). So (2) yields \( q \subset \bigcup_{p \in \Sigma} p \). But \( \Sigma \) is finite. So \( q \subset p \) for some \( p \in \Sigma \) by Prime Avoidance (6.14). Hence \( q = p \) since \( p \) is minimal. But \( K = S^{-1}R \). Therefore, by (14.20), \( p \in \Sigma \), the extensions \( pK \) are the only primes of \( K \), and they all are both maximal and minimal.

Fix \( p \in \Sigma \). Then \( K/pK = S^{-1}(R/p) \) by (14.22). So \( S^{-1}(R/p) \) is a field. But clearly \( S^{-1}(R/p) \subset \text{Frac}(R/p) \). Therefore, \( K/pK \text{Frac}(R/p) \) by (2.24). Further, \( S \subset R - p \). Hence (14.24) yields \( p = \varphi^{-1}_S(pK) \). Therefore, \( \varphi^{-1}_S(K - pK) = R - p \). So \( K/pK = R_p \) by (14.24). But \( R_p = \text{Frac}(R/p) \) by hypothesis. Thus \( K \) has only finitely many primes, the \( pK \); each \( pK \) is minimal, and each \( K/pK \) is a domain. Therefore, (14.31) yields \( K = \prod_{p \in \Sigma} K/pK \). Thus (3) holds.

Assume (3) holds. Then \( K \) is a finite product of fields, and fields are reduced. But clearly, a product of reduced ring is reduced. Further, \( R \subset K \), and trivially, a subring of a reduced ring is reduced. Thus (1) holds.

**Exercise (14.14).** — Let \( A \) be a reduced local ring with residue field \( k \) and a finite set \( \Sigma \) of minimal primes. For each \( p \in \Sigma \), set \( K(p) := \text{Frac}(A/p) \). Let \( P \) be a finitely generated module. Show that \( P \) is free of rank \( r \) if and only if \( \text{dim}_k(P \otimes_A k) = r \) and \( \text{dim}_{K(p)}(P \otimes_A K(p)) = r \) for each \( p \in \Sigma \).
SOLUTION: If \( P \) is free of rank \( r \), then \( \dim(P \otimes k) = r \) and \( \dim(P \otimes K(p)) = r \) owing to (14.15).

Conversely, suppose \( \dim(P \otimes k) = r \). As \( P \) is finitely generated, (14.16) implies \( P \) is generated by \( r \) elements. So (17.4) yields an exact sequence

\[
0 \to M \to A^r \to P \to 0.
\]

Momentarily, fix a \( p \in \Sigma \). Since \( A \) is reduced, \( K(p) = R_p \) by (15.3). So \( K(p) \) is flat by (17.4). So the induced sequence is exact:

\[
0 \to M \otimes K(p) \to K(p)^r \to P \otimes K(p) \to 0.
\]

Suppose \( \dim(P \otimes K(p)) = r \) too. It then follows that \( M \otimes A K(p) = 0 \).

Let \( K \) be the total quotient ring of \( A \), and form this commutative square:

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha} & A^r \\
\varphi_M & & \varphi_{A^r} \\
M \otimes K & \to & K^r
\end{array}
\]

Here \( \alpha \) is injective. And \( \varphi_{A^r} \) is injective as \( \varphi_A : A \to K \) is. Hence, \( \varphi_M \) is injective.

By hypothesis, \( A \) is reduced and \( \Sigma \) is finite; so \( K = \prod_{p \in \Sigma} K(p) \) by (15.12). So \( M \otimes K = \prod (M \otimes K(p)) \). But \( M \otimes A K(p) = 0 \) for each \( p \in \Sigma \). So \( M \otimes K = 0 \). But \( \varphi_A : A \to M \otimes K \) is injective. So \( M = 0 \). Thus \( A^r \to P \), as desired. \( \Box \)

**Exercise (14.15).** — Let \( A \) be a reduced semilocal ring with a finite set of minimal primes. Let \( P \) be a finitely generated \( A \)-module, and \( B \) an \( A \)-algebra such that \( \text{Spec}(B) \to \text{Spec}(A) \) is surjective. For each prime \( q \subset B \), set \( L(q) = \text{Frac}(B/q) \). Given \( r \), assume \( \dim((P \otimes_A B) \otimes_B L(q)) = r \) whenever \( q \) is either maximal or minimal. Show that \( P \) is a free \( A \)-module of rank \( r \).

**Solution:** Let \( p \subset A \) be a prime. Since \( \text{Spec}(B) \to \text{Spec}(A) \) is surjective, there is a prime \( q \subset B \) whose contraction is \( p \). Then the cancellation law yields

\[
(P \otimes_A K(p)) \otimes_{K(p)} L(q) = (P \otimes_A B) \otimes_B L(q). \tag{14.15.1}
\]

If \( p \) is minimal, take a minimal prime \( q' \subset q \). Then the contraction of \( q' \) is contained in \( p \), so equal to \( p \). Replace \( q \) by \( q' \). If \( p \) is maximal, take a maximal ideal \( q' \supset q \). Then the contraction of \( q' \) contains \( p \), so is equal to \( p \). Again, replace \( q \) by \( q' \). Either way, \( \dim((P \otimes_A B) \otimes_B L(q)) = r \) by hypothesis. So (14.15.1) yields \( \dim((P \otimes_A K(p)) \otimes_{K(p)} L(q)) = r \). Hence \( \dim(P \otimes_A K(p)) = r \).

If \( A \) is local, then \( P \) is a free \( A \)-module of rank \( r \) by (15.12). In general, let \( m \subset A \) be a maximal ideal. Then \( \text{Spec}(B_m) \to \text{Spec}(A_m) \) is surjective by an argument like one in the proof of (14.15.12), using (15.14)(2). Hence \( P_m \) is a free \( A_m \)-module of rank \( r \) by the preceding case. Thus \( P \) is free of rank \( r \) by (14.16).

**Exercise (14.17).** — Let \( R \) be a ring, \( p_1, \ldots, p_r \) all its minimal primes, and \( K \) the total quotient ring. Prove that these three conditions are equivalent:

1. \( R \) is normal.
2. \( R \) is reduced and integrally closed in \( K \).
3. \( R \) is a finite product of normal domains \( R_i \).

Assume the conditions hold. Prove the \( R_i \) are equal to the \( R/p_j \) in some order.
SOLUTION: Assume (1). Let \( m \) any maximal ideal. Then \( R_m \) is a normal domain. So \( R \) is reduced by (13.31).

Let \( S_0 \) be the set of nonzerodivisors of \( R \), so that \( K := S_0^{-1}R \). Set \( S := R - m \), so that \( R_m := S^{-1}R \). But \( S^{-1}S_0^{-1}RS_0^{-1}S^{-1}R \) by (11.29)(2). So \( S^{-1}K = S_0^{-1}R_m \).

Let \( t \in S_0 \). Then \( t/1 \neq 0 \) in \( R_m \); else, there’s \( s \in S \) with \( st = 0 \), a contradiction as \( s \neq 0 \) and \( t \in S_0 \). Thus (11.24) and (11.3) yield \( S_0^{-1}R_m \subset \text{Frac}(R_m) \).

Let \( x \in K \) be integral over \( R \). Then \( x/1 \in S^{-1}K \) is integral over \( S^{-1}R \) by (11.21). But \( S^{-1}R = R_m \), and \( R_m \) is a normal domain. So \( x/1 \in R_m \). Hence \( x \in R \) by (13.31). Thus (2) holds.

Assume (2). Set \( R_i := R/p_i \) and \( K_i := \text{Frac}(R_i) \). Then \( K = \prod K_i \) by (13.13). Let \( R'_i \) be the normalization of \( R_i \). Then \( R \subset \prod R_i \subset \prod R'_i \). Further, the first extension is integral by (11.29), and the second, by (11.31); whence, \( R \subset \prod R'_i \) is integral by the tower property (11.27). However, \( R \) is integrally closed in \( K \) by hypothesis. Hence \( R = \prod R_i = \prod R'_i \). Thus (3) holds.

Assume (3). Let \( p \) be any prime of \( R \). Then \( R_p = \prod (R_i)_p \) by (12.21), and each \((R_i)_p\) is normal by (11.32). But \( R_p \) is local. So \( R_p = (R_i)_p \) for some \( i \) by (11.7). Hence \( R_p \) is a normal domain. Thus (1) holds.

Finally, the last assertion results from (13.31)(2).

\[ \square \]

15. Noether Normalization

EXERCISE (15.2). — Let \( k := \mathbb{F}_q \) be the finite field with \( q \) elements, and \( k[X,Y] \) the polynomial ring. Set \( f := X^3 - 2XY^2 \) and \( R := k[X,Y]/(f) \). Let \( x, y \in R \) be the residues of \( X, Y \). For every \( a \in k \), show that \( R \) is not module finite over \( P := k[y-ax] \). (Thus, in (15.3), no \( k \)-linear combination works.) First, take \( a = 0 \).

SOLUTION: Take \( a = 0 \). Then \( P = k[y] \). Any algebraic relation over \( P \) satisfied by \( x \) is given by a polynomial in \( k[X,Y] \), which is a multiple of \( f \). However, no multiple of \( f \) is monic in \( X \). So \( x \) is not integral over \( P \). By (11.29), \( R \) is not module finite over \( P \).

Consider an arbitrary \( a \). Since \( a^q = a \), after the change of variable \( Y' := Y - aX \), our \( f \) still has the same form. Thus, we have reduced to the previous case.

EXERCISE (15.3). — Let \( k \) be a field, and \( X, Y, Z \) variables. Set 
\[
R := k[X, Y, Z]/(X^2 - Y^3 - 1, XZ - 1),
\]
and let \( x, y, z \in R \) be the residues of \( X, Y, Z \). Fix \( a, b \in k \), and set \( t := x + ay + bz \) and \( P := k[t] \). Show that \( x \) and \( y \) are integral over \( P \) for any \( a, b \) and that \( z \) is integral over \( P \) if and only if \( b \neq 0 \).

SOLUTION: To see \( x \) is integral, notice \( xz = 1 \), so \( x^2 - tx + b = -axy \). Raising both sides of the latter equation to the third power, and using the equation \( y^3 = x^2 - 1 \), we obtain an equation of integral dependence of degree 6 for \( x \) over \( P \). Now, \( y^3 - x^2 + 1 = 0 \), so \( y \) is integral over \( P[x] \). Hence, the Tower Property, (11.27), implies that \( y \) too is integral over \( P \).

If \( b \neq 0 \), then \( z = b^{-1}(t - x - ay) \in P[x, y] \), and so \( z \) is integral over \( P \) by (11.28).

Assume \( b = 0 \) and \( z \) is integral over \( P \). Now, \( P \subset k[x, y] \). So \( z \) is integral over \( k[x, y] \) as well. But \( y^3 - x^2 + 1 = 0 \). So \( y \) is integral over \( k[x] \). Hence \( z \) too.

However, \( k[x] \) is a polynomial ring, so integrally closed in its fraction field \( k(x) \) by
Solutions: (15.19) 215

\((\text{15.21})\) (1). Moreover, \(z = 1/x \in k(x)\). Hence, \(1/x \in k[x]\), which is absurd. Thus \(z\) is not integral over \(P\) if \(b = 0\). □

**Exercise (15.8).** — Let \(K\) be a field, \(K\) an algebraically closed extension field. (So \(K\) contains a copy of every finite extension field.) Let \(P := k[X_1, \ldots, X_n]\) be the polynomial ring, and \(f, f_1, \ldots, f_r \in P\). Assume \(f\) vanishes at every zero in \(K^n\) of \(f_1, \ldots, f_r\); in other words, if \((a_1, \ldots, a_n) \in K^n\) and \(f_1(a) = 0, \ldots, f_r(a) = 0\), then \(f(a) = 0\) too. That there are polynomials \(g_1, \ldots, g_r \in P\) and an integer \(N\) such that \(f^N g_1 f_1 + \cdots + g_r f_r\).

**Solution:** Set \(a := (f_1, \ldots, f_r)\). We have to show \(f \in \sqrt{a}\). But, by the Hilbert Nullstellensatz, \(\sqrt{a}\) is equal to the intersection of all the maximal ideals \(m\) containing \(a\). So given an \(m\), we have to show that \(f \in m\).

Set \(L := P/m\). By the weak Nullstellensatz, \(L\) is a finite extension field of \(k\). So we may embed \(L/k\) as a subextension of \(K/k\). Let \(a_i \in K\) be the image of the variable \(X_i \in P\), and set \((a) := (a_1, \ldots, a_n) \in K^n\). Then \(f_1(a) = 0, \ldots, f_r(a) = 0\). Hence \(f(a) = 0\) by hypothesis. Therefore, \(f \in m\), as desired. □

**Exercise (15.11).** — Let \(R\) be a domain of \((\text{finite})\) dimension \(r\), and \(p\) a nonzero prime. Prove that \(\dim(R/p) < r\).

**Solution:** Every chain of primes of \(R/p\) is of the form \(p_0 \supseteq \cdots \supseteq p_s/p\) where \(0 \supseteq p_0 \supseteq \cdots \supseteq p_s\) is a chain of primes of \(R\). So \(s < r\). Thus \(\dim(R/p) < r\). □

**Exercise (15.12).** — Let \(R'/R\) be an integral extension of rings. Prove that \(\dim(R) = \dim(R')\).

**Solution:** Let \(p_0 \subseteq \cdots \subseteq p_r\) be a chain of primes of \(R\). Set \(p'_1 := 0\). Given \(p'_{i-1}\) for \(0 \leq i \leq r\), Going up, \(\text{(15.33)}\) (4), yields a prime \(p'_i\) of \(R'\) with \(p'_{i-1} \subset p'_i\) and \(p'_i \cap R = p_i\). Then \(p'_0 \subseteq \cdots \subseteq p'_r\), as \(p_0 \subseteq \cdots \subseteq p_r\). Thus \(\dim(R) \leq \dim(R')\).

Conversely, let \(p'_0 \subseteq \cdots \subseteq p'_r\) be a chain of primes of \(R'\). Set \(p_i := p'_i \cap R\). Then \(p_0 \subseteq \cdots \subseteq p_r\), by Incomparability, \(\text{(15.33)}\) (2). Thus \(\dim(R) \geq \dim(R')\). □

**Exercise (15.17).** — Let \(k\) be a field, \(R\) a finitely generated \(k\)-algebra, \(f \in R\) nonzero. Assume \(R\) is a domain. Prove that \(\dim(R) = \dim(R_f)\).

**Solution:** Note that \(R_f\) is a finitely generated \(R\)-algebra by \(\text{(15.10)}\), as \(R_f\) is, by \(\text{(15.13)}\), obtained by adjoining \(1/f\). So since \(R\) is a finitely generated \(k\)-algebra, \(R_f\) is one too. Moreover, \(R\) and \(R_f\) have the same fraction field \(K\). Hence both \(\dim(R)\) and \(\dim(R_f)\) are equal to \(\text{tr} \deg_k(K)\) by \(\text{(15.14)}\). □

**Exercise (15.18).** — Let \(k\) be a field, \(P := k[f]\) the polynomial ring in one variable \(f\). Set \(p := (f)\) and \(R := P_p\). Find \(\dim(R)\) and \(\dim(R_f)\).

**Solution:** In \(P\), the chain of primes \(0 \subseteq p\) is of maximal length by \(\text{(15.6)}\) and \(\text{(15.10)}\) or \(\text{(15.13)}\). So \((0)\) and \(pR\) are the only primes in \(R\) by \(\text{(15.2)}\). Thus \(\dim(R) = 1\).

Set \(K := \text{Frac}(P)\). Then \(R_f = K\) since, if \(a/(bf^n) \in K\) with \(a, b \in P\) and \(f \not\mid b\), then \(a/b \in R\) and so \((a/b)/f^n \in R_f\). Thus \(\dim(R_f) = 0\). □

**Exercise (15.19).** — Let \(R\) be a ring, \(R[X]\) the polynomial ring. Prove

\[
1 + \dim(R) \leq \dim(R[X]) \leq 1 + 2 \dim(R).
\]

(In particular, \(\dim(R[X]) = \infty\) if and only if \(\dim(R) = \infty\).)
SOLUTION: Let \( p_0 \subseteq \cdots \subseteq p_n \) be a chain of primes in \( R \). Then
\[
p_0 R[X] \subseteq \cdots \subseteq p_n R[X] \subseteq p_n R[X] + \langle X \rangle
\]
is a chain of primes in \( R[X] \) by (15.23). Thus \( 1 + \dim(R) \leq \dim(R[X]) \).

Let \( p \) be a prime of \( R \), and \( q_0 \subseteq \cdots \subseteq q_r \) be a chain of primes of \( R[X] \) with \( q_i \cap R = p \) for each \( i \). Then (15.24) yields a chain of primes of length \( r \) in \( R[X]/pR[X] \).

Further, as \( q_i \cap R = p \) for each \( i \), the latter chain gives rise to a chain of primes of length \( r \) in \( k(p)[X] \) where \( k(p) = (R/p)_p \) by (15.23) and (15.24). But \( k(p)[X] \) is a PID. Hence \( r \leq 1 \).

Take any chain \( p_0 \subseteq \cdots \subseteq p_n \) of primes in \( R[X] \). It contracts to a chain \( p_0 \subseteq \cdots \subseteq p_n \) in \( R \). At most two \( p_j \) contract to a given \( p_i \) by the above discussion. So \( m + 1 \leq 2(n + 1) \), or \( m \leq 2n + 1 \). Thus \( \dim(R[X]) \leq 1 + 2 \dim(R) \).

EXERCISE (15.23). — Let \( X \) be a topological space. We say a subset \( Y \) is **locally closed** if \( Y \) is the intersection of an open set and a closed set; equivalently, \( Y \) is open in its closure \( \overline{Y} \); equivalently, \( Y \) is closed in an open set containing it.

We say a subset \( X_0 \) of \( X \) is **very dense** if \( X_0 \) meets every nonempty locally closed subset \( Y \). We say \( X \) is **Jacobson** if its set of closed points is very dense.

Show that the following conditions on a subset \( X_0 \) of \( X \) are equivalent:

1. \( X_0 \) is very dense.
2. Every closed set \( F \) of \( X \) satisfies \( F \cap X_0 = F \).
3. The map \( U \mapsto U \cap X_0 \) from the open sets of \( X \) to those of \( X_0 \) is bijective.

SOLUTION: Assume (1). Given a closed set \( F \), take any \( x \in F \), and let \( U \) be an open neighborhood of \( x \) in \( X \). Then \( F \cup U \) is locally closed, so meets \( X_0 \). Hence \( x \in F \cap X_0 \). Thus \( F \subseteq F \cap X_0 \). The opposite inclusion is trivial. Thus (2) holds.

Assume (2). In (3), the map is trivially surjective. To check it’s injective, suppose \( U \cap X_0 = V \cap X_0 \). Then \( (X-U) \cap X_0 = (X-V) \cap X_0 \). So (2) yields \( X-U = X-V \). So \( U = V \). Thus (3) holds.

Assume (3). Then the map \( F \mapsto F \cap X_0 \) of closed sets is bijective too; whence, so is the map \( Y \mapsto Y \cap X_0 \) of locally closed sets. In particular, if a locally closed set \( Y \) is nonempty, then \( Y \cap X_0 \) is nonempty. Thus (1) holds.

EXERCISE (15.24). — Let \( R \) be a ring, \( X := \text{Spec}(R) \), and \( X_0 \) the set of closed points of \( X \). Show that the following conditions are equivalent:

1. \( R \) is a Jacobson ring.
2. \( X \) is a Jacobson space.
3. If \( y \in X \) is a point such that \( \{y\} \) is locally closed, then \( y \in X_0 \).

SOLUTION: Assume (1). Let \( F \subseteq X \) be closed. Trivially, \( F \supseteq \overline{F \cap X_0} \). To prove \( F \subseteq \overline{F \cap X_0} \), say \( F = \text{V}(a) \) and \( \overline{F \cap X_0} = \text{V}(b) \). Then \( F \cap X_0 \) is the set of maximal ideals \( m \) containing \( a \) by (15.23), and every such \( m \) contains \( b \). So (1) implies \( b \subseteq \sqrt{a} \). But \( \text{V}(\sqrt{a}) = F \). Thus \( F \subseteq \overline{F \cap X_0} \). Thus (15.24) yields (2).

Assume (2). Let \( y \in X \) be a point such that \( \{y\} \) is locally closed. Then \( \{y\} \cap X_0 \) is nonempty by (2). So \( \{y\} \cap X_0 \supset y \). Thus (3) holds.

Assume (3). Let \( p \) be a prime ideal of \( R \) such that \( pR_f \) is maximal for some \( f \notin p \). Then \( \{p\} \) is closed in \( D(f) \) by (15.23). So \( \{p\} \) is locally closed in \( X \). Hence \( \{p\} \) is closed in \( X \) by (3). Thus \( p \) is maximal. Thus (15.24) yields (1).
Exercise (15.28). — Let $P := \mathbb{Z}[X_1, \ldots, X_n]$ be the polynomial ring. Assume $f \in P$ vanishes at every zero in $K^n$ of $f_1, \ldots, f_r \in P$ for every finite field $K$; that is, if $(a) := (a_1, \ldots, a_n) \in K^n$ and $f_1(a) = 0, \ldots, f_r(a) = 0$ in $K$, then $f(a) = 0$ too. Prove there are $g_1, \ldots, g_r \in P$ and $N \geq 1$ such that $f^N = g_1 f_1 + \cdots + g_r f_r$. 

Solution: Set $a := (f_1, \ldots, f_r)$. Suppose $f \notin \sqrt{a}$. Then $f$ lies outside some maximal ideal $m$ containing $a$ by (15.20)(2) and (15.24). Set $K := P/m$. Then $K$ is a finite extension of $\mathbb{F}_p$ for some prime $p$ by (15.20)(1). So $K$ is finite. Let $a_i$ be the residue of $X_i$; set $(a) := (a_1, \ldots, a_n) \in K^n$. Then $f_1(a) = 0, \ldots, f_r(a) = 0$. So $f(a) = 0$ by hypothesis. Thus $f \in m$, a contradiction. Thus $f \in \sqrt{a}$. ∎

Exercise (15.29). — Let $R$ be a ring, $R'$ an algebra. Prove that if $R'$ is integral over $R$ and $R$ is Jacobson, then $R'$ is Jacobson.

