Problem 1: Let $\mathbb{R}$ be a 3-dimensional Euclidean vector space and let $\{f_1, f_2, f_3\}$ be an arbitrary (not necessarily orthonormal) basis for $\mathbb{R}$. Define a set of vectors $\{e_1, e_2, e_3\}$ by
\[
\begin{align*}
e_1 &= f_1, \\
e_2 &= f_2 + c_{21}e_1, \\
e_3 &= f_3 + c_{31}e_1 + c_{32}e_2.
\end{align*}
\]
i) Calculate the values of the scalars $c_{21}, c_{31}$ and $c_{32}$ that makes $\{e_1, e_2, e_3\}$ a mutually orthogonal set of vectors.

ii) Is the set $\{e_1, e_2, e_3\}$ linearly independent?

iii) Does $\{e_1, e_2, e_3\}$ form an orthonormal basis for $\mathbb{R}$?

Problem 2: Let $\mathbb{R}$ be the Euclidean vector space consisting of “trigonometric polynomials”, a typical vector $p$ having the form
\[
p = p(t) = \sum_{n=0}^{2} \alpha_n \cos nt \quad \text{where the } \alpha \text{'s span all real numbers.}
\]
The natural operations of addition and multiplication by a scalar are in force. The scalar product between two vectors $p$ and $q$ is taken to be
\[
p \cdot q = \int_0^{2\pi} p(t)q(t) \, dt.
\]
Let $A$ be the tensor that carries a vector $p = p(t)$ into its second derivative:
\[
Ap = p''(t).
\]
i) Is $A$ singular or nonsingular?

ii) Is $A$ symmetric?

iii) Determine the eigenvalues of $A$.

Problem 3: Let $\mathbb{R}$ be an arbitrary 3-dimensional vector space and let $A$ be a linear transformation on $\mathbb{R}$. (Note that $\mathbb{R}$ might not be Euclidean and $A$ might not be symmetric.) Suppose that $A$ has three real eigenvalues $\alpha_1, \alpha_2, \alpha_3$, and suppose that they are distinct: $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_1$. Let $a_1, a_2, a_3$ be the corresponding eigenvectors.

i) Show that any pair of these eigenvectors, e.g. $\{a_1, a_2\}$, is a linearly independent pair of vectors.

ii) Next show that $\{a_1, a_2, a_3\}$ is a linearly independent set of vectors.