MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.151 Advanced System Dynamics and Control

Review of First- and Second-Order System Response[1]

1 First-Order Linear System Transient Response

The dynamics of many systems of interest to engineers may be represented by a simple model containing one independent energy storage element. For example, the braking of an automobile, the discharge of an electronic camera flash, the flow of fluid from a tank, and the cooling of a cup of coffee may all be approximated by a first-order differential equation, which may be written in a standard form as

$$\tau \frac{dy}{dt} + y(t) = f(t) \tag{1}$$

where the system is defined by the single parameter τ, the system time constant, and $f(t)$ is a forcing function. For example, if the system is described by a linear first-order state equation and an associated output equation:

$$\dot{x} = ax + bu \tag{2}$$

$$y = cx + du. \tag{3}$$

and the selected output variable is the state-variable, that is $y(t) = x(t)$, Eq. (3) may be rearranged

$$\frac{dy}{dt} - ay = bu, \tag{4}$$

and rewritten in the standard form (in terms of a time constant $\tau = -1/a$), by dividing through by $-a$:

$$-\frac{1}{a}\frac{dy}{dt} + y(t) = -\frac{b}{a}u(t) \tag{5}$$

where the forcing function is $f(t) = (-b/a)u(t)$.

If the chosen output variable $y(t)$ is not the state variable, Eqs. (2) and (3) may be combined to form an input/output differential equation in the variable $y(t)$:

$$\frac{dy}{dt} - ay = d\frac{du}{dt} + (bc - ad)\,u. \tag{6}$$

To obtain the standard form we again divide through by $-a$:

$$-\frac{1}{a}\frac{dy}{dt} + y(t) = -\frac{d}{a}\frac{du}{dt} + \frac{ad - bc}{a}u(t). \tag{7}$$

Comparison with Eq. (1) shows the time constant is again $\tau = -1/a$, but in this case the forcing function is a combination of the input and its derivative

$$f(t) = -\frac{d}{a}\frac{du}{dt} + \frac{ad - bc}{a}u(t). \tag{8}$$

In both Eqs. (5) and (7) the left-hand side is a function of the time constant $\tau = -1/a$ only, and is independent of the particular output variable chosen.

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■ Example 1

A sample of fluid, modeled as a thermal capacitance C_t, is contained within an insulating vacuum flask. Find a pair of differential equations that describe 1) the temperature of the fluid, and 2) the heat flow through the walls of the flask as a function of the external ambient temperature. Identify the system time constant.

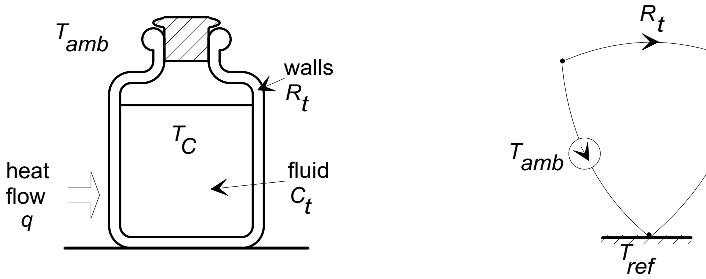

Figure 1: A first-order thermal model representing the heat exchange between a laboratory vacuum flask and the environment.

Solution: The walls of the flask may be modeled as a single lumped thermal resistance R_t and a linear graph for the system drawn as in Fig. 1. The environment is assumed to act as a temperature source $T_{amb}(t)$. The state equation for the system, in terms of the temperature T_C of the fluid, is

$$\frac{dT_C}{dt} = -\frac{1}{R_tC_t}T_C + \frac{1}{R_tC_t}T_{amb}(t). \tag{i}$$

The output equation for the flow q_R through the walls of the flask is

$$\begin{aligned} q_R &= \frac{1}{R_t}T_R \\ &= -\frac{1}{R_t}T_C + \frac{1}{R_t}T_{amb}(t). \end{aligned} \tag{ii}$$

The differential equation describing the dynamics of the fluid temperature T_C is found directly by rearranging Eq. (i):

$$R_tC_t\frac{dT_C}{dt} + T_C = T_{amb}(t). \tag{iii}$$

from which the system time constant τ may be seen to be $\tau = R_tC_t$.