Solution: Given an ideal $a' \subseteq R'$ and an $f$ outside $\sqrt{a}$, set $R' := R[f]$. Then $R''$ is Jacobson by (15.20)(2). So $R''$ has a maximal ideal $m''$ that avoids $f$ and contains $a' \cap R''$. But $R'$ is integral over $R''$. So $R'$ contains a prime $m'$ that contains $a'$ and that contracts to $m''$ by Going Up (14.3)(4). Then $m'$ avoids $f$ as $m''$ does, and $m'$ is maximal by Maximality, (14.3)(1). Thus $R'$ is Jacobson. ∎

Exercise (15.30). — Let $R$ be a Jacobson ring, $S$ a multiplicative subset, $f \in R$. True or false: prove or give a counterexample to each of the following statements.

1. The localized ring $R_f$ is Jacobson.
2. The localized ring $S^{-1}R$ is Jacobson.
3. The filtered direct limit $\lim\limits_{\rightarrow} R_\lambda$ of Jacobson rings is Jacobson.
4. In a filtered direct limit of rings $R_\lambda$, necessarily $\lim\limits_{\rightarrow} \text{rad}(R_\lambda) = \text{rad}(\lim\limits_{\rightarrow} R_\lambda)$.

Solution: (1) True: $R_f = R[1/f]$ by (14.13); so $R_f$ is Jacobson by (15.20)(1).
(2) False: by (15.21), $Z$ is Jacobson, but $Z_{(p)}$ isn’t for any prime number $p$.
(3) False: $Z_{(p)}$ isn’t Jacobson by (15.21), but $Z_{(p)} = \lim\limits_{\rightarrow} Z$ by (12.3).
(4) False: $\text{rad}(Z_{(p)}) = pZ_{(p)}$, but $\text{rad}(Z) = \langle 0 \rangle$, so $\lim\limits_{\rightarrow} \text{rad}(Z) = \langle 0 \rangle$. ∎

Exercise (15.31). — Let $R$ be a reduced Jacobson ring with a finite set $\Sigma$ of minimal primes, and $P$ a finitely generated module. Show that $P$ is locally free of rank $r$ if and only if $\dim_{R/m}(P/mP) = r$ for any maximal ideal $m$.

Solution: Suppose $P$ is locally free of rank $r$. Then given any maximal ideal $m$, there is an $f \in R - m$ such that $P_f$ is a free $R_f$-module of rank $r$ by (15.24). But $P_m$ is a localization of $P_f$ by (14.23). So $P_m$ is a free $R_m$-module of rank $r$ by (14.24). But $P_m/mP_m = (P/mP)_m$ by (14.22). Also $R_m/mR_m = R/m$ by (14.23). Thus $\dim_{R/m}(P/mP) = r$.

Consider the converse. Given a $p \in \Sigma$, set $K := \text{Frac}(R/p)$. Then $P \otimes_R K$ is a $K$-vector space, say of dimension $n$. Since $R$ is reduced, $K = R_p$ by (14.13). So by (14.24), there is an $h \in R - p$ with $P_h$ free of rank $n$. As $R$ is Jacobson, there is a maximal ideal $m$ avoiding $h$, by (14.21). Hence, as above, $\dim_{R/m}(P/mP) = n$. But, by hypothesis, $\dim_{R/m}(P/mP) = r$. Thus $n = r$.

Given a maximal ideal $m$, set $A := R_m$. Then $A$ is reduced by (14.23). Each minimal prime of $A$ is of the form $pA$ where $p \in \Sigma$ by (14.24)(2). Further, it’s not hard to see, essentially as above, that $P_m \otimes \text{Frac}(A/pA) = P \otimes \text{Frac}(R/p)$. Hence (14.24) implies $P_m$ is a free $A$-module of rank $r$. Finally, (13.52) implies $P$ is locally free of rank $r$. ∎

Solutions: (15.31) 217
16. Chain Conditions

Exercise (16.2). — Let $M$ be a finitely generated module over an arbitrary ring. Show every set that generates $M$ contains a finite subset that generates.

Solution: Say $M$ is generated by $x_1, \ldots, x_n$ and also by the $y_\lambda$ for $\lambda \in \Lambda$. Say $x_i = \sum_j z_j y_{\lambda_j}$. Then the $y_{\lambda_j}$ generate $M$. □

Exercise (16.8). — Let $R$ be a ring, $X$ a variable, $R[X]$ the polynomial ring. Prove this statement or find a counterexample: if $R[X]$ is Noetherian, then so is $R$.

Solution: It’s true. Since $R[X]$ is Noetherian, so is $R[X]/\langle 1 \rangle$ by (16.7). But the latter ring is isomorphic to $R$ by (16.3); so $R$ is Noetherian. □

Exercise (16.9). — Let $R \subset R'$ be a ring extension with an $R$-linear retraction $\rho: R' \to R$. Assume $R'$ is Noetherian, and prove $R$ is too.

Solution: Let $a \subset R$ be an ideal. As $R'$ is Noetherian, $aR'$ is finitely generated. But, by definition, $a$ generates $aR'$. So by (16.7) there are $a_1, \ldots, a_n$ that generate $aR'$. Hence, given any $a \in a$, there are $x' \in R'$ such that $a = a_1 x'_1 + \cdots + a_n x'_n$. Applying $\rho$ yields $a = a_1 x_1 + \cdots + a_n x_n$ with $x_i := \rho(x'_i) \in R$. Thus $a$ is finitely generated. Thus $R$ is Noetherian.

Alternatively, let $a_1 \subset a_2 \subset \cdots$ be an ascending chain of ideals of $R$. Then $a_1 R' \subset a_2 R' \subset \cdots$ stabilizes as $R'$ is Noetherian. So $\rho(a_1 R') \subset \rho(a_2 R') \subset \cdots$ stabilizes too. But $\rho(a, R') = a, \rho(R') = a_1$. Thus by (16.7), $R$ is Noetherian. □

Exercise (16.15). — Let $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ be a short exact sequence of $R$-modules, and $M_1, M_2$ two submodules of $M$. Prove or give a counterexample to this statement: if $\beta(M_1) = \beta(M_2)$ and $\alpha^{-1}(M_1) = \alpha^{-1}(M_2)$, then $M_1 = M_2$.

Solution: The statement is false: form the exact sequence

$$0 \to \mathbb{R} \xrightarrow{\alpha} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\beta} \mathbb{R} \to 0$$

with $\alpha(r) := (r, 0)$ and $\beta(r, s) := s$, and take

$$M_1 := \{(t, 2t) \mid t \in \mathbb{R}\} \quad \text{and} \quad M_2 := \{(2t, t) \mid t \in \mathbb{R}\}.$$

(Geometrically, we can view $M_1$ as the line determined by the origin and the point $(1,2)$, and $M_2$ as the line determined by the origin and the point $(2,1)$. Then $\beta(M_1) = \beta(M_2) = \mathbb{R}$, and $\alpha^{-1}(M_1) = \alpha^{-1}(M_2) = 0$, but $M_1 \neq M_2$ in $\mathbb{R} \oplus \mathbb{R}$.) □

Exercise (16.18). — Let $R$ be a ring, $a_1, \ldots, a_r$ ideals such that each $R/a_i$ is a Noetherian ring. Prove (1) that $\bigoplus R/a_i$ is a Noetherian $R$-module, and (2) that, if $\bigcap a_i = 0$, then $R$ too is a Noetherian ring.

Solution: Any $R$-submodule of $R/a_i$ is an ideal of $R/a_i$. Since $R/a_i$ is a Noetherian ring, such an ideal is finitely generated as an $(R/a_i)$-module, so as an $R$-module as well. Thus $R/a_i$ is a Noetherian $R$-module. So $\bigoplus R/a_i$ is a Noetherian $R$-module by (16.17). Thus (1) holds.

To prove (2), note that the kernel of the natural map $R \to \bigoplus R/a_i$ is $\bigcap a_i$, which is 0 by hypothesis. So $R$ can be identified with a submodule of the Noetherian $R$-module $\bigoplus R/a_i$. Hence $R$ itself is a Noetherian $R$-module by (16.17)(2). So $R$ is a Noetherian ring by (16.3). □
**Exercise (16.20).** — Let $R$ be a Noetherian ring, $M$ and $N$ finitely generated modules. Show that $\text{Hom}(M, N)$ is finitely generated.

**Solution:** Say $M$ is generated $m$ elements. Then (14.10) yields a surjection $R^\oplus m \to M$. It yields an inclusion $\text{Hom}(M, N) \hookrightarrow \text{Hom}(R^\oplus m, N)$ by (5.15). But $\text{Hom}(R^\oplus m, N) = \text{Hom}(R, N)^\oplus m = N^\oplus m$ by (16.15.2) and (16.3). Plainly $N^\oplus m$ is finitely generated as $N$ is. Hence $\text{Hom}(R^\oplus m, N)$ is finitely generated, so Noetherian by (16.15). Thus $\text{Hom}(M, N)$ is finitely generated. \hfill \Box

**Exercise (16.24).** — Let $R$ be a domain, $R'$ an algebra, and set $K := \text{Frac}(R)$. Assume $R$ is Noetherian.

1. [1] Thm. 3] Assume $R'$ is a field containing $R$. Show $R'/R$ is algebra finite if and only if $K'/R'$ is algebra finite and $R'/K'$ is (module) finite.

2. [1] bot. p. 77] Let $K' \supset R$ be a field that embeds in $R'$. Assume $R'/R$ is algebra finite. Show $K/R$ is algebra finite and $K'/K$ is finite.

**Solution:** For (1), first assume $R'/R$ is algebra finite. Now, $R \subset K \subset R'$. So $R'/K$ is algebra finite. Thus $R'/K$ is (module) finite by (16.21) or (16.7.21), and so $K/R$ is algebra finite by (16.7.41).

Conversely, say $x_1, \ldots, x_m$ are algebra generators for $K/R$, and say $y_1, \ldots, y_n$ are module generators for $R'/K$. Then clearly $x_1, \ldots, x_m, y_1, \ldots, y_n$ are algebra generators for $R'/R$. Thus (1) holds.

For (2), let $m$ be any maximal ideal of $R'$, and set $L := R'/m$. Then $L$ is a field, $R \subset K \subset K' \subset L$, and $R/K$ is algebra finite. So $K/R$ is algebra finite and $L/K$ is finite by (1); whence, $K'/K$ is finite too. Thus (2) holds. \hfill \Box

**Exercise (16.28).** — Let $k$ be a field, $R$ an algebra. Assume that $R$ is finite dimensional as a $k$-vector space. Prove that $R$ is Noetherian and Artinian.

**Solution:** View $R$ as a vector space, and ideals as subspaces. Now, by a simple dimension argument, any ascending or descending chain of subspaces of $R$ stabilizes. Thus $R$ is Noetherian by (15.39) and is Artinian by definition. \hfill \Box

**Exercise (16.29).** — Let $p$ be a prime number, and set $M := \mathbb{Z}[1/p]/\mathbb{Z}$. Prove that any $\mathbb{Z}$-submodule $N \subset M$ is either finite or all of $M$. Deduce that $M$ is an Artinian $\mathbb{Z}$-module, and that it is not Noetherian.

**Solution:** Given $q \in N$, write $q = n/p^e$ where $n$ is relatively prime to $p$. Then there is an $m \in \mathbb{Z}$ with $nm \equiv 1 \pmod{p^e}$. Hence $N \ni m(n/p^e) = 1/p^e$, and so $1/p^e = p^{e-r}(1/p^e) \in N$ for any $0 \leq r \leq e$. Therefore, either $N = M$, or there is a largest integer $e \geq 0$ with $1/p^e \in N$. In the second case, $N$ is finite.

Let $M \supseteq N_1 \supset N_2 \supset \cdots$ be a descending chain. By what we just proved, each $N_i$ is finite, say with $n_i$ elements. Then the sequence $n_1 \geq n_2 \geq \cdots$ stabilizes; say $n_i = n_{i+1} = \cdots$. But $N_i \supset N_{i+1} \supset \cdots$, so $N_i = N_{i+1} = \cdots$. Thus $M$ is Artinian.

Finally, suppose $m_1, \ldots, m_r$ generate $M$, say $m_i = n_i/p^{e_i}$. Set $e := \max e_i$. Then $1/p^e$ generates $M$, a contradiction since $1/p^{e+1} \in M$. Thus $M$ is not finitely generated, and so not Noetherian. \hfill \Box

**Exercise (16.30).** — Let $R$ be an Artinian ring. Prove that $R$ is a field if it is a domain. Deduce that in general every prime ideal $\mathfrak{p}$ of $R$ is maximal.
220 Solutions: (17.22)

**Solution:** Take any nonzero element $x \in R$, and consider the chain of ideals $\langle x \rangle \supset \langle x^2 \rangle \supset \cdots$. Since $R$ is Artinian, the chain stabilizes; so $\langle x^e \rangle = \langle x^{e+1} \rangle$ for some $e$. Hence $x^{e+1}a = 1$ for some $a \in R$. If $R$ is a domain, then we can cancel to get $1 = ax$; thus $R$ is then a field.

In general, $R/p$ is Artinian by (17.7.7)(2). Now, $R/p$ is also a domain by (2.13). Hence, by what we just proved, $R/p$ is a field. Thus $p$ is maximal by (17.7.7). □

17. Associated Primes

**Exercise (17.6).** — Given modules $M_1, \ldots, M_r$, set $M := M_1 \oplus \cdots \oplus M_r$. Prove $\text{Ass}(M) = \text{Ass}(M_1) \cup \cdots \cup \text{Ass}(M_r)$.

**Solution:** Set $N := M_2 \oplus \cdots \oplus M_r$. Then $N, M_1 \subset M$. Also, $M/N = M_1$. So (17.3) yields $\text{Ass}(N), \text{Ass}(M_1) \subset \text{Ass}(M) \subset \text{Ass}(N) \cup \text{Ass}(M_1)$. So $\text{Ass}(M) = \text{Ass}(N) \cup \text{Ass}(M_1)$. The assertion follows by induction on $r$. □

**Exercise (17.7).** — Take $R := \mathbb{Z}$ and $M := \mathbb{Z}/(2) \oplus \mathbb{Z}$. Find $\text{Ass}(M)$ and find two submodules $L, N \subset M$ with $L + N = M$ but $\text{Ass}(L) \cup \text{Ass}(N) \subsetneq \text{Ass}(M)$.

**Solution:** First, we have $\text{Ass}(M) = \{\{0\}, \langle 2 \rangle\}$ by (17.7)(2) and (17.4)(2). Next, take $L := R \cdot (1, 1)$ and $N := R \cdot (0, 1)$. Then the canonical maps $R \to L$ and $R \to N$ are isomorphisms. Hence both $\text{Ass}(L)$ and $\text{Ass}(N)$ are $\{\{0\}\}$ by (17.7)(2). Finally, $L + N = M$ because $(a, b) = a \cdot (1, 1) + (b - a) \cdot (0, 1)$. □

**Exercise (17.8).** — If a prime $p$ is sandwiched between two primes in $\text{Ass}(M)$, is $p$ necessarily in $\text{Ass}(M)$ too?

**Solution:** No, for example, let $R := k[X, Y]$ be the polynomial ring over a field. Set $M := R \oplus (R/(X, Y))$ and $p := \langle X \rangle$. Then $\text{Ass}(M) = \text{Ass}(R) \cup \text{Ass}(R/(X, Y))$ by (17.7)(2). Further, $\text{Ass}(R) = \{0\}$ and $\text{Ass}(R/(X, Y)) = \langle X, Y \rangle$ by (17.7)(2). □

**Exercise (17.11).** — Let $R$ be a ring, and suppose $R_p$ is a domain for every prime $p$. Prove every associated prime of $R$ is minimal.

**Solution:** Let $p \in \text{Ass}(R)$. Then $pR_p \in \text{Ass}(R_p)$ by (17.7)(10). By hypothesis, $R_p$ is a domain. So $pR_p = \{0\}$ by (17.7)(2). Hence $p$ is a minimal prime of $R$ by (17.7)(2).

Alternatively, say $p = \text{Ann}(x)$ with $x \in R$. Then $x/1 \neq 0$ in $R_p$; otherwise, there would be some $s \in R - p$ such that $sx = 0$, contradicting $p = \text{Ann}(x)$. However, for any $y \in p$, we have $xy/1 = 0$ in $R_p$. Since $R_p$ is a domain and since $x/1 \neq 0$, we must have $y/1 = 0$ in $R_p$. So there exists some $t \in R - p$ such that $ty = 0$. Now, $p \supset q$ for some minimal prime $q$ by (17.13). Suppose $p \neq q$. Then there is some $y \in p - q$. So there exists some $t \in R - p$ such that $ty = 0 \in q$, contradicting the primeness of $q$. Thus $p = q$; that is, $p$ is minimal. □

**Exercise (17.16).** — Let $R$ be a Noetherian ring, $M$ a module, $N$ a submodule, $x \in R$. Show that, if $x \notin p$ for any $p \in \text{Ass}(M/N)$, then $xM \cap N = xN$.

**Solution:** Trivially, $xN \subset xM \cap N$. Conversely, take $m \in M$ with $xm \in N$. Let $m'$ be the residue of $m$ in $M/N$. Then $xm' = 0$. By (17.7)(10), $x \notin z \cdot \text{div}(M/N)$. So $m' = 0$. So $m \in N$. So $xm \in xN$. Thus $xM \cap N \subset xN$, as desired. □
EXERCISE (17.22). — Let $R$ be a Noetherian ring, $a$ an ideal. Prove the primes minimal containing $a$ are associated to $a$. Prove such primes are finite in number.

SOLUTION: Since $a = \text{Ann}(R/a)$, the primes in question are the primes minimal in $\text{Supp}(R/a)$ by (17.24)(3). So they are associated to $a$ by (17.25), and they are finite in number by (17.24). \hfill \square

EXERCISE (17.23). — Take $R := \mathbb{Z}$ and $M := \mathbb{Z}$ in (17.24). Determine when a chain $0 \subset M_1 \subseteq M$ is acceptable, and show that then $p_2 \notin \text{Ass}(M)$.

SOLUTION: If the chain is acceptable, then $M_1 \neq 0$ as $M_1/0 \simeq R/p_1$, and $M_1$ is a prime ideal as $M_1 = \text{Ann}(M/M_1) = p_2$. Conversely, the chain is acceptable if $M_1$ is a nonzero prime ideal $p$, as then $M_1/0 \simeq R/0$ and $M/M_1 \simeq R/p$.

Finally, $\text{Ass}(M) = 0$ by (17.22). Further, as just observed, given any acceptable chain, $p_2 = M_1 \neq 0$. So $p_2 \notin \text{Ass}(M)$. \hfill \square

EXERCISE (17.24). — Take $R := \mathbb{Z}$ and $M := \mathbb{Z}/(12)$ in (17.24). Find all three acceptable chains, and show that, in each case, $\{p_1\} = \text{Ass}(M)$.

SOLUTION: An acceptable chain in $M$ corresponds to a chain

$$\langle 12 \rangle \subset \langle a_1 \rangle \subset \langle a_2 \rangle \subset \cdots \subset \langle a_n \rangle = \mathbb{Z}.$$ 

Here $\langle a_1 \rangle/\langle 12 \rangle \simeq \mathbb{Z}/(p_1)$ with $p_1$ prime. So $a_1p_1 = 12$. Hence the possibilities are $p_1 = 2$, $a_1 = 6$ and $p_1 = 3$, $a_1 = 4$. Further, $\langle a_2 \rangle/\langle a_1 \rangle \simeq \mathbb{Z}/(p_2)$ with $p_2$ prime. So $a_2p_2 = a_1$. Hence, if $a_1 = 6$, then the possibilities are $p_2 = 2$, $a_2 = 3$ and $p_2 = 3$, $a_2 = 2$; if $a_1 = 4$, then the only possibility is $p_2 = 2$ and $a_2 = 2$. In each case, $a_2$ is prime; hence, $n = 3$, and these three chains are the only possibilities. Conversely, each of these three possibilities, clearly, does arise.

In each case, $\{p_1\} = \{(2), (3)\}$. Hence (17.24) yields $\text{Ass}(M) \subset \{(2), (3)\}$. For any $M$, if $0 \subset M_1 \subset \cdots \subset M$ is an acceptable chain, then (17.23) and (17.24)(2) yield $\text{Ass}(M) \supset \text{Ass}(M_1) = \{p_1\}$. Here, there’s one chain with $p_1 = (2)$ and another with $p_1 = (3)$; hence, $\text{Ass}(M) \supset \{(2), (3)\}$. Thus $\text{Ass}(M) = \{(2), (3)\}$. \hfill \square

EXERCISE (17.26). — Let $R$ be a Noetherian ring, $a$ an ideal, and $M$ a finitely generated module. Show that the following conditions are equivalent:

1. $V(a) \cap \text{Ass}(M) = \emptyset$;
2. $\text{Hom}(N, M) = 0$ for all finitely generated modules $N$ with $\text{Supp}(N) \subset V(a)$;
3. $\text{Hom}(N, M) = 0$ for some finitely generated module $N$ with $\text{Supp}(N) = V(a)$;
4. $a \not\subset z\cdot \text{div}(M)$; that is, there is a nonzerodivisor $x$ on $M$ in $a$;
5. $a \not\subset p$ for any $p \in \text{Ass}(M)$.

SOLUTION: Assume (1). Then $\text{Supp}(N) \cap \text{Ass}(M) = \emptyset$ for any module $N$ with $\text{Supp}(N) \subset V(a)$. Hence $\text{Ass}(\text{Hom}(N, M)) = \emptyset$ by (17.22). So $\text{Hom}(N, M) = 0$ by (17.13). Thus (2) holds. Clearly (2) with $N := R/a$ implies (3).

Assume (3). Then $\text{Ass}(\text{Hom}(N, M)) = \emptyset$ by (17.25). So $V(a) \cap \text{Ass}(M) = \emptyset$ by (17.14). Thus (1) holds. Clearly (1) and (5) are equivalent.

Finally, $z\cdot \text{div}(M) = \bigcup_{p \in \text{Ass}(M)} p$ by (17.15). So (4) implies (5). Moreover, the union is finite by (17.24); so (3.15) and (5) yield (4). \hfill \square

18. Primary Decomposition
Exercise (18.6). — Let $R$ be a ring, and $p = \langle p \rangle$ a principal prime generated by a nonzerodivisor $p$. Show every positive power $p^n$ is $p$-primary, and conversely, if $R$ is Noetherian, then every $p$-primary ideal $q$ is equal to some power $p^n$.

