MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.151 Advanced System Dynamics and Control

Controllability, Observability and the Transfer Function[1]

In this brief note we examine some additional conclusions on system controllability and observability based on the transfer function matrix.

Consider a linear system of order n with r inputs and m outputs:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

with distinct eigenvalues. By taking the Laplace transform of the state-equations, we obtain

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s) = \mathbf{H}_1(s)\mathbf{U}(s) \tag{1}$$

where $\mathbf{H}_1(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}$ is a ($n \times r$) transfer function matrix relating the state-variable responses to the system inputs. Each row is the Laplace domain relationship between a state-variable $X_i(s)$ and the input vector $\mathbf{U}(s)$.

Controllability is determined by the ability to manipulate the system state $\mathbf{x}$ from an initial state to any arbitrary value in finite time, through a suitable (but undefined) choice of the input $\mathbf{U}(s)$. We therefore can make the following additional conclusions concerning system controllability from $\mathbf{H}_1(s)$.

A system is uncontrollable if any of the following conditions are met:

- Any row of the matrix $[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}$ is $\mathbf{0}$. If a zero row exists, it implies that a state is unaffected by any component of the input vector $\mathbf{U}(s)$.
- There exists a linear dependence between the rows of $[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}$. This condition implies that the forced responses of two or more state-variables are linearly related, and therefore cannot be independently manipulated by the input vector $\mathbf{U}(t)$.
- There exists a linear dependence between the rows of the matrix $e^{\mathbf{A}t}\mathbf{B}$. This follows directly because $e^{\mathbf{A}t}\mathbf{B} = \mathcal{L}^{-1}\left\{[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\right\}$, and since the Laplace transform is a one-to-one operator the linear dependence in one domain translates directly to the other.
- A necessary and sufficient condition for controllability is that no single pole of the system is cancelled by a zero in all of the elements of the transfer-function matrix $[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}$. If such cancellation occurs, the system cannot be controlled in the direction of the cancelled mode.
- We state without proof that a system is uncontrollable if for any eigenvalue $\lambda_i, i = 1 \dots n$, of $\mathbf{A}$, the rank of the $(n \times (n + r))$ matrix $[(\lambda_i\mathbf{I} - \mathbf{A}) \mid \mathbf{B}] < n$. (The proof of this is beyond the scope of this handout.)

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Observability defines the ability to determine the initial state of a system from finite observation of the output. Because the effect of the input may be subtracted out, it is sufficient to consider the homogeneous response to determine observability. The response of the system from initial condition $x(0)$ is

$$\begin{aligned} \mathbf{y} &= \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) \\ &= [\mathbf{Q}_1(t) \mid \mathbf{Q}_2(t) \mid \ldots \mid \mathbf{Q}_n(t)] \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix} \end{aligned} \tag{2}$$

where $\mathbf{Q}_i(t)$ is the ith column of $\mathbf{C}e^{\mathbf{A}t}$. It is easy to show that the system is unobservable if the columns of $\mathbf{C}e^{\mathbf{A}t}$, that is $\mathbf{Q}_i(t)$, are linearly dependent. For example, suppose $\mathbf{Q}_2(t) = k\mathbf{Q}_1(t)$, then the output equation becomes

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{Q}_1(t)x_1(0) + k\mathbf{Q}_1(t)x_2(0) + \mathbf{Q}_3(t)x_3(0) + \ldots + \mathbf{Q}_n(t)x_n(0) \\ &= \mathbf{Q}_1(t)(x_1(0) + kx_2(0)) + \mathbf{Q}_3(t)x_3(0) + \ldots + \mathbf{Q}_n(t)x_n(0) \end{aligned}$$

where it can be seen that the two initial condition components $x_1(0)$ and $x_2(0)$ are bound together as a weighted sum in all output equations, and therefore cannot be estimated from measurements on $\mathbf{y}$. We therefore can make the following additional conclusions concerning system observability:

A system is unobservable if any of the following conditions are met:

- Any column of the matrix $\mathbf{C}e^{\mathbf{A}t}$ is $\mathbf{0}$. If a zero column exists, it implies that a state does not appear in any of the output equations, and is thus unobservable. The equivalent Laplace domain statement is that a system is unobservable if any column of the matrix $\mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}$ is $\mathbf{0}$

- There exists a linear dependence between the columns of the matrix $\mathbf{C}e^{\mathbf{A}t}$. This is demonstrated above in Eq. (2). The equivalent Laplace domain condition is that a system is unobservable if there exists a linear dependence between the columns of $\mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}$.