The differential equation relating the heat flow through the flask is

$$\frac{dq_R}{dt} + \frac{1}{R_tC_t}q_R = \frac{1}{R_t}\frac{dT_{amb}}{dt}. \tag{iv}$$

This equation may be written in the standard form by dividing both sides by $1/R_tC_t$,

$$R_tC_t\frac{dq_R}{dt} + q_R = C_t\frac{dT_{amb}}{dt}, \tag{v}$$

and by comparison with Eq. (7) it can be seen that the system time constant $\tau = R_tC_t$ and the forcing function is $f(t) = C_t dT_{amb}/dt$. Notice that the time constant is independent of the output variable chosen.

1.1 The Homogeneous Response and the First-Order Time Constant

The standard form of the homogeneous first-order equation, found by setting $f(t) \equiv 0$ in Eq. (1), is the same for all system variables:

$$\tau \frac{dy}{dt} + y = 0 \tag{9}$$

and generates the characteristic equation:

$$\tau\lambda + 1 = 0 \tag{10}$$

which has a single root, $\lambda = -1/\tau$. The system response to an initial condition $y(0)$ is

$$y_h(t) = y(0)e^{\lambda t} = y(0)e^{-t/\tau}. \tag{11}$$

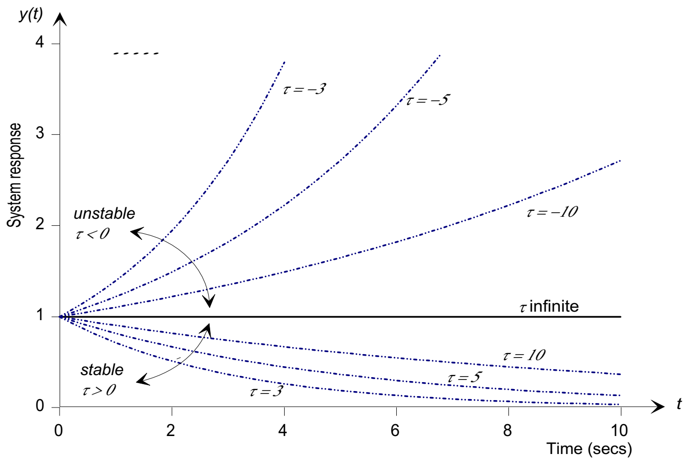

Figure 2: Response of a first-order homogeneous equation $\tau\dot{y} + y(t) = 0$. The effect of the system time constant τ is shown for stable systems ($\tau > 0$) and unstable systems ($\tau < 0$).

A physical interpretation of the time constant τ may be found from the initial condition response of any output variable $y(t)$. If $\tau > 0$, the response of any system variable is an exponential decay from the initial value $y(0)$ toward zero, and the system is *stable*. If $\tau < 0$ the response grows exponentially for any finite value of y_0, as shown in Fig. 1.1, and the system is *unstable*. Although energetic systems containing only sources and passive linear elements are usually stable, it is possible to create instability when an active control system is connected to a system. Some sociological and economic models exhibit inherent instability. The time-constant τ, which has units of time, is the system parameter that establishes the time scale of system responses in a first-order system. For example a resistor-capacitor circuit in an electronic amplifier might have a time constant of a few microseconds, while the cooling of a building after sunset may be described by a time constant of many hours.

It is common to use a normalized time scale, t/τ, to describe first-order system responses. The homogeneous response of a stable system is plotted in normalized form in Fig. 3, using both the normalized time and also a normalized response magnitude $y(t)/y(0)$:

$$y(t)/y(0) = e^{-(t/\tau)}. \tag{12}$$

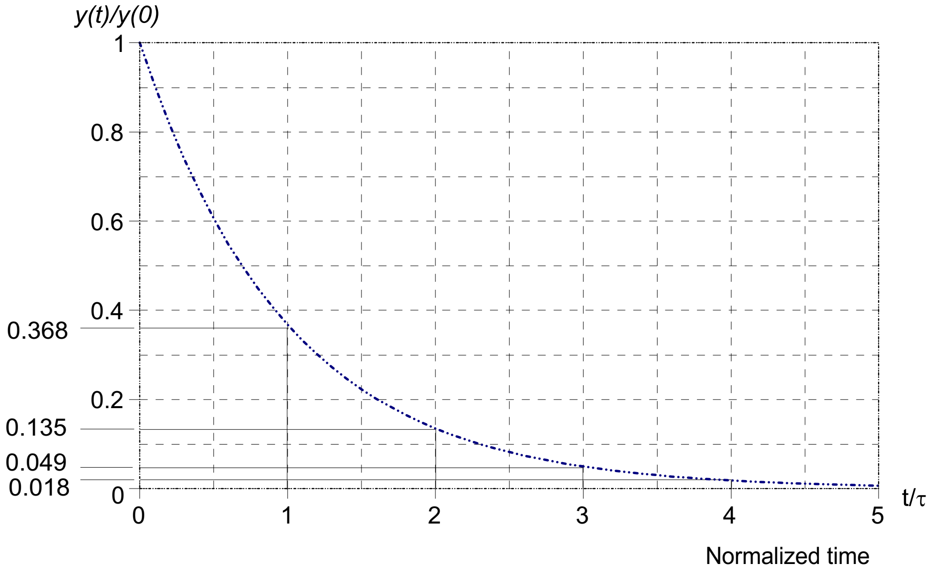

Figure 3: Normalized unforced response of a stable first-order system.

Time	$-t/\tau$	$y(t)/y(0) = e^{-t/\tau}$	$y_s(t) = 1 - e^{-t/\tau}$
0	0.0	1.0000	0.0000
τ	1.0	0.3679	0.6321
2τ	2.0	0.1353	0.8647
3τ	3.0	0.0498	0.9502
4τ	4.0	0.0183	0.9817

Table 1: Exponential components of first-order system responses in terms of normalized time t/τ.

The third column of Table 1 summarizes the homogeneous response after periods $t = \tau, 2\tau, \ldots$ After a period of one time constant ($t/\tau = 1$) the output has decayed to $y(\tau) = e^{-1}y(0)$ or 36.8% of its initial value, after two time constants the response is $y(2\tau) = 0.135y(0)$.

Several first-order mechanical and electrical systems and their time constants are shown in Fig. 4. For the mechanical mass-damper system shown in Fig. 4a, the velocity of the mass decays from any initial value in a time determined by the time constant $\tau = m/B$, while the unforced deflection of the spring shown in Fig. 4b decays with a time constant $\tau = B/K$. In a similar manner the voltage on the capacitor in Fig. 4c will decay with a time constant $\tau = RC$, and the current in the inductor in Fig. 4d decays with a time constant equal to the ratio of the inductance to the resistance $\tau = L/R$. In all cases, if SI units are used for the element values, the units of the time constant will be seconds.