Solution: Let’s proceed by induction. Form the exact sequence

$$0 \to p^n/p^{n+1} \to R/p^{n+1} \to R/p^n \to 0.$$ 

Consider the map $R \to p^n/p^{n+1}$ given by $x \mapsto xp^n$. It is surjective, and its kernel is $p$ as $p$ is a nonzerodivisor. Hence $R/p \xrightarrow{\sim} p^n/p^{n+1}$. But $\text{Ass}(R/p) = \{p\}$ by (17.14)(2). Hence (17.2) yields $\text{Ass}(R/p^n) = \{p\}$ for every $n \geq 1$, as desired.

Conversely, $p = \sqrt{q}$ by (18.12). So $p^n \in q$ for some $n$; take $n$ minimal. Then $p^n \subset q$. Suppose there is an $x \in q - p^n$. Say $x = yp^m$ for some $y$ and $m \geq 0$. Then $m < n$ as $x \notin p^n$. Take $m$ maximal. Now, $p^m \notin q$ as $n$ is minimal. So (18.13) yields $y \in q \subset p$. Hence $y = zp$ for some $z$. Then $x = yp^{m+1}$, contradicting the maximality of $m$. Thus $q = p^n$. □

Exercise (18.7). — Let $k$ be a field, and $k[X,Y]$ the polynomial ring. Let $a$ be the ideal $\langle X^2, XY \rangle$. Show $a$ is not primary, but $\sqrt{a}$ is prime. Show $a$ satisfies this condition: $ab \in a$ implies $a^2 \in a$ or $b^2 \in a$.

Solution: First, $\langle X \rangle$ is prime by (7.10). But $\langle X^2 \rangle \subset a \subset \langle X \rangle$. So $\sqrt{a} = \langle X \rangle$ by (18.25). On the other hand, $XY \in a$, but $X \notin a$ and $Y \notin \sqrt{a}$; thus $a$ is not primary by (18.25). If $ab \in a$, then $X | a$ or $X | b$, so $a^2 \in a$ or $b^2 \in a$. □

Exercise (18.8). — Let $\varphi: R \to R'$ be a homomorphism of Noetherian rings, and $q \subset R'$ a $p$-primary ideal. Show that $\varphi^{-1}q \subset R$ is $\varphi^{-1}p$-primary. Show that the converse holds if $\varphi$ is surjective.

Solution: Let $xy \in \varphi^{-1}q$, but $x \notin \varphi^{-1}q$. Then $\varphi(x)\varphi(y) \in q$, but $\varphi(x) \notin q$. So $\varphi(y)^n \in q$ for some $n \geq 1$ by (18.8). Hence, $y^n \in \varphi^{-1}q$. So $\varphi^{-1}q$ is primary by (18.8). Its radical is $\varphi^{-1}p$ as $p = \sqrt{q}$, and taking the radical commutes with taking the inverse image by (7.28). The converse can be proved similarly. □

Exercise (18.17). — Let $k$ be a field, $R := k[X,Y,Z]$ be the polynomial ring. Set $a := \langle XY, X - YZ \rangle$, set $q_1 := \langle X, Z \rangle$, and set $q_2 := \langle Y^2, X - YZ \rangle$. Show that $a = q_1 \cap q_2$ holds and that this expression is an irredundant primary decomposition.

Solution: First, $XY = Y(X - YZ) + Y^2Z \in q_2$. Hence $a \subset q_1 \cap q_2$. Conversely, take $F \in q_1 \cap q_2$. Then $F \in q_2$, so $F = GY^2 + H(X - YZ)$ with $G, H \in R$. But $F \in q_1$, so $G = AX + BZ$ with $A, B \in R$. Then

$$F = (AY + B)XY + (H - BY)(X - YZ) \in a.$$ 

Thus $a \supset q_1 \cap q_2$. Thus $a = q_1 \cap q_2$ holds.

Finally, $q_1$ is prime by (4.11). Now, using (18.8), let’s show $q_2$ is $\langle X, Y \rangle$-primary. Form $\varphi: k[X,Y,Z] \to k[Y,Z]$ with $\varphi(X) := YZ$. Clearly, $q_2 = \varphi^{-1}\langle Y^2 \rangle$ and $\langle X, Y \rangle = \varphi^{-1}\langle Y \rangle$; also, $(Y^2)$ is $\langle Y \rangle$-primary by (18.8). Thus $a = q_1 \cap q_2$ is a primary decomposition. It is irredundant as $q_1$ and $\langle X, Y \rangle$ are distinct. □

Exercise (18.18). — Let $R := R' \times R''$ be a product of two domains. Find an irredundant primary decomposition of $\langle 0 \rangle$.

Solution: Set $p' := \langle 0 \rangle \times R''$ and $p'' := R' \times \langle 0 \rangle$. Then $p'$ and $p''$ are prime by (7.11), so primary by (17.2)(2). Clearly $\langle 0 \rangle = p' \cap p''$. Thus this representation is a primary decomposition; it is irredundant as both $p'$ and $p''$ are needed. □
Exercise (18.22). — Let $R$ be a Noetherian ring, $a$ an ideal, and $M$ a finitely generated module. Consider the following submodule of $M$:

$$
\Gamma_a(M) := \bigcup_{n \geq 1} \{ m \in M \mid a^n m = 0 \text{ for some } n \geq 1 \}.
$$

(1) For any decomposition $0 = \bigcap Q_i$ with $Q_i$, $p_i$-primary, show $\Gamma_a(M) = \bigcap_{a \nmid p_i} Q_i$.

(2) Show $\Gamma_a(M)$ is the set of all $m \in M$ such that $m/1 \in M_p$ vanishes for every prime $p$ with $a \nmid p$. (Thus $\Gamma_a(M)$ is the set of all $m$ whose support lies in $V(a)$.)

Solution: For (1), given $m \in \Gamma_a(M)$, say $a^n m = 0$. Given $i$ with $a \nmid p_i$, take $a \in a - p_i$. Then $a^n m = 0 \in Q_i$. Hence $m \in Q_i$ by (18.24). Thus $m \in \bigcap_{a \nmid p_i} Q_i$.

Conversely, given $m \in \bigcap_{a \nmid p_i} Q_i$, take any $j$ with $a \subset p_j$. Then $p_j = \text{nil}(M/Q_j)$ by (18.3). So there is $n_j$ with $a^{n_j} m \subset Q_j$. Set $n := \max\{n_j\}$. Then $a^n m \subset Q_i$ for all $i$, whether $a \subset p$, or not. Hence $a^n m \in \bigcap Q_i = 0$. Thus $m \in \Gamma_a(M)$.

For (2), given $m \in \Gamma_a(M)$, say $a^n m = 0$. Given a prime $p$ with $a \nmid p$, take $a \in a - p$. Then $a^n m = 0$ and $a^n \nmid p$. So $m/1 \in M_p$ vanishes.

Conversely, given an $m \in M$ such that $m/1 \in M_p$ vanishes for every prime $p$ with $a \nmid p$, consider a decomposition $0 = \bigcap Q_i$ with $Q_i$, $p_i$-primary; one exists by (18.21). By (1), it suffices to show $m \in Q_i$ if $a \nmid p_i$. But $m/1 \in M_{p_i}$ vanishes. So there’s an $a \in R - p_i$ with $am = 0 \in Q_i$. So (18.24) yields $m \in Q_i$, as desired. □

Exercise (18.26). — Let $R$ be a Noetherian ring, $M$ a finitely generated module, $N$ a submodule. Prove $N = \bigcap_{p \in \text{Ass}(M/N)} \varphi^{-1}_p (N_p)$.

Solution: (18.21) yields an irredundant primary decomposition $N = \bigcap_{i=1}^r Q_i$. Say $Q_i$ is $p_i$-primary. Then $\{p_i\}_i = \text{Ass}(M/N)$ by (18.21). Also, (18.24) yields $\varphi^{-1}_p (N_p) = \bigcap_{p \subset p_i} Q_j$. Thus $\bigcap_{i=1}^r \varphi^{-1}_p (N_p_i) = \bigcap_{i=1}^r \bigcap_{p \subset p_i} Q_j = \bigcap_{i=1}^r Q_i = N$. □

Exercise (18.27). — Let $R$ be a Noetherian ring, $p$ a prime. Its $n$th symbolic power $p^{(n)}$ is defined as the saturation $(p^n)^S$ where $S := R - p$.

(1) Show $p^{(n)}$ is the $p$-primary component of $p^n$.

(2) Show $p^{(m+n)}$ is the $p$-primary component of $p^{(m)} p^{(n)}$.

(3) Show $p^{(n)} = p^n$ if and only if $p^n$ is $p$-primary.

(4) Given a $p$-primary ideal $q$, show $q \supset p^{(n)}$ for all large $n$.

Solution: Note $p$ is minimal in $V(p^n)$. But $V(p^n) \text{Supp} (R/p^n)$ by (18.22)(3). Hence $p$ is minimal in $\text{Ass}(R/p^n)$ by (17.13) and (17.3). Thus (18.24) yields (1). Notice (18.27)(3) yields $(p^{(m)} p^{(n)})^S = p^{(m+n)}$. Thus (18.24) yields (2).

If $p^{(n)} = p^n$, then $p^n$ is $p$-primary by (1). Conversely, if $p^n$ is $p$-primary, then $p^{(n)} = p^{(n)}$ because primary ideals are saturated by (18.24). Thus (3) holds.

For (4), recall $p = \sqrt{q}$ by (18.2). So $q \supset p^n$ for all large $n$ by (18.3). Hence $q^S \supset p^{(n)}$. But $q^S = q$ by (18.24) since $p \cap (R - p) = \emptyset$. Thus (4) holds. □

Exercise (18.28). — Let $R$ be a Noetherian ring, $(0) = q_1 \cap \cdots \cap q_n$, an irredundant primary decomposition. Set $p_i := \sqrt{q_i}$ for $i = 1, \ldots, n$.

(1) Suppose $p_i$ is minimal for some $i$. Show $q_i = p_i^{(r)}$ for all large $r$.

(2) Suppose $p_i$ is not minimal for some $i$. Show that replacing $q_i$ by $p_i^{(r)}$ for large $r$ gives infinitely many distinct irredundant primary decompositions of $(0)$.
SOLUTION: Set $A := R_{p_i}$ and $m := p_iA$. Then $A$ is Noetherian by (18.77). Suppose $p_i$ is minimal. Then $m$ is the only prime in $A$. So $m = \sqrt{(0)}$ by the Scheinnullstellensatz (18.24). So $m^{r} = 0$ for all large $r$ by (18.28). So $p_i^{(r)} = q_i$ by Lemma (18.29) and the Second Uniqueness Theorem (18.28). Thus (1) holds.

Suppose $p_i$ is not minimal. Assume $m^{r} = m^{r+1}$ for some $r$. Then $m^{r} = 0$ by Nakayama’s Lemma (10.11). Hence $m$ is minimal. So $p_i$ is too, contrary to hypothesis. Thus by (18.14) (1), the powers $p_i^{(r)}$ are distinct.

However, $q_i \supset p_i^{(r)}$ for all large $r$ by (18.29) (4). Hence $(0) = p_i^{(r)} \cap \bigcap_{j \neq i} q_j$. But $p_i^{(r)}$ is $p_i$-primary by (18.29) (1). Thus replacing $q_i$ by $p_i^{(r)}$ for large $r$ gives infinitely many distinct primary decompositions of $(0)$.

These decompositions are irredundant owing to two applications of (18.14). A first yields $\{p_i\} = \text{Ass}(R)$ as $(0)q_1 \cap \cdots \cap q_n$ is irredundant. So a second yields the desired irredundancy. \qed

Exercise (18.30). — Let $R$ be a Noetherian ring, $m \subset \text{rad}(R)$ an ideal, $M$ a finitely generated module, and $M'$ a submodule. Considering $M/M'$, show that

$$M' = \bigcap_{n \geq 0}(m^n M + M').$$

Solution: Set $N := \bigcap_{n \geq 0} m^n(M/M')$. Then by (18.24), there is $x \in m$ such that $(1 + x)N = 0$. By (3.2), $1 + x$ is a unit since $m \subset \text{rad}(R)$. So $N = 0$. But $m^n(M/M')(m^n M + M')/M'$. Thus $\bigcap(m^n M + M')/M' = 0$, as desired. \qed

19. Length

Exercise (19.2). — Let $R$ be a ring, $M$ a module. Prove these statements:

1. If $M$ is simple, then any nonzero element $m \in M$ generates $M$.
2. $M$ is simple if and only if $M \cong R/m$ for some maximal ideal $m$, and if so, then $m = \text{Ann}(M)$.
3. If $M$ has finite length, then $M$ is finitely generated.

Solution: Obviously, $\text{Rm}$ is a nonzero submodule. So it is equal to $M$, because $M$ is simple. Thus (1) holds.

Assume $M$ is simple. Then $M$ is cyclic by (1). So $M \cong R/m$ for $m := \text{Ann}(M)$ by (18.77). Since $M$ is simple, $m$ is maximal owing to the bijective correspondence of (18.33). By the same token, if, conversely, $M \cong R/m$ with $m$ maximal, then $M$ is simple. Thus (2) holds.

Assume $\ell(M) < \infty$. Let $M = M_0 \supset M_1 \supset \cdots \supset M_m = 0$ be a composition series. If $m = 0$, then $M = 0$. Assume $m \geq 1$. Then $M_1$ has a composition series of length $m - 1$. So, by induction on $m$, we may assume $M_1$ is finitely generated. Further, $M/M_1$ is simple, so finitely generated by (1). Hence $M$ is finitely generated by (18.16) (1). Thus (3) holds. \qed

Exercise (19.4). — Let $R$ be a Noetherian ring, $M$ a finitely generated module. Prove the equivalence of the following three conditions:

1. that $M$ has finite length;
2. that $\text{Supp}(M)$ consists entirely of maximal ideals;
3. that $\text{Ass}(M)$ consists entirely of maximal ideals.

Prove that, if the conditions hold, then $\text{Ass}(M)$ and $\text{Supp}(M)$ are equal and finite.
Assume (1). Then (16.24) yields (2).
Assume (2). Then (16.22) and (16.24) yield (1). Further, (16.23) yields (3).
Finally, assume (3). Then (16.22) and (16.24) imply that $\text{Ass}(M)$ and $\text{Supp}(M)$ are equal and consist entirely of maximal ideals. In particular, (2) holds. However, $\text{Ass}(M)$ is finite by (16.24). Thus the last assertion holds.

**Exercise (19.5).** — Let $R$ be a Noetherian ring, $q$ a $p$-primary ideal. Consider chains of primary ideals from $q$ to $p$. Show (1) all such chains have length at most $\ell(A) − 1$ where $A := (R/q)_p$ and (2) all maximal chains have length exactly $\ell(A) − 1$.

**Solution:** There is a natural bijective correspondence between the $p$-primary ideals containing $q$ and the $(p/q)$-primary ideals of $R/q$, owing to (16.23). In turn, there is one between the latter ideals and the ideals of $A$ primary for its maximal ideal $m$, owing to (16.8) again and also to (16.26) with $M := A$.

However, $p = \sqrt{q}$ by (16.3). So $m = \sqrt{(0)}$. Hence every ideal of $A$ is $m$-primary by (16.10). Further, $m$ is the only prime of $A$; so $\ell(A)$ is finite by (16.3) with $M := A$. Hence (19.3) with $M := A$ yields (1) and (2).

**Exercise (19.8).** — Let $k$ be a field, $R$ an algebra-finite extension. Prove that $R$ is Artinian if and only if $R$ is a finite-dimensional $k$-vector space.

**Solution:** As $k$ is Noetherian by (16.11) and as $R/k$ is algebra-finite, $R$ is Noetherian by (16.24). Assume $R$ is Artinian. Then $\ell(R) < \infty$ by (19.6). So $R$ has a composition series. The successive quotients are isomorphic to residue class fields by (16.22)(2). These fields are finitely generated $k$-algebras, as $R$ is. Hence these fields are finite extension fields of $k$ by the Zariski Nullstellensatz. Thus $R$ is a finite-dimensional $k$-vector space. The converse holds by (16.25).

**Exercise (19.10).** — Let $k$ be a field, $A$ a local $k$-algebra. Assume the map from $k$ to the residue field is bijective. Given an $A$-module $M$, prove $\ell(M) = \dim_k(M)$.

**Solution:** If $M = 0$, then $\ell(M) = 0$ and $\dim_k(M) = 0$. If $M \cong_k k$, then $\ell(M) = 1$ and $\dim_k(M) = 1$. Assume $1 \leq \ell(M) < \infty$. Then $M$ has a submodule $M'$ with $M/M' \cong_k k$. So $\text{Additivity of Length}$, (19.4), yields $\ell(M') = \ell(M) − 1$ and $\dim_k(M') = \dim_k(M) − 1$. Hence $\ell(M') = \dim_k(M')$ by induction on $\ell(M)$.

Thus $\ell(M) = \dim_k(M)$.

If $\ell(M) = \infty$, then for every $m \geq 1$, there exists a chain of submodules,

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_m = 0.$$  

Hence $\dim_k(M) = \infty$.

**Exercise (19.12).** — Prove these conditions on a Noetherian ring $R$ equivalent:

1. $R$ is Artinian;
2. $\text{Spec}(R)$ is discrete and finite;
3. $\text{Spec}(R)$ is discrete.

**Solution:** Condition (1) holds, by (19.11), if and only if $\text{Spec}(R)$ consists of finitely points and each is a maximal ideal. But a prime $p$ is a maximal ideal if and only if $\{p\}$ is closed in $\text{Spec}(R)$ by (13.2). It follows that (1) and (2) are equivalent.

Trivially, (2) implies (3). Conversely, (3) implies (2), since $\text{Spec}(R)$ is quasi-compact by (13.20). Thus all three conditions are equivalent.

**Exercise (19.13).** — Let $R$ be an Artinian ring. Show that $\text{rad}(R)$ is nilpotent.
SOLUTION: Set \( \mathfrak{m} := \text{rad}(R) \). Then \( \mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \cdots \) is a descending chain. So \( \mathfrak{m}^r = \mathfrak{m}^{r+1} \) for some \( r \). But \( R \) is Noetherian by (13.11). So \( \mathfrak{m} \) is finitely generated. Thus Nakayama’s Lemma (10.11) yields \( \mathfrak{m}^r = 0 \).

Alternatively, \( R \) is Noetherian and \( \dim R = 0 \) by (13.11). So \( \text{rad}(R) \) is finitely generated and \( \text{rad}(R) = \text{nil}(R) \). Thus (13.32) implies \( \mathfrak{m}^r = 0 \) for some \( r \).

Exercise (19.16). — Let \( R \) be a ring, \( p \) a prime ideal, and \( R' \) a module-finite \( R \)-algebra. Show that \( R' \) has only finitely many primes \( p' \) over \( p \), as follows: reduce to the case that \( R \) is a field by localizing at \( p \) and passing to the residue rings.

SOLUTION: First note that, if \( p' \subset R' \) is a prime lying over \( p \), then \( p'R'_p \subset R'_p \) is a prime lying over the maximal ideal \( pR_p \). Hence, by (11.22)(2), it suffices to show that \( R'_p \) has only finitely many such primes. Note also that \( R'_p \) is module-finite over \( R_p \). Hence we may replace \( R \) and \( R' \) by \( R_p \) and \( R'_p \), and thus assume that \( p \) is the unique maximal ideal of \( R \). Similarly, we may replace \( R \) and \( R' \) by \( R/p \) and \( R'/pR' \), and thus assume that \( R \) is a field.

There are a couple of ways to finish. First, \( R' \) is now Artinian by (11.16) or by (11.25); hence, \( R' \) has only finitely many primes by (11.14). Alternatively, every prime is now minimal by incomparability (14.4) (2). Further, \( R' \) is Noetherian by (14.12); hence, \( R' \) has only finitely many minimal primes by (14.22). \( \square \)

Exercise (19.18). — Let \( R \) be a Noetherian ring, and \( M \) a finitely generated module. Prove the following four conditions are equivalent:

1. that \( M \) has finite length;
2. that \( M \) is annihilated by some finite product of maximal ideals \( \prod m_i \);
3. that every prime \( p \) containing \( \text{Ann}(M) \) is maximal;
4. that \( R/\text{Ann}(M) \) is Artinian.

SOLUTION: Assume (1). Let \( M = M_0 \supseteq \cdots \supseteq M_n = 0 \) be a composition series; set \( m_i := \text{Ann}(M_{i-1}/M_i) \). Then \( m_i \) is maximal by (13.2)(2). Also, \( m_iM_{i-1} \subset M_i \). Hence \( m_i \mid m_{i-1} \subset M_i \). Thus (2) holds.

Assume (2). Let \( p \) be a prime containing \( \text{Ann}(M) \). Then \( p \supseteq \prod m_i \). So \( p \supseteq m_i \) for some \( i \) by (2.26). So \( p = m_i \) as \( m_i \) is maximal. Thus (3) holds.

Assume (3). Then \( \dim(R/\text{Ann}(M)) = 0 \). But, by (11.7), any quotient of \( R \) is Noetherian. Thus (11.11) yields (4).

If (4) holds, then (11.13) yields (1), because \( M \) is a finitely generated module over \( R/\text{Ann}(M) \). \( \square \)

20. Hilbert Functions

Exercise (20.5). — Let \( k \) be a field, \( k[X, Y] \) the polynomial ring. Show \( \langle X, Y^2 \rangle \) and \( \langle X^2, Y^2 \rangle \) have different Hilbert Series, but the same Hilbert Polynomial.

