

## Linear Graph Modeling: State Equation Formulation<sup>1</sup>

### 1 State Variable System Representation

Linear graph system models provide a graphical representation of a system model and the interconnection of its elements. A set of differential and algebraic equations which completely define the system may be derived directly from the linear graph model. In this handout we develop a procedure for deriving a specific set of differential equations, known as the *state equations*, from the system linear graph. These equations are expressed in terms of a set of *state variables* and provide a basis for determining the system response to external inputs.

#### 1.1 State Equation Based Modeling Procedure

The complete system model for a linear time-invariant system consists of (i) a set of  $n$  state equations, defined in terms of the matrices  $\mathbf{A}$  and  $\mathbf{B}$ , and (ii) a set of output equations that relate any output variables of interest to the state variables and inputs, and expressed in terms of the  $\mathbf{C}$  and  $\mathbf{D}$  matrices. The task of modeling the system is to derive the elements of the matrices, and to write the system model in the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (1)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}. \quad (2)$$

The matrices  $\mathbf{A}$  and  $\mathbf{B}$  are properties of the system and are determined by the system structure and elements. The output equation matrices  $\mathbf{C}$  and  $\mathbf{D}$  are determined by the particular choice of output variables.

The overall modeling procedure developed in this chapter is based on the following steps:

1. Determination of the system order  $n$  and selection of a set of state variables from the linear graph system representation.
2. Generation of a set of state equations and the system  $\mathbf{A}$  and  $\mathbf{B}$  matrices using a well defined methodology. This step is also based on the linear graph system description.
3. Determination of a suitable set of output equations and derivation of the appropriate  $\mathbf{C}$  and  $\mathbf{D}$  matrices.

### 2 Linear Graphs and System Structural Properties

#### 2.1 Linear Graph Properties

The derivation of the state equations in this chapter is based on the use of the system linear graph model. A linear graph with  $B$  branches represents  $B$  system elements, each with a known elemental

---

<sup>1</sup>D. Rowell - Revised: 9/16/04

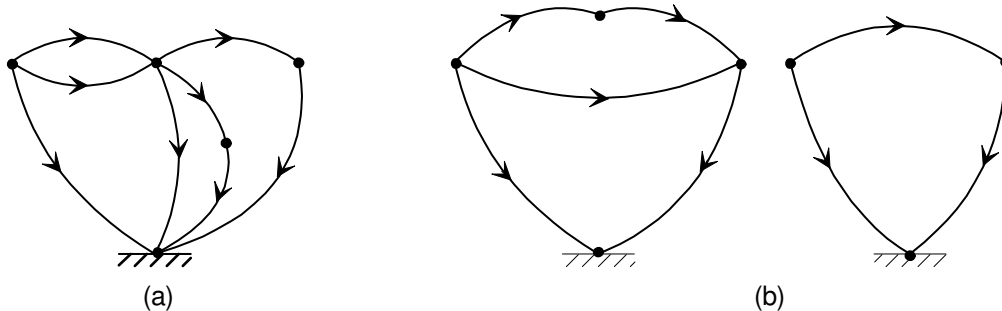


Figure 1: Examples of (a) a connected system graph, and (b) an unconnected system graph.

equation or source function. The graph also represents the structure of the element interconnections, in terms of the continuity and compatibility constraint equations. In the following sections we use the properties of linear graphs to (i) derive the system structural constraints, (ii) define the set of state variables, and (iii) provide a systematic technique for deriving the system state equations [4-8]. The following definitions are introduced:

**System Graph:** The oriented linear graph model of a system.

**Connected Graph** A system graph in which a path exists between all pairs of nodes. A path is said to exist if the node pair is joined by a series of branches.

Figure 1 shows a connected graph along with a system graph which is not connected. System graphs for systems consisting of one-port elements are usually connected graphs, while systems that include two-port elements introduced in a separate handout may not generate connected graphs. In this chapter we assume that all system graphs are connected graphs.

Consider a system represented by a connected linear graph with  $B$  branches of which  $S$  are active source elements and the remaining  $B - S$  branches represent passive one-port elements. Each branch has an across-variable and a through-variable, giving a total of  $2B$  variables within the system. Of these  $S$  are prescribed source variables, so that there are  $2B - S$  unknowns in the system; we therefore require a total of  $2B - S$  independent equations in these unknowns in order to determine all system variables.

There are  $B - S$  elemental equations relating the across and through-variables for the passive branches. In addition the system structure, defined by the compatibility and continuity conditions, may be used to generate an additional  $B$  linearly independent constraint equations. The total set of  $2B - S$  elemental and structural equations may be algebraically manipulated to produce the state and output equations. The following example illustrates these relationships for an electrical system model:

---

### ■ Example

The electrical system shown in Fig. 2 consists of a capacitor  $C$ , inductor  $L$ , and a resistor  $R$  connected as shown and driven by a voltage source  $V_s(t)$ . Its linear graph contains four branches each of which has an across-variable and a through-variable, giving a total of eight system variables of which seven ( $i_C$ ,  $v_C$ ,  $i_L$ ,  $v_L$ ,  $i_R$ ,  $v_R$ , and  $i_s$ ) are unknown. Each of the three passive elements is described by an elemental equation:

$$\frac{dv_C}{dt} = \frac{1}{C}i_C \quad (3)$$

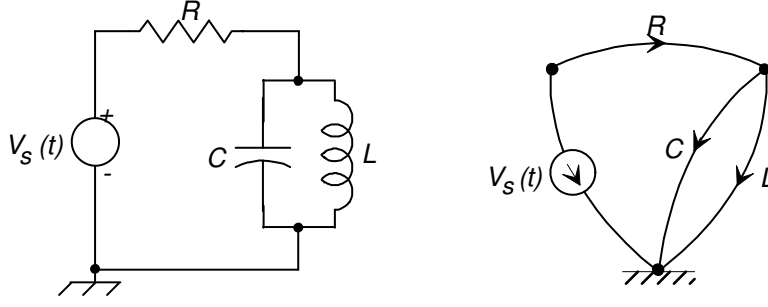


Figure 2: An electrical system and its linear graph

$$\frac{di_L}{dt} = \frac{1}{L}v_L \quad (4)$$

$$i_R = \frac{1}{R}v_R \quad (5)$$

giving three equations in six unknowns. The system structure defined by the linear graph imposes additional constraints; the three variables on the right-hand side may be eliminated by using (i) a continuity equation

$$i_C = i_R - i_L \quad (6)$$

to define  $i_C$ , and (ii) two compatibility equations

$$v_L = v_C \quad (7)$$

$$v_R = V_s - v_C \quad (8)$$

to define  $v_L$  and  $v_R$ . Substitution of Eqs. (iv) through (vi) into Eqs. (i) through (iii) and some algebraic manipulation, produces a pair of coupled differential equations:

$$\frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L + \frac{1}{RC}V_s \quad (9)$$

$$\frac{di_L}{dt} = \frac{1}{L}v_C \quad (10)$$

which are a pair of coupled first-order differential equations in the form of Eqs. (??). These are a set of two state equations for this system in terms of state variables  $v_C$  and  $i_L$ . Eqs. (vii) and (viii) allow the system **A** and **B** matrices to be written:

$$\mathbf{A} = \begin{bmatrix} -1/RC & -1/C \\ 1/L & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1/RC \\ 0 \end{bmatrix}. \quad (11)$$

Equations (i) through (vi) may be used to generate output equations in the form of Eqs. (??). For example, if the variables  $i_R$ ,  $v_R$ ,  $v_L$ , and  $i_C$  are of interest:

$$i_R = -\frac{1}{R}v_C + \frac{1}{R}V_s \quad (12)$$

$$v_R = -v_C + V_s \quad (13)$$

$$v_L = v_C \quad (14)$$

$$i_C = -\frac{1}{R}v_C - i_L + \frac{1}{R}V_s \quad (15)$$

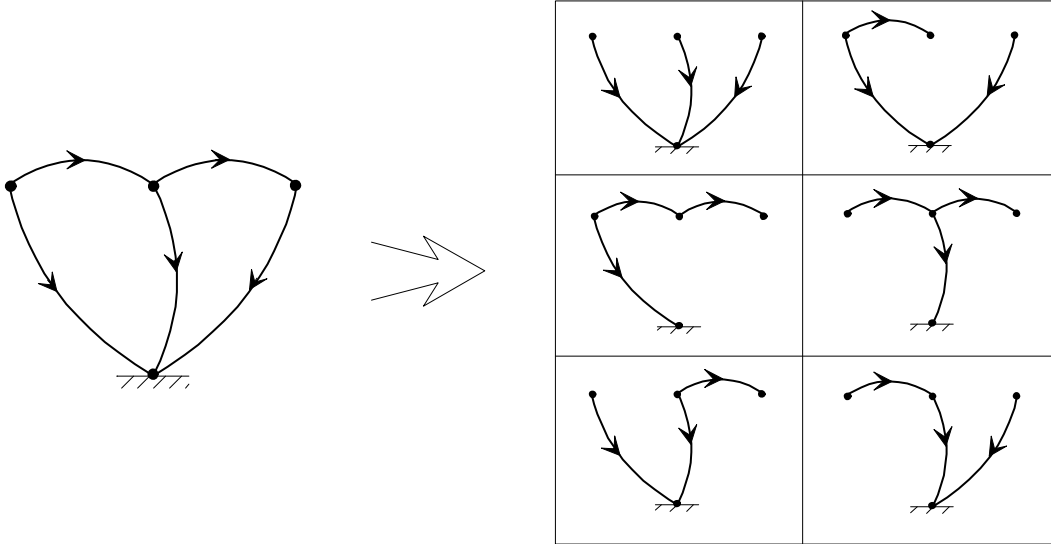


Figure 3: A system graph with four nodes and five branches, and several trees derived from the graph.

so that if the output vector is defined to be  $\mathbf{y} = [i_R, v_R, v_L, i_C]^T$ , the  $\mathbf{C}$  and  $\mathbf{D}$  matrices are:

$$\mathbf{C} = \begin{bmatrix} -1/R & 0 \\ -1 & 0 \\ 1 & 0 \\ -1/R & -1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1/R \\ 1 \\ 0 \\ 1/R \end{bmatrix}. \quad (16)$$

## 2.2 Graph Trees

The procedure for generating the system equations is based on relationships defined by a *graph tree*. A tree is defined to be a subgraph of the system graph which contains:

- (1) all of the graph nodes, and
- (2) the maximum number of branches of the system graph that be included without creating any closed loops.

Branches of the system graph that are not included in a tree are known as *links*. In general several different graph trees may be formed from any linear graph, as shown in Fig. 3. The number of branches in a tree  $B_T$ , and the number of tree links  $B_L$ , depends on the number of nodes  $N$  and the number of branches  $B$  in the graph. For a connected graph, the first branch entered into a tree connects two nodes, while all of the subsequent branches connect one additional node until the maximum number ( $N - 1$ ) of branches is entered without forming any closed loops. The number of tree links is simply the total number of branches less those in the graph tree:

$$B_T = N - 1 \quad (17)$$

$$B_L = B - B_T = B - N + 1 \quad (18)$$

Each tree link is a system graph branch that forms a closed loop when added to a graph tree.

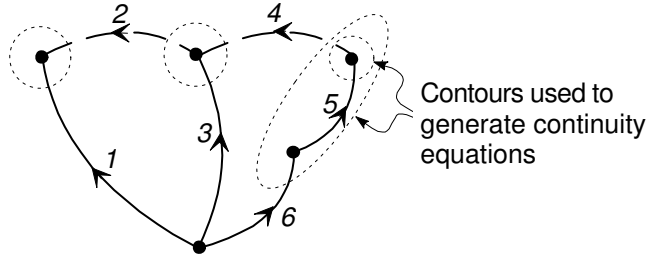


Figure 4: A system graph tree, with the links shown as dotted lines, and a set of contours used to generate continuity equations.

### 2.3 System Graph Structural Constraints

A graph tree may be used to generate a set of  $B$  linearly independent compatibility and continuity equations that are particularly useful in system equation development. We define the following classes of variables based on a given tree:

**Primary variables:** The system primary variables are the across-variables on tree branches, and the through-variables on the tree links.

**Secondary variables:** Conversely, the secondary variables are the through-variables on the tree branches, and the across-variables on tree links.

The compatibility constraint equations relate across-variables around any closed loop in the system graph. Consider the tree of a linear graph with  $N = 5$  and  $B = 6$  shown in Fig. 4. There are  $N - 1 = 4$  branches and  $B - N + 1 = 2$  links in any tree of this graph. Each time a link is placed in the tree a closed loop is formed, and a compatibility equation may be written for the loop. For the tree shown in Fig. 4 the two compatibility equations generated by replacing the links are:

$$\begin{aligned} v_2 &= v_1 - v_3 \\ v_4 &= v_3 - v_5 - v_6. \end{aligned}$$

In this tree structure  $v_2$  and  $v_4$  are secondary variables; each of the above compatibility equation specifies one of these secondary variables in terms of the primary across-variables. In general, each time a loop is formed by placing a link in a tree, the across-variable on that link may be written in terms of the across-variables on the tree branches; in other words the resulting compatibility equation contains only one secondary variable.