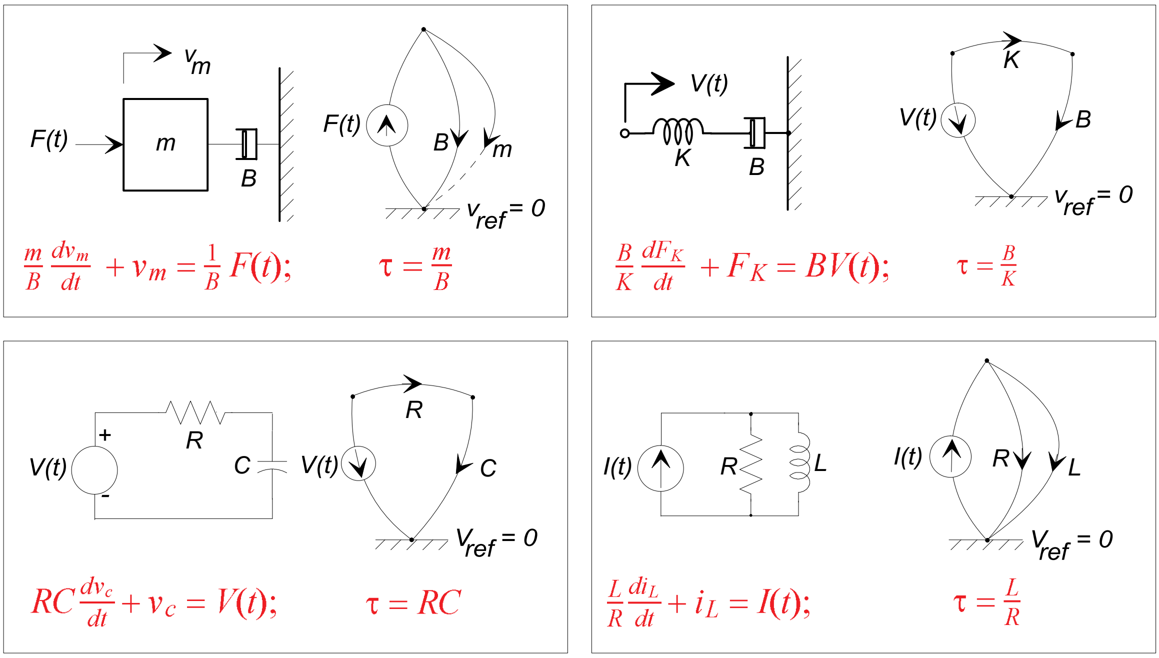

Figure 4: Time constants of some typical first-order systems.

■ Example 2

A water tank with vertical sides and a cross-sectional area of 2 m^2, shown in Fig. 5, is fed from a constant displacement pump, which may be modeled as a flow source $Q_{in}(t)$. A valve, represented by a linear fluid resistance R_f, at the base of the tank is always open and allows water to flow out. In normal operation the tank is filled to a depth of

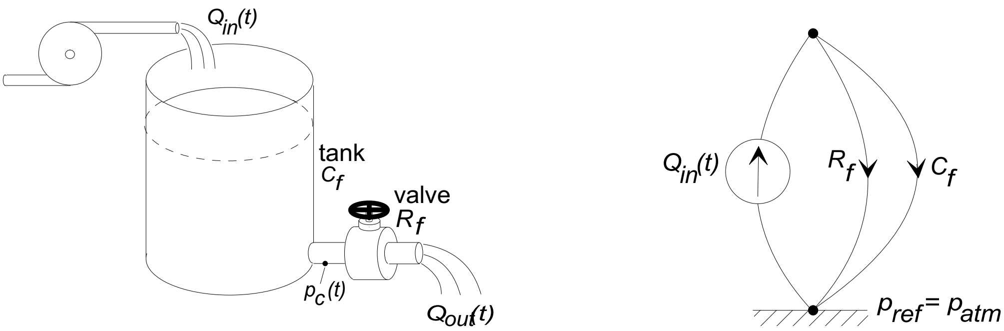

Figure 5: Fluid tank example

1.0 m. At time $t = 0$ the power to the pump is removed and the flow into the tank is disrupted.

If the flow through the valve is 10^{-6} m^3/s when the pressure across it is 1 N/m^2, determine the pressure at the bottom of the tank as it empties. Estimate how long it takes for the tank to empty.

Solution: The tank is represented as a fluid capacitance C_f with a value:

$$C_f = \frac{A}{\rho g} \tag{i}$$

where A is the area, g is the gravitational acceleration, and ρ is the density of water. In this case $C_f = 2/(1000 \times 9.81) = 2.04 \times 10^{-4}$ m^5/n and $R_f = 1/10^{-6} = 10^6$ N-s/m^5.

The linear graph generates a state equation in terms of the pressure across the fluid capacitance $P_C(t)$:

$$\frac{dP_C}{dt} = -\frac{1}{R_f C_f} P_C + \frac{1}{C_f} Q_{in}(t) \tag{ii}$$

which may be written in the standard first-order form

$$R_f C_f \frac{dP_C}{dt} + P_C = R_f Q_{in}(t). \tag{iii}$$

The time constant is $\tau = R_f C_f$. When the pump fails the input flow Q_{in} is set to zero, and the system is described by the homogeneous equation

$$R_f C_f \frac{dP_C}{dt} + P_C = 0. \tag{iv}$$

The homogeneous pressure response is (from Eq. (11)):

$$P_C(t) = P_C(0) e^{-t/R_f C_f}. \tag{v}$$

With the given parameters the time constant is $\tau = R_f C_f = 204$ seconds, and the initial depth of the water $h(0)$ is 1 m; the initial pressure is therefore $P_C(0) = \rho g h(0) = 1000 \times 9.81 \times 1$ N/m^2. With these values the pressure at the base of the tank as it empties is

$$P_C(t) = 9810 e^{-t/204} \text{ N/m}^2 \tag{vi}$$

which is the standard first-order form shown in Fig. 3.