SOLUTION: Let \( m := \langle X, Y \rangle \) and \( a := \langle X, Y^2 \rangle \) and \( b := \langle X^2, Y^2 \rangle \). They are graded by degree. So \( \ell(a_1) = 1 \), and \( \ell(a_n) = \ell(m_n) \) for all \( n \geq 2 \). Further, \( \ell(b_1) = 0 \), \( \ell(b_2) = 2 \), and \( \ell(b_n) = \ell(m_n) \) for \( n \geq 3 \). Thus the three ideals have the same Hilbert Polynomial, namely \( h(n) = n + 1 \), but different Hilbert Series. \( \square \)
EXERCISE (20.6). — Let $R = \bigoplus R_n$ be a graded ring, $M = \bigoplus M_n$ a graded $R$-module. Let $N = \bigoplus N_n$ be a homogeneous submodule; that is, $N_n = N \cap M_n$. Assume $R_0$ is Artinian, $R$ is a finitely generated $R_0$-algebra, and $M$ is a finitely generated $R$-module. Set

$$N' := \left\{ m \in M \mid \text{there is } k_0 \text{ such that } R_k m \subset N \text{ for all } k \geq k_0 \right\}.$$ 

(1) Prove that $N'$ is a homogeneous submodule of $M$ with the same Hilbert Polynomial as $N$, and that $N'$ is the largest such submodule containing $N$.

(2) Let $N = \bigcap Q_i$ be a decomposition with $Q_i$ $p_i$-primary. Set $R_+ := \bigoplus_{n > 0} R_n$. Prove that $N' = \bigcap_{p_i \not\supset R_+} Q_i$.

SOLUTION: Given $m = \sum m_i \in N'$, say $R_k m \subset N$. Then $R_k m_i \subset N$ since $N$ is homogeneous. Hence $m_i \in N'$. Thus $N'$ is homogeneous.

By (19.11) and (16.2), $R$ is Noetherian. So $N'$ is finitely generated by (15.15). Let $n_1, \ldots, n_m$ be homogeneous generators of $N'$ with $n_i \in N_{k_i}$; set $k' := \max\{k_i\}$. There is $k$ such that $R_k n_i \subset N$ for all $i$. Given $\ell \geq k + k'$, take $n \in N'_\ell$, and write $n = \sum y_i n_i$ with $y_i \in R_{\ell-k_i}$. Then $y_i n_i \in N_{\ell}$ for all $i$. So $n \in N_{\ell}$. Thus $N'_\ell = N_{\ell}$ for all $\ell \geq k + k'$. Thus $N$ and $N'$ have the same Hilbert polynomial.

Say $N'' \subset N$, and both have the same Hilbert Polynomial. Then there is $k_0$ with $\ell(N''_k) = \ell(N_k)$ for all $k \geq k_0$. So $N''_k = N_k$ for all $k \geq k_0$. So, if $n \in N''$, then $R_k n \subset N$ for all $k \geq k_0$. Thus $N'' \subset N'$. Thus (1) holds.

To prove (2), note $0 = \bigcap (Q_i/N)$ in $M/N$. By (15.22),

$$\Gamma_{R_+}(M/N) = \bigcap_{p_i \not\supset R_+} (Q_i/N).$$

But clearly $\Gamma_{R_+}(M/N) = N'/N$. Thus $N' = \bigcap_{p_i \not\supset R_+} Q_i$. \qed

EXERCISE (20.9). — Let $k$ be a field, $P := k[X, Y, Z]$ the polynomial ring in three variables, $f \in P$ a homogeneous polynomial of degree $d \geq 1$. Set $R := P/(f)$. Find the coefficients of the Hilbert Polynomial $h(R, n)$ explicitly in terms of $d$.

SOLUTION: Clearly, the following sequence is exact:

$$0 \to P(-d) \xrightarrow{\mu} P \to R \to 0.$$ 

Hence, Additivity of Length. (14.8), yields $h(R, n) = h(P, n) - h(P(-d), n)$. But $P(-d)_n = P(n-d)$, so $h(P(-d), n) = h(P, n-d)$. Therefore, (20.3) yields

$$h(R, n) = \binom{d+n}{2} - \binom{d+n}{d} = dn - (d-3)d/2.$$ \qed

EXERCISE (20.10). — Under the conditions of (20.3), assume there is a homogeneous nonzerodivisor $f \in R$ with $M_f = 0$. Prove $\deg h(R, n) > \deg h(M, n)$; start with the case $M := R/(f^k)$.

SOLUTION: Suppose $M := R/(f^k)$. Set $c := k \deg f$. Form the exact sequence

$$0 \to R(-c) \xrightarrow{\mu} R \to M \to 0$$

where $\mu$ is multiplication by $f^k$. Then Additivity of Length (15.8) yields $h(M, n) = h(R, n) - h(R, n-c)$. But

$$h(R, n) = \binom{e(1)}{(d-1)!} n^{d-1} + \cdots \quad \text{and} \quad h(R, n-c) = \binom{e(1)}{(d-1)!} (n-c)^{d-1} + \cdots$$

by (20.3). Thus $\deg h(R, n) > \deg h(M, n)$.

In the general case, there is $k$ with $f^k M = 0$ by (14.4). Set $M' := R/(f^k)$. Then generators $m_i \in M_{c_i}$ for $1 \leq i \leq r$ yield a surjection $\bigoplus_i M'(-c_i) \to M$. Hence $\sum_i \ell(M'_{n-c_i}) \geq \ell(M_n)$ for all $n$. But $\deg h(M'(c_i), n) = \deg h(M', n)$.
Hence $\deg h(M', n) \geq \deg h(M, n)$. But $\deg h(R, n) > \deg h(M', n)$ by the first case. Thus $\deg h(R, n) > \deg h(M, n)$.

**Exercise (20.15).** — Let $R$ be a Noetherian ring, $q$ an ideal, and $M$ a finitely generated module. Assume $\ell(M/qM) < \infty$. Set $m := \sqrt{q}$. Show

$$\deg p_m(M, n) = \deg p_q(M, n).$$

**Solution:** There is an $m$ such that $m \supset q \supset m^m$ by (4.24). Hence

$$m^m M \supset q^m M \supset m^{mn} M$$

for all $n \geq 0$. Dividing into $M$ and extracting lengths yields

$$\ell(M/m^m M) \leq \ell(M/q^m M) \leq \ell(M/m^{mn} M).$$

Therefore, for large $n$, we get

$$p_m(M, n) \leq p_q(M, n) \leq p_m(M, nm).$$

The two extremes are polynomials in $n$ with the same degree, say $d$, (but not the same leading coefficient). Dividing by $n^d$ and letting $n \to \infty$, we conclude that the polynomial $p_q(M, n)$ also has degree $d$.

**Exercise (20.19).** — Derive the Krull Intersection Theorem, (18.29), from the Artin–Rees Lemma, (20.18).

**Solution:** In the notation of (18.29), we must prove that $N = aN$. So apply the Artin–Rees Lemma to $N$ and the $a$-adic filtration of $M$; we get an $m$ such that $a(N \cap a^n M) = N \cap a^{n+1} M$. But $N \cap a^n M = N$ for all $n \geq 0$. Thus $N = aN$.

20. Appendix: Homogeneity

**Exercise (20.22).** — Let $R = \bigoplus R_n$ be a graded ring, $M = \bigoplus_{n \geq n_0} M_n$ a graded module, $a \subset \bigoplus_{n > 0} R_n$ a homogeneous ideal. Assume $M = aM$. Show $M = 0$.

**Solution:** Suppose $M \neq 0$; say $M_{n_0} \neq 0$. Note $M = aM \subset \bigoplus M_n$; hence $M_{n_0} = 0$, a contradiction. Thus $M = 0$.

**Exercise (20.23).** — Let $R = \bigoplus R_n$ be a Noetherian graded ring, $M = \bigoplus M_n$ a finitely generated graded $R$-module, $N = \bigoplus N_n$ a homogeneous submodule. Set

$$N' := \{ m \in M \mid R_n m \in N \text{ for all } n \gg 0 \}.$$

Show that $N'$ is the largest homogeneous submodule of $M$ containing $N$ and having, for all $n \gg 0$, its degree-$n$ homogeneous component $N'_n$ equal to $N_n$.

**Solution:** Given $m, m' \in N'$, say $R_n m, R_n m' \in N$ for $n \gg 0$. Let $x \in R$. Then $R_n (m + m')$, $R_n x m \in N$ for $n \gg 0$. So $N' \subset M$ is a submodule. Trivially $N \subset N'$. Let $m_i$ be a homogeneous component of $m$. Then $R_n m_i \in N$ for $n \gg 0$ as $N$ is homogeneous. Thus $N' \subset M$ is a homogeneous submodule containing $N$.

Since $R$ is Noetherian and $M$ is finitely generated, $N'$ is finitely generated, say by $g, g', \ldots, g^{(r)}$. Then there is $n_0$ with $R_n g, R_n g', \ldots, R_n g^{(r)} \in N$ for $n \geq n_0$. Replace $g, g', \ldots, g^{(r)}$ by their homogeneous components. Say $g, g', \ldots, g^{(r)}$ are now of degrees $d, d', \ldots, d^{(r)}$ with $d \geq d' \geq \cdots \geq d^{(r)}$. Set $n_1 := d + n_0$.

Given $m \in N'_n$ with $n \geq n_1$, say $m = xg + x'g' + \cdots$ with $x \in R_{n-d}$ and $x' \in R_{n-d'}$ and so on. Then $n_0 \leq n - d \leq n - d' \leq \cdots$. Hence $m \in N_n$. Thus
Let $R$ be a graded ring, $\mathfrak{a}$ a homogeneous ideal, and $M$ a graded module. Prove that $\sqrt{\mathfrak{a}}$ and $\text{Ann}(M)$ and $\text{nil}(M)$ are homogeneous.

**Solution:** Take $x = \sum_{i \geq r} x_i \in R$ with the $x_i$ the homogeneous components.

First, suppose $x \in \sqrt{\mathfrak{a}}$. Either $x^k$ vanishes or it is the initial component of $x^k$. But $\mathfrak{a}$ is homogeneous. So $x^k \in \mathfrak{a}$. So $x - x_i \in \sqrt{\mathfrak{a}}$ by (13.4.1). So all the $x_i$ are in $\sqrt{\mathfrak{a}}$ by induction on $n$. Thus $\sqrt{\mathfrak{a}}$ is homogeneous.

Second, suppose $x \in \text{Ann}(M)$. Let $m \in M$. Then $0 = xm = \sum x_i m$. If $m$ is homogeneous, then $x_i m = 0$ for all $i$, since $M$ is graded. But $M$ has a set of homogeneous generators. Thus $x_i \in \text{Ann}(M)$ for all $i$, as desired.

Finally, $\text{nil}(M)$ is homogeneous, as $\text{nil}(M) = \sqrt{\text{Ann}(M)}$ by (13.2.4).

**Exercise (20.26).** --- Let $R$ be a Noetherian graded ring, $M$ a finitely generated graded module, $Q$ a submodule. Let $Q^* \subset Q$ be the submodule generated by the homogeneous elements of $Q$. Assume $Q$ is primary. Then $Q^*$ is primary too.

**Solution:** Let $x \in R$ and $m \in M$ be homogeneous with $xm \in Q^*$. Assume $x \notin \text{nil}(M/Q^*)$. Then, given $\ell \geq 1$, there is $m' \in M$ with $x^\ell m' \notin Q^*$. So $m'$ has a homogeneous component $m''$ with $x^\ell m'' \notin Q^*$. Then $x^\ell m'' \notin Q$ by definition of $Q^*$. Thus $x \notin \text{nil}(M/Q)$. Since $Q$ is primary, $m \in Q$ by (11.5.3). Since $m$ is homogeneous, $m \in Q^*$. Thus $Q^*$ is primary by (21.1.2).

**Exercise (20.30).** --- Under the conditions of (24.1.2), assume that $R$ is a domain and that its integral closure $\overline{R}$ in $\text{Frac}(R)$ is a finitely generated $R$-module.

1. Prove that there is a homogeneous $f \in R$ with $R_f = \overline{R}_f$.
2. Prove that the Hilbert Polynomials of $R$ and $\overline{R}$ have the same degree and same leading coefficient.

**Solution:** Let $x_1, \ldots, x_r$ be homogeneous generators of $\overline{R}$ as an $R$-module. Write $x_i = a_i/b_i$ with $a_i, b_i \in R$ homogeneous. Set $f := \prod b_i$. Then $fx_i \in R$ for each $i$. So $\overline{R_f} = \overline{R}_f$. Thus (1) holds.

Consider the short exact sequence $0 \to R \to \overline{R} \to \overline{R}/R \to 0$. Then $(\overline{R}/R)_f = 0$ by (24.1.10). So $\deg h(\overline{R}/R, n) < \deg h(\overline{R}, n)$ by (24.4.11) and (1). But $h(\overline{R}, n) = h(R, n) + h(\overline{R}/R, n)$ by (11.3.9) and (24.1.2). Thus (2) holds.

21. Dimension

**Exercise (21.6).** --- Let $A$ be a Noetherian local ring, $N$ a finitely generated module, $y_1, \ldots, y_t$ a sop for $N$. Set $N_i := N/\langle y_1, \ldots, y_i \rangle$. Show $\dim(N_i) = r - i$.

**Solution:** First, $\dim(N) = r$ by (21.2). Then $\dim(N_i) \geq \dim(N_{i-1}) - 1$ for all $i$ by (19.1.28), and $\dim(N_r) = 0$ by (19.1.28). So $\dim(N_i) = r - i$ for all $i$.

**Exercise (21.9).** --- Let $R$ be a Noetherian ring, and $\mathfrak{p}$ be a prime minimal containing $x_1, \ldots, x_r$. Given $r'$ with $1 \leq r' \leq r$, set $R' := R/\langle x_1, \ldots, x_r \rangle$ and $\mathfrak{p}' := \mathfrak{p}/\langle x_1, \ldots, x_r \rangle$. Assume $\text{ht}(\mathfrak{p}) = r$. Prove $\text{ht}(\mathfrak{p}') = r - r'$.
SOLUTION: Let $x'_i \in R'$ be the residue of $x_i$. Then $p'$ is minimal containing $x'_{i+1}, \ldots, x'_i$ by (11.11) and (21.7). So $\text{ht}(p') \leq r - r'$ by (21.8).

On the other hand, $R'_p = R'_p$ by (11.23), and $R'_p = R_p/(x_1, \ldots, x_{r-1})$ by (17.22). Hence $\dim(R'_p) \geq \dim(R_p) - r'$ by repeated application of (21.5) with $R_p$ for both $R$ and $M$. So $\text{ht}(p') \geq r - r'$ by (21.4), as required.

EXERCISE (21.11). — Let $R$ be a Noetherian ring, $p$ a prime of height at least 2. Prove that $p$ is the union of height-1 primes, but not of finitely many.

SOLUTION: If $p$ were the union of finitely many height-1 primes, then by Prime Avoidance (21.10), one would be equal to $p$, a contradiction.

To prove $p$ is the union of height-1 primes, we may replace $R$ by $R/q$ where $q \subset p$ is a minimal prime, as preimage commutes with union. Thus we may assume $R$ is a domain. Given a nonzero $x \in p$, let $q_x \subset p$ be a minimal prime of $\langle x \rangle$. Then $\text{ht}(q_x) = 1$ by the Krull Principal Theorem (21.10). Plainly $\bigcup q_x = p$. □

EXERCISE (21.12). — Let $R$ be a Noetherian ring. Prove the following equivalent:

1. $R$ has only finitely many primes.
2. $R$ has only finitely many height-1 primes.
3. $R$ is semilocal of dimension 1.

SOLUTION: Trivially, (1) implies (2).

Assume (2). By (21.11), there’s no prime of height at least 2. Thus $\dim(R) \leq 1$. So every prime is either of height 1 or of height 0. But the height-0 primes are minimal, so finite in number by (11.22). Hence $R$ is semilocal. Thus (3) holds.

Finally, assume (3). Again, every prime is either of height 1 or of height 0, and the height-0 primes are finite in number. But the height-1 primes are maximal, so finite in number. Thus (1) holds. □

EXERCISE (21.13) (Artin–Tate [10, Thm. 4]). — Let $R$ be a Noetherian domain, and set $K := \text{Frac}(R)$. Prove the following equivalent:

1. $K = R_f$ for some nonzero $f \in R$.
2. $K$ is algebra finite over $R$.
3. Some nonzero $f \in R$ lies in every nonzero prime.
4. $R$ has only finitely many height-1 primes.
5. $R$ is semilocal of dimension 1.

SOLUTION: By (11.13), (1) implies (2).

Assume (2), and say $K = R[x_1, \ldots, x_n]$. Let $f$ be a common denominator of the $x_i$. Then given any $y \in K$, clearly $f^m y \in R$ for some $m \geq 1$.

Let $p \subset R$ be a nonzero prime. Take a nonzero $z \in p$. By the above, $f^m(1/z) \in R$ for some $m \geq 1$. So $f^m(1/z)z \in p$. So $f \in p$. Thus (2) implies (3).

Assume (3). Given $0 \neq y \in R$, the Scheinnullstellensatz (5.24) yields $f \in \sqrt{\langle y \rangle}$. So $f^m = xy$ for some $n \geq 1$ and $x \in R$. So $1/y = x/f^m$. Thus (3) implies (1).

Again assume (3). Let $p$ be a height-1 prime. Then $f \in p$. So $p$ is minimal containing $(f)$. So $p$ is one of finitely many primes by (11.22). Thus (4) holds.

Conversely, assume (4). Take a nonzero element in each height-1 prime, and let $f$ be their product. Then $f$ lies in every height-1 prime. But every nonzero prime contains a height-1 prime owing to the Dimension Theorem (21.7). Thus (3) holds.

Finally, (4) and (5) are equivalent by (21.12). □
Exercise (21.14). — Let $R$ be a domain. Prove that, if $R$ is a UFD, then every height-1 prime is principal, and that the converse holds if $R$ is Noetherian.

Solution: Let $p$ be a height-1 prime. Then there’s a nonzero $x \in p$. Factor $x$. One prime factor $q$ must lie in $p$ as $p$ is prime. Then $(p)$ is a prime ideal as $p$ is a prime element by (2.10). But $(p) \subset p$ and $ht(p) = 1$. Thus, $(p) = p$.

Conversely, assume every height-1 prime is principal and assume $R$ is Noetherian. To prove $R$ is a UFD, it suffices to prove every irreducible element $p$ is prime (see [2, Ch. 11, Sec. 2, pp. 392–396]). Let $p$ be a prime minimal containing $p$. By Krull’s Principal Ideal Theorem, $ht(p) = 1$. So $p = (x)$ for some $x$. Then $x$ is prime by (2.6). And $p = xy$ for some $y$ as $p \in p$. But $p$ is irreducible. So $y$ is a unit. Thus $p$ is prime, as desired.

Exercise (21.15). — (1) Let $A$ be a Noetherian local ring with a principal prime $p$ of height at least 1. Prove $A$ is a domain by showing any prime $q \unlhd p$ is $(0)$.

(2) Let $k$ be a field, $P := k[[X]]$ the formal power series ring in one variable. Set $R := P \times P$. Prove that $R$ is Noetherian and semi-local, and that $R$ contains a principal prime $p$ of height 1, but that $R$ is not a domain.

Solution: To prove (1), say $p = (x)$. Take $y \in q$. Then $y = ax$ for some $a$. But $x \notin q$ since $q \unlhd p$. Hence $a \in q$. Thus $q = qx$. But $x$ lies in the maximal ideal of the local ring $A$, and $q$ is finitely generated since $A$ is Noetherian. Hence Nakayama’s Lemma (10.14) yields $q = (0)$. Thus $(0)$ is prime, and so $A$ is a domain.

Alternatively, as $a \in q$, also $a = a_1x$ with $a_1 \in q$. Repeating yields an ascending chain of ideals $(a) \subset (a_1) \subset (a_2) \subset \cdots$. It stabilizes as $A$ is Noetherian: there’s a $k$ such that $a_k \in (a_{k-1})$. Then $a_k = ba_{k-1} = ba_kx$ for some $b$. So $a_k(1 - bx) = 0$. But $1 - bx$ is a unit by (4.2) as $A$ is local. So $a_k = 0$. Thus $y = 0$, so $A$ is a domain.

As to (2), every nonzero ideal of $P$ is of the form $(X^n)$ by (6.11). Hence $P$ is Noetherian. Thus $R$ is Noetherian by (15.17).

The primes of $R$ are of the form $q \times P$ or $P \times q$ where $q$ is a prime of $P$ by (2.11). Further, $m := (X)$ is the unique maximal ideal by (5.11). Hence $R$ has just two maximal ideals $m \times P$ and $P \times m$. Thus $R$ is semi-local.

Set $p := ((X, 1))$. Then $p = m \times P$. So $p$ is a principal prime. Further, $p$ contains just one other prime $0 \times P$. Thus $ht(p) = 1$.

Finally, $R$ is not a domain as $1, 0) \cdot (0, 1) = 0$.

Exercise (21.16). — Let $R$ be a finitely generated algebra over a field. Assume $R$ is a domain of dimension $r$. Let $x \in R$ be neither 0 nor a unit. Set $R' := R/(x)$. Prove that $r - 1$ is the length of any chain of primes in $R'$ of maximal length.

Solution: A chain of primes in $R'$ of maximal length lifts to a chain of primes $p_i$ in $R$ of maximal length with $(x) \subset p_1 \subset \cdots \subset p_d$. As $x$ is not a unit, $d \geq 1$. As $x \neq 0$, also $p_1 \neq 0$. But $R$ is a domain. So Krull’s Principal Ideal Theorem, (21.31), yields $ht(p_1) = 1$. So $0 \subset p_1 \subset \cdots \subset p_d$ is of maximal length in $R$. But $R$ is a finitely generated algebra over a field. Hence $d = dim R$ by (15.2).

Exercise (21.18). — Let $R$ be a Noetherian ring. Show that

$$\dim(R[[X]]) = \dim(R) + 1.$$
\section*{22. Completion}

\textbf{Exercise (22.3).} — In the 2-adic integers, evaluate the sum $1 + 2 + 4 + 8 + \cdots$.  

\textbf{Solution:} In the 2-adic integers, $1 + 2 + 4 + 8 + \cdots = 1/(1 - 2) = -1$.  

\textbf{Exercise (22.4).} — Let $R$ be a ring, $\mathfrak{a}$ an ideal, and $M$ a module. Prove that the following three conditions are equivalent:

(1) $\kappa: M \to \widehat{M}$ is injective;  
(2) $\bigcap a^n M = \{0\}$;  
(3) $M$ is separated.  

Assume $R$ is Noetherian and $M$ finitely generated. Assume either (a) $\mathfrak{a} \subseteq \text{rad}(R)$ or (b) $R$ is a domain, $\mathfrak{a}$ is proper, and $M$ is torsionfree. Conclude $M \subseteq \widehat{M}$.  