The continuity equations represent a set of constraints on through-variables passing through any closed contour on the graph. If a closed contour is drawn on a tree such that it cuts only one tree branch, then the through-variable on that branch (a secondary variable), may be expressed in terms of the through-variables on tree links, which are primary variables. In Fig. 4 a set of four contours has been drawn so that each cuts a single tree branch. For the contours shown the following continuity equations may be written:

$$\begin{aligned} f_1 &= -f_2 \\ f_3 &= f_2 - f_4 \\ f_5 &= f_4 \\ f_6 &= f_4. \end{aligned}$$

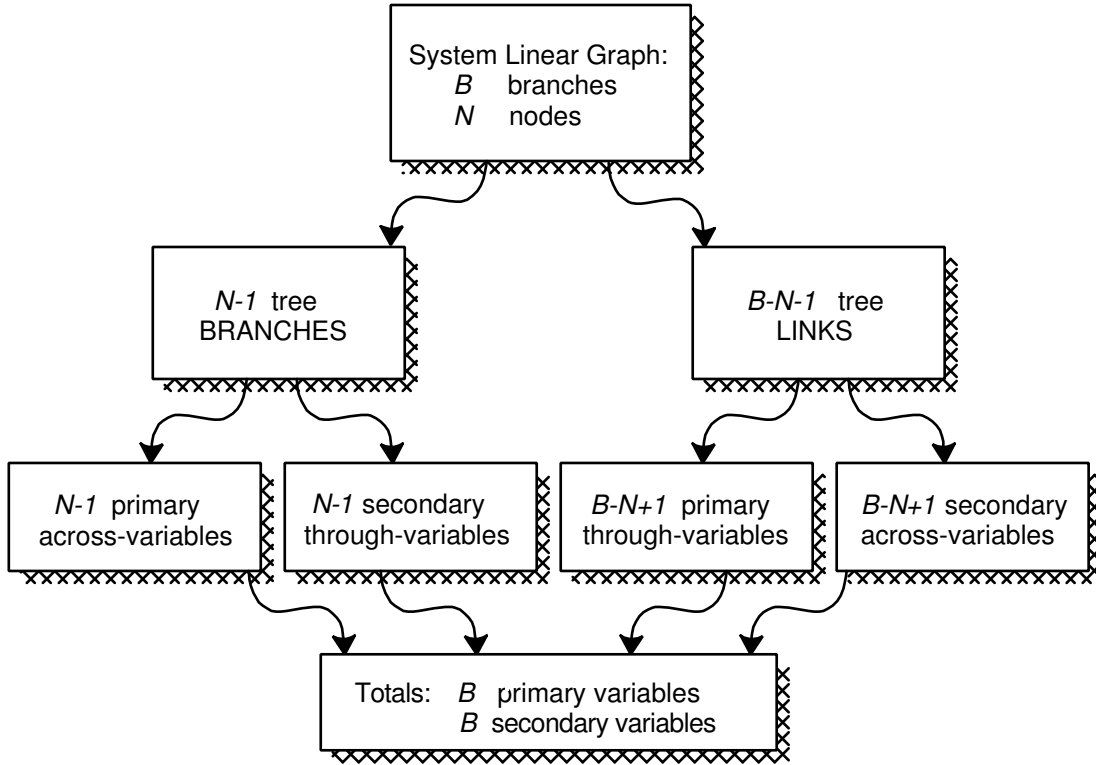


Figure 5: System graph variables.

The secondary through-variables on the tree branches  $f_1$ ,  $f_3$ ,  $f_5$ , and  $f_6$  are expressed directly in terms of the through-variables on the links ( $f_2$  and  $f_4$ ). This particular set of contours has generated a set of continuity equations that express the secondary through-variables in terms of the primary through-variables.

In summary, each of the  $B - N + 1$  tree links generates a compatibility equation involving a single secondary across-variable, and each of the  $N - 1$  tree branches generates a continuity equation that expresses its secondary through-variable in terms of the link primary through-variables. The tree therefore generates a set of  $B$  constraint equations that can be used in the equation formulation method to eliminate secondary variables from elemental equations. Figure 5 shows the division of the system variables into primary and secondary variables using a graph tree.

## 2.4 The System Normal Tree

We may form a special graph tree, known as a *normal tree*, and use it to define (i) the system primary and secondary variables, (ii) the system order  $n$ , (iii) the state variables, and (iv) a set of independent compatibility and continuity equations. A normal tree for a connected system graph is formed by the following steps:

**Step 1:** Draw the system graph nodes.

**Step 2:** Include *all* across-variable sources as tree branches. (If all across-variable sources cannot be included in the normal tree, then the across-variable sources must form a loop and compatibility is violated.)

**Step 3:** Include as many as possible of the A-type energy storage elements as tree branches. (Any

A-type element which cannot be included in the normal tree is a *dependent* energy storage element as described below.)

**Step 4:** Attempt to complete the tree, which must contain  $N - 1$  branches, by including as many as possible D-type dissipative elements in the tree. It may not be possible to include all D-type elements.

**Step 5:** If the tree is not complete after the addition of D-type elements, add the minimum number of T-type energy storage elements required to complete the tree. (Any T-type element included in the tree at this point is a *dependent* energy storage element.)

**Step 6:** Examine the tree to determine if any through-variable sources are required to complete the tree. If any through-variable source can be inserted into the normal tree, then that source cannot be independently specified and continuity is violated.

The normal tree effectively defines a set of *independent* energy storage elements, that is the set of elements whose stored energy may be independently set and controlled. If in step (3) above, an A-type element cannot be included in a normal tree it implies that a loop has been formed from a combination of across-variable sources and A-type elements. The compatibility equation for that loop specifies that the across-variable on the A-type element link is dependent on that of across-variables on other A-type elements and sources. This is illustrated in Fig. 6a, where an electrical system and a normal tree are shown. It is not possible to include both capacitors in the normal tree; if capacitor  $C_2$  is placed in the tree the resulting compatibility equation is:

$$v_{C_2} = V_s - v_{C_1}.$$

The energy storage variables (voltage) on the two capacitors are therefore linearly dependent. In this case either  $C_1$  or  $C_2$ , but not both can be included in the normal tree.

Similarly, in step (5) any T-type element that is a branch of a normal tree is a dependent energy storage element. When the tree is complete the resulting continuity equation generated by a contour cutting this branch expresses its through-variable in terms of link primary variables. The continuity equation for that branch therefore specifies a linear dependence between its through-variable and other T-type or source through-variables. This is illustrated for a mechanical system in Fig. 6b where a spring  $K$  is used to transmit energy between a force source  $F_s(t)$  and a mass  $m$ . The spring can be inserted into the normal tree. Continuity applied to the spring element requires that:

$$f_k = F_s(t),$$

or in other words the energy stored in the spring is defined entirely by the force source  $F_s(t)$ . The spring  $K$  is a dependent energy storage element.

Dependent energy storage elements do not generate independent state equations. From the normal tree we make the following definitions:

- (a) The system primary variables are the across-variables on the normal tree branches and the through-variables on its links.
- (b) The system secondary variables are the through-variables on the normal tree branches and the across-variables on its links.
- (c) The system order is the number of independent energy storage elements in the system, that is the sum of the number of A-type elements in the normal tree branches and the number of T-type elements among the links.

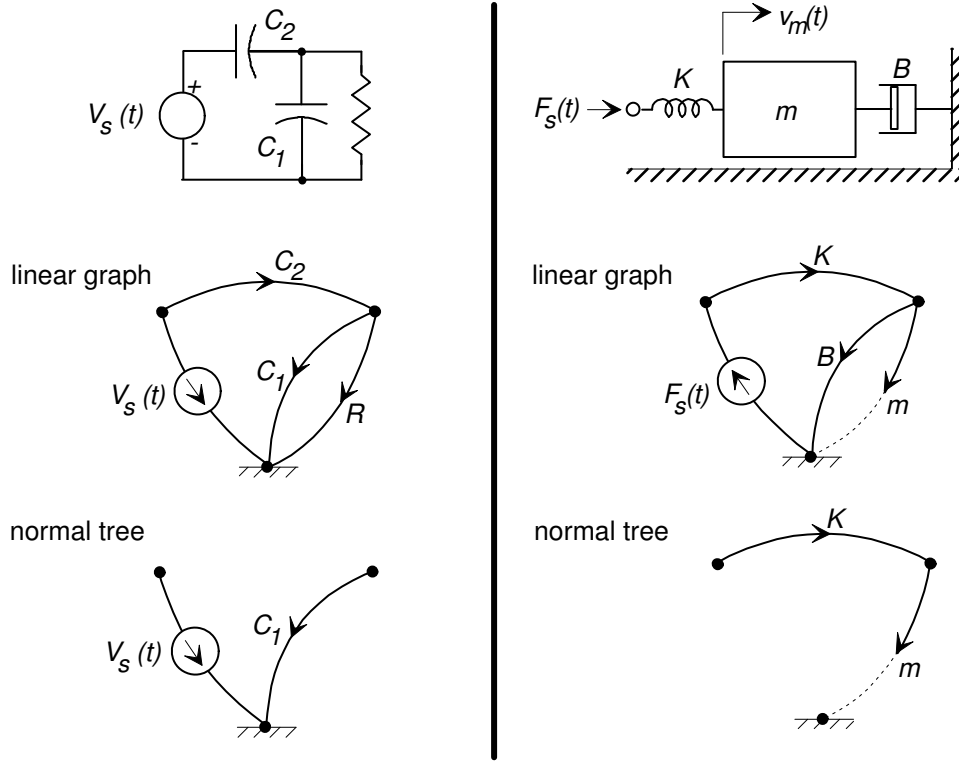


Figure 6: Two systems with dependent energy storage elements and their normal trees; (a) an electrical system with dependent capacitors, and (b) a mechanical system with a dependent spring.

- (d) The  $n$  state variables are selected as the energy storage variables among the primary variables, that is the across-variables on A-type elements in the normal tree branches, and the through-variables on T-type elements in the links.

Figure 7 shows two systems along with their system graphs and normal trees. The mechanical system in Fig. 7a has four energy storage elements, the normal tree indicates that they are all independent, and the system order is four. The normal tree defines the state variables to be the velocities ( $v_{m_1}$  and  $v_{m_2}$ ) of the two mass elements, and the forces ( $f_{K_1}$  and  $f_{K_2}$ ) on the two springs. The electrical system in Fig. 7b contains five energy storage elements, but the normal tree indicates that capacitors  $C_2$  and  $C_3$  are dependent. The order of this system is four, and the state variables for the normal tree shown are  $v_{C_1}$ ,  $v_{C_2}$ ,  $i_{L_1}$ , and  $i_{L_2}$ .

#### 2.4.1 Special Cases in which the Procedure Results in “Excess” State Variables

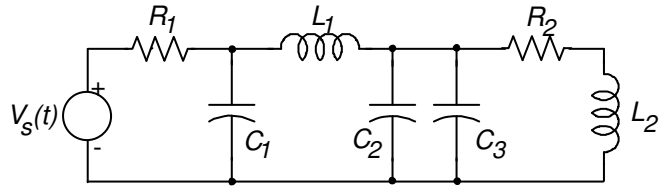
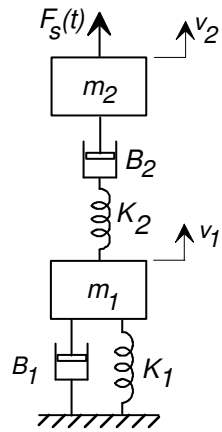
Two special cases in which the procedure outlined above <sup>2</sup> results in a formulation in which the number of state variables exceeds the number of independent energy storage elements are:

- (1) Systems which contain two or more A-type elements in direct series connection.
- (2) Systems which contain two or more T-type elements in direct parallel connection.

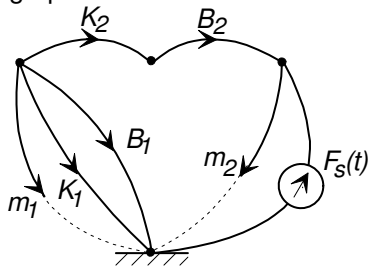
Illustration of these two special cases is provided in Fig. 8. In the two cases the number of state

<sup>2</sup>In these special cases, the procedure results in a state variable formulation which is *uncontrollable*, that is the across-variables on each of the direct series A-type elements and the through-variable on each of the direct parallel T-type elements may not be independently specified and controlled [2,3].

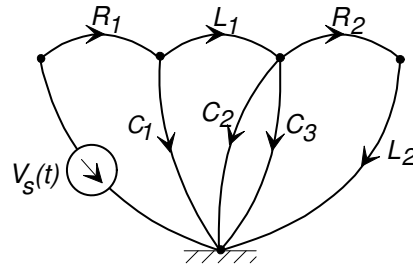




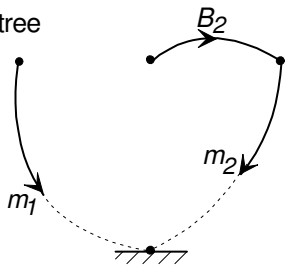
Linear graph



Linear graph



Normal tree



Normal tree

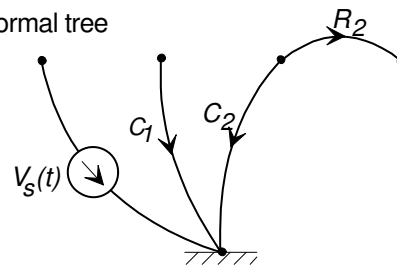


Figure 7: Two systems with their linear graphs and normal trees.