- A necessary and sufficient condition for observability is that no single pole of the system is cancelled by a zero in all of the elements of the matrix $\mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}$. If such cancellation occurs, the cancelled mode cannot be observed in the output.

- A system is unobservable if for any eigenvalue $\lambda_i, i = 1 \ldots n$, of $\mathbf{A}$ the rank of the $(n + m \times n))$ matrix

  $$\begin{bmatrix} \lambda_i\mathbf{I} - \mathbf{A} \\ \cdots \\ \mathbf{C} \end{bmatrix} < n.$$

  The proof of this is beyond the scope of this handout.

Pole/Zero Cancellation and Controllability/Observability: The previous discussion implies that a system in which any pole is cancelled in all of the elements of the matrix $\mathbf{C}\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B}$ will be unobservable. A system will be uncontrollable if a pole is cancelled in all of elements the transfer function matrix $\mathbf{H}_1(s) = \left[s\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B}$ that relates the state vector $\mathbf{X}(s)$ to the input $\mathbf{U}(s)$.

The Effect of Redundant State-Variables: While nobody would intentionally use more state-variables than is necessary to describe the dynamics of a physical system, most modeling methods will occasionally generate linearly dependent state equations. Consider the system with a minimum set of state equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

and suppose, for some reason that k more state-variables $\mathbf{z}$, proportional to those present in $\mathbf{x}$, have also been defined. The linear dependence is described by

$$\mathbf{z} = \mathbf{F}\mathbf{x}$$

where $\mathbf{F}$ is $(k \times n)$. Then

$$\dot{\mathbf{z}} = \mathbf{F}\dot{\mathbf{x}} = \mathbf{F}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})$$

The system model, including the superfluous variables, is then

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u}$$

where $\hat{\mathbf{x}} = [\mathbf{x} \mid \mathbf{z}]^T$, and

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{FA} & \mathbf{0} \end{bmatrix} \qquad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ \mathbf{FB} \end{bmatrix}$$

Now consider a transformation of variables given by $\hat{\mathbf{x}} = \mathbf{P}\bar{\mathbf{x}}$ where the $(n+k) \times (n+k)$ transformation matrix $\mathbf{P}$ is

$$\mathbf{P} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{F} & \mathbf{I}_k \end{bmatrix} \qquad \text{and} \qquad \mathbf{P}^{-1} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{F} & \mathbf{I}_k \end{bmatrix}$$

where $\mathbf{I}_n$ and $\mathbf{I}_k$ are the $(n \times n)$ and $(k \times k)$ identity matrices, so that

$$\bar{\mathbf{A}} = \mathbf{P}^{-1}\hat{\mathbf{A}}\mathbf{P} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{F} & \mathbf{I}_k \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{FA} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{F} & \mathbf{I}_k \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

and

$$\bar{\mathbf{B}} = \mathbf{P}^{-1}\hat{\mathbf{B}} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{F} & \mathbf{I}_k \end{bmatrix} \begin{bmatrix} \mathbf{B} \\ \mathbf{FB} \end{bmatrix} = \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}$$

Then in the transformed states

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}\mathbf{u}$$

and if we write $\bar{\mathbf{x}} = [\mathbf{x}' \mid \mathbf{z}']^T$, the transformed state equations are

$$\begin{aligned} \dot{\mathbf{x}}' &= \mathbf{A}\mathbf{x}' + \mathbf{B}\mathbf{u} \\ \dot{\mathbf{z}}' &= \mathbf{0} \end{aligned}$$

so that the transformation $\mathbf{P}$ has reduced the k excess state equations to a vector $\mathbf{z}'$ containing k integrators with no inputs driving them - these excess states cannot be controlled, and the system is therefore uncontrollable.