\textbf{Solution:} A constant sequence $(m)$ has 0 as a limit if and only if $m \in a^n M$ for every $n$. So $\text{Ker}(\kappa) = \bigcap a^n M$. Thus (1) and (2) are equivalent. Moreover, (2) and (3) were proved equivalent in (22.31).  

Set $N := \bigcap a^n M$. Assume $R$ is Noetherian and $M$ finitely generated. By the Krull Intersection Theorem, (15.24) or (20.40), there’s $x \in \mathfrak{a}$ with $(1 + x)N = \{0\}$.  

Assume (a). Then $1 + x$ is a unit by (6.19). Thus (2) holds and (1) follows.  

Finally, assume (b). Then $1 + x \neq 0$ as $\mathfrak{a}$ is proper. Let $m \in M$. If $(1 + x)m = 0$, then $m = 0$ as $M$ is torsionfree. Thus again (2) holds and (1) follows.  

Exercise (22.6). — Let $R$ be a ring. Given $R$-modules $Q_n$ equipped with linear maps $a_n: Q_{n+1} \to Q_n$ for $n \geq 0$, set $m := a_n^{n+1} \cdots a_m$ for $m > n$. We say the $Q_n$ satisfy the \textit{Mittag-Leffler Condition} if the descending chain

$$Q_n \supset a_n^{n+1}Q_{n+1} \supset a_n^{n+2}Q_{n+2} \supset \cdots \supset a_mQ_m \supset \cdots$$

stabilizes; that is, $a_n^mQ_m = a_n^{m+k}Q_{m+k}$ for all $k > 0$.

1. Assume for each $n$, there is $m > n$ with $a_n^n = 0$. Show $\varprojlim Q_n = 0$.

2. Assume $a_n^{n+1}$ is surjective for all $n$. Show $\varprojlim Q_n = 0$.

3. Assume the $Q_n$ satisfy the Mittag-Leffler Condition. Set $P_n := \bigcap_{m \geq n} a_m^nQ_m$, which is the stable submodule. Show $a_n^{n+1}P_{n+1} = P_n$.

4. Assume the $Q_n$ satisfy the Mittag-Leffler Condition. Show $\varprojlim Q_n = 0$.

Solution: For (1), given $(q_n) \in \prod Q_n$, for each $k \geq n$, set $q_k' := a_k^kq_k$ and $p_n := q_n + q_n' + \cdots + q_m$. Then $\theta(np_n) = p_n - a_n^{n+1}p_{n+1} = q_n$ as $a_n^{m+k} = 0$ for all $k \geq 0$ owing to the hypothesis. So $\theta$ is surjective. Thus (1) holds.

For (2), given $(q_n) \in \prod Q_n$, solve the equations $p_n - a_n^{n+1}(p_{n+1}) = q_n$ recursively, starting with $p_0 = 0$, to get $(p_n) \in \prod Q_n$ with $\theta(p_n) = q_n$. Thus (2) holds.

For (3), there is $m > n + 1$ such that $P_n = a_n^mQ_m$ and $P_{n+1} = a_{n+1}^mQ_{n+1}$. But $a_n^mQ_m = a_n^{n+1}a_{n+1}Q_m$ by definition of $a_n^n$. Thus (3) holds.

For (4), form the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & \to & \prod P_n \to \prod Q_n \to \prod(Q_n/P_n) \to 0 \\
\varprojlim & \downarrow & \varprojlim & \downarrow & \varprojlim \\
0 & \to & \prod P_n \to \prod Q_n \to \prod(Q_n/P_n) \to 0
\end{array}
\]

Apply the Snake Lemma \((6.13)\). It yields the following exact sequence of cokernels:

$$\lim_{\leftarrow} P_n \to \lim_{\leftarrow} Q_n \to \lim_{\leftarrow}(Q_n/P_n).$$

For each $n$, the restriction $a_n^{n+1}|P_n$ is surjective by (3). So $\lim_{\leftarrow} P_n = 0$ by (1). Further, for each $n$, there is $m > n$ such that $a_n^mQ_m = P_n$. So the induced map $(Q_m/P_m) \to (Q_n/P_n)$ is 0. So $\lim_{\leftarrow}(Q_n/P_n) = 0$ by (1). Thus (4) holds.

Exercise (22.10). — Let $A$ be a ring, and $m_1, \ldots, m_n$ be maximal ideals. Set $m := \bigcap m_i$, and give $A$ the $m$-adic topology. Prove that $\hat{A} = \prod \hat{A}_m$.

Solution: For each $n > 0$, the $m^n$ are pairwise comaximal by \((1.23)(3)\). Hence $m^n = \prod_{i=1}^m m^n_i$ by \((1.23)(4b)\), and so $A/m^n = \prod_{i=1}^m A/m^n_i$ by \((1.23)(4c)\). But $A/m^n_i$ is local with maximal ideal $m_i/m^n_i$. So $(A/m^n_i)m_i = A/m^n_i$ by \((1.23)(4d)\). Further, $(A/m^n_i)m_i = A/m^n_iA_m$ by \((2.22)\). So $A/m^n = \prod_{i=1}^m (A/m^n_iA_m)$. Taking inverse limits, we obtain the assertion by \((2.28)\), because inverse limit commutes with finite product by the construction of the limit.

Exercise (22.11). — Let $R$ be a ring, $M$ a module, $F^*M$ a filtration, and $N \subseteq M$ a submodule. Give $N$ and $M/N$ the induced filtrations:

$$F^nN := N \cap F^nM \quad \text{and} \quad F^n(M/N) := F^nM/F^nN.$$

1. Prove $\hat{N} \subset \hat{M}$ and $\hat{M}/\hat{N} = (\hat{M}/\hat{N})$.

2. Also assume $N \supset F^nM$ for $n \gg 0$. Prove $\hat{M} = \hat{N} = M/N$ and $G^*\hat{M} = G^*M$. 
Solution: For (1), set $P := M/N$. Form the following commutative diagram:

$$
\begin{array}{ccc}
0 & \rightarrow & N/F^{n+1}N \rightarrow M/F^{n+1}M \rightarrow P/F^{n+1}P \rightarrow 0 \\
\kappa_n \downarrow & & \downarrow \\
0 & \rightarrow & N/F^nN \rightarrow M/F^nM \rightarrow P/F^nP \rightarrow 0
\end{array}
$$

Its rows are exact, and $\kappa_n$ is surjective. So the induced sequence

$$0 \rightarrow \hat{N} \rightarrow \hat{M} \rightarrow \hat{P} \rightarrow 0$$

is exact by (22.11) and (22.13). Thus (1) holds.

For (2), notice $F^nP = 0$ for $n \gg 0$. So plainly $P = \hat{P}$. Thus $\hat{M}/\hat{N} = M/N$.

In particular, fix $n$ and take $N := F^nM$. Then $\hat{M}/\hat{N}M = M/F^nM$. But $n$ is arbitrary. Hence $F^n\hat{M}/F^n\hat{N}M = F^nM/F^nM$. Thus $G^*\hat{M} = G^*M$. □

Exercise (22.12). — (1) Let $R$ be a ring, $a$ an ideal. If $G^*R$ is a domain, show $\hat{R}$ is an domain. If also $\bigcap_{n \geq 0} a^n = 0$, show $\hat{R}$ is a domain.

(2) Use (1) to give an alternative proof that a regular local ring $A$ is a domain.

Solution: Consider (1). Let $x, y \in \hat{R}$ be nonzero. Since $\hat{R}$ is separated there are positive integers $r$ and $s$ with $x \in \hat{a}^r - \hat{a}^{r+1}$ and $y \in \hat{a}^s - \hat{a}^{s+1}$. Let $x' \in G^*R$ and $y' \in G^*\hat{R}$ denote the images of $x$ and $y$. Then $x' \neq 0$ and $y' \neq 0$. Now, $G^*\hat{R} = G^*R$ by (22.11). Assume $G^*R$ is a domain. Then $x'y' \neq 0$. Hence $x'y' \in G^*\hat{R}$ is the image of $xy \in \hat{a}^{r+s}$. Hence $xy \neq 0$. Thus $\hat{R}$ is a domain.

If $\bigcap_{n \geq 0} a^n = 0$, then $R \subset \hat{R}$ by (22.13); so $\hat{R}$ is a domain if $\hat{R}$ is. Thus (1) holds.

As to (2), denote the maximal ideal of $A$ by $m$. Then $\bigcap_{n \geq 0} m^n = 0$ by the Krull Intersection Theorem (15.17), and $G^*A$ is a polynomial ring by (22.12), so a domain. Hence $A$ is a domain, by (1). Thus (2) holds. □

Exercise (22.14). — Let $A$ be a Noetherian local ring, $m$ the maximal ideal, $M$ a finitely generated module. Prove (1) that $\hat{A}$ is a Noetherian local ring with $\hat{m}$ as maximal ideal, (2) that $\dim(M) = \dim(\hat{M})$, and (3) that $A$ is regular if and only if $\hat{A}$ is regular.

Solution: First, $\hat{A}$ is Noetherian by (22.31), and it’s local with $\hat{m}$ as maximal ideal by (22.13). Thus (1) holds.

Second, $M/m^\infty M = \hat{M}/\hat{m}^\infty \hat{M}$ by (22.11) and (22.22). So $d(M) = d(\hat{M})$ by (22.14). Thus (2) holds by (22.3).

Third, $m/m^2 = \hat{m}/\hat{m}^2$ by (22.11). Hence $m$ and $\hat{m}$ have generating sets with the same number of elements by (11.11). Thus (3) holds. □

Exercise (22.15). — Let $A$ be a ring, and $m_1, \ldots, m_m$ maximal ideals. Set $m := \bigcap m_i$ and give $A$ the $m$-adic topology. Prove that $\hat{A}$ is a semilocal ring, that $\hat{m}_1, \ldots, \hat{m}_m$ are all its maximal ideals, and that $\hat{m} = \operatorname{rad}(\hat{A})$.

Solution: First, (22.11) yields $\hat{A}/\hat{m} = A/m$ and $\hat{A}/\hat{m}_i = A/m_i$. So $\hat{m}_i$ is maximal. By hypothesis, $m = \bigcap m_i$; so $A/m \subset \bigcap (A/m_i)$. Hence $\hat{A}/\hat{m} \subset \bigcap (\hat{A}/\hat{m}_i)$; so $\hat{m} = \bigcap \hat{m}_i$. So $\hat{m} \supset \operatorname{rad}(\hat{A})$. But $\hat{m} \subset \operatorname{rad}(\hat{A})$ by (22.24). Thus $\hat{m} = \operatorname{rad}(\hat{A})$.

Finally, let $m'$ be any maximal ideal of $\hat{A}$. Then $m' \supset \operatorname{rad}(\hat{A}) = \bigcap \hat{m}_i$. Hence $m' \supset \hat{m}_i$ for some $i$ by (22.24). But $\hat{m}_i$ is maximal. So $m' = \hat{m}_i$. Thus $\hat{m}_1, \ldots, \hat{m}_m$ are all the maximal ideals of $\hat{A}$, and so $\hat{A}$ is semilocal. □
Exercise (22.18). — Let $A$ be a Noetherian semilocal ring. Prove that an element $x \in A$ is a nonzerodivisor on $A$ if and only if its image $\tilde{x} \in \hat{A}$ is one on $\hat{A}$.

Solution: Assume $x$ is a nonzerodivisor. Then the multiplication map $\mu_x$ is injective on $A$. So by Exactness of Completion (22.17), the induced map $\tilde{\mu}_x$ is injective on $\hat{A}$. But $\tilde{\mu}_x = \mu_{\tilde{x}}$. Thus $\tilde{x}$ is a nonzerodivisor.

Conversely, assume $\tilde{x}$ is a nonzerodivisor and $A$ is semilocal. Then $\tilde{\mu}_x$ is injective on $\hat{A}$. So its restriction is injective on the image of the canonical map $A \to \hat{A}$. But this map is injective by (22.3), as the completion is taken with respect to the Jacobson radical; further, $\tilde{\mu}_x$ induces $\mu_x$. Thus $x$ is a nonzerodivisor.

Exercise (22.19). — Let $p \in \mathbb{Z}$ be prime. For $n > 0$, define a $\mathbb{Z}$-linear map

$$\alpha_n: \mathbb{Z}/\langle p \rangle \to \mathbb{Z}/\langle p^n \rangle \quad \text{by} \quad \alpha_n(1) = p^{n-1}.$$ Set $A := \bigoplus_{n \geq 1} \mathbb{Z}/\langle p \rangle$ and $B := \bigoplus_{n \geq 1} \mathbb{Z}/\langle p^n \rangle$. Set $\alpha := \bigoplus \alpha_n$; so $\alpha: A \to B$.

(1) Show that the $p$-adic completion $\hat{A}$ is just $A$.

(2) Show that, in the topology on $A$ induced by the $p$-adic topology on $B$, the completion $\hat{A}$ is equal to $\prod_{n=1}^{\infty} \mathbb{Z}/\langle p \rangle$.

(3) Show that the natural sequence of $p$-adic completions

$$\hat{A} \xrightarrow{\alpha} \hat{B} \xrightarrow{\kappa} (B/A)^\wedge$$

is not exact at $\hat{B}$. (Thus $p$-adic completion is neither left exact nor right exact.)

Solution: For (1), note $pA = 0$. So every Cauchy sequence is constant. Hence $\hat{A} = A$. Thus (1) holds.

For (2), set $A_k := \alpha^{-1}(p^k B)$. These $A_k$ are the fundamental open neighborhoods of 0 in the topology induced from the $p$-adic topology of $B$. So

$$A_k = \alpha^{-1}(0 \oplus \cdots \oplus 0 \oplus \bigoplus_{n > k} \langle p^k \rangle / \langle p^n \rangle) = (0 \oplus \cdots \oplus 0 \oplus \bigoplus_{n > k} \mathbb{Z}/\langle p \rangle).$$

Hence $A/A_k = \bigoplus_{n=1}^{k} \mathbb{Z}/\langle p \rangle = \prod_{n=1}^{k} \mathbb{Z}/\langle p \rangle$. But by (22.3), in the induced topology, the completion $\hat{A}$ is equal to $\lim_{\leftarrow k} A/A_k$. Thus

$$\hat{A} = \lim_{\leftarrow k} \prod_{n=1}^{k} \mathbb{Z}/\langle p \rangle.$$ Given any sequence of modules $M_1, M_2, \ldots$, let $\pi_{k+1}^k: \prod_{n=1}^{k+1} M_n \to \prod_{n=1}^{k} M_n$ be the projections. Then (22.5) yields $\lim_{\leftarrow k} \prod_{n=1}^{k} M_n = \prod_{n=1}^{\infty} M_n$. Thus (2) holds.

For (3), note that, by (2) and (22.4), the following sequence is exact:

$$0 \to \hat{A} \xrightarrow{\alpha} \hat{B} \xrightarrow{\kappa} (B/A)^\wedge.$$

But $\hat{A} = A$ by (1), and $A \neq \hat{A}$ as $A$ is countable yet $\hat{A}$ is not. Thus $\text{Im}(\alpha) \neq \text{Ker}(\kappa)$; that is, (3) holds.

Exercise (22.21). — Let $R$ be a ring, $a$ an ideal. Show that $M \mapsto \hat{M}$ preserves surjections, and that $\hat{R} \otimes M \to \hat{M}$ is surjective if $M$ is finitely generated.

Solution: The first part of the proof of (22.14) shows that $M \mapsto \hat{M}$ preserves surjections. So (22.13) yields the desired surjectivity.

Exercise (22.24). — Let $R$ be a Noetherian ring, $a$ an ideal. Prove that $\hat{R}$ is faithfully flat if and only if $a \subset \text{rad}(R)$.
SOLUTION: First, \( \hat{R} \) is flat over \( R \) by (22.25.6). Next, let \( m \) be a maximal ideal of \( R \). Then \( \hat{R} \otimes_{R} R/m = (R/m) \) by (22.25.6). But \( (R/m) = \lim \frac{R}{a^r} \) by (22.25.6). Plainly \( \frac{R}{a^r} = R/(a^r + m) \). Hence \( \hat{R} \otimes_{R} (R/m) \neq 0 \) if and only if \( a \subset m \). Thus, the assertion follows from (22.25.6).

Exercise (22.25). — Let \( R \) be a Noetherian ring, and \( a \) and \( b \) ideals. Assume \( a \subset \text{rad}(R) \), and use the \( a \)-adic topology. Prove \( b \) is principal if \( b \hat{R} \) is.

Solution: Since \( R \) is Noetherian, \( b \) is finitely generated. But \( a \subset \text{rad}(R) \), hence \( b \) is principal if \( b/ab \) is a cyclic \( R \)-module by (10.4.12)(2). But \( b/ab \) is a cyclic \( R \)-module, as desired.

Exercise (22.28) (Nakayama’s Lemma for a complete ring). — Let \( R \) be a ring, \( a \) an ideal, and \( M \) a module. Assume \( R \) is complete, and \( M \) separated. Show \( m_1, \ldots, m_n \in M \) generate assuming their images \( m'_1, \ldots, m'_n \) in \( M/aM \) generate.

Solution: Note that \( m'_1, \ldots, m'_n \) generate \( G \cdot M \) over \( G \cdot R \). Thus \( m_1, \ldots, m_n \) generate \( M \) over \( R \) by the proof of (22.27).

Alternatively, \( M \) is finitely generated over \( R \) and complete by the statement of (22.27). As \( M \) is also separated, \( M = \hat{M} \). Hence \( M \) is also an \( \hat{R} \)-module. As \( R \) is complete, \( \kappa_R : R \to \hat{R} \) is surjective. Now, \( a \) is closed by (22.24); so \( \hat{a} \) is complete; whence, \( \kappa_\hat{a} : a \to \hat{a} \) is surjective too. Hence \( aM = \hat{a}M \). Thus \( M/aM = M/\hat{a}M \). So the \( m_i \) generate \( M/\hat{a}M \). But \( \hat{a} \subset \text{rad}(\hat{R}) \) by (22.25). So by Nakayama’s Lemma (10.4.13)(2), the \( m_i \) generate \( M \) over \( \hat{R} \), so also over \( R \) as \( \kappa_R \) is surjective.

23. Discrete Valuation Rings

Exercise (23.6). — Let \( R \) be a ring, \( M \) a module, and \( x, y \in R \).

1. Assume that \( x, y \) form an \( M \)-sequence. Prove that, given any \( m, n \in M \) with \( xm = yn \), there exists \( p \in M \) with \( m = yp \) and \( n = xp \).

2. Assume that \( x, y \) form an \( M \)-sequence and that \( y \notin z \cdot \text{div}(M) \). Prove that \( y, x \) form an \( M \)-sequence too.

3. Assume that \( R \) is local, that \( x, y \) lie in its maximal ideal \( m \), and that \( M \) is nonzero and Noetherian. Assume that, given any \( m, n \in M \) with \( xm = yn \), there exists \( p \in M \) with \( m = yp \) and \( n = xp \). Prove that \( x, y \) and \( x, x \) form \( M \)-sequences.

Solution: Consider (1). Let \( n_1 \) be the residue of \( n \) in \( M_1 := M/xM \). Then \( yn_1 = 0 \), but \( y \notin z \cdot \text{div}(M_1) \). Hence \( n_1 = 0 \). So there exists \( p \in M \) with \( n = xp \). So \( x(m - yp) = 0 \). But \( x \notin z \cdot \text{div}(M) \). Thus \( m = yp \).

Consider (2). First, \( M/(y, x)M \neq 0 \) as \( x, y \) form an \( M \)-sequence. Next, set \( M_1 := M/yM \). We must prove \( x \notin z \cdot \text{div}(M_1) \). Given \( m_1 \in M_1 \) with \( xm_1 = 0 \), lift \( m_1 \) to \( m \in M \). Then \( xm = yn \) for some \( n \in M \). By (1), there is \( p \in M \) with \( m = yp \). Thus \( m_1 = 0 \), as desired.

Consider (3). The statement is symmetric in \( x \) and \( y \). So let’s prove \( x, y \) form an \( M \)-sequence. First, \( M/(x, y)M \neq 0 \) by Nakayama’s Lemma.

Next, we must prove \( x \notin z \cdot \text{div}(M) \). Given \( m \in M \) with \( xm = 0 \), set \( n := 0 \).
Then \(xm = yn\); so there exists \(p \in M\) with \(m = yp\) and \(n = xp\). Repeat with \(p\) in place of \(m\), obtaining \(p_i \in M\) such that \(p = yp_1\) and \(0 = xp_1\). Induction yields \(p_i \in M\) for \(i \geq 2\) such that \(p_{i-1} = yp_i\) and \(0 = xp_i\).

Then \(Rp_1 \subset Rp_2 \subset \cdots\) is an ascending chain. It stabilizes as \(M\) is Noetherian. Say \(Rp_n = Rp_{n+1}\). So \(p_{n+1} = zp_n\) for some \(z \in R\). Then \(p_n = yp_{n+1} = yzp_n\). So \((1-yz)p_n = 0\). But \(y \in m\). So \(1-yz\) is a unit. Hence \(p_n = 0\). But \(m = y^{n+1}p_n\). Thus \(m = 0\), as desired. Thus \(x \notin z\cdot\text{div}(M)\).

Finally, set \(M_1 := M/xM\). We must prove \(y \notin z\cdot\text{div}(M_1)\). Given \(n_1 \in M_1\) with \(yn_1 = 0\), lift \(n_1\) to \(n \in M\). Then \(yn = xm\) for some \(m \in M\). So there exists \(p \in M\) with \(n = xp\). Thus \(n_1 = 0\), as desired. Thus \(x, y\) form an \(M\)-sequence.

Exercise (23.8). — Let \(R\) be a local ring, \(m\) its maximal ideal, \(M\) a Noetherian module, \(x_1, \ldots, x_n \in m\), and \(\sigma\) a permutation of \(1, \ldots, n\). Assume \(x_1, \ldots, x_n\) form an \(M\)-sequence, and prove \(x_{\sigma 1}, \ldots, x_{\sigma n}\) do too; first, say \(\sigma\) transposes \(i\) and \(i + 1\).