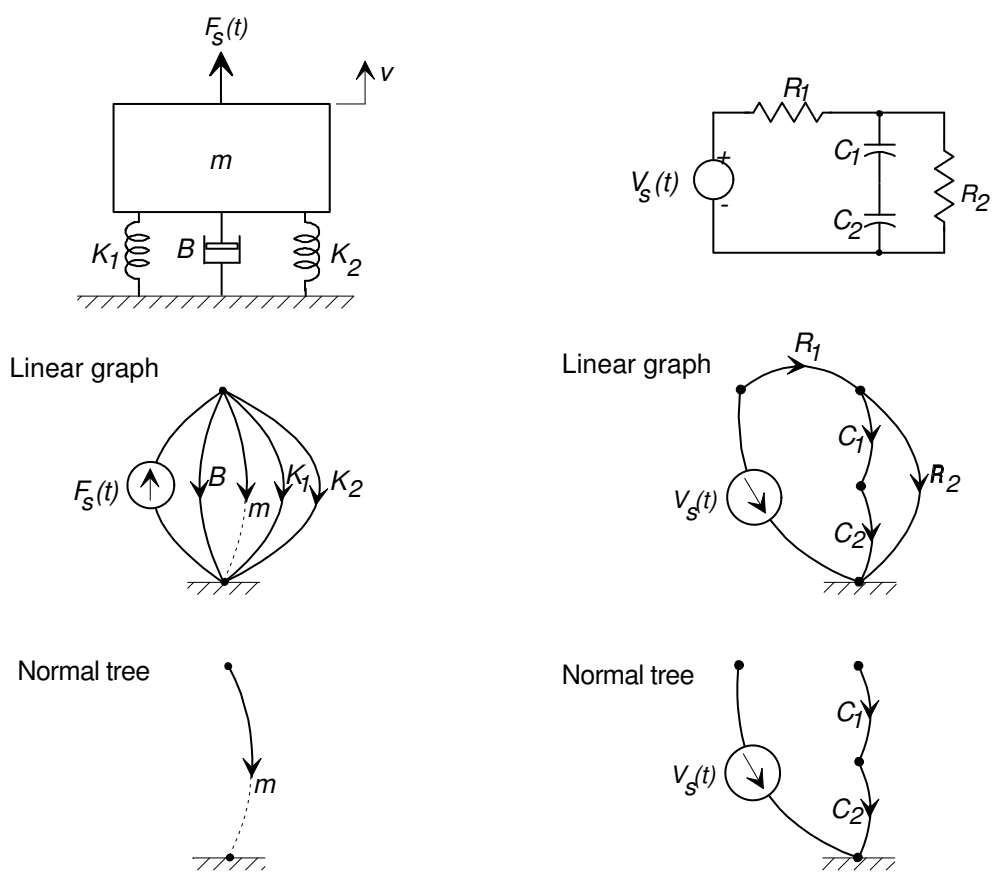


Figure 8: Examples of two systems with "excess" state variables.

variables identified by forming a normal tree exceeds the number of independent energy storage elements. The difficulty represented by the two special cases described may be avoided by combining all A-type energy storage elements directly in series into a single equivalent A-type element, and all T-type elements directly in parallel into a single equivalent element. The general expressions for the equivalent elements are indicated in Fig. 9. When the elements are combined and the equivalent system linear graph is used to generate the normal tree, the number of state variables in the combined system is equal to the number of independent energy storage elements in the combined system. Since *controllable* representations of systems are desired, all A-type elements directly in series and all T-type elements directly in parallel should be combined before determining the system normal tree.

### 3 State Equation Formulation

The system normal tree may be used to generate a set of state equations in terms of the energy storage variables on the  $n$  independent energy storage elements. In a system graph with  $B$  branches, of which  $S$  represent ideal source elements, there are  $2(B - S)$  system variables associated with the passive branches; one across and one through-variable on each branch. On each branch one of these variables is a primary variable while the other is a secondary variable. The desired  $n$  state variables are a subset of the  $B - S$  primary variables. There are  $(B - S)$  elemental equations relating these primary and secondary variables for the passive branches; the normal tree is used to generate  $B - S$  continuity and compatibility equations that can be used to eliminate the secondary variables associated with the passive elements.

The state equations are formulated in two steps:

- (1) Derivation of a set of  $B - S$  differential and algebraic equations in terms of primary variables only by starting with the passive elemental equations, and using  $B - S$  compatibility and continuity equations to eliminate all secondary variables.
- (2) Algebraic manipulation of this set of  $B - S$  equations to produce  $n$  differential equations in the  $n$  state variables and the  $S$  specified source variables.

Since sources have one variable independently specified, only  $B - S$  elemental equations for the passive elements need to be written. It is convenient to divide the number of sources  $S$  into across-variable sources  $S_A$  and through-variable sources  $S_T$ , so that  $S = S_A + S_T$ . The secondary variables may be eliminated from these equations by using a total of  $B - S$  independent compatibility and continuity equations formed from (i)  $N - 1 - S_A$  continuity equations, and (ii)  $B - N + 1 - S_T$  compatibility equations. (The secondary variables associated with sources do not enter directly into the state equation formulation, therefore  $S_A$  continuity and  $S_T$  compatibility equations do not need to be considered.)

A systematic procedure for deriving the  $n$  state equations is given below:

**Step 1:** Generate a normal tree from the connected system graph.

**Step 2:** From the branches and links of the normal tree identify the primary and secondary variables. Define the system order  $n$  as the number of independent energy storage elements. Select the state variables as across-variables on A-type energy storage elements in the normal tree branches, and through-variables on T-type energy storage elements in the links.

**Step 3:** Write the  $B - S$  elemental equations for the passive (non-source) elements explicitly in terms of their primary variables, that is with the primary variable on the left-hand side of the

equations. Note that the derivatives of the  $n$  state variables should appear on the left-hand side of the elemental equations for all independent energy storage elements.

**Step 4:** Write  $N - 1 - S_A$  independent continuity equations involving only one secondary through-variable (a tree branch through-variable) by applying the continuity condition to a set of  $N - 1 - S_A$  contours that each “cut” only one passive branch of the normal tree. Express each equation explicitly in terms of the secondary through-variable.

**Step 5:** Write  $B - N + 1 - S_T$  independent compatibility equations involving only one secondary across-variable (a tree link across-variable) by placing each of the passive links back in the tree to form a loop and writing the resulting compatibility equation. Express each equation explicitly in terms of the secondary across variable.

**Step 6:** Use the  $N - 1 - S_A$  continuity and the  $B - N + 1 - S_T$  compatibility equations (a total of  $B - S$  equations) to eliminate all secondary variables from the  $B - S$  elemental equations by direct substitution.

**Step 7:** Reduce the resulting  $B - S$  equations in the primary variables to  $n$  state equations in the  $n$  state variables and the  $S$  source variables.

**Step 8:** Write the resulting state equations in the standard form.

The application of this procedure is illustrated in the examples that follow, which show that after the normal tree is formed the subsequent steps in the procedure may be completed in a straight forward manner.

### ■ Example

A set of state equations describing the mechanical system shown in Fig. 10 may be derived directly from the system graph. The system graph has three branches ( $B = 3$ ), two nodes ( $N = 2$ ), and a single force source ( $S = 1$ ). The normal tree therefore contains  $N - 1 = 1$  branch. The rules for constructing the tree specify that the branch should contain the mass element  $m$ . The spring element  $K$  and the source element  $F$  therefore form the tree links. From the normal tree in Fig. 10c:

Primary variables:  $F_s(t), v_m, F_K$   
 Secondary variables:  $v_s, F_m, v_K$   
 System order:  $2$   
 State variables:  $v_m, F_K$

1. The  $B - S = 2$  elemental equations may be written in terms of the primary variables as:

$$\text{primary variables} \left\{ \begin{array}{l} \frac{dv_m}{dt} = \frac{1}{m} F_m \\ \frac{dF_K}{dt} = K v_K \end{array} \right\} \text{secondary variables} \quad (19)$$

2. The single continuity equation ( $N - 1 - S_A = 2 - 1 - 0 = 1$ ) required to eliminate  $F_m$  may be written using the closed contour shown on Fig. 10c:

$$F_m = F_s - F_K. \quad (20)$$

3. The single compatibility equation ( $B - N + 1 - S_T = 3 - 2 + 1 - 1 = 1$ ) required to eliminate the secondary variable  $v_K$  may be found by placing the spring element in the normal tree, Fig. 10c, and writing the loop equation:

$$v_K = v_m. \quad (21)$$

4. The secondary variables  $F_m$  and  $v_K$  may be eliminated by substituting Eqs. (ii) and (iii) into the the elemental equations to yield:

$$\begin{aligned} \frac{dv_m}{dt} &= \frac{1}{m} (F_s - F_K) \\ \frac{dF_K}{dt} &= kv_m \end{aligned} \quad (22)$$

Equations (iv) express the derivatives of the state variables directly in terms of the state variables and the input. They are the desired pair of state equations.

5. The pair of state equations may be written in the standard matrix notation as:

$$\begin{bmatrix} \dot{v}_m \\ \dot{F}_K \end{bmatrix} = \begin{bmatrix} 0 & -1/m \\ K & 0 \end{bmatrix} \begin{bmatrix} v_m \\ F_K \end{bmatrix} + \begin{bmatrix} 1/m \\ 0 \end{bmatrix} F_s. \quad (23)$$

If the output variable of interest is the velocity of the mass, then an output equation may be written in matrix form:

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_m \\ F_K \end{bmatrix} + [0] F_s \quad (24)$$

## ■ Example

A mechanical system and its oriented linear graph are shown in Fig. 11. In the graph two T-type energy storage elements are connected directly in parallel; therefore to eliminate the generation of an excess state variable, the two spring elements are combined into one equivalent spring with an effective spring constant  $K = K_1 + K_2$ . The equivalent linear graph is shown in Fig. 11b, with the system normal tree in Fig. 11c with  $N - 1 = 1$  branch containing the mass  $m$ . From the normal tree:

Primary variables:  $F_s(t), v_m, F_K, F_B$   
 Secondary variables:  $v_s, F_m, v_K, v_B$   
 System order: 2  
 State variables:  $v_m, F_K$

1. The  $B - S = 3$  elemental equations written in terms of the primary variables are:

$$\text{primary variables} \left\{ \begin{array}{l} \frac{dv_m}{dt} = \frac{1}{m} F_m \\ \frac{dF_K}{dt} = K v_K \\ F_B = B v_B \end{array} \right\} \text{secondary variables} \quad (25)$$

2. The single ( $N-1-S_A = 1$ ) continuity equation required to eliminate the secondary through variable,  $F_m$ , is found by drawing a closed contour that cuts the mass tree branch as shown in Fig. 11:

$$F_m = F_s - F_B - F_K \quad (26)$$

3. The  $B-N+1-S_T = 2$  compatibility equations required to eliminate the secondary variables  $v_K$  and  $v_B$  are written by considering the loops formed by adding the two links back into the tree shown in Fig. 11c:

$$v_K = v_m \quad (27)$$

$$v_B = v_m \quad (28)$$

4. All secondary variables may be eliminated from the elemental equations by direct substitution of the three continuity and compatibility equations to yield:

$$\frac{dv_m}{dt} = \frac{1}{m} (F_s - F_B - F_K) \quad (29)$$

$$\frac{dF_K}{dt} = K v_m \quad (30)$$

$$F_B = B v_m \quad (31)$$

5. These  $B-S = 3$  equations in terms of primary variables may be combined to give the two desired state equations:

$$\frac{dv_m}{dt} = \frac{1}{m} (-B v_m - F_K + F_s) \quad (32)$$

$$\frac{dF_K}{dt} = K v_m \quad (33)$$

$$(34)$$

6. Finally the pair of state equations may be written in the standard matrix form:

$$\begin{bmatrix} \dot{v}_m \\ \dot{F}_K \end{bmatrix} = \begin{bmatrix} -B/m & -1/m \\ K & 0 \end{bmatrix} \begin{bmatrix} v_m \\ F_K \end{bmatrix} + \begin{bmatrix} 1/m \\ 0 \end{bmatrix} F_s(t) \quad (35)$$

## ■ Example

A mechanical rotational system has the linear graph shown in Fig. 12b. The linear graph contains four elements ( $B = 4$ ) and three nodes ( $N = 3$ ). The normal tree, shown in Fig. 12c, contains the angular velocity source and the rotary inertia. From the normal tree:

Primary variables:	$\Omega_s(t)$ ,	$\Omega_J$ ,	$T_{B_1}$ ,	$T_{B_2}$
Secondary variables:	$T_s$ ,	$T_J$ ,	$\Omega_{B_1}$ ,	$\Omega_{B_2}$
System order:	1			
State variable:	$\Omega_J$			

1. The  $B - S = 4 - 1 = 3$  elemental equations written in terms of the primary variables are:

$$\text{primary variables} \left\{ \begin{array}{l} \frac{d\Omega_J}{dt} = \frac{1}{J} T_J \\ T_{B_1} = B_1 \Omega_{B_1} \\ T_{B_2} = B_2 \Omega_{B_2} \end{array} \right\} \text{secondary variables} \quad (36)$$

2. The secondary variable  $T_J$  may be eliminated by the single ( $N - 1 - S_A = 1$ ) continuity equation found from the closed contour shown in Fig. 12c:

$$T_J = T_{B_2} - T_{B_1}. \quad (37)$$

3. The two secondary variables  $\Omega_{B_1}$  and  $\Omega_{B_2}$  may be eliminated from the  $B - N + 1 - S_T = 2$  compatibility equations formed by placing the two links in the normal tree:

$$\begin{array}{l} \Omega_{B_1} = \Omega_s - \Omega_J \\ \Omega_{B_2} = \Omega_J. \end{array} \quad (38)$$

4. The secondary variables may be eliminated from the elemental equations (i) using (ii) and (iii):

$$\begin{array}{l} \frac{d\Omega_J}{dt} = \frac{1}{J} (T_{B_2} - T_{B_1}) \\ T_{B_1} = B_1 \Omega_J. \\ T_{B_2} = B_2 (\Omega_s - \Omega_J) \end{array} \quad (39)$$

5. By combining the elemental equations in (iv) to eliminate  $T_{B_1}$  and  $T_{B_2}$ , the single state equation may be written:

$$\frac{d\Omega_J}{dt} = \frac{1}{J} [-(B_1 + B_2) \Omega_J + B_2 \Omega_s] \quad (40)$$

6. The state equation may be expressed in matrix form as:

$$[\dot{\Omega}_J] = \left[ -\frac{(B_1 + B_2)}{J} \right] [\Omega_J] + \left[ \frac{B_2}{J} \right] [\Omega_s]. \quad (41)$$

7. If the flywheel angular velocity  $\Omega_J$  is selected as the output variable, the output equation in matrix form is:

$$\Omega_J = [1] [\Omega_J] + [0] [\Omega_s]. \quad (42)$$

## ■ Example

An electrical circuit is shown in Fig. 13a together with its oriented linear graph model in Fig. 13b. For the model the number of nodes and branches are  $N = 4$  and  $B = 4$ . The normal tree, shown in Fig. 13c, has  $N - 1 = 3$  branches which include, the voltage source  $V_s$ , the capacitor  $C$  and the resistor  $R$ . The inductor branch  $L$  remains in the tree links.