The time for the tank to drain cannot be simply stated because the pressure asymptotically approaches zero. It is necessary to define a criterion for the complete decay of the response; commonly a period of $t = 4\tau$ is used since $y(t)/y(0) = e^{-4} < 0.02$ as shown in Table 1. In this case after a period of $4\tau = 816$ seconds the tank contains less than 2% of its original volume and may be approximated as empty.

1.2 The Characteristic Response of First-Order Systems

In standard form the input/output differential equation for any variable in a linear first-order system is given by Eq. (1):

$$\tau \frac{dy}{dt} + y = f(t). \tag{13}$$

The only system parameter in this differential equation is the time constant τ. The solution with the given $f(t)$ and the initial condition $y(0) = 0$ is defined to be the *characteristic first-order response.*

The first-order *homogeneous solution* is of the form of an exponential function $y_h(t) = e^{-\lambda t}$ where $\lambda = 1/\tau$. The total response $y(t)$ is the sum of two components

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= Ce^{-t/\tau} + y_p(t) \end{aligned} \tag{14}$$

where C is a constant to be found from the initial condition $y(0) = 0$, and $y_p(t)$ is a *particular solution* for the given forcing function $f(t)$. In the following sections we examine the form of $y(t)$ for the ramp, step, and impulse singularity forcing functions.

1.2.1 The Characteristic Unit Step Response

The unit step $u_s(t)$ is commonly used to characterize a system's response to sudden changes in its input. It is discontinuous at time $t = 0$:

$$f(t) = u_s(t) = \begin{cases} 0 & t < 0, \\ 1 & t \geq 0. \end{cases}$$

The characteristic step response $y_s(t)$ is found by determining a particular solution for the step input using the method of undetermined coefficients. From Table 8.2, with a constant input for $t > 0$, the form of the particular solution is $y_p(t) = K$, and substitution into Eq. (13) gives $K = 1$. The complete solution $y_s(t)$ is

$$y_s(t) = Ce^{-t/\tau} + 1. \tag{15}$$

The characteristic response is defined when the system is initially *at rest*, requiring that at $t = 0$, $y_s(0) = 0$. Substitution into Eq. (14) gives $0 = C + 1$, so that the resulting constant $C = -1$. The unit step response of a system defined by Eq. (13) is:

$$y_s(t) = 1 - e^{-t/\tau}. \tag{16}$$

Equation (16) shows that, like the homogeneous response, the time dependence of the step response depends only on τ and may expressed in terms of a normalized time scale t/τ. The unit step characteristic response is shown in Fig. 6, and the values at normalized time increments are summarized in the fourth column of Table 1. The response asymptotically approaches a steady-state value

$$y_{ss} = \lim_{t\to\infty} y_s(t) = 1. \tag{17}$$

It is common to divide the step response into two regions,

(a) a *transient* region in which the system is still responding dynamically, and

(b) a *steady-state* region, in which the system is assumed to have reached its final value y_{ss}.

There is no clear division between these regions but the time $t = 4\tau$, when the response is within 2% of its final value, is often chosen as the boundary between the transient and steady-state responses.

The initial slope of the response may be found by differentiating Eq. (16) to yield:

$$\left.\frac{dy}{dt}\right|_{t=0} = \frac{1}{\tau}. \tag{18}$$

The step response of a first-order system may be easily sketched with knowledge of (1) the system time constant τ, (2) the steady-state value y_{ss}, (3) the initial slope $\dot{y}(0)$, and (4) the fraction of the final response achieved at times equal to multiples of τ.