Solution: Say \(\sigma\) transposes \(i\) and \(i + 1\). Set \(M_j := M/\langle x_1, \ldots, x_j \rangle\). Then \(x_i, x_{i+1}\) form an \(M_{i-1}\)-sequence; so \(x_{i+1}, x_i\) do too owing to (23.9). So

\[
x_1, \ldots, x_{i-1}, x_{i+1}, x_i
\]

form an \(M\)-sequence. But \(M/\langle x_1, \ldots, x_{i-1}, x_{i+1}, x_i \rangle\) form an \(M\)-sequence. In general, \(\sigma\) is a composition of transpositions of successive integers; hence, the general assertion follows.

Exercise (23.7). — Let \(A\) be a Noetherian local ring, \(M\) and \(N\) nonzero finitely generated modules, \(F: ((R\text{-mod})) \rightarrow ((R\text{-mod}))\) a left-exact functor that preserves the finitely generated modules (such as \(F(\bullet) := \text{Hom}(M, \bullet)\) by (16.20)). Show that, if \(N\) has depth at least 2, then so does \(F(N)\).

Solution: An \(N\)-sequence \(x, y\) yields a commutative diagram with exact rows:

\[
0 \rightarrow N \xrightarrow{\mu_y} N \rightarrow N/xN
\]

Applying the left-exact functor \(F\) yields this commutative diagram with exact rows:

\[
0 \rightarrow F(N) \xrightarrow{\mu_y} F(N) \rightarrow F(N/xN)
\]

Thus \(x\) is a nonzerodivisor on \(F(N)\). Further \(F(N)/xF(N) \hookrightarrow F(N/xN)\).

As \(N/xN \xrightarrow{\mu_y} N/xN\) is injective and \(F\) is left exact, the right-hand vertical map \(\mu_y\) is injective. So its restriction

\[
F(N)/xF(N) \xrightarrow{\mu_y} F(N)/xF(N)
\]

is also injective. Thus \(x, y\) is an \(F(N)\)-sequence.

Exercise (23.9). — Prove that a Noetherian local ring \(A\) of dimension \(r \geq 1\) is regular if and only if its maximal ideal \(m\) is generated by an \(A\)-sequence. Prove that, if \(A\) is regular, then \(A\) is Cohen–Macaulay.
SOLUTION: Assume $A$ is regular. Given a regular sop $x_1, \ldots, x_r$, let’s show it’s an $A$-sequence. Set $A_1 := A/x_1$. Then $A_1$ is regular of dimension $r - 1$ by (21.22). So $x_1 \neq 0$. But $A$ is a domain by (21.23). So $x_1 \notin z \cdot \text{div}(A)$. Further, if $r \geq 2$, then the residues of $x_2, \ldots, x_r$ form a regular sop of $A_1$; so we may assume they form an $A_1$-sequence by induction on $r$. Thus $x_1, \ldots, x_r$ is an $A$-sequence.

Conversely, if $m$ is generated by an $A$-sequence $x_1, \ldots, x_n$, then $n \leq \text{depth}(A) \leq r$ by (23.3) and (23.3)(3), and $n \geq r$ by (21.19). Thus then $n = \text{depth}(A) = r$, and so $A$ is regular and Cohen–Macaulay. □

EXERCISE (23.11). — Let $A$ be a DVR with fraction field $K$, and $f \in A$ a nonzero nonunit. Prove $A$ is a maximal proper subring of $K$. Prove $\dim(A) \neq \dim(A_f)$.

SOLUTION: Let $R$ be a ring, $A \subseteq R \subseteq K$. Then there’s an $x \in R - A$. Say $x = ut^n$ where $u \in A^*$ and $t$ is a uniformizing parameter. Then $n < 0$. Set $y := u^{-1}t^{-n-1}$. Then $y \in A$. So $t^{-1} = xy \in R$. Hence $wt^m \in R$ for any $w \in A^*$ and $m \in \mathbb{Z}$. Thus $R = K$, as desired.

Since $f$ is a nonzero nonunit, $A \subseteq A_f \subseteq K$. Hence $A_f = K$ by the above. So $\dim(A_f) = 1$. But $\dim(A) = 1$ by (23.1).

EXERCISE (23.12). — Let $k$ be a field, $P := k[X, Y]$ the polynomial ring in two variables, $f \in P$ an irreducible polynomial. Say $f = \ell(X, Y) + g(X, Y)$ with $\ell(X, Y) = aX + bY$ for $a, b \in k$ and with $g \in \langle X, Y \rangle^2$. Set $R := P/\langle f \rangle$ and $p := \langle X, Y \rangle/\langle f \rangle$. Prove that $R_p$ is a DVR if and only if $\ell \neq 0$. (Thus $R_p$ is a DVR if and only if the plane curve $C : f = 0 \subseteq k^2$ is nonsingular at $(0, 0)$.)

SOLUTION: Set $A := R_p$ and $m := pA$. Then (14.72) and (14.73) yield $A/m = (R/p)_p = k$ and $m/m^2 = p/p^2$.

First, assume $\ell \neq 0$. Now, the $k$-vector space $m/m^2$ is generated by the images $x$ and $y$ of $X$ and $Y$ in $A$. Clearly, the image of $f$ is 0 in $m/m^2$. Also, $g \in \langle X, Y \rangle^2$; so its image in $m/m^2$ is also 0. Hence, the image of $\ell$ is 0 in $m/m^2$; that is, $x$ and $y$ are linearly dependent. Now, $f$ cannot generate $\langle X, Y \rangle$, so $m \neq 0$; hence, $m/m^2 \neq 0$ by Nakayama’s Lemma, (10.11). Therefore, $m/m^2$ is 1-dimensional over $k$; hence, $m$ is principal by (10.13)(2). Now, since $f$ is irreducible, $A$ is a domain. Hence, $A$ is a DVR by (23.1).

Conversely, assume $\ell = 0$. Then $f = g \in \langle X, Y \rangle^2$. So $m/m^2 = p/p^2 = \langle X, Y \rangle/\langle X, Y \rangle^2$.

Hence, $m/m^2$ is 2-dimensional. Therefore, $A$ is not a DVR by (23.1).

EXERCISE (23.13). — Let $k$ be a field, $A$ a ring intermediate between the polynomial ring and the formal power series ring in one variable: $k[X] \subset A \subset k[[X]]$. Suppose that $A$ is local with maximal ideal $\langle X \rangle$. Prove that $A$ is a DVR. (Such local rings arise as rings of power series with curious convergence conditions.)

SOLUTION: Let’s show that the ideal $\mathfrak{a} := \bigcap_{n \geq 0}(X^n)$ of $A$ is zero. Clearly, $\mathfrak{a}$ is a subset of the corresponding ideal $\bigcap_{n \geq 0}(X^n)$ of $k[[X]]$, and the latter ideal is clearly zero. Hence (23.10) implies $A$ is a DVR. □

EXERCISE (23.14). — Let $L/K$ be an algebraic extension of fields, $X_1, \ldots, X_n$ variables, $P$ and $Q$ the polynomial rings over $K$ and $L$ in $X_1, \ldots, X_n$.

(1) Let $q$ be a prime of $Q$, and $p$ its contraction in $P$. Prove $\text{ht}(p) = \text{ht}(q)$.
Let $f, g \in P$ be two polynomials with no common prime factor in $P$. Prove that $f$ and $g$ have no common prime factor $q \in Q$.

**Solution:** Since $L/K$ is algebraic, $Q/P$ is integral. Furthermore, $P$ is normal, and $Q$ is a domain. Hence we may apply the Going Down Theorem (15.16). So given any chain of primes $q_0 \subsetneq \cdots \subsetneq q_r = q$ with $q_i \cap P = p_i$. Thus $ht (p) \leq ht (q)$. Conversely, any chain of primes $q_0 \subsetneq \cdots \subsetneq q_r = q$ contracts to a chain of primes $p_0 \subsetneq \cdots \subsetneq p_r = p$, and $p_i \neq p_{i+1}$ by Incomparability, (13.20); whence, $ht (p) = ht (q)$. Thus (1) holds.

Alternatively, by (15.14), $ht (p) + \dim (P/p) = n$ and $ht (q) + \dim (Q/q) = n$ as both $P$ and $Q$ are polynomial rings in $n$ variables over a field. However, by (15.14), $\dim P/p = tr. \deg K Frac (P/p)$ and $\dim Q/q = tr. \deg L Frac (Q/q)$, and these two transcendence degrees are equal as $Q/P$ is an integral extension. Thus again, (1) holds.

Suppose $f$ and $g$ have a common prime factor $q \in Q$, and set $q := Qq$. Then the maximal ideal $qQ_q$ of $Q_q$ is principal and nonzero. Hence $Q_q$ is a DVR by (14.13). Thus $ht (q) = 1$. Set $p := q \cap P$. Then $p$ contains $f$; whence, $p$ contains some prime factor $p$ of $f$. Then $p \supsetneq Pq$, and $Pq$ is a nonzero prime. Hence $p = Pq$ since $ht (p) = 1$ by (1). However, $p$ contains $q$ too. Therefore, $p \mid q$, contrary to the hypothesis. Thus (2) holds. (Caution: if $f := X_1$ and $g := X_2$, then $f$ and $g$ have no common factor, yet there are no $\varphi$ and $\psi$ such that $\varphi f + \psi g = 1$.)

**Exercise (23.16).** Let $R$ be a Noetherian domain, $M$ a finitely generated module. Show that $M$ is torsionfree if and only if it satisfies $(S_1)$.

**Solution:** Assume $M$ satisfies $(S_1)$. By (23.14), the only prime in $Ass(M)$ is $\langle 0 \rangle$. Hence $z.\text{div}(M) = \langle 0 \rangle$ by (12.17). Thus $M$ is torsionfree.

Conversely, assume $M$ is torsionfree. Suppose $p \in Ass(M)$. Then $p = Ann(m)$ for some $m \in M$. But $Ann(m) = \langle 0 \rangle$ for all $m \in M$. So $p = \langle 0 \rangle$ is the only associated prime. Thus $M$ satisfies $(S_1)$ by (23.16).

**Exercise (23.17).** Let $R$ be a Noetherian ring. Show that $R$ is reduced if and only if $(R_0)$ and $(S_1)$ hold.

**Solution:** Assume $(R_0)$ and $(S_1)$ hold. Consider any irredundant primary decomposition $\langle 0 \rangle = \bigcap q_i$. Set $p_i := \sqrt{q_i}$. Then $p_i \in Ass(R)$ by (18.5) and (18.20). So $p_i$ is minimal by $(S_1)$. Hence the localization $R_{p_i}$ is a field by $(R_0)$. So $p_iR_{p_i} = 0$. But $p_iR_{p_i} \supsetneq q_iR_{p_i}$. Hence $p_iR_{p_i} = q_iR_{p_i}$. Therefore, $p_i = q_i$ by (18.23). So $\langle 0 \rangle = \bigcap p_i = \sqrt{\langle 0 \rangle}$. Thus $R$ is reduced.

Conversely, assume $R$ is reduced. Then $R_p$ is reduced for any prime $p$ by (18.24). So if $p$ is minimal, then $R_p$ is a field. Thus $(R_0)$ holds. But $\langle 0 \rangle = \bigcap p_i$ minimal $p$. So $p$ is minimal whenever $p \in Ass(R)$ by (18.24). Thus $R$ satisfies $(S_1)$.

**Exercise (23.22).** Prove that a Noetherian domain $R$ is normal if and only if, given any prime $p$ associated to a principal ideal, $pR_p$ is principal.

**Solution:** Assume $R$ normal. Say $p \in Ass(R/\langle x \rangle)$. Then $pR_p \in Ass(R_p/\langle x/1 \rangle)$ by (17.21). So $\text{depth}(R_p) = 1$ by (12.25). But $R_p$ is normal by (12.22). Hence $pR_p$ is principal by (12.25).

Conversely, assume that, given any prime $p$ associated to a principal ideal, $pR_p$ is principal. Given any prime $p$ of height 1, take a nonzero $x \in p$. Then $p$ is minimal
containing \(x\). So \(p \in \text{Ass}(R/(x))\) by (17.13). So, by hypothesis, \(pR_p\) is principal. So \(R_p\) is a DVR by (23.21). Thus \(R\) satisfies (R1).

Given any prime \(p\) with \(\text{depth}(R_p) = 1\), say \(pR_p \in \text{Ass}(R_p/(x/s))\) with \(x \neq 0\) by (23.3)(2). Then \((x/s) = (x/1) \subset R_p\). So \(p \in \text{Ass}(R/(x))\) by (17.11). So, by hypothesis, \(pR_p\) is principal. So \(\dim(R_p) = 1\) by (23.11). Thus \(R\) also satisfies (S2). So \(R\) is normal by Serre’s Criterion. (23.21).

**Exercise (23.23).** — Let \(R\) be a Noetherian ring, \(K\) its total quotient ring, Set

\[
\Phi := \{ p \text{ prime } | \text{ht}(p) = 1 \} \quad \text{and} \quad \Sigma := \{ p \text{ prime } | \text{depth}(R_p) = 1 \}.
\]

Assuming (S1) holds for \(R\), prove \(\Phi \subset \Sigma\), and prove \(\Phi = \Sigma\) if and only if (S2) holds.

Further, without assuming (S1) holds, prove this canonical sequence is exact:

\[
R \rightarrow K \rightarrow \prod_{p \in \Sigma} K_p/R_p.
\]

**Solution:** Assume (S1) holds. Then, given \(p \in \Phi\), there exists a nonzerodivisor \(x \in p\) owing to (17.13) and (23.13). Clearly, \(p\) is minimal containing \((x)\). So \(p \in \text{Ass}(R/(x))\) by (17.13). Hence \(\text{depth}(R_p) = 1\) by (23.3)(2). Thus \(\Phi \subset \Sigma\).

However, as (S1) holds, (S2) holds if and only if \(\Phi \supset \Sigma\). Thus \(\Phi = \Sigma\) if and only if \(R\) satisfies (S2).

Further, without assuming (S1), consider (23.21.1). Trivially, the composition is zero. Conversely, take an \(x \in K\) that vanishes in \(\prod_{p \in \Sigma} K_p/R_p\). Say \(x = a/b\) with \(a, b \in R\) and \(b\) a nonzerodivisor. Then \(a/1 \in bR_p\) for all \(p \in \Sigma\). But \(b/1 \in R_p\) is, clearly, a nonzerodivisor for any prime \(p\). Hence, if \(p \in \text{Ass}(R_p/bR_p)\), then \(p \in \Sigma\) by (23.3)(2). Therefore, \(a \in bR\) by (18.20). Thus \(x \in R\). Thus (23.23.1) is exact.

**Exercise (23.24).** — Let \(R\) be a Noetherian ring, and \(K\) its total quotient ring. Set \(\Phi := \{ p \text{ prime } | \text{ht}(p) = 1 \}\). Prove these three conditions are equivalent:

1. \(R\) is normal.
2. (R1) and (S2) hold.
3. (R1) and (S1) hold, and \(R \rightarrow K \rightarrow \prod_{p \in \Phi} K_p/R_p\) is exact.

**Solution:** Assume (1). Then \(R\) is reduced by (15.7). So (23.14) yields (R0) and (S1). But \(R_p\) is normal for any prime \(p\) by (13.16). Thus (2) holds by (23.11).

Assume (2). Then (R1) and (S1) hold trivially. Thus (23.23) yields (3).

Assume (3). Let \(x \in K\) be integral over \(R\). Then \(x/1 \in K\) is integral over \(R_p\) for any prime \(p\). Now, \(R_p\) is a DVR for all \(p\) of height 1 as \(R\) satisfies (R1). Hence, \(x/1 \in R_p\) for all \(p \in \Phi\). So \(x \in R\) by the exactness of the sequence in (3). But \(R\) is reduced by (23.17). Thus (15.9) yields (1).

23. Appendix: Cohen–Macaulayness

**Exercise (23.25).** — Let \(R \rightarrow R'\) be a flat map of Noetherian rings, \(a \subset R\) an ideal, \(M\) a finitely generated \(R\)-module, and \(x_1, \ldots, x_r\) an \(M\)-sequence in \(a\). Set \(M' := M \otimes_R R'\). Assume \(M'/aM' \neq 0\). Show \(x_1, \ldots, x_r\) is an \(M'\)-sequence in \(aR'\).

**Solution:** For all \(i\), set \(M_i := M/(x_1, \ldots, x_i)M\) and \(M_i' := M'/(x_1, \ldots, x_i)M'\). Then \(M_i = M_i \otimes_R R'\) by right exactness of tensor product (5.13). Moreover, by hypothesis, \(x_{i+1}\) is a nonzerodivisor on \(M_i\). Thus the multiplication map \(\mu_{x_{i+1}}: M_i \rightarrow M_i\) is injective. Hence \(\mu_{x_{i+1}}: M_i' \rightarrow M_i'\) is injective by flatness. Finally \((x_1, \ldots, x_r) \subset a\), so \(M_r' \neq 0\). Thus the assertion holds.
EXERCISE (23.26). — Let $R$ be a Noetherian ring, $a$ an ideal, and $M$ a finitely generated module with $M/aM \neq 0$. Let $x_1, \ldots, x_r$ be an $M$-sequence in $a$ and $p \in \text{Supp}(M/aM)$. Prove the following statements:

1. $x_1/1, \ldots, x_r/1$ is an $M_p$-sequence in $a_p$, and
2. $\text{depth}(a, M) \leq \text{depth}(a_p, M_p)$.

SOLUTION: First, (23.31) yields $p \in \text{Supp}(M) \cap V(a)$. So $M_p \neq 0$ and $p \in V(a)$. Hence $M_p/a_pM_p \neq 0$ by Nakayama’s Lemma (21.31). But $R_p$ is $R$-flat by (17.41). Thus (23.34) yields (1). Hence $r \leq \text{depth}(a_p, M_p)$. Thus (23.3) yields (2). □

EXERCISE (23.29). — Let $R$ be a Noetherian ring, $a$ an ideal, and $M$ a finitely generated module with $M/aM \neq 0$. Let $x \in a$ be a nonzerodivisor on $M$. Show

$$\text{depth}(a, M/xM) = \text{depth}(a, M) - 1.$$  

(23.29.1)

SOLUTION: There’s a finished $M/xM$-sequence $x_2, \ldots, x_r$ in $a$ by (23.23). Then $x, x_2, \ldots, x_r$ is a finished $M$-sequence in $a$. Thus (23.24) yields (23.29.1). □

EXERCISE (23.30). — Let $A$ be a Noetherian local ring, $M$ a finitely generated module, $x \notin \text{z.div}(M)$. Show $M$ is Cohen–Macaulay if and only if $M/xM$ is so.

SOLUTION: First (23.22) yields $\text{depth}(M/xM) = \text{depth}(M) - 1$. Also (21.5) yields $\text{dim}(M/xM) = \text{dim}(M) - 1$. The assertion follows. □

EXERCISE (23.32). — Let $A$ be a Noetherian local ring, and $M$ a nonzero finitely generated module. Prove the following statements:

1. $\text{depth}(M) = \text{depth}(\hat{M})$.
2. $M$ is Cohen–Macaulay if and only if $\hat{M}$ is Cohen–Macaulay.

SOLUTION: The completion $\hat{A}$ is faithfully flat by (21.7), and the maximal ideal of $\hat{A}$ extends to the maximal ideal of $\hat{A}$ by (24.31) (1) and (24.22) (2). So (23.31) yields (1). Further, $\text{dim}(M) = \text{dim}(\hat{M})$ by (24.3) (2); so (1) implies (2). □

EXERCISE (23.33). — Let $R$ be a Noetherian ring, $a$ an ideal, and $M$ a finitely generated module with $M/aM \neq 0$. Show that there is $p \in \text{Supp}(M/aM)$ with

$$\text{depth}(a, M) = \text{depth}(a_p, M_p).$$  

(23.33.1)

SOLUTION: There exists a finished $M$-sequence $x_1, \ldots, x_r$ in $a$ by (24.32), and (24.22) (1) implies $x_1/1, \ldots, x_r/1$ is an $M_p$-sequence. Set $M_r := M/\langle x_1, \ldots, x_r \rangle M$. Then $a \subseteq \text{z.div}(M_r)$ by finiteness. So $a \subseteq p$ for some $p \in \text{Ass}(M_r)$ by (17.48). So $pR_p \subseteq \text{z.div}(M_r)_p$. Hence $x_1/1, \ldots, x_r/1$ is finished in $pR_p$. So (23.22) yields (23.33.1). □

EXERCISE (23.37). — Prove that a Cohen–Macaulay local ring $A$ is catenary.

SOLUTION: Take primes $q \subseteq p$ in $A$. Fix a maximal chain from $p$ up to the maximal ideal and a maximal chain from $q$ down to a minimal prime. Now, all maximal chains of primes in $A$ have length $\text{dim}(A)$ by (23.30). Hence all maximal chains of primes from $q$ to $p$ have the same length. Thus $A$ is catenary. □

24. Dedekind Domains
Exercise (24.5). — Let $R$ be a domain, $S$ a multiplicative subset.

(1) Assume $\dim(R) = 1$. Prove $\dim(S^{-1}R) = 1$ if and only if there is a nonzero prime $p$ with $p \cap S = \emptyset$.

(2) Assume $\dim(R) \geq 1$. Prove $\dim(R) = 1$ if and only if $\dim(R_p) = 1$ for every nonzero prime $p$.

Solution: Consider (1). Suppose $\dim(S^{-1}R) = 1$. Then there’s a chain of primes $0 \subseteq p' \subset S^{-1}R$. Set $p := p' \cap R$. Then $p$ is as desired by (24.5).

Conversely, suppose there’s a nonzero $p$ with $p \cap S = \emptyset$. Then $0 \subseteq pS^{-1}R$ is a chain of primes by (24.5): so $\dim(S^{-1}R) \geq 1$. Now, given a chain of primes $0 = p_0 \subset \ldots \subset p_r \subset S^{-1}R$, set $p_i := p_i \cap R$. Then $0 = p_0 \subset \ldots \subset p_r \subset R$ is a chain of primes by (24.5). So $r \leq 1$ as $\dim(R) = 1$. Thus $\dim(S^{-1}R) = 1$.

Consider (2). If $\dim(R) = 1$, then (1) yields $\dim(R_p) = 1$ for every nonzero $p$.

Conversely, let $0 = p_0 \subset \ldots \subset p_r \subset R$ be a chain of primes. Set $p_i := p_iR_{p_i}$. Then $0 = p_0 \subset \ldots \subset p_r$, is a chain of primes by (24.5). So if $\dim(R_{p_i}) = 1$, then $r \leq 1$. Thus, if $\dim(R_p) = 1$ for every nonzero $p$, then $\dim(R) \leq 1$.