From the normal tree:

$$\begin{array}{ll} \text{Primary variables:} & V_s(t), \quad v_C, \quad v_R, \quad i_L \\ \text{Secondary variables:} & i_s, \quad i_C, \quad i_R, \quad v_L \\ \text{System order:} & 2 \\ \text{State variables:} & v_C, \quad i_L \end{array}$$

The state equations may be derived in the following steps:

1. The  $B - S = 3$  elemental equations written in terms of the primary variables:

$$\text{primary variables} \left\{ \begin{array}{l} \frac{dv_C}{dt} = \frac{1}{C} i_C \\ \frac{di_L}{dt} = \frac{1}{L} v_L \\ v_R = R i_R \end{array} \right\} \text{secondary variables} \quad (43)$$

2. The  $N - 1 - S_A = 2$  continuity equations required to eliminate the two secondary through variables,  $i_C$ , and  $i_R$ , are found by drawing a pair of closed contours that cut only one normal tree branch, giving:

$$i_C = i_L \quad (44)$$

$$i_R = i_L \quad (45)$$

3. The  $B - N + 1 - S_T = 1$  compatibility equation to eliminate the secondary variable  $v_L$  is written by adding the inductor back into the tree and writing the equation:

$$v_L = V_s - v_R - v_C \quad (46)$$

4. The secondary variables may be eliminated from the right hand side of the elemental equations by direct substitution from the continuity and compatibility equations to generate three equations in the primary variables:

$$\frac{dv_C}{dt} = \frac{1}{C} i_L \quad (47)$$

$$\frac{di_L}{dt} = \frac{1}{L} (V_s - v_R - v_C) \quad (48)$$

$$v_R = R i_L \quad (49)$$

5. The  $B - S = 3$  equations in the primary variables may be combined to yield two first-order differential equations in terms of the state and source variables:

$$\frac{dv_C}{dt} = \frac{1}{C} i_L \quad (50)$$

$$\frac{di_L}{dt} = \frac{1}{L} (-v_C - R i_L + V_s) \quad (51)$$



6. The pair of state equations may be written in the standard matrix form:

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V_s(t) \quad (52)$$

If the system variables  $v_L$ ,  $i_C$ , and  $v_c$  are defined to be output variables for this example, a set of algebraic output equations may be formed in the state and input variables by using the elemental, compatibility, and continuity equations. In particular:

$$\begin{aligned} v_L &= V_s - v_R - v_C \\ &= V_s - Ri_L - v_C \end{aligned} \quad (53)$$

$$i_C = i_L \quad (54)$$

$$v_C = v_C \quad (55)$$

The output equations  $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$  become:

$$\begin{bmatrix} v_L \\ i_C \\ v_C \end{bmatrix} = \begin{bmatrix} -1 & -R \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} V_s(t) \quad (56)$$

## ■ Example

A fluid system is shown in Fig. 14 together with its linear graph.

The linear graph contains five elements ( $B = 5$ ) with two energy storage elements and one source ( $S_A = 1$ ). The graph contains four nodes ( $N = 4$ ), thus  $N - 1 = 3$  elements must be placed in the normal tree. The order of placement of elements into the tree is (i) the pressure source (an across variable source), (ii) the fluid capacitance  $C$  (an A-type element), and (iii) the resistive element  $R_p$ . At this point the tree is complete, and is as shown in Fig. 14c. The independent energy storage elements are then the fluid capacitance  $C$  and the fluid inductance of the pipe  $I_p$ :

$$\begin{array}{ll} \text{Primary variables:} & P_s(t), \quad P_C, \quad P_{R_p}, \quad Q_{I_p}, \quad Q_{R_f} \\ \text{Secondary variables:} & Q_s, \quad Q_C, \quad i_{R_p}, \quad P_{I_p}, \quad P_{R_f} \\ \text{System order:} & 2 \\ \text{State variables:} & P_C, \quad Q_{I_p} \end{array}$$

1. The  $B - S = 4$  elemental equations are:

$$\text{primary variables} \left\{ \begin{array}{l} \frac{dP_C}{dt} = \frac{1}{C} Q_C \\ \frac{dQ_{I_p}}{dt} = \frac{1}{I_p} P_{I_p} \\ P_{R_p} = R_p Q_{R_p} \\ Q_{R_f} = \frac{1}{R_f} P_{R_f} \end{array} \right\} \text{secondary variables} \quad (57)$$

2. The two secondary flow variables,  $Q_C$  and  $Q_{R_p}$  may be expressed in terms of primary variables using the  $N - 1 - S_A = 2$  continuity equations. Using the contours shown in Fig. 14c:

$$\begin{aligned} Q_C &= Q_{I_p} - Q_{R_f} \\ Q_{R_p} &= Q_{I_p} \end{aligned} \quad (58)$$

3. The two secondary pressure variables  $P_{I_p}$  and  $P_{R_p}$  may be expressed in terms of primary variables from the  $B - N + 1 - S_T = 2$  compatibility equations formed by placing the links back into the tree:

$$\begin{aligned} P_{I_p} &= P_s - P_{R_p} - P_C \\ P_{R_f} &= P_C \end{aligned} \quad (59)$$

4. The secondary variables in the elemental equations (i) may be eliminated by direct substitution from the continuity and compatibility equations:

$$\frac{dP_C}{dt} = \frac{1}{C} (Q_{I_p} - Q_{R_f}) \quad (60)$$

$$\frac{dQ_{I_p}}{dt} = \frac{1}{I_p} (P_s - P_{R_p} - P_C) \quad (61)$$

$$P_{R_p} = R_p Q_{I_p} \quad (62)$$

$$Q_{R_f} = \frac{1}{R_f} P_C \quad (63)$$

5. Equations (vi) and (vii) may be used to eliminate  $P_{R_p}$  and  $Q_{R_f}$  from Eqs. (iv) and (v) to yield two state equations:

$$\frac{dP_C}{dt} = \frac{1}{C} \left( Q_{I_p} - \frac{1}{R_f} P_C \right) \quad (64)$$

$$\frac{dQ_{I_p}}{dt} = \frac{1}{I_p} (P_s - P_C - R_p Q_{I_p}) \quad (65)$$

6. The pair of state equations may be written in the standard matrix form:

$$\begin{bmatrix} \dot{P}_C \\ \dot{Q}_{I_p} \end{bmatrix} = \begin{bmatrix} -1/R_f C & 1/C \\ -1/I_p & -R_p/I_p \end{bmatrix} \begin{bmatrix} P_C \\ Q_{I_p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/I_p \end{bmatrix} P_s(t) \quad (66)$$

### ■ Example

A thermal system model and its linear graph shown in Fig. 15. the graph consists of four elements ( $B = 4$ ) with one energy storage element (a thermal capacitance) and a through-variable source ( $S_T = 1$ ), and three nodes  $N = 3$ .

The order of entry of elements into the normal tree is (i) the capacitance  $C$ , and (ii) one of the two thermal resistances  $R_1$  or  $R_2$ . It is arbitrary which one is chosen; in this example we select  $R_2$  as a normal tree branch.

Primary variables:  $Q_s(t), T_C, q_{R_1}, T_{R_2}$   
 Secondary variables:  $T_s, q_C, T_{R_1}, q_{R_2}$   
 System order: 1  
 State variable:  $T_C$

1. The  $B - S = 3$  elemental equations are:

$$\text{primary variables} \left\{ \begin{array}{l} \frac{dT_C}{dt} = \frac{1}{C}q_C \\ T_{R_2} = R_2q_{R_2} \\ q_{R_1} = \frac{1}{R_1}T_{R_1} \end{array} \right\} \text{secondary variables} \quad (67)$$

2. The secondary heat flow variables,  $q_C$  and  $q_{R_2}$  may be expressed in terms of primary variables using the  $N - 1 - S_A = 2$  continuity equations derived from the contours shown in Fig. 15c:

$$\begin{aligned} q_C &= q_{R_1} \\ q_{R_2} &= q_s - q_{R_1} \end{aligned} \quad (68)$$

3. The secondary temperature variable  $T_{R_1}$  may be expressed in terms of primary variables from the  $B - N + 1 - S_T = 1$  compatibility equations formed by placing the link back into the tree:

$$T_{R_1} = T_{R_2} - T_C \quad (69)$$

4. Using the continuity (ii) and compatibility (iii) equations the secondary variables may be eliminated from the elemental equations (i):

$$\frac{dT_C}{dt} = \frac{1}{C}q_{R_1} \quad (70)$$

$$T_{R_2} = R_2(Q_s - q_{R_1}) \quad (71)$$

$$q_{R_1} = \frac{1}{R_1}(T_{R_2} - T_C) \quad (72)$$

5. Equations (v) and (vi) may be used to eliminate all but state and source variables from (iv). Combining (v) and (vi):

$$q_{R_1} = \frac{1}{R_1 + R_2}(R_2Q_s - T_C) \quad (73)$$

and substituting into Eq. (iv):

$$\frac{dT_C}{dt} = \frac{1}{C(R_1 + R_2)}(R_2Q_s - T_C) \quad (74)$$

6. If temperature is selected as the output variable, the matrix form of the state and output equations are:

$$\begin{bmatrix} \dot{T}_C \end{bmatrix} = \begin{bmatrix} -1 \\ C(R_1 + R_2) \end{bmatrix} T_C + \begin{bmatrix} R_2 \\ C(R_1 + R_2) \end{bmatrix} Q_s(t) \quad (75)$$

$$T_C = [1] T_C + [0] Q_s(t) \quad (76)$$

## 4 Systems with Non-standard State Equations

### 4.1 Input Derivative Form

Occasionally system models generate a set of state and output equations that contain terms in the derivative of the system inputs. In such cases, which generally result from models containing dependent energy storage elements, the standard form of the state equations must be extended to include these derivative terms:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\dot{\mathbf{u}} \quad (77)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{F}\dot{\mathbf{u}}. \quad (78)$$

The  $n \times r$  matrix  $\mathbf{E}$  and the  $m \times r$  matrix  $\mathbf{F}$  are introduced to include the input derivative terms  $\dot{\mathbf{u}}(t)$ .

Assume, for example, that a system contains a *dependent* T-type element  $L$ . The element is a branch in the normal tree and its primary variable is the across-variable  $v$ . The elemental equation is written with the derivative of the through-variable  $f$  on the right hand side:

$$v = L \frac{df}{dt}. \quad (79)$$

If the continuity equation used to eliminate the secondary variable  $f$  contains any through-variable source terms  $F_s(t)$ , substitution into Eq. (79) generates a term involving the derivative of the input:

$$v = L \frac{d}{dt} (\dots + F_s + \dots). \quad (80)$$

This input derivative term remains throughout the subsequent algebraic manipulations that generate the state and output equations.

The most common situations that generate the extended form of the state equations are:

1. When a compatibility equation includes the across-variable on a *dependent* A-type element, and an across-variable source term, or
2. When a continuity equation includes the through-variable on a *dependent* T-type element and a through-variable source term.

---

### ■ Example

A “lead-lag” electrical circuit, used in control systems, is shown in Fig. 16 together with its system graph and normal tree. Find a state equation describing the system and an output equation for the current in the capacitor  $C_1$ .

**Solution:** The system is first-order, and contains a dependent capacitor. Two normal trees are possible; with the choice of the normal tree as shown:

Primary variables:	$V_s(t), v_{C_2}, i_{R_1}, i_{R_2}, i_{C_1}$
Secondary variables:	$i_s, i_{C_2}, v_{R_1}, v_{R_2}, v_{C_1}$
System order:	1
State variable:	$v_{C_2}$

The elemental equations are:

$$\text{primary variables} \left\{ \begin{array}{l} \frac{dv_{C_2}}{dt} = \frac{1}{C_2} i_{C_2} \\ i_{R_1} = \frac{1}{R_1} v_{R_1} \\ i_{R_2} = \frac{1}{R_2} v_{R_2} \\ i_{C_1} = C_1 \frac{dv_{C_1}}{dt}. \end{array} \right\} \text{secondary variables.} \quad (81)$$

The normal tree generates one continuity equation:

$$i_{C_2} = i_{R_1} + i_{C_1} - i_{R_2} \quad (82)$$

and three compatibility equations:

$$v_{R_2} = v_{C_2} \quad (83)$$

$$v_{R_1} = V_s(t) - v_{C_2} \quad (84)$$

$$v_{C_1} = V_s(t) - v_{C_2}. \quad (85)$$

When Eq. (v) is substituted into Eqs. (i) to eliminate the secondary variable  $v_{C_1}$ , the elemental equation becomes:

$$i_{C_1} = C_1 \left[ \frac{dV_s(t)}{dt} - \frac{dv_{C_2}}{dt} \right]. \quad (86)$$

The derivative propagates through the subsequent algebraic steps and the state equation is:

$$\dot{v}_{C_2} = -\frac{R_1 + R_2}{R_1 R_2 (C_1 + C_2)} v_{C_2} + \frac{1}{R_1 (C_1 + C_2)} V_s(t) + \frac{C_1}{C_1 + C_2} \dot{V}_s(t). \quad (87)$$

The output equation for  $i_{C_1}$  may be found from Eqs. (i) and (v):

$$i_{C_1} = C_1 \frac{d}{dt} (V_s(t) - v_{C_2}) \quad (88)$$

and substituting Eq. (vii)

$$i_{C_1} = \frac{(R_1 + R_2) C_1}{R_1 R_2 (C_1 + C_2)} v_{C_2} - \frac{C_1}{R_1 (C_1 + C_2)} V_s(t) + \frac{C_1 C_2}{C_1 + C_2} \dot{V}_s(t). \quad (89)$$

Both the state and output equations include the derivative of the input  $V_s(t)$ .