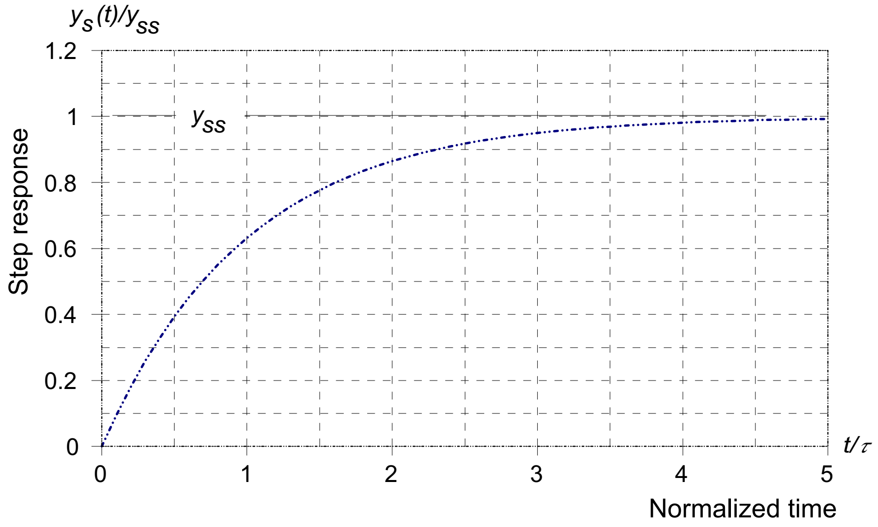

Figure 6: The step response of a first-order system described by $\tau\dot{y} + y = u_s(t)$.

1.2.2 The Characteristic Impulse Response

The impulse function $\delta(t)$ is defined as the limit of a pulse of duration T and amplitude $1/T$ as T approaches zero, and is used to characterize the response of systems to brief transient inputs. The impulse may be considered as the derivative of the unit step function.

The derivative property of linear systems allows us to find the characteristic impulse response $y_\delta(t)$ by simply differentiating the characteristic step response $y_s(t)$. When the forcing function $f(t) = \delta(t)$ the characteristic response is

$$\begin{aligned} y_\delta(t) &= \frac{dy_s}{dt} = \frac{d}{dt}\left(1 - e^{-t/\tau}\right) \\ &= \frac{1}{\tau}e^{-t/\tau} \qquad \text{for } t \geq 0. \end{aligned} \tag{19}$$

The characteristic impulse response is an exponential decay, similar in form to the homogeneous response. It is discontinuous at time $t = 0$ and has an initial value $y(0^+) = 1/\tau$, where the superscript 0^+ indicates a time incrementally greater than zero. The response is plotted in normalized form in Fig. 7.

1.2.3 The Characteristic Ramp Response

The unit ramp $u_r(t) = t$ for $t \geq 0$ is the integral of the unit step function $u_s(t)$:

$$u_r(t) = \int_0^t u_s(t)dt. \tag{20}$$

The integration property of linear systems (Section 8.4.4) allows the characteristic response $y_r(t)$ to a ramp forcing function $f(t) = u_r(t)$ to be found by integrating the step response $y_s(t)$:

$$\begin{aligned} y_r(t) &= \int_0^t y_s(t)dt = \int_0^t \left(1 - e^{-t/\tau}\right)dt \\ &= t - \tau\left(1 - e^{-t/\tau}\right) \end{aligned} \tag{21}$$

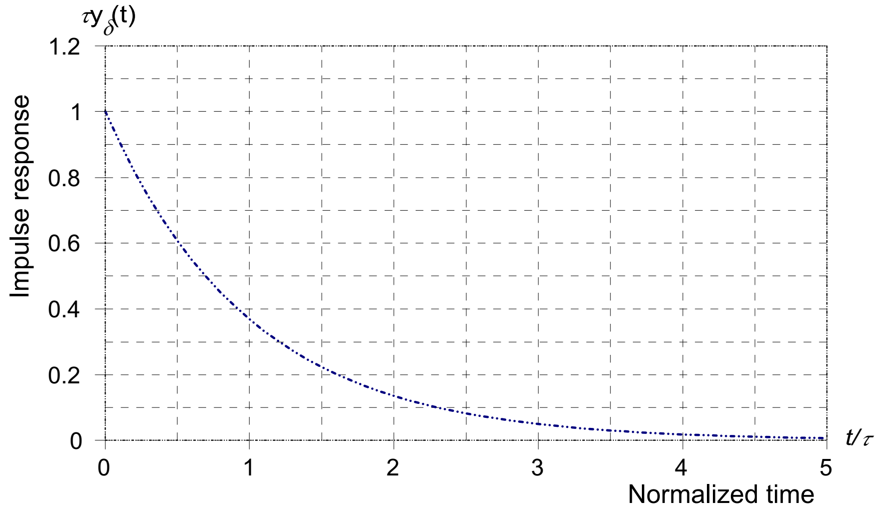

Figure 7: The impulse response of a first-order system described by $\tau\dot{y} + y = \delta(t)$.

and is plotted in Fig. 8. As t becomes large the exponential term decays to zero and the response becomes

$$y_r(t) \approx t - \tau \qquad \text{for } t \gg \tau. \tag{22}$$

1.3 System Input/Output Transient Response

In the previous section we examined the system response to particular forms of the forcing function $f(t)$. We now return to the solution of the complete most general first-order differential equation, Eq. (7):

$$\tau\frac{dy}{dt} + y(t) = q_1\frac{du}{dt} + q_0 u(t) \tag{23}$$

where $\tau = -1/a$, $q_1 = -d/a$ and $q_2 = (ad - bc)/a$ are constants defined by the system parameters. The forcing function in this case is a superposition of the system input $u(t)$ and its derivative:

$$f(t) = q_1\frac{du}{dt} + q_0 u(t).$$

The superposition principle for linear systems allows us to compute the response separately for each term in the forcing function, and to combine the component responses to form the overall response $y(t)$. In addition, the differentiation property of linear systems allows the response to the derivative of an input to be found by differentiating the response to that input. These two properties may be used to determine the overall input/output response in two steps:

(1) Find the characteristic response $y_u(t)$ of the system to the forcing function $f(t) = u(t)$, that is solve the differential equation:

$$\tau\frac{dy_u}{dt} + y_u(t) = u(t), \tag{24}$$

(2) Form the output as a combination of the output and its derivative:

$$y(t) = q_1\frac{dy_u}{dt} + q_0 y_u(t). \tag{25}$$

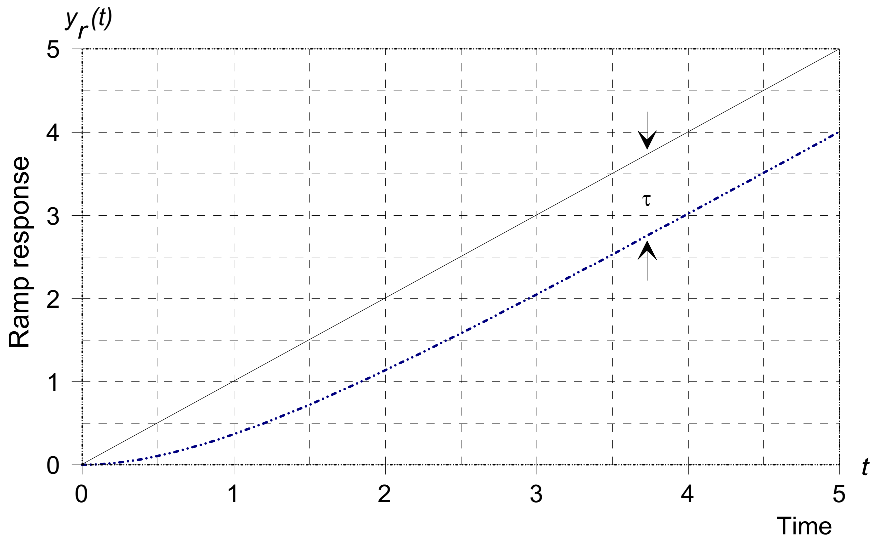

Figure 8: The ramp response of a first-order system described by $\tau\dot{y} + y = u_r(t)$.

The characteristic responses $y_u(t)$ are by definition zero for time $t < 0$. If there is a discontinuity in $y_u(t)$ at $t = 0$, as in the case for the characteristic impulse response $y_\delta(t)$ (Eq. (19)), the derivative dy_u/dt contains an impulse component, for example

$$\frac{d}{dt}y_\delta(t) = \frac{1}{\tau}\delta(t) - \frac{1}{\tau^2}e^{-t/\tau} \tag{26}$$

and if $q_1 \neq 0$ the response $y(t)$ will contain an impulse function.

1.3.1 The Input/Output Step Response

The characteristic response for a unit step forcing function, $f(t) = u_s(t)$, is (Eq. (16)):

$$y_s(t) = \left(1 - e^{-t/\tau}\right) \qquad \text{for } t > 0.$$

The system input/output step response is found directly from Eq. (25):

$$\begin{aligned} y(t) &= q_1 \frac{d}{dt}\left(1 - e^{-t/\tau}\right) + q_0\left(1 - e^{-t/\tau}\right) \\ &= q_0\left[1 - \left(1 - \frac{q_1}{q_0\tau}\right)e^{-t/\tau}\right]. \end{aligned} \tag{27}$$

If $q_1 \neq 0$ the output is discontinuous at $t