Exercise (24.6). — Let $R$ be a Dedekind domain, $S$ a multiplicative subset. Prove $S^{-1}R$ is a Dedekind domain if and only if there’s a nonzero prime $p$ with $p \cap S = \emptyset$.

Solution: Suppose there’s a prime nonzero $p$ with $p \cap S = \emptyset$. Then $0 \subsetneq S$. So $S^{-1}R$ is a domain by (24.3). And $S^{-1}R$ is normal by (24.2). Further, $S^{-1}R$ is Noetherian by (24.7). Also, $\dim(S^{-1}R) = 1$ by (24.10)(1). Thus $S^{-1}R$ is Dedekind.

The converse results directly from (24.10)(1).

Exercise (24.8). — Let $R$ be a Dedekind domain, and $a$, $b$, $c$ ideals. By first reducing to the case that $R$ is local, prove that

$$a \cap (b + c) = (a \cap b) + (a \cap c),$$

$$a + (b \cap c) = (a + b) \cap (a + c).$$

Solution: By (24.7), it suffices to establish the two equations after localizing at each maximal ideal $p$. But localization commutes with sum and intersection by (24.12)(4),(5). So the localized equations look like the original ones, but with $a$, $b$, $c$ replaced by $a_p$, $b_p$, $c_p$. Thus replacing $R$ by $R_p$, we may assume $R$ is a DVR.

Referring to (24.11), take a uniformizing parameter $t$. Say $a = (t^i)$ and $b = (t^j)$ and $c = (t^k)$. Then the two equations in questions are equivalent to these two:

$$\max\{i, \min\{j, k\}\} = \min\{\max\{i, j\}, \max\{i, k\}\},$$

$$\min\{i, \max\{j, k\}\} = \max\{\min\{i, j\}, \min\{i, k\}\}.$$  

However, these two equations are easy to check for any integers $i$, $j$, $k$.

Exercise (24.12). — Prove that a semilocal Dedekind domain $A$ is a PID. Begin by proving that each maximal ideal is principal.

Solution: Let $p_1, \ldots, p_r$ be the maximal ideals of $A$. Let’s prove they are principal, starting with $p_1$. By Nakayama’s lemma (24.11), $p_1A_{p_1} \neq p_1^2A_{p_1}$; so $p_1 \neq p_1^2$. Take $y \in p_1 - p_1^2$. The ideals $p_1^2$, $p_2$, $\ldots$, $p_r$ are pairwise comaximal because no two of them lie in the same maximal ideal. Hence, by the Chinese Remainder Theorem, (24.9), there is an $x \in A$ with $x \equiv y \mod p_1^2$ and $x \equiv 1 \mod p_i$ for $i \geq 2$.  

Solutions: (25.2)  243

The Main Theorem of Classical Ideal Theory, \((24.11)\), yields \((x)p_1^{n_1}p_2^{n_2}\cdots p_r^{n_r}\) with \(n_i \geq 0\). But \(x \not\in p_i\) for \(i \geq 2\); so \(n_i = 0\) for \(i \geq 2\). Further, \(x \in p_1 - p_1^2\); so \(n_1 = 1\). Thus \(p_1 = \langle x \rangle\). Similarly, all the other \(p_i\) are principal.

Finally, let \(a\) be any nonzero ideal. Then the Main Theorem, \((24.11)\), yields \(a = \prod p_i^{m_i}\) for some \(m_i\). Say \(p_i = \langle x_i \rangle\). Then \(a = \langle \prod x_i^{m_i} \rangle\), as desired. \(\square\)

Exercise (24.13). — Let \(R\) be a Dedekind domain, \(a\) and \(b\) two nonzero ideals.

Prove (1) every ideal in \(R/a\) is principal, and (2) \(b\) is generated by two elements.

Solution: To prove (1), let \(p_1, \ldots, p_r\) be the associated primes of \(a\), and set \(S := \bigcap\langle R - p_i \rangle\). Then \(S\) is multiplicative. Set \(R' := S^{-1}R\). Then \(R'\) is a Dedekind domain by \((24.10)\). Let’s prove \(R'\) is semilocal.

Let \(q\) be a maximal ideal of \(R'\), and set \(p := q \cap R\). Then \(q = pR'\) by \((11.20)\). So \(p\) is nonzero, whence maximal since \(R\) has dimension 1. Suppose \(p\) is distinct from all the \(p_i\). Then \(p\) and the \(p_i\) are pairwise comaximal. So, by the Chinese Remainder Theorem, \((11.23)\), there is a \(u \in R\) that is congruent to 0 modulo \(p\) and to 1 modulo each \(p_i\). Hence, \(u \in p \cap S\), but \(q = pR'\), a contradiction. Thus \(p_1R', \ldots, p_rR'\) are all the maximal ideals of \(R'\).

So \(R'\) is a PID by \((24.12)\); so every ideal in \(R'/aR'\) is principal. But by \((12.22)\), \(R'/aR' = S^{-1}(R/a)\). Finally, \(S^{-1}(R/a) = R/a\) by \((11.6)\), as every \(u \in S\) maps to a unit in \(R/a\) since the image lies in no maximal ideal of \(R/a\). Thus (1) holds.

Alternatively, we can prove (1) without using \((12.22)\), as follows. The Main Theorem of Classical Ideal Theory, \((24.11)\), yields \(a = p_1^{n_1} \cdots p_k^{n_k}\) for distinct maximal ideals \(p_i\). The \(p_i^{n_i}\) are pairwise comaximal. So, by the Chinese Remainder Theorem, \((11.3)\), there’s a canonical isomorphism:

\[
R/a \rightarrow R/p_1^{n_1} \times \cdots \times R/p_k^{n_k}.
\]

Next, let’s prove each \(R/p_i^{n_i}\) is a Principal Ideal Ring (PIR); that is, every ideal is principal. Set \(S := R - p_i\). Then \(S^{-1}(R/p_i^{n_i}) = R_{p_i}/p_i^{n_i}R_{p_i}\), and the latter ring is a PIR because \(R_{p_i}\) is a DVR. But \(R/p_i^{n_i} = S^{-1}(R/p_i^{n_i})\) by \((11.6)\), as every \(u \in S\) maps to a unit in \(R/p_i^{n_i}\) since \(p/p_i^{n_i}\) is the only prime in \(R/p_i^{n_i}\).

Finally, given finitely many PIRs \(R_1, \ldots, R_k\), we must prove their product is a PIR. Consider an ideal \(b \subset R_1 \times \cdots \times R_k\). Then \(b = b_1 \times \cdots \times b_k\) where \(b_i \subset R_i\) is an ideal by \((11.2)\). Say \(b_i = \langle a_i \rangle\). Then \(b = \langle a_1, \ldots, a_k \rangle\). Thus again, (1) holds.

Consider (2). Let \(x \in b\) be nonzero. By (1), there is a \(y \in b\) whose residue generates \(b/\langle x \rangle\). Then \(b = \langle x, y \rangle\). \(\square\)

25. Fractional Ideals

Exercise (25.2). — Let \(R\) be a domain, \(M\) and \(N\) nonzero fractional ideals. Prove that \(M\) is principal if and only if there exists some isomorphism \(M \cong R\).

Construct the following canonical surjection and canonical isomorphism:

\[
\pi: M \otimes N \rightarrow MN \quad \text{and} \quad \varphi: (M : N) \rightarrow \text{Hom}(N, M).
\]

Solution: If \(M \cong R\), let \(x\) correspond to 1; then \(M = Rx\). Conversely, assume \(M = Rx\). Then \(x \neq 0\) as \(M \neq 0\). Form the map \(R \rightarrow M\) with \(a \mapsto ax\). It’s surjective as \(M = Rx\). It’s injective as \(x \neq 0\) and \(M \subset \text{Frac}(R)\).

Form the canonical \(M \times N \rightarrow MN\) with \((x, y) \mapsto xy\). It’s bilinear. So it induces a map \(\pi: M \otimes N \rightarrow MN\), and clearly \(\pi\) is surjective.

Define \(\varphi\) as follows: given \(z \in (M : N)\), define \(\varphi(z): N \rightarrow M\) by \(\varphi(z)(y) := yz\).
Clearly, $\varphi$ is $R$-linear. Say $y \neq 0$. Then $yz = 0$ implies $z = 0$; thus, $\varphi$ is injective.

Finally, given $\theta : N \to M$, fix a nonzero $n \in N$, and set $z := \theta(n)/n$. Given $y \in N$, say $y = a/b$ and $n = c/d$ with $a, b, c, d \in R$. Then $bcy = ad\theta(n)$. Hence $\theta(y) = yz$. Therefore, $z \in (M : N)$ as $y \in N$ is arbitrary and $\theta(y) \in M$; further, $\theta = \varphi(z)$. Thus, $\varphi$ is surjective, as desired. \hfill $\square$

**EXERCISE (25.6).** — Let $R$ be a domain, $M$ and $N$ fractional ideals. Prove that the map $\pi : M \otimes N \to MN$ is an isomorphism if $M$ is locally principal.

**Solution:** By (1.), it suffices to prove that, for each maximal ideal $m$, the localization $\pi_m : (M \otimes N)_m \to (MN)_m$ is bijective. But $(M \otimes N)_m = M_m \otimes N_m$ by (25.4), and $(MN)_m = M_mN_m$ by (25.3). By hypothesis, $M_m = R_mx$ for some $x$. Clearly $R_mx \simeq R_m$. And $R_m \otimes N_m = N_m$ by (8.7)(2). Thus $\pi_m \simeq 1_{N_m}$. \hfill $\square$

**EXERCISE (25.9).** — Let $R$ be a domain, $M$ and $N$ fractional ideals.

1. Assume $N$ is invertible, and show that $(M : N) = M \cdot N^{-1}$.
2. Show that both $M$ and $N$ are invertible if and only if their product $MN$ is, and that if so, then $(MN)^{-1} = N^{-1}M^{-1}$.

**Solution:** For (1), note that $N^{-1} = (R : N)$ by (25.8). So $M(R : N)N = M$. Thus $M(R : N) \subset (M : N)$. Conversely, note that $(M : N)N \subset M$. Hence $(M : N) = (M : N)N(R : N) \subset M(R : N)$. Thus (1) holds.

In (2), if $M$ and $N$ are invertible, then $(MN)N^{-1}M^{-1} = MM^{-1} = R$; thus $MN$ is invertible, and $N^{-1}M^{-1}$ is its inverse. Conversely, if $MN$ is invertible, then $R = (MN)(MN)^{-1} = M(N(MN)^{-1})$; thus, $M$ is invertible. Similarly, $N$ is invertible. Thus (2) holds. \hfill $\square$

**EXERCISE (25.12).** — Let $R$ be a UFD. Show that a fractional ideal $M$ is invertible if and only if $M$ is principal and nonzero.

**Solution:** By (4.6)(1), a nonzero principal ideal is always invertible.

Conversely, assume $M$ is invertible. Then trivially $M \neq 0$. Say $1 = \sum m_in_i$ with $m_i \in M$ and $n_i \in M^{-1}$. Fix a nonzero $m \in M$.

Then $m = \sum m_i n_i m_i$. But $n_i m \in R$ as $m \in M$ and $n_i \in M^{-1}$. Set

$$d := \gcd\{n_i m\} \in R \quad \text{and} \quad x := \sum (n_i m/d) n_i \in M.$$ 

Then $m = dx$.

Given $m' \in M$, write $m'/m = a/b$ where $a, b \in R$ are relatively prime. Then

$$d' := \gcd\{n_i m'\} = \gcd\{n_i ma/b\} = a \gcd\{n_i m\}/b = ad/b.$$ 

So $m' = (a/b)m = (ad/b)x = d'x$. But $d' \in R$. Thus $M = Rx$. \hfill $\square$

**EXERCISE (25.15).** — Show that a ring is a PID if and only if it’s a Dedekind domain and a UFD.

**Solution:** A PID is Dedekind by (25.6), and is a UFD by (4.9).

Conversely, let $R$ be a Dedekind UFD. Then every nonzero fractional ideal is invertible by (4.8) and (4.7), so is principal by (4.13). Thus $R$ is a PID.

Alternatively and more directly, every nonzero prime is of height 1 as $\dim R = 1$, so is principal by (4.13). But, by (4.13), every nonzero ideal is a product of nonzero prime ideals. Thus again, $R$ is a PID. \hfill $\square$

**EXERCISE (25.17).** — Let $R$ be a ring, $M$ an invertible module. Prove that $M$ is finitely generated, and that, if $R$ is local, then $M$ is free of rank 1.
SOLUTION: Say \( \alpha: M \otimes N \rightarrow R \) and \( 1 = \alpha(\sum m_i \otimes n_i) \) with \( m_i \in M \) and \( n_i \in N \). Given \( m \in M \), set \( a_i := \alpha(m \otimes n_i) \). Form this composition:

\[
\beta: M = M \otimes R \rightarrow M \otimes M \otimes N = M \otimes N \otimes M \rightarrow R \otimes M = M.
\]

Then \( \beta(m) = \sum a_i m_i \). But \( \beta \) is an isomorphism. Thus the \( m_i \) generate \( M \).

Suppose \( R \) is local. Then \( R - Rx \) is an ideal. So \( u := \alpha(m_i \otimes n_i) \in R \) for some \( i \). Set \( m := u^{-1}m_i \) and \( n := n_i \). Then \( \alpha(m \otimes n) = 1 \). Define \( \nu: M \rightarrow R \) by \( \nu(m') := \alpha(m' \otimes n) \). Then \( \nu(m) = 1 \); so \( \nu \) is surjective. Define \( \mu: R \rightarrow M \) by \( \mu(x) := x_m \). Then \( \mu \nu(m') = \nu(m')m = \beta(m') \), or \( \mu \nu = \beta \). But \( \beta \) is an isomorphism. So \( \nu \) is injective. Thus \( \nu \) is an isomorphism, as desired. \( \Box \)

EXERCISE (25.18). — Show these conditions on an \( R \)-module \( M \) are equivalent:

1. \( M \) is invertible.
2. \( M \) is finitely generated, and \( M_m \simeq R_m \) at each maximal ideal \( m \).
3. \( M \) is locally free of rank 1.

Assuming the conditions, show \( M \) is finitely presented and \( M \otimes \text{Hom}(M, R) \).

SOLUTION: Assume (1). Then \( M \) is finitely generated by \( (25.17) \). Further, say \( M \otimes N \simeq R \). Let \( m \) be a maximal ideal. Then \( M_m \otimes N_m \simeq R_m \). Hence \( M_m \simeq R_m \) again by \( (25.17) \). Thus (2) holds.

Conditions (2) and (3) are equivalent by \( (13.52) \).

Assume (3). Then (2) holds; so \( M_m \simeq R_m \) at any maximal ideal \( m \). Also, \( M \) is finitely presented by \( (13.51) \); so \( \text{Hom}_R(M, R)_m = \text{Hom}_R(M_m, R_m) \) by \( (14.28) \).

Consider the evaluation map

\[
ev(M, R): M \otimes \text{Hom}(M, R) \rightarrow R \text{ defined by } ev(M, R)(m, \alpha) := \alpha(m).
\]

Clearly \( ev(M, R)_m = ev(M_m, R_m) \). Clearly \( ev(R_m, R_m) \) is bijective. Hence \( ev(M, R) \) is bijective by \( (13.23) \). Thus the last assertions hold; in particular, (1) holds. \( \Box \)

26. Arbitrary Valuation Rings

EXERCISE (26.3). — Let \( V \) be a domain. Show that \( V \) is a valuation ring if and only if, given any two ideals \( a \) and \( b \), either \( a \) lies in \( b \) or \( b \) lies in \( a \).

SOLUTION: First, suppose \( V \) is a valuation ring. Suppose also \( a \not\subseteq b \); say \( x \in a \), but \( x \not\in b \). Take \( y \in b \). Then \( x/y \not\in V \); else \( x = (x/y)y \in b \). So \( y/x \in V \). Hence \( y = (y/x)x \in a \). Thus \( b \subseteq a \).

Conversely, let \( x, y \in V - \{0\} \), and suppose \( x/y \not\in V \). Then \( \langle x \rangle \not\subseteq \langle y \rangle \); else, \( x = wy \) with \( w \in V \). Hence \( \langle y \rangle \subseteq \langle x \rangle \) by hypothesis. So \( y = zx \) for some \( z \in V \); in other words, \( y/x \in V \). Thus \( V \) is a valuation ring. \( \Box \)

EXERCISE (26.4). — Let \( V \) be a valuation ring of \( K \), and \( V \subset W \subset K \) a subring. Prove that \( W \) is also a valuation ring of \( K \), that its maximal ideal \( p \) lies in \( V \), that \( V/p \) is a valuation ring of the field \( W/p \), and that \( W = V_p \).

SOLUTION: First, let \( x \in K - W \subset K - V \). Then \( 1/x \in V \subset W \). Thus \( V \) is a valuation ring of \( K \).

Second, let \( y \in p \). Then \( (26.2) \) implies \( 1/y \in K - W \subset K - V \). So \( y \in V \).

Third, \( x \in W - V \) implies \( 1/x \in V \); whence, \( V/p \) is a valuation ring of \( W/p \).

Fourth, \( V_p \subset W_p = W \). Conversely, let \( x \in W - V \). Then \( 1/x \in V \). But \( 1/x \not\in p \) as \( p \) is the maximal ideal of \( W \). So \( x \in V_p \). Thus \( W = V_p \). \( \Box \)
Exercise (26.5). — Prove that a valuation ring $V$ is normal.

Solution: Let $K := \text{Frac}(V)$, and let $m$ be the maximal ideal. Take $x \in K$ integral over $V$, say $x^n + a_1x^{n-1} + \cdots + a_n = 0$ with $a_i \in V$. Then
\[
1 + a_1x^{-1} + \cdots + a_nx^{-n} = 0. \tag{26.5.1}
\]
If $x \notin V$, then $x^{-1} \in m$ by (26.12). So (26.5.1) yields $1 \in m$, a contradiction. Hence $x \in V$. Thus $V$ is normal. $\square$

Exercise (26.9). — Let $K$ be a field, $S$ the set of local subrings ordered by domination. Show that the valuation rings of $K$ are the maximal elements of $S$.

Solution: Let $V$ be a valuation ring of $K$. Then $V \in S$ by (26.2). Let $V' \in S$ dominate $V$. Let $m$ and $m'$ be the maximal ideals of $V$ and $V'$. Take any nonzero $x \in V'$. Then $1/x \notin m'$ as $1 \notin m'$; so also $1/x \notin m$. So $x \in V$ by (26.2). Hence, $V' = V$. Thus $V$ is maximal.

Conversely, let $V \in S$ be maximal. By (26.2), $V$ is dominated by a valuation ring $V'$ of $K$. By maximality, $V = V'$. $\square$

Exercise (26.14). — Let $V$ be a valuation ring, such as a DVR, whose value group $\Gamma$ is Archimedean; that is, given any nonzero $\alpha, \beta \in \Gamma$, there’s $n \in \mathbb{Z}$ such that $n\alpha > \beta$. Show that $V$ is a maximal proper subring of its fraction field $K$.

Solution: Let $R$ be a subring of $K$ strictly containing $V$, and fix $a \in R - V$. Given $b \in K$, let $\alpha$ and $\beta$ be the values of $a$ and $b$. Then $\alpha < 0$. So, as $\Gamma$ is Archimedean, there’s $n > 0$ such that $-n\alpha > -\beta$. Then $v(b/a^n) > 0$. So $b/a^n \in V$. So $b = (b/a^n)a^n \in R$. Thus $R = K$. $\square$

Exercise (26.15). — Let $V$ be a valuation ring. Show that
1. every finitely generated ideal $a$ is principal, and
2. $V$ is Noetherian if and only if $V$ is a DVR.

Solution: To prove (1), say $a = (x_1, \ldots, x_n)$ with $x_i \neq 0$ for all $i$. Let $v$ be the valuation. Suppose $v(x_1) \leq v(x_i)$ for all $i$. Then $x_i/x_1 \in V$ for all $i$. So $x_1 \in (x_1)$. Hence $a = (x_1)$. Thus (1) holds.

To prove (2), first assume $V$ is Noetherian. Then $V$ is local by (26.2), and by (1) its maximal ideal $m$ is principal. Hence $V$ is a DVR by (26.11). Conversely, assume $V$ is a DVR. Then $V$ is a PID by (26.1), so Noetherian. Thus (2) holds. $\square$

Exercise (26.20). — Let $R$ be a Noetherian domain, $K := \text{Frac}(R)$, and $L$ a finite extension field (possibly $L = K$). Prove the integral closure $\overline{R}$ of $R$ in $L$ is the intersection of all DVRs $V$ of $L$ containing $R$ by modifying the proof of (26.11): show $y$ is contained in a height-1 prime $p$ of $R[y]$ and apply (26.13) to $R[y]_p$.

Solution: Every DVR $V$ is normal by (26.11). So if $V$ is a DVR of $L$ and $V \supset R$, then $V \supset \overline{R}$. Thus $\bigcap_{V \supset R} V \supset \overline{R}$.

To prove the opposite inclusion, take any $x \in K - \overline{R}$. To find a DVR $V$ of $L$ with $V \supset R$ and $x \notin V$, set $y := 1/x$. If $1/y \in R[y]$, then for some $n$,
\[
1/y = a_0y^n + a_1y^{n-1} + \cdots + a_n \quad \text{with} \quad a_\lambda \in R.
\]
Multiplying by $x^n$ yields $x^{n+1} - a_nx^n - \cdots - a_0 = 0$. So $x \in \overline{R}$, a contradiction.

Thus $y$ is a nonzero nonunit of $R[y]$. Also, $R[y]$ is Noetherian by the Hilbert Basis Theorem (26.1). So $y$ lies in a height-1 prime $p$ of $R[y]$ by the Krull Principal
Ideal Theorem (21.10). Then $R[y]_p$ is Noetherian of dimension 1. But $L/K$ is a finite field extension, so $L/\text{Frac}(R[y])$ is one too. Hence the integral closure $R'$ of $R[y]_p$ in $L$ is a Dedekind domain by (26.18). So by the Going-up Theorem (14.3), there's a prime $q$ of $R'$ lying over $pR[y]_p$. Then as $R'$ is Dedekind, $R'_q$ is a DVR of $L$ by (24.7). Further, $y \in qR'_q$. Thus $x \notin R'_q$, as desired. 