## 4.2 Transformation to the Standard Form

If the modeling procedure generates state equations in the form of Eq. (77) an algebraic transformation may be used to generate a new set of state variables and state equations that are in the standard form of Eq. (??). For a system with state equations:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} + \mathbf{E}\dot{\mathbf{u}} \quad (90)$$

a new set of state variables  $\mathbf{x}'$  may be defined:

$$\mathbf{x}' = \mathbf{x} - \mathbf{E}\mathbf{u}. \quad (91)$$

Differentiation of Eq. (91) gives:

$$\begin{aligned} \dot{\mathbf{x}}' &= \dot{\mathbf{x}} - \mathbf{E}\dot{\mathbf{u}} = (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\dot{\mathbf{u}}) - \mathbf{E}\dot{\mathbf{u}} \\ &= \mathbf{A}(\mathbf{x}' + \mathbf{E}\mathbf{u}) + \mathbf{B}\mathbf{u} \\ &= \mathbf{A}\mathbf{x}' + \mathbf{B}'\mathbf{u} \end{aligned} \quad (92)$$

where  $\mathbf{B}' = \mathbf{A}\mathbf{E} + \mathbf{B}$ . Although the elements of the new state vector  $\mathbf{x}'$  are no longer the physical variables that were used to generate the original equations, the complete vector  $\mathbf{x}'$  satisfies all of the requirements of a state vector. Any physical variable in the system may be found from a modified set of output equations written in terms of the transformed state vector:

$$\begin{aligned} \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{F}\dot{\mathbf{u}} \\ &= \mathbf{C}(\mathbf{x}' + \mathbf{E}\mathbf{u}) + \mathbf{D}\mathbf{u} + \mathbf{F}\dot{\mathbf{u}} \\ &= \mathbf{C}\mathbf{x}' + \mathbf{D}'\mathbf{u} + \mathbf{F}\dot{\mathbf{u}} \end{aligned} \quad (93)$$

where  $\mathbf{D}' = \mathbf{C}\mathbf{E} + \mathbf{D}$ .

### ■ Example

Figure 17 illustrates a mechanical translational system. The system graph and normal tree demonstrate that although there are three energy storage elements, the two springs are dependent and the system order is two. Find a set of state equations that does not involve the derivative of the input vector and an accompanying output equation for the velocity of the dashpot  $B_1$ .

**Solution:** With the choice of normal tree as shown, the state variables are  $v_m$  and  $F_{K_1}$  and the elemental equations are:

$$\frac{dv_m}{dt} = \frac{1}{m}F_m \quad (94)$$

$$\frac{dF_{K_1}}{dt} = K_1v_{K_1} \quad (95)$$

$$F_{B_2} = B_2v_{B_2} \quad (96)$$

$$v_{B_1} = \frac{1}{B_1}F_{B_1} \quad (97)$$

$$v_{K_2} = \frac{1}{K_2}\frac{dF_{K_2}}{dt}. \quad (98)$$

There are two compatibility equations:

$$v_{K_1} = v_{K_2} - v_{B_1} \quad (99)$$

$$v_{B_2} = v_m, \quad (100)$$

and three continuity equations:

$$F_{K_2} = F_s(t) - F_{K_1} \quad (101)$$

$$F_{B_1} = F_{K_1} \quad (102)$$

$$F_m = F_s(t) - F_{B_2}. \quad (103)$$

Equation (viii) relates the through variable  $F_{K_2}$  on a dependent T-type element  $K_2$  to the input  $F_s(t)$ . When the secondary variables are eliminated and the two state equations are written in matrix form, the result is:

$$\begin{bmatrix} \dot{v}_m \\ \dot{f}_{K_1} \end{bmatrix} = \begin{bmatrix} -B_2/m & 0 \\ 0 & -K_1K_2/(B_1(K_1 + K_2)) \end{bmatrix} \begin{bmatrix} v_m \\ F_{K_1} \end{bmatrix} + \begin{bmatrix} 1/m \\ 0 \end{bmatrix} F_s(t) \\ + \begin{bmatrix} 0 \\ K_1/(K_1 + K_2) \end{bmatrix} \dot{F}_s(t) \quad (104)$$

showing the dependence on the derivative of the input  $F_s(t)$ . The output equation for  $v_{B_1}$  is found from Eqs. (iv) and (ix):

$$v_{B_1} = \frac{1}{B_1} F_{K_1} \quad (105)$$

and in matrix form

$$v_{B_1} = \begin{bmatrix} 0 & 1/B_1 \end{bmatrix} \begin{bmatrix} v_m \\ F_{K_1} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} F_s(t). \quad (106)$$

The state variables may be transformed using Eq. (91), that is:

$$\dot{\mathbf{x}}' = \mathbf{A}\mathbf{x}' + (\mathbf{A}\mathbf{E} + \mathbf{B})\mathbf{u}$$

or:

$$\begin{bmatrix} \dot{x}'_1 \\ \dot{x}'_2 \end{bmatrix} = \begin{bmatrix} -B_2/m & 0 \\ 0 & -K_1K_2/(B_1(K_1 + K_2)) \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} \\ + \begin{bmatrix} 1/m \\ K_1^2K_2/(B_1(K_1 + K_2)^2) \end{bmatrix} F_s(t) \quad (107)$$

and the corresponding output equation in terms of the transformed state vector is found directly from Eq. (93):

$$\begin{aligned} v_{B_1} &= \mathbf{C}\mathbf{x}' + (\mathbf{C}\mathbf{E} + \mathbf{D})\mathbf{u} + \mathbf{F}\dot{\mathbf{u}} \\ &= \begin{bmatrix} 0 & 1/B_1 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} + \begin{bmatrix} 0 & 1/B_1 \end{bmatrix} \begin{bmatrix} 0 \\ K_1/(K_1 + K_2) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} F_s(t) \\ &\quad + \begin{bmatrix} 0 \end{bmatrix} \dot{F}_s(t) \\ &= \frac{1}{B_1} x'_2 + \frac{K_1}{B_1(K_1 + K_2)} F_s(t). \end{aligned} \quad (108)$$

## 5 State Equation Generation Using Linear Algebra

For linear systems, the state equations may be found directly from the equations in the primary variables and inputs by matrix methods. After the elemental equations have been written with causality define by the normal tree, and the substitutions for the secondary variables made through the continuity and compatibility equations, the primary variables may be divided into three groups, and expressed in three vectors:

- $\mathbf{x}$  - the vector of state variables (primary variables associated with independent energy storage elements).
- $\mathbf{d}$  - the vector of primary variables associated with dependent energy storage elements.
- $\mathbf{p}$  - the vector of primary variables associated with non-energy storage elements (D-type elements and two-port branches (Chapter 6)).

The set of equations in the primary variables and inputs can be written as three matrix equations in  $\mathbf{x}$ ,  $\mathbf{d}$ , and  $\mathbf{p}$ .

$$\dot{\mathbf{x}} = \mathbf{P}\mathbf{x} + \mathbf{Q}\mathbf{p} + \mathbf{R}\mathbf{d} + \mathbf{S}\mathbf{u} \quad (109)$$

$$\mathbf{d} = \mathbf{M}\dot{\mathbf{x}} + \mathbf{N}\dot{\mathbf{u}} \quad (110)$$

$$\mathbf{p} = \mathbf{H}\mathbf{x} + \mathbf{J}\mathbf{p} + \mathbf{K}\mathbf{d} + \mathbf{L}\mathbf{u} \quad (111)$$

The state equations are found by solving explicitly for  $\mathbf{p}$  and  $\mathbf{d}$  and substitution into Eq. (109).

If there are no dependent energy storage elements in the system, that is  $\mathbf{d} = \mathbf{0}$  and  $\mathbf{M} = \mathbf{N} = \mathbf{K} = \mathbf{R} = \mathbf{0}$ , the state equations are found by solving Eq. (111) for  $\mathbf{p}$  and substituting into Eq. (109). In the simplest case, the matrix  $\mathbf{J} = \mathbf{0}$  and the state equations may be found by substituting Eq. (111) directly into Eq. (109)

$$\dot{\mathbf{x}} = [\mathbf{P} + \mathbf{QH}]\mathbf{x} + [\mathbf{S} + \mathbf{QL}]\mathbf{u}. \quad (112)$$

Otherwise  $\mathbf{p}$  must be found by rearranging and solving Eq. (111):

$$[\mathbf{I} - \mathbf{J}]\mathbf{p} = \mathbf{H}\mathbf{x} + \mathbf{L}\mathbf{u} \quad (113)$$

where  $\mathbf{I}$  is the identity matrix, or

$$\mathbf{p} = [\mathbf{I} - \mathbf{J}]^{-1}\mathbf{H}\mathbf{x} + [\mathbf{I} - \mathbf{J}]^{-1}\mathbf{L}\mathbf{u} \quad (114)$$

$$= \mathbf{H}'\mathbf{x} + \mathbf{L}'\mathbf{u} \quad (115)$$

where  $\mathbf{H}' = [\mathbf{I} - \mathbf{J}]^{-1}\mathbf{H}$  and  $\mathbf{L}' = [\mathbf{I} - \mathbf{J}]^{-1}\mathbf{L}$ . Substitution of Eq. (115) into Eq. (109) generates the state equations

$$\dot{\mathbf{x}} = (\mathbf{P} + \mathbf{QH}')\mathbf{x} + (\mathbf{S} + \mathbf{QL}')\mathbf{u}. \quad (116)$$

## ■ Example

The electrical circuit shown in Fig. 18, together with its linear graph and a normal tree, generates the following equations in the primary variables:

$$\dot{v}_C = -\frac{1}{C}i_{R_1} + \frac{1}{C}I_s \quad (117)$$

$$i_L = \frac{1}{L}v_{R_2} \quad (118)$$

$$v_{R_2} = R_2i_{R_1} - R_2i_L \quad (119)$$

$$i_{R_1} = \frac{1}{R_1}v_C - \frac{1}{R_1}v_{R_2} \quad (120)$$

Use the matrix method to find the state equations in the variables  $v_C$  and  $i_L$ .



**Solution:** Divide the primary variables into two vectors,  $x = [v_C \ i_L]^T$  and  $p = [v_{R_2} \ i_{R_1}]^T$ . Then the primary equations (Eqs. (i - iv)) can be written

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 & -1/C \\ 1/L & 0 \end{bmatrix} \begin{bmatrix} v_{R_2} \\ i_{R_1} \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} I_s(t) \quad (121)$$

$$\begin{bmatrix} v_{R_2} \\ i_{R_1} \end{bmatrix} = \begin{bmatrix} 0 & -R_2 \\ 1/R_1 & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 & R_2 \\ -1/R_1 & 0 \end{bmatrix} \begin{bmatrix} v_{R_2} \\ i_{R_1} \end{bmatrix} \quad (122)$$

Equation (vi) may be rearranged

$$\begin{bmatrix} 1 & -R_2 \\ 1/R_1 & 1 \end{bmatrix} \begin{bmatrix} v_{R_2} \\ i_{R_1} \end{bmatrix} = \begin{bmatrix} 0 & -R_2 \\ 1/R_1 & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} \quad (123)$$

and solved for  $\mathbf{p} = [v_{R_2} \ i_{R_1}]^T$

$$\begin{bmatrix} v_{R_2} \\ i_{R_1} \end{bmatrix} = \begin{bmatrix} 1 & -R_2 \\ 1/R_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -R_2 \\ 1/R_1 & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} \quad (124)$$

$$= \frac{R_1}{R_1 + R_2} \begin{bmatrix} 1 & R_2 \\ -1/R_1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -R_2 \\ 1/R_1 & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} \quad (125)$$

$$= \begin{bmatrix} -R_2/(R_1 + R_2) & R_1 R_2 / (R_1 + R_2) \\ -1/(R_1 + R_2) & -R_2 / (R_1 + R_2) \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} \quad (126)$$

When Eq. (ix) is substituted into Eq. (v) the state equations are found to be

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} 1/(R_1 + R_2)C & R_2/(R_1 + R_2)C \\ -R_2/(R_1 + R_2)L & R_1 R_2 / (R_1 + R_2)L \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} I_s(t). \quad (127)$$

In the most general case, when there are dependent energy storage elements in the system, so that  $\mathbf{d} \neq \mathbf{0}$ , the state equations are found by substituting Eqs. (115) and (110) into Eq. (109), giving

$$[\mathbf{I} - (\mathbf{QK}' + \mathbf{R}) \mathbf{M}] \dot{\mathbf{x}} = (\mathbf{P} + \mathbf{QH}') \mathbf{x} + (\mathbf{S} + \mathbf{QL}') \mathbf{u} + (\mathbf{R} + \mathbf{QK}') \mathbf{N} \dot{\mathbf{u}} \quad (128)$$

which may be solved by pre-multiplying both sides by  $[\mathbf{I} - (\mathbf{QK}' + \mathbf{R}) \mathbf{M}]^{-1}$  to generate the state equations.

## 6 Nonlinear Systems

### 6.1 General Considerations

In the general case of nonlinear state-determined system models, relationships may exist in which a single system variable is a nonlinear function of several other variables. For example the torque output of an automobile engine may be represented as a nonlinear function of the fuel/air ratio, spark advance and engine speed. Systematic equation formulation procedures are not generally available for such systems. However, for nonlinear systems which can be represented by linear graph models containing combinations of pure (nonlinear) and ideal (linear) one and two-port elements, the equation formulation procedures described in Sections 1 to 4 provide an effective starting point

for the formulation of a set of nonlinear state equations. For such models, the system constraints represented by a set of compatibility and continuity equations are valid. Thus, the procedure of forming a normal tree, and identifying the independent variables and state variables, is unchanged from the method described for linear system models, as are the procedures for generating the compatibility and continuity equations.