□
Bibliography

Disposition of the Exercises in [3]

Chapter 1, pp. 10–16
1. — Essentially (1.2), II, owing to (1.3), II.
2. — Essentially (1.6), II.
3. — Essentially (1.7), II.
4. — Essentially (1.9), II.
5. — Essentially (1.10), II.
6. — Part of (1.12), II.
7. — Essentially (1.12), II.
8. — Follows easily from (1.13), II.
9. — Essentially (1.14), II.
10. — Essentially (1.15), II.
11. — Essentially (1.16), II, and (1.17), II.
12. — Essentially (1.17), II.
13. — Standard; see II, Theorem 2.5, p. 231.
14. — Follows easily from (1.18), II, and (1.19), II.
15. — Part of (1.20), II.
16. — Best done by hand.
17. — Part of (1.21), II.
18. — Part of (1.22), II.
19. — Essentially (1.23), II.
20. — Essentially (1.24), II.
21. — Part of (1.25), II, and (1.26), II.
22. — Essentially (1.27), II, and (1.28), II.
23. — Essentially (1.29), II, and (1.30), II.
24. — About lattices, which we don’t treat.
25. — Essentially (1.31), II.
26. — Essentially (1.32), II.
27. — Essentially (1.33), II.
28. — Essentially (1.34), II.

Chapter 3, pp. 43–49
1. — Part of (3.1), II.
2. — Essentially (3.2), II.
3. — Essentially (3.3), II.
4. — Part of (3.4), II.
5. — Essentially (3.5), II.
6. — Essentially (3.6), II.
7. — i) Part of (3.7), II.
8. — Essentially (3.8), II.
9. — Essentially (3.9), II.
10. — Essentially (3.10), II.
11. — Essentially (3.11), II.
12. — Essentially (3.12), II.
13. — Essentially (3.13), II.
14. — Essentially (3.14), II.
15. — Essentially (3.15), II.
16. — Essentially (3.16), II.
17. — Essentially (3.17), II.
18. — Essentially (3.18), II.
19. — i) Essentially (3.19), II.
20. — Essentially (3.20), II.
21. — i) Essentially (3.21), II.
22. — Essentially (3.22), II.
23. — Essentially (3.23), II.
24. — Covered in (3.24), II, and (3.25), II.

Chapter 2, pp. 31–35
1. — Essentially (2.1), III.
2. — Essentially (2.2), III.
3. — Essentially (2.3), III.
4. — Part of (2.4), III.
5. — Part of (2.5), III.
6. — Essentially (2.6), III.
7. — Part of (2.7), III.
8. — i) Part of (2.8), III.
8. — ii) Part of (2.9), III.
9. — Part of (2.10), III.
10. — Essentially (2.11), III.
11. — Mostly in (2.12), II, and (2.13), III.
12. — Immediate from (2.14), III, and (2.15), III.
13. — Essentially (2.16), III.
14. — Part of (2.17), III.
15. — Part of (2.18), III.
16. — Essentially (2.19), III.
17. — Essentially (2.20), III, and (2.21), III.
18. — Essentially (2.22), III.
19. — Essentially (2.23), III.
20. — Essentially (2.24), III.
21. — i) Essentially (2.25), III.
22. — Essentially (2.26), III.
23. — Essentially (2.27), III.
24. — Covered in (2.28), III, and (2.29), III.
(3.7), p. 199
25. — Essentially (2.30), III.

250
Disposition of the Exercises in [3]  251

Chapter 5, pp. 67-73
1.—To be done
2.—To be done
3.—To be done
4.—To be done
5.—To be done
6.—Part of \((1.8.4.11), (1.8.10)\)
7.—To be done
8.—Part of \((1.8.71), (1.8)\)
9.—To be done
10.—To be done
11.—Essentially \((1.8.81), (1.8)\)
12.—To be done
13.—To be done
14.—To be done
15.—To be done
16.—Essentially \((1.8.91), (1.8)\)
17.—Part of \((1.8.8), (1.8)\)
18.—Essentially \((1.8.3), (1.8)\)
19.—Part of \((1.8.13), (1.8)\)
20.—To be done
21.—To be done
22.—To be done
23.—To be done
24.—Essentially \((1.8.15), (1.8), \) and \((1.8.27), (1.8)\),
25.—To be done
26.—Essentially \((1.8.27), (1.8), \) and \((1.8.28), (1.8)\),
27.—Essentially \((1.8.15), (1.8)\)
28.—Essentially \((1.8.27), (1.8), \) and part of
\((1.8.29), (1.8)\)
29.—Part of \((1.8.29), (1.8)\)
30.—Part of \((1.8.30), (1.8)\)
31.—Part of \((1.8.31), (1.8)\)
32.—To be done
33.—Essentially \((1.8.32), (1.8)\)
34.—To be done
35.—To be done

Chapter 4, pp. 55-58
1.—To be done
2.—To be done
3.—To be done
4.—To be done
5.—To be done
6.—Analysis, continuing Ex. 26, p. 3 in \(c\)
7.—To be done
8.—To be done
9.—To be done
10.—To be done
11.—To be done
12.—To be done
13.—Part of \((1.8.84), (1.8)\)
14.—Essentially \((1.8.11), (1.8)\)
15.—To be done
16.—Covered in \((1.8.24), (1.8)\)
17.—Technical conditions for primary decomposition; solution sketched in place
18.—Technical conditions for primary decomposition; solution sketched in place
19.—To be done
20.—To be done
21.—Essentially \((1.8.16), (1.8), \) and \((1.8.20), (1.8)\),
22.—Essentially \((1.8.16), (1.8)\)
23.—Essentially \((1.8.17), (1.8), \) and \((1.8.21), (1.8)\),
and \((1.8.25), (1.8), \) and \((1.8.28), (1.8)\),
24.—Essentially \((1.8.17), (1.8), \) and \((1.8.21), (1.8)\),
25.—To be done
26.—Essentially \((1.8.22), (1.8), \) and \((1.8.28), (1.8)\),
27.—Essentially \((1.8.17), (1.8)\)
28.—Essentially \((1.8.22), (1.8), \) and part of
\((1.8.23), (1.8)\)
29.—Part of \((1.8.23), (1.8)\)
30.—Part of \((1.8.24), (1.8)\)
31.—Part of \((1.8.25), (1.8)\)
32.—To be done
33.—Essentially \((1.8.26), (1.8)\)
34.—To be done
35.—To be done

Chapter 6, pp. 78-79
1.—Essentially \((1.8.29), (1.8)\), and to be done
2.—Part of \((1.8.30), (1.8)\)
3.—To be done
4.—To be done
5.—To be done
6.—To be done
7.—To be done
8.—To be done
9.—To be done
10.—To be done
11.—To be done
12.—To be done

Chapter 7, pp. 84-88
1.—Essentially \((1.8.31), (1.8)\)
2.—To be done
3.—To be done
4.—To be done
5.—Essentially \((1.8.32), (1.8)\)
6.—To be done
7.—Follows easily from \((1.8.33), (1.8)\)
8.—Essentially \((1.8.34), (1.8)\)
9.—To be done
10.—To be done
11.—To be done
12.—To be done
13.—To be done
14.—Essentially \((1.8.35), (1.8)\)
15.—Essentially \((1.8.36), (1.8)\), and \((1.8.27), (1.8)\)
16.—Essentially \((1.8.37), (1.8)\)
17.—To be done
18.—Essentially \((1.8.38), (1.8), \) and \((1.8.29), (1.8)\),
252 Disposition of the Exercises in [3]

19.— Follows easily from (18.27), 19.
20.— Essentially EGA, IV1, 1.8.5), p. 239
21.— Essentially EGA, IV2, 2.4.6), p. 20
22.— Essentially EGA, IV1, 1.10.4), p. 250
23.— Essentially EGA, IV1, 1.10.4), p. 250
24.— Essentially EGA, IV2, 2.4.6), p. 20
25.— Trivial K-theory
26.— Trivial K-theory

Chapter 8, pp. 91–92
1.— Essentially (15.13), 15.
2.— Essentially (15.17), 15.
3.— Essentially (15.15), 15.
4.— To be done
5.— To be done
6.— To be done

Chapter 9, p. 99
1.— To be done
2.— To be done
3.— Part of (26.15), 246
4.— Essentially (23.3), 239
5.— Part of (25.21), 255
6.— To be done
7.— To be done
8.— To be done
9.— To be done

Chapter 10, pp. 113–115
1.— Essentially (22.19), 235
2.— Essentially (22.9), 133
3.— To be done
4.— To be done
5.— To be done
6.— To be done
7.— Essentially (22.24), 135
8.— To be done
9.— To be done
10.— To be done
11.— To be done
12.— To be done

Chapter 11, pp. 125–126
1.— To be done
2.— To be done
3.— Essentially (15.13), 239
4.— To be done
5.— Trivial K-theory
6.— Essentially (15.19), 215
7.— Essentially (21.18), 231
8.— To be done
9.— To be done
10.— To be done
11.— To be done
12.— To be done
13.— To be done
Index

algebra: (1.1.1), 1
algebra finite: (1.5.9), 1
algebra map: (1.2.4), 6
coproduct: (2.7.1), 8
faithfully flat: (1.3.1), 6
finitely generated: (1.2.2), 6
flat: (2.8.2), 1
group algebra: (2.8.2), 6
homomorphism: (1.1.1), 6
integral over a ring: (1.1.1), 6
localization: (1.1.1), 6
module finite: (1.1.1), 6
structure map: (1.1.1), 6
subalgebra: (1.1.1), 6
generated by: (1.1.1), 6
tensor product: (1.2.2), 8

canonical: (1.1.1), 1

category theory
coequalizer: (1.5.9), 6
colimit: (1.5.9), 6
composition law: (1.1.1), 6
associative: (2.1.1), 6
coproduct: (1.2.4), 8
direct limit: (1.2.2), 6
has: (1.2.2), 6
indexed by: (1.2.2), 6

dually: (1.2.2), 6
filtered direct limit: (2.1.1), 6
identity: (1.2.2), 6
unitary: (1.2.2), 6
inclusion: (1.2.2), 6
initial object: (1.2.2), 6
insertion: (1.2.2), 6
inverse: (1.2.2), 6
isomorphism: (1.1.1), 6
map: (1.2.2), 6
morphism: (1.2.2), 6
object: (1.2.2), 6
pushout: (1.2.2), 6
transition map: (1.2.2), 6
category: (1.1.1), 6

directed set: (1.1.1), 6

discrete: (1.1.1), 6
functor: (1.2.1), 6
has direct limits: (1.4.1), 6
product: (1.2.2), 6
small: (1.2.2), 8

diagram
chase: (1.5.9), 6
commutative: (1.1.1), 1

element
annihilator: (1.2.1), 6
Cauchy sequence: (1.2.1), 6
complementary idempotents: (1.1.1), 6
composite: (1.2.1), 6
equation of integral dependence: (1.1.1), 6
formal power series: (1.2.1), 6
free: (1.2.1), 6
generators: (1.2.1), 6
homogeneous: (1.1.1), 6
homogeneous component: (1.1.1), 6
homogeneous of degree n: (1.1.1), 6
idempotent: (1.1.1), 6
initial component: (1.2.1), 6
integral over a ring: (1.1.1), 6
integally independent on a ring: (1.1.1), 6
irreducible: (1.1.1), 6
Kronecker delta function: (1.2.1), 6
lift: (1.1.1), 6
limit: (1.2.1), 6
linear combination: (1.1.1), 6
linearly independent: (1.1.1), 6
nilpotent: (1.2.1), 6
nonzerodivisor: (1.2.1), 6
p-adic integer: (1.2.1), 6
prime: (1.2.1), 6
reciprocal: (1.1.1), 6
relatively prime: (1.2.1), 6
residue of: (1.1.1), 6
restricted vectors: (1.2.1), 6
restricted vectors: (1.2.1), 6
uniformizing parameter: (1.2.1), 6
unit: (1.1.1), 6
zerodivisor: (1.2.1), 6
zerodivisor over a PIR: (1.2.1), 6

field: (1.1.1), 6
discrete valuation: (1.2.1), 6
discrete valuation: (1.2.1), 6
fraction field: (1.1.1), 6
p-adic valuation: (1.2.1), 6
rational functions: (1.1.1), 6
Trace Pairing: (1.2.1), 6
trace: (1.2.1), 6
functor: (1.2.1), 6
additive: (1.2.1), 6
adjoint: (1.2.1), 6
adjoint pair: (1.2.1), 6
adjoint: (1.2.1), 6

count: (1.2.1), 6

universal: (1.2.1), 6

253
cofinal: (13.1), 8
constant: (13.3), 3
contravariant: (13.1), 6
covariant: (13.2), 6
diagonal: (13.3), 3
direct system: (13.7), 10
exact: (13.6), 8
faithful: (13.8), 10
forgetful: (13.8), 10
isomorphic: (13.5), 7
left adjoint: (13.3), 3
left exact: (13.7), 3
linear: (13.4), 9
natural bijection: (13.5), 7
natural transformation: (13.6), 9
right adjoint: (13.3), 3
right exact: (13.7), 3
ideal: (13.1), 2
associated prime: (13.2), 4
chain stabilizes: (13.5), 7
comaximal: (13.3), 3
contraction: (13.1), 6
extension: (13.3), 3
Fitting: (13.8), 10
fractional: (13.4), 9
invertible: (13.7), 9
locally principal: (13.4), 9
minimal: (13.4), 9
principal: (13.4), 9
product: (13.4), 9
quotient: (13.4), 9
generated: (13.1), 3
idempotent: (13.1), 3
intersection: (13.2), 6
length of chain: (13.5), 7
lie over: (13.5), 7
maximal: (13.4), 9
nested: (13.4), 9
nilradical: (13.4), 9
parameter: (13.4), 9
prime: (13.4), 9
height: (13.4), 9
maximal chain: (13.4), 9
minimal: (13.4), 9
principal: (13.4), 9
product: (13.4), 9
proper: (13.4), 9
radical: (13.4), 9
saturated: (13.4), 9
saturation: (13.4), 9
sum: (13.3), 3
symbolic power: (13.4), 9
variety: (13.4), 9

Lemma
Artin Character: (13.2), 10

Artin–Rees: (13.2), 10
Artin–Tate: (13.2), 10
Equational Criterion for Flatness: (13.2), 13
Equational Criterion for Vanishing: (13.2), 13
Five: (13.2), 10
Ideal Criterion for Flatness: (13.2), 11
Ideal Criterion for Flatness: (13.1), 11
Nakayama: (13.1), 7
Nonunit Criterion: (13.2), 10
Prime Avoidance: (13.2), 10
Schanuel: (13.2), 10
Snake: (13.2), 10
Zorn’s: (13.1), 6

map
R-linear: (13.6), 6
automorphism: (13.1), 3
bilinear: (13.6), 6
bimodule homomorphism: (13.6), 6
endomorphism: (13.1), 3
homogeneous: (13.1), 3
isomorphism: (13.1), 3
lift: (13.6), 6
local homomorphism: (13.1), 3
Noether Isomorphisms: (13.1), 3
quotient map: (13.6), 6
retraction: (13.6), 6
section: (13.1), 3
trilinear: (13.1), 3

matrix of cofactors: (13.2), 8

module: (13.1), 8
S-torsion: (13.2), 8
a-dic topology: (13.3), 8
ascending chain condition (acc): (13.4), 9
annihilator: (13.3), 8
Artinian: (13.2), 8
associated graded: (13.4), 9
associated prime: (13.4), 9
bimodule: (13.2), 8
bimodule homomorphism: (13.2), 8
chain stabilizes: (13.5), 7
characteristic polynomial: (13.4), 9
closed: (13.4), 9
Cohen–Macaulay: (13.2), 8
Cohen–Macaulay: (13.2), 8
coimage: (13.3), 8
cokernel: (13.3), 8
complete: (13.2), 8
composition series: (13.2), 8
cyclic: (13.2), 8
depth: (13.2), 8
descending chain condition (dcc): (15.4.10),

finishing: (45.4.10), 18

Noetherian: (16.1.10), 26

presentation: (45.4.10), 18

projective: (45.4.10), 18

quotient: (3.4.11), 4

quotient map: (3.4.11), 18

$R$-linear map: (17.1.11), 5

radical: (17.1.11), 5

Rees Module: (17.1.11), 5

residue: (11.10), 18

restriction of scalars: (15.4.10), 26

scalar multiplication: (15.4.10), 26

semilocal: (17.1.11), 5

separation: (17.1.11), 5

separated completion: (17.1.11), 5

Serre's Condition: (18.2.10), 22

simple: (17.1.11), 5

standard basis: (11.15), 18

sum: (11.10), 18

support: (11.15), 18

system of parameters (sop): (11.16), 18

tensor product, see also

torsion: (11.15), 18

torsionfree: (11.15), 18

notation

$a + b$: (11.10), 18

$a \cap b$: (11.10), 18

$ab$: (11.10), 18

$\alpha R \otimes \alpha': (4.8.12), 18$

$\alpha N$: (4.8.12), 18

$\alpha R'$: (4.8.12), 18

$\alpha^\circ$: (4.8.12), 18

$M = N$: (11.15), 18

$M^\wedge$: (11.15), 18

$M^n$: (11.15), 18

$P(M, t)$: (4.2.10), 41

$p^{(n)}$: (4.2.10), 41

$G^* M$: (4.5.10), 18

$\prod M_i$: (11.10), 18

$R \cong R'$: (11.10), 18

$\sum R \alpha_i$: (11.10), 18

$\sum \alpha_i$: (11.10), 18

$p(F^* M, n)$: (4.5.10), 18

$\rho(M, n)$: (4.2.10), 41

$M \cong N$: (11.10), 18

($(R\text{-alg})$: (4.1.11), 18

($(R\text{-mod})$: (4.1.11), 18

($(\text{Rings})$: (4.1.11), 18

($(\text{Sets})$: (4.1.11), 18

Ann($M$): (11.10), 18

Ann($n$): (11.10), 18

Ann($M$): (11.10), 18
algebra, see also Artinian, ascending chain condition (acc), associated graded, Boolean, catenary, coefficient field, Cohen–Macaulay, Dedekind domain, dimension, Discrete Valuation Ring (DVR), domain, dominates, extension, factor ring, field, see also formal power series ring, homomorphism, Ideal Class Group, integral closure, integral domain, integrally closed, Jacobson, Jacobson radical, kernel, Laurent series, local, local homomorphism, localization, localization at f, localization at p, map, maximal condition (maxc), minimal prime, modulo, Noetherian, nonzerodivisor, normal, normalization, p-adic integers, Picard Group, polynomial ring, Principal Ideal Domain (PID), Principal Ideal Domain (PID), Principal Ideal Ring (PIR), product ring, quotient map, quotient ring, radical, reduced, regular local, regular system of parameters, residue ring, ring of fractions, semilocal, Serre's Conditions, spectrum, subring, total quotient ring, Unique Factorization Domain (UFD), valuation, sequence, Cauchy, exact, M-sequence, finished, regular sequence, short exact, isomorphism of, split exact, submodule, p-primary component, homogeneous, p-primary, primary, primary decomposition, irredundant, minimal, saturated, saturation, subset, characteristic function, multiplicative, saturated, symmetric difference, system of parameters (sop), regular, tensor product, adjoint associativity, associative law, cancellation, commutative law, unitary law, Theorem, Additivity of Length
Akizuki–Hopkins: (11.27, 18)
Characterization of DVRs: (4.5, 11), 11
Cayley–Hamilton: (11.11), 11
Cohen: (11.4, 11), 11
Cohen Structures: (11.4, 11), 11
Determinant Trick: (11.20), 11
Dimension: (11.4, 11), 11
Direct limits commute: (11.20), 11
Exactness of Localization: (21.10), 11
Exactness of Completion: (21.10), 11
Exactness of Filtered Direct Limits: (21.10), 11
Finite closure: (11.10), 11
Finite closure of integral closure: (11.10), 11
First Uniqueness: (11.13), 11
Gauss: (11.27), 11
Generalized Hilbert Nullstellensatz: (11.19), 11
Going down
for flat algebras: (11.13), 11
for integral extensions: (11.13), 11
Going up: (11.9), 11
Hilbert Basis: (11.27), 11
Hilbert Nullstellensatz: (11.27), 11
Hilbert–Serre: (11.27), 11
Incomparability: (11.27), 11
Jordan–Hölder: (11.9), 11
Krull Intersection: (11.27), 11
Krull Principal Ideal: (11.27), 11
Lazard: (11.27), 11
Left exactness of hom: (11.20), 11
Lying over: (11.20), 11
Main of Classical Ideal Theory: (11.20), 11
Main Theorem of Classical
Maximality: (11.27), 11
Noether on Invariants: (11.27), 11
Noether on Finiteness of Closure: (11.27), 11
Scheinnullstellensatz: (11.27), 11
Second Uniqueness: (11.27), 11
Serre's: (11.27), 11
Stone’s: (11.27), 11
Tower Law for Integrality: (11.27), 11
Watts: (11.27), 11
Zariski Nullstellensatz: (11.27), 11

Topology
closed point: (11.15), 11
compact: (11.15), 11
irreducible: (11.15), 11
irreducible component: (11.15), 11
locally closed subset: (11.15), 11
quasi-compact: (11.15), 11
totally disconnected: (11.23), 11
very dense subset: (11.23), 11
topology
a-adic: (11.3), 11
separated: (11.3), 11
Zariski: (11.3), 11
totally ordered group: (11.3), 11
Archimedean: (11.3), 11
value group: (11.3), 11

Unitary: (11.3), 11
Universal Mapping Property (UMP)
coequalizer: (11.3), 11
cokernel: (11.3), 11
colimit: (11.3), 11
coproduct: (11.3), 11
direct limit: (11.3), 11
direct product: (11.3), 11
direct sum: (11.3), 11
Formal power series: (11.3), 11
Fraction field: (11.3), 11
free module: (11.3), 11
inverse limit: (11.3), 11
localization: (11.3), 11
polynomial ring: (11.3), 11
pushout: (11.3), 11
residue module: (11.3), 11
residue ring: (11.3), 11
tensor product: (11.3), 11

Valuation
discrete: (11.4), 11
general: (11.4), 11
p-adic: (11.4), 11

258 Index