With models containing nonlinear elements the algebraic steps in the subsequent elimination of secondary variables are not always straightforward, and may not always be accomplished with simple analytical techniques. However, for many nonlinear systems containing pure elements, the methods developed for linear system models may be used with some modification to derive a set of nonlinear state equations.

The formulation of nonlinear state equations is illustrated in the following section, where a set of nonlinear state equations is derived in the general form of Eqs. (??), that is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

where the derivative of each state is related by a nonlinear function to the system state variables and inputs. Some nonlinear systems models may not be written explicitly as a set of nonlinear state equations in the standard form of Eqs. (??) because it may not be possible to directly solve for the derivative of each state variable independently.

## 6.2 Examples of Nonlinear System Model Formulation

To illustrate the equation derivation procedures for systems containing nonlinear elements, state equations are derived for three representative systems in the following examples. Each example contains one or more nonlinear dissipative or energy storage element.

---

### ■ Example

An automobile, with mass  $m$ , traveling on a flat, straight section of road is shown in Fig. 19. Derive a model for the forward speed  $v_m$  of the car, in response to a time varying propulsive force  $F_p(t)$  generated by the engine. The model is restricted to the moderate to high speed range where the two primary resistance forces (in the absence of braking) are road resistance  $F_r$  and aerodynamic drag  $F_d$ , each approximated as:

$$F_r = F_o \tag{129}$$

$$F_d = c_d v_d^2. \tag{130}$$

**Solution:** The road resistance is represented as a constant resistance force  $F_0$  and the aerodynamic drag  $F_d$  is represented as a resistive force which increases with the square of vehicle velocity with the constant  $c_d$  dependent on the automobile drag coefficient.

A system graph model for the car is shown in Fig. 19b, in which the car mass  $m$  is represented as an ideal element, the propulsive force is represented as an ideal force source  $F_p(t)$ , the road resistance is represented as an ideal force source  $F_o$ , and the aerodynamic drag is represented as a nonlinear damper element. The system contains one independent energy storage element, as shown by the normal tree. From the normal tree (Fig. 19c):

Primary variables:  $v_m, F_p, F_r, F_d$   
 Secondary variables:  $F_m, v_p, v_r, v_d$   
 System order: 1  
 State variable:  $v_m$ .

The elemental equations for the two passive elements are:

$$\frac{dv_m}{dt} = \frac{1}{m} F_m \quad (131)$$

$$F_d = c_d V_d^2 \quad (132)$$

The normal tree generates one continuity equation:

$$F_m = F_P - F_o - F_d \quad (133)$$

and one compatibility equation:

$$v_d = v_m. \quad (134)$$

Substitution to eliminate the secondary variables from the elemental equations yields:

$$\frac{dv_m}{dt} = \frac{1}{m} [F_p - F_o - F_d] \quad (135)$$

$$F_d = c_d v_m^2. \quad (136)$$

Combining these two equations generates the system nonlinear state equation:

$$\frac{dv_m}{dt} = \frac{1}{m} [F_p - F_o - c_d v_m^2] \quad (137)$$

## ■ Example

A part of a sensitive torque measuring instrument, shown in Fig. 20a, consists of a pendulum of length  $l$  with proof mass  $m$  mounted in support bearings in a gravity field  $g$ .

The input to the instrument is a time varying torque  $T_s(t)$  which acts on the shaft to displace the pendulum by an angle  $\theta$ . Under static input conditions where  $T_s(t) = T_0$ , the gravitational restoring torque is a function of the angular displacement:

$$T_0 = mgl \sin \theta \quad (138)$$

This is an algebraic relationship between a torque (through-variable) and a displacement (integrated across-variable) and represents the constitutive relationship of a nonlinear torsional spring. The elemental equation for the equivalent spring  $K$  is found by differentiating Eq. (i)

$$\frac{dT_K}{dt} = (mgl \cos \theta) \frac{d\theta}{dt}. \quad (139)$$

The support bearings for the shaft have a resistive torque  $T_B$  proportional to the shaft angular velocity  $\Omega_B$ :

$$T_B = B\Omega_B. \quad (140)$$

In addition to the gravitational restoring torque, the pendulum also has a rotational inertia with a point mass  $m$  concentrated at a radius  $l$  from the center of rotation, so that the moment of inertia is

$$J = ml^2. \quad (141)$$

By considering the inertia, the equivalent spring, the damper and the input source, a system graph model may be formulated, as shown in Fig. 20b, which has the normal tree shown in Fig. 20c. From the linear graph and normal tree:

$$\begin{aligned} \text{Primary variables:} & \quad \Omega_J, \quad T_s, \quad T_K, \quad T_B \\ \text{Secondary variable:} & \quad T_J, \quad \Omega_s, \quad \Omega_K, \quad \Omega_B \\ \text{System order:} & \quad 2 \\ \text{State variables:} & \quad \Omega_J, \quad T_K. \end{aligned}$$

There are three elemental equations:

$$\frac{d\Omega_J}{dt} = \frac{1}{J}T_J \quad (142)$$

$$\frac{dT_K}{dt} = (mgl \cos \theta) \frac{d\theta}{dt} = (mgl \cos \theta) \Omega_K \quad (143)$$

$$T_B = B\Omega_B. \quad (144)$$

The normal tree generates a single continuity equation:

$$T_K = T_J + T_B + T_S \quad (145)$$

and two compatibility equations:

$$\Omega_B = \Omega_J \quad (146)$$

$$\Omega_K = \Omega_J \quad (147)$$

Elimination of the secondary variables yields:

$$\frac{d\Omega_J}{dt} = \frac{1}{J}(T_s - B\Omega_J - T_K) \quad (148)$$

$$\frac{dT_K}{dt} = (mgl \cos \theta) \Omega_J \quad (149)$$

The equations may be expressed directly in terms of the state variables alone by substituting directly for  $\theta$  from the constitutive equation (Eq. (i)):

$$\theta = \sin^{-1} \left( \frac{T_K}{mgl} \right) \quad (150)$$

and Eq. (xii) may be written

$$\frac{dT_K}{dt} = mgl \cos \left[ \sin^{-1} (T_K/mgl) \right] \Omega_J. \quad (151)$$

Equations (xi) and (xiv) represent a pair of state equations for the nonlinear system, while Eq. (xiii) allows the computation of  $\theta$ , the system output. The nonlinearity in this system arises from the trigonometric relationship of the pendulum restoring torque to its displacement. While the basic derivation method used is similar to that for linear systems, additional manipulation is required to eliminate the secondary variables and to form the state equations.

---



---

### ■ Example

The fluid distribution system shown in Fig. 21 consists of a flow source  $Q_s(t)$  which feeds a storage tank with non-vertical walls. The output from the tank is distributed into a fluid network consisting of a short pipe discharging through an orifice, and a long pipe discharging through another orifice.

The fluid flow through an orifice obeys a quadratic relationship:

$$Q = C_0 \sqrt{|\Delta P|} \text{sgn}(\Delta P) \quad (152)$$

where  $Q$  is the flow through the orifice,  $\Delta P$  is the pressure drop across the orifice, and  $C_0$  is an orifice coefficient that is dependent upon the geometry of the orifice. The signum function,  $\text{sgn}()$ , is used to indicate that the flow changes sign when the sign of  $\Delta P$  changes.

The tank is shaped as the frustum of a cone, and therefore has a volume which is a nonlinear function of the height of the fluid in the tank. From the tank geometry shown in Fig. 21, the volume is

$$\begin{aligned} V &= \int_0^h \pi r^2 dh = \int_0^h \pi (r_1 + K_C h)^2 dh \\ &= \int_0^h \pi (r_1^2 + 2r_1 K_C h + K_C^2 h^2) dh \\ &= \pi (r_1^2 h + r_1 K_C h^2 + K_C^2 h^3 / 3) \end{aligned} \quad (153)$$

In an open tank in a gravity field, the pressure at the base is directly related to the height of fluid

$$P_C = \rho g h \quad (154)$$

so that the following constitutive relationship may be written

$$V = \frac{\pi r_1^2}{\rho g} P_C + \frac{\pi r_1 K_C}{(\rho g)^2} P_C^2 + \frac{\pi K_C^2}{3(\rho g)^3} P_C^3 \quad (155)$$

$$= K_{T0} P_C + \frac{K_{T1}}{2} P_C^2 + \frac{K_{T2}}{3} P_C^3 \quad (156)$$

where  $K_{T0} = \pi r_1^2 / (\rho g)$ ,  $K_{T1} = 2\pi r_1 K_C / (\rho g)^2$ , and  $K_{T2} = \pi K_C^2 / (\rho g)^3$ . The tank has a constitutive relationship of a pure nonlinear fluid capacitance.

The system graph model for the system, including the nonlinear capacitance, the two nonlinear resistances of the orifices, and a fluid inertance  $I$  of the long fluid line (assuming that the pipe resistance is small compared to the orifice resistance) is shown in Fig. 21b. From this graph and the normal tree shown in Fig. 21c the system model includes the following:

Primary variables:	$P_C, P_{R_2}, Q_s, Q_I, Q_{R_1}$
Secondary variables:	$Q_C, Q_{R_2}, P_s, P_I, P_{R_1}$
System order:	2
State variables:	$P_C, Q_I.$

The system has four elemental equations. For the fluid inertance

$$\frac{dQ_I}{dt} = \frac{1}{I}P_I. \quad (157)$$

For the nonlinear fluid capacitance the elemental equation must be found by differentiating the constitutive equation (v):

$$\frac{dV}{dt} = Q_C = [K_{T0} + K_{T1}P_C + K_{T2}P_C^2] \frac{dP_C}{dt} \quad (158)$$

or

$$\frac{dP_C}{dt} = \left[ \frac{1}{K_{T0} + K_{T1}P_C + K_{T2}P_C^2} \right] Q_C. \quad (159)$$

For orifice  $R_1$  the elemental equation is

$$Q_{R_1} = K_1 \sqrt{|P_{R_1}|} \text{sgn}(P_{R_1}), \quad (160)$$

while for orifice  $R_2$  which has the pressure drop  $P_{R_2}$  as its primary variable the equation may be written:

$$P_{R_2} = \frac{1}{K_2^2} Q_{R_2} |Q_{R_2}|, \quad (161)$$

where  $Q_{R_2} |Q_{R_2}|$  is the absquare, or absolute square, which ensures that the pressure drop changes sign when the flow changes direction through an orifice.

The two continuity equations are:

$$Q_C = Q_s - Q_{R_1} \quad (162)$$

$$Q_{R_2} = Q_I \quad (163)$$

and the two compatibility equations are:

$$P_{R_1} = P_C \quad (164)$$

$$P_I = P_C - P_{R_2}. \quad (165)$$

Elimination of the secondary variables yields

$$\frac{dQ_I}{dt} = \frac{1}{I} [P_C - P_{R_2}] \quad (166)$$

$$\frac{dP_C}{dt} = \left[ \frac{1}{K_{T0} + K_{T1}P_C + K_{T2}P_C^2} \right] [Q_s - Q_{R_1}] \quad (167)$$

$$Q_{R_1} = K_1 \sqrt{|P_C|} \text{sgn}(P_C) \quad (168)$$

$$P_{R_2} = \frac{1}{K_2^2} Q_I |Q_I|. \quad (169)$$

and substituting for  $Q_{R_1}$  and  $P_{R_2}$  results in two nonlinear state equations:

$$\frac{dQ_I}{dt} = \frac{1}{I} \left[ P_C - \frac{1}{K_2^2} Q_I |Q_I| \right] \quad (170)$$

$$\frac{dP_C}{dt} = \left[ \frac{1}{K_{T0} + K_{T1}P_C + K_{T2}P_C^2} \right] \left[ Q_s - K_1 \sqrt{|P_C|} \text{sgn}(P_C) \right]. \quad (171)$$

## 7 Linearization of State Equations

A set of nonlinear state equations for a system may be linearized directly to obtain a set of linear state equations that approximate the system response over a limited range of operation. In the process of linearization a new set of variables is defined that describe the response about a system equilibrium point. The method is useful for systems containing nonlinear elements where the excursion of the system variables about the equilibrium point is small. Because the response of a linearized model only approximates that of the nonlinear system, care must be taken in interpreting any analyses and judgement must be used in defining the applicable range of operation.

To illustrate the linearization process, it is convenient to initially consider a first-order nonlinear system with a single state equation

$$\dot{x} = f(x, u). \quad (172)$$

Assume that with a known constant input  $u_0$ , the system ultimately reaches a steady response  $x_0$ , after which time the derivative  $\dot{x}$  is identically zero. Any such *equilibrium* value for the state variable must therefore satisfy the algebraic equation

$$f(x_0, u_0) = 0. \quad (173)$$

In general more than one, or no values of  $x$  may satisfy a nonlinear equilibrium equation, and the solution of Eq. (173) may prove to be a difficult task. However, for many nonlinear systems containing only ideal and pure elements an equilibrium condition can be found. The linearization of  $f(x, u)$  may be performed by expanding the function as a Taylor series in two variables  $x$  and  $u$  about the equilibrium condition as

$$f(x, u) = f(x_0, u_0) + \left. \frac{\partial f(x, u)}{\partial x} \right|_{\substack{x=x_0 \\ u=u_0}} (x - x_0) + \left. \frac{\partial f(x, u)}{\partial u} \right|_{\substack{x=x_0 \\ u=u_0}} (u - u_0) + \dots \quad (174)$$

By definition,  $f(x_0, u_0) = 0$ . If the deviations  $x^* = x - x_0$ , and  $u^* = u - u_0$  are small, high order terms may be neglected and a linear approximation to the first-order state equation may be written

$$\dot{x}^* = \left. \frac{\partial f(x, u)}{\partial x} \right|_{\substack{x=x_0 \\ u=u_0}} x^* + \left. \frac{\partial f(x, u)}{\partial u} \right|_{\substack{x=x_0 \\ u=u_0}} u^*. \quad (175)$$

Equation (175) is a linear first-order state equation with constant coefficients which may be written as a standard state equation

$$\dot{x}^* = ax^* + bu^* \quad (176)$$

where

$$a = \left. \frac{\partial f(x, u)}{\partial x} \right|_{\substack{x=x_0 \\ u=u_0}} \quad \text{and} \quad b = \left. \frac{\partial f(x, u)}{\partial u} \right|_{\substack{x=x_0 \\ u=u_0}}. \quad (177)$$

The procedure for linearizing a higher order system about an equilibrium point is analogous to that described for a first-order system. The set of  $n$  state equations with  $r$  inputs

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \quad (178)$$

is first solved for an equilibrium state  $\mathbf{x}_0$  with the given input  $\mathbf{u}_0$ , by setting the derivative of the state vector to zero and solving the resulting set of nonlinear algebraic equations

$$\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) = \mathbf{0}. \quad (179)$$

Each of the  $n$  state equations

$$\dot{x}_i = f_i(x_1, \dots, x_n, u_1, \dots, u_r) \quad (180)$$

may then be linearized by generating a Taylor series expansion in the  $(n + r)$  variables about the operating point and ignoring all but the linear terms. A set of incremental variables  $x_i^* = x_i - (x_0)_i$  and  $u_i^* = u_i - (u_0)_i$  is defined. Because there are  $(n + r)$  such variables,  $(n + r)$  partial derivatives are required in each equation. The result is:

$$\begin{aligned} \dot{x}_i^* &= \frac{\partial f_1(\mathbf{x}, \mathbf{u})}{\partial x_1} \Big|_{\substack{x=x_0 \\ u=u_0}} x_1^* + \frac{\partial f_1(\mathbf{x}, \mathbf{u})}{\partial x_2} \Big|_{\substack{x=x_0 \\ u=u_0}} x_2^* + \dots + \frac{\partial f_1(\mathbf{x}, \mathbf{u})}{\partial x_n} \Big|_{\substack{x=x_0 \\ u=u_0}} x_n^* \\ &= \frac{\partial f_1(\mathbf{x}, \mathbf{u})}{\partial u_1} \Big|_{\substack{x=x_0 \\ u=u_0}} u_1^* + \frac{\partial f_1(\mathbf{x}, \mathbf{u})}{\partial u_2} \Big|_{\substack{x=x_0 \\ u=u_0}} u_2^* + \dots + \frac{\partial f_1(\mathbf{x}, \mathbf{u})}{\partial u_r} \Big|_{\substack{x=x_0 \\ u=u_0}} u_r^* \end{aligned} \quad (181)$$

which is a linear state equation of the form

$$\dot{x}_i^* = a_{i1}x_1^* + \dots + a_{in}x_n^* + b_{i1}u_1^* + \dots + b_{ir}u_r^* \quad (182)$$

The collection of all such equations can be written in matrix form:

$$\dot{\mathbf{x}}^* = \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^* \quad (183)$$

where the elements of the matrices  $\mathbf{A}$  and  $\mathbf{B}$  are:

$$a_{ij} = \frac{\partial f_i(\mathbf{x}, \mathbf{u})}{\partial x_j} \Big|_{\substack{x=x_0 \\ u=u_0}}, \quad \text{and} \quad b_{ij} = \frac{\partial f_i(\mathbf{x}, \mathbf{u})}{\partial u_j} \Big|_{\substack{x=x_0 \\ u=u_0}}. \quad (184)$$

Matrices containing partial derivatives in the form of  $\mathbf{A}$  and  $\mathbf{B}$  above are known as Jacobian matrices.

### ■ Example

In Example 6.2 a nonlinear state equation was derived for a vehicle of mass  $m$  traveling at moderate speed  $v_m$  with propulsive force  $F_p(t)$ . The model, including the effects of aerodynamic drag ( $F_d = c_d v_m^2$ ), and constant road resistance ( $F_r$ ), generated a single nonlinear state equation

$$\dot{v}_m = \frac{1}{m} [F_p - F_r - c_d v_m^2] = F(v_m, F_p, F_r).$$

Derive a linearized state equation to describe the dynamics of the vehicle for small deviations in speed from a nominal operating speed  $v_m = v_0$ .

**Solution:** First the equilibrium condition is found by solving the equation

$$\dot{v}_m = 0 = \frac{1}{m} [(F_p)_0 - (F_r)_0 - c_d (v_m)_0^2] \quad (185)$$

and at equilibrium

$$(F_p)_0 = (F_r)_0 + c_d (v_m)_0^2. \quad (186)$$



The linearized equations may be derived in terms of incremental variables  $v_m^*$ ,  $F_p^*$ , and  $F_r^*$

$$\dot{v}_m^* = \left. \frac{\partial F}{\partial v_m} \right|_0 v_m^* + \left. \frac{\partial F}{\partial F_p} \right|_0 F_p^* + \left. \frac{\partial F}{\partial F_r} \right|_0 F_r^* \quad (187)$$

and when each term is evaluated and the substitution  $(v_m)_0 = v_0$  is made

$$\dot{v}_m^* = \left[ -\frac{2c_d v_0}{m} \right] v_m^* + \left[ \frac{1}{m} \right] F_p^* + \left[ \frac{1}{m} \right] F_r^*. \quad (188)$$

When it is noted that  $F_r^*$  is zero because  $F_r$  is constant, the linearized equation may be written

$$\dot{v}_m^* = -\frac{B}{m} v_m^* + \frac{1}{m} F_p^* \quad (189)$$

where  $B = 2c_d v_0$  is the equivalent damper coefficient. In interpreting and using this equation, it is noted that the solution  $v_m^*$  is the deviation from the equilibrium speed, and that the input  $F_p^*(t)$  is the deviation from the equilibrium propulsive force.

## ■ References:

1. Kalman R. E., "On the General Theory of Control Systems", Proc. First IFAC Congress, Butterworth, London, 1960
2. Schultz D. G., Melsa J. L., *State Functions and Linear Control Systems*, McGraw-Hill Book Company, New York NY, 1967
3. Timothy L. K., Bona B. E., *State Space Analysis*, McGraw-Hill Book Company, New York NY, 1969
4. Koenig H. E., Tokad Y., Kesavan H. K., Hedges H. G., *Analysis of Discrete Physical Systems*, McGraw-Hill Book Company, New York NY, 1967
5. Blackwell W. A., *Mathematical Modeling of Physical Networks*, Macmillan Publishing Company, New York NY, 1967
6. Chan S. P., Chan S. Y., Chan S. G., *Analysis of linear Networks and Systems*, Addison-Wesley Publishing Company, Reading MA, 1972
7. Shearer J. L., Murphy A. T., Richardson H. H., *Introduction to System Dynamics*, Addison-Wesley Publishing Company, Reading MA, 1967
8. Durfee W. K., Wall M. B., Rowell D., Abott F. K., "Interactive Software for Dynamic System Modeling Using linear Graphs", IEEE Control Systems Magazine, 11, 4, pp 60–66, June 1991

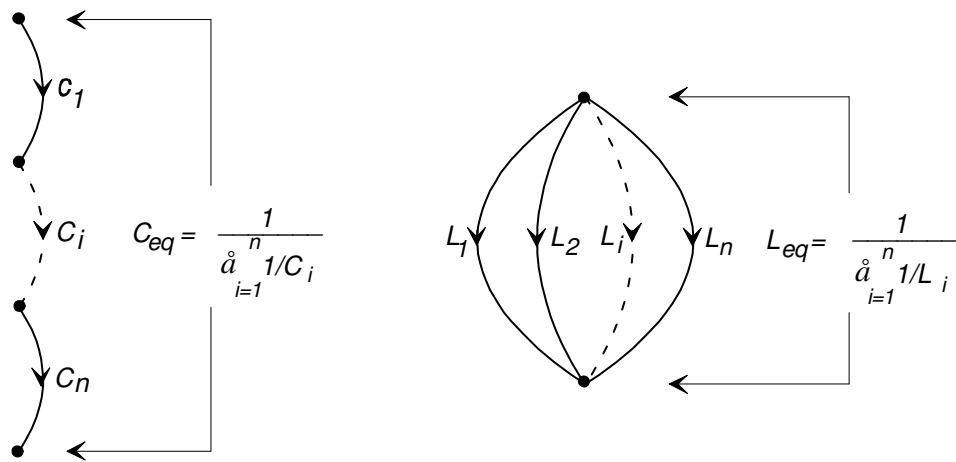


Figure 9: Combination of elements in series and in parallel to eliminate excess state variables.

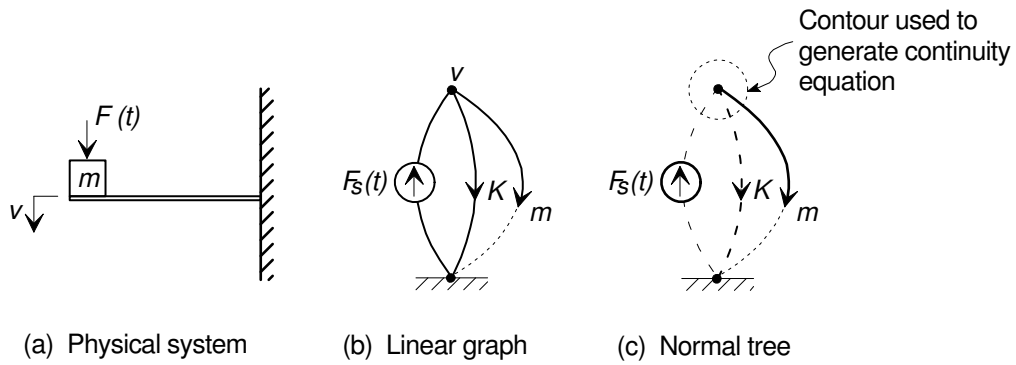


Figure 10: Mechanical system model and its normal tree.

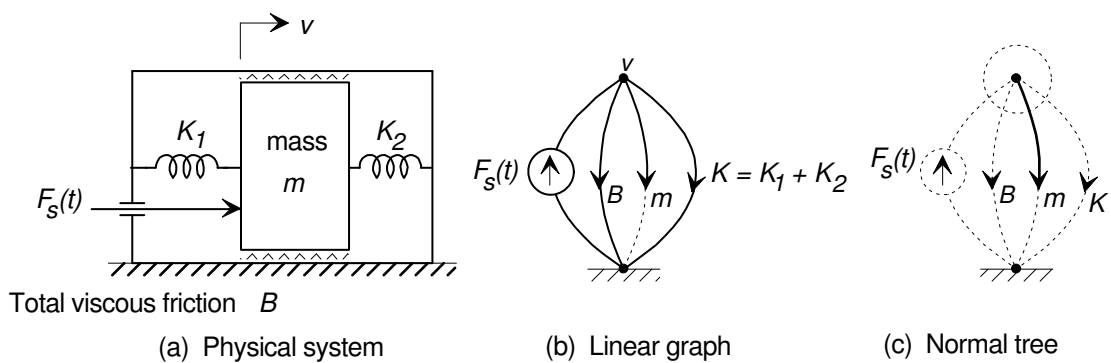


Figure 11: A mechanical system with two parallel springs, its linear graph and a normal tree after combining the two springs.

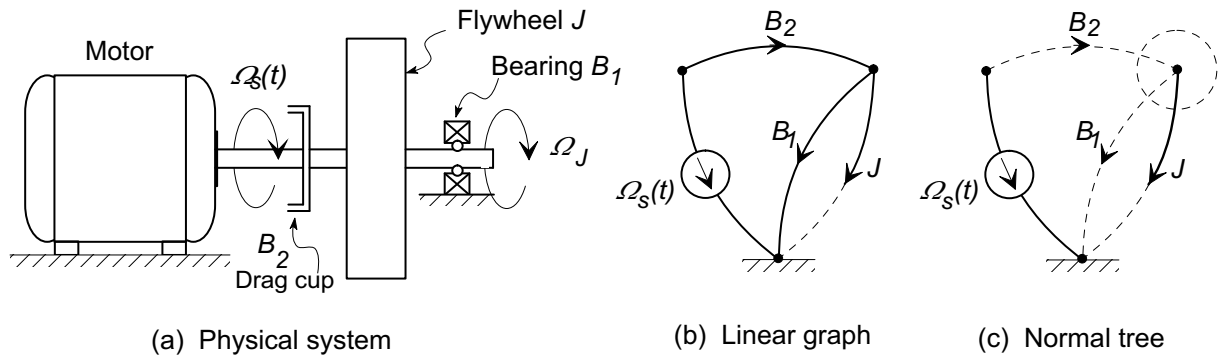


Figure 12: A rotational system, its linear graph and normal tree.

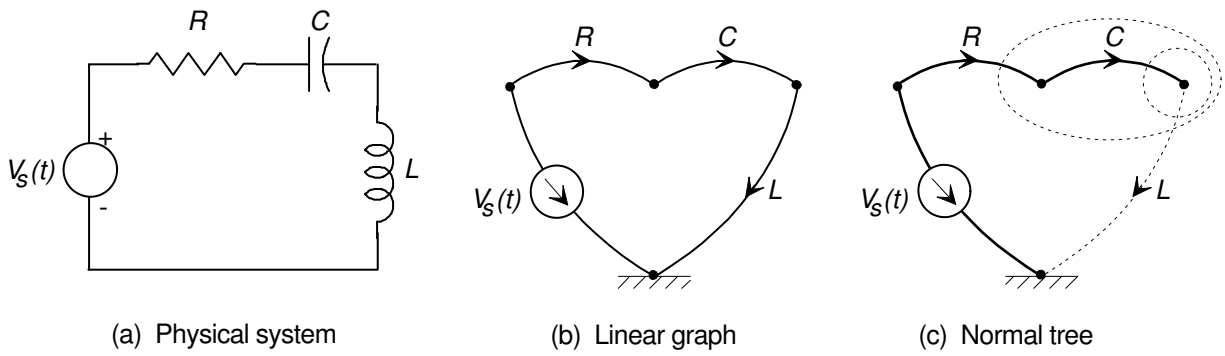


Figure 13: An electrical circuit, its linear graph and normal tree.

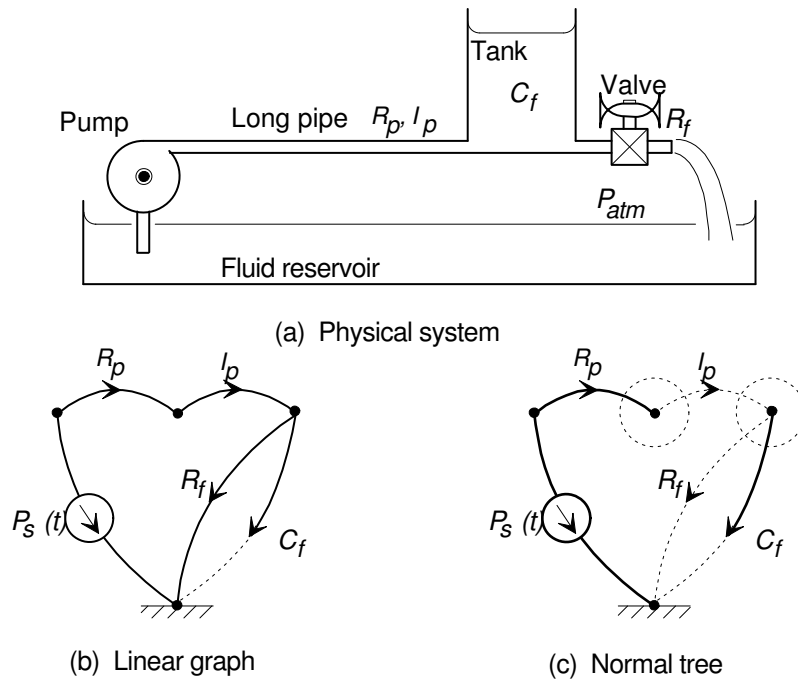


Figure 14: A fluid system, its system graph and normal tree.

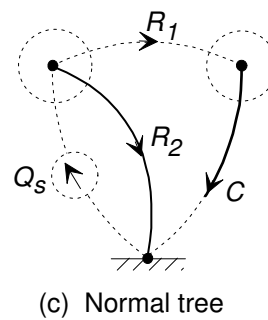
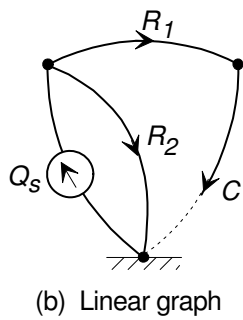
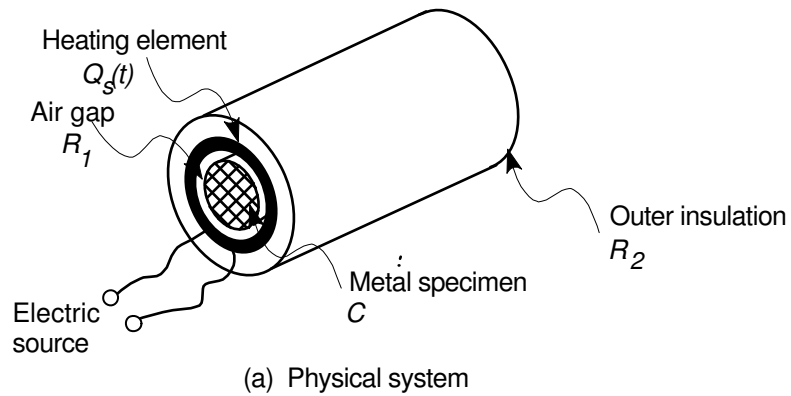


Figure 15: A thermal system, its linear graph and normal tree.

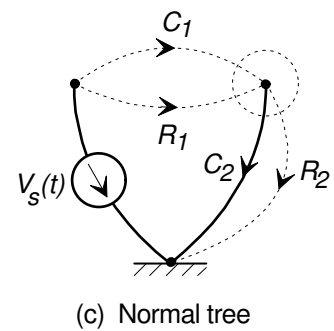
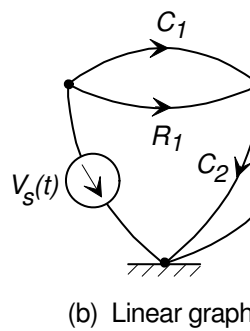
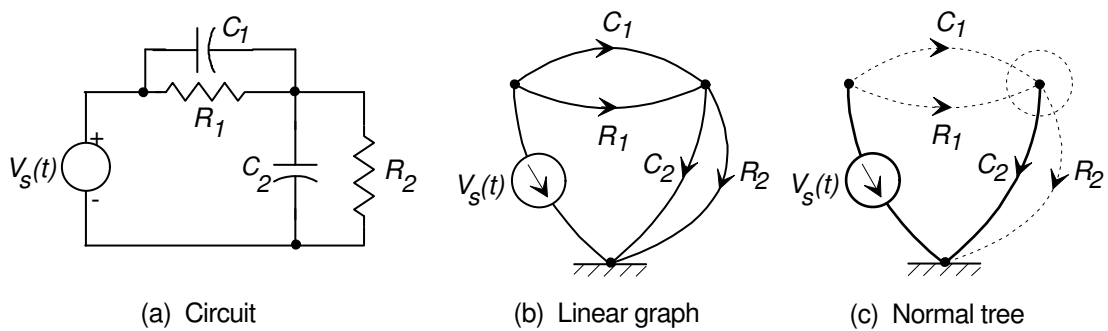


Figure 16: Simple first-order electrical lead-lag network with a dependent energy storage element.

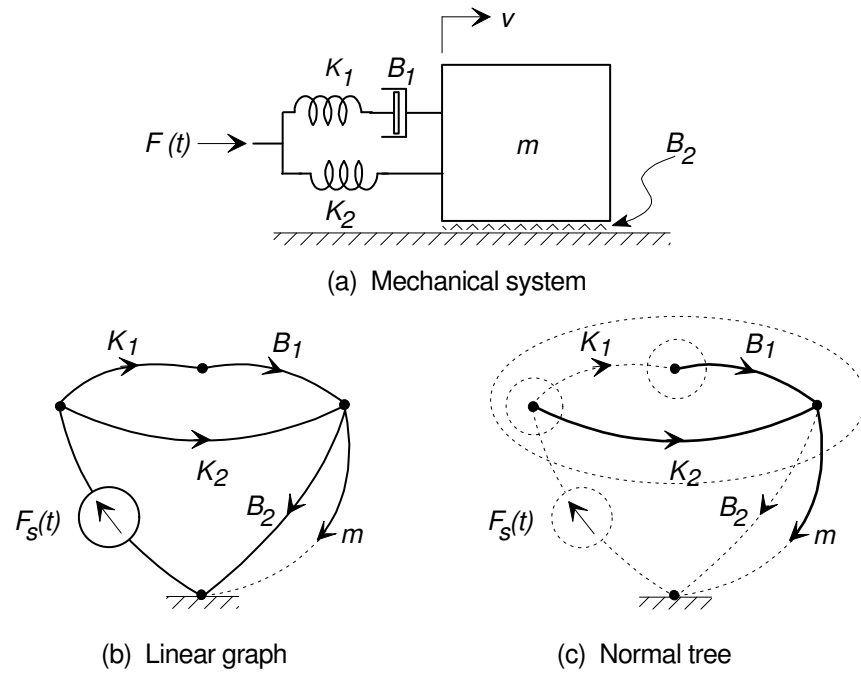


Figure 17: Second-order mechanical system with dependent springs.

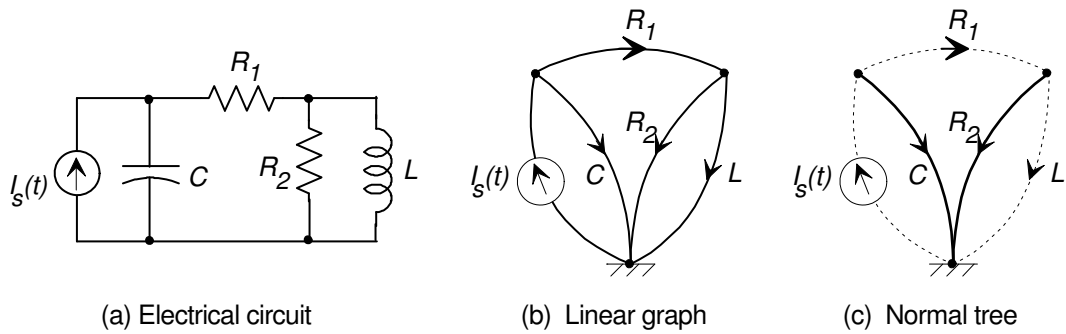


Figure 18: An electrical system, its linear graph and a normal tree.

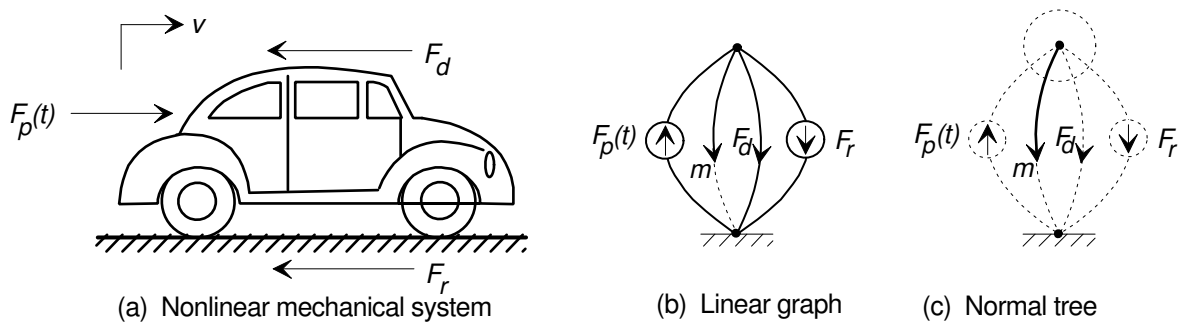


Figure 19: Linear graph model of automobile with nonlinear drag forces.

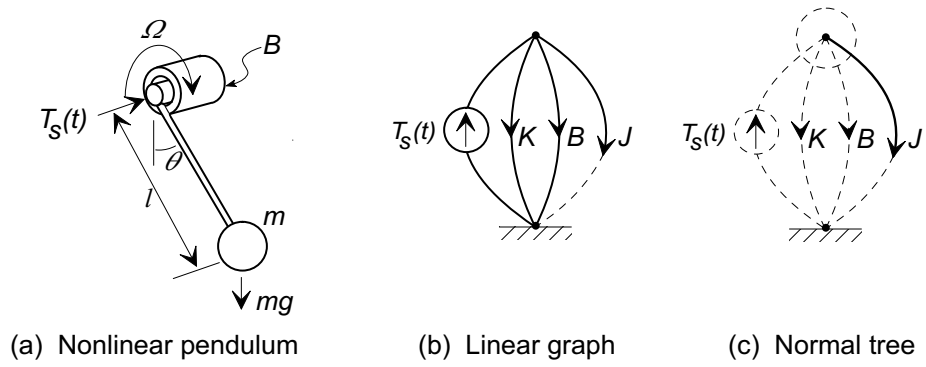


Figure 20: A mechanical torque measuring instrument based on a pendulum.

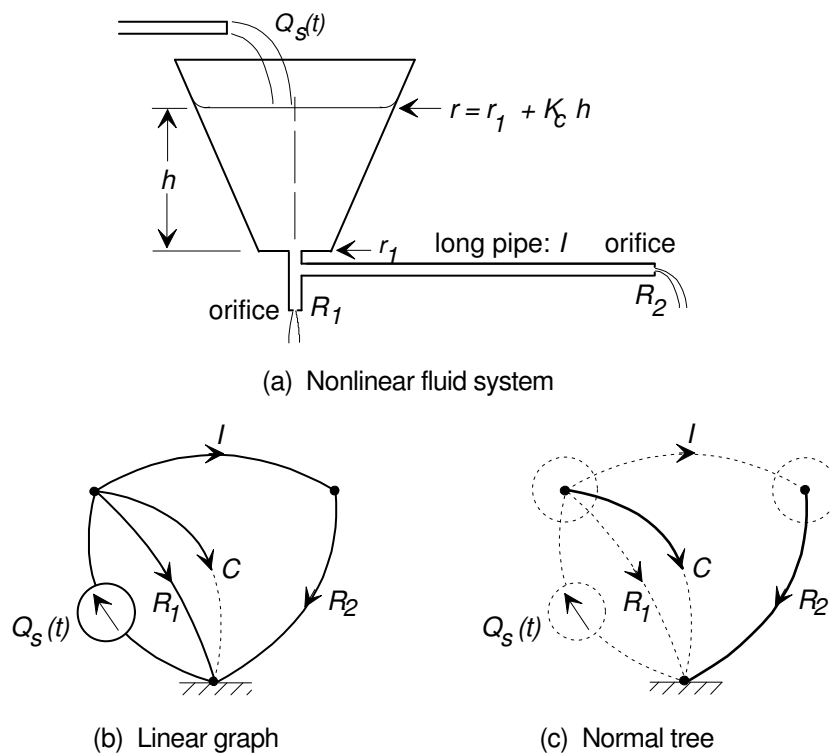


Figure 21: Fluid distribution system with nonlinear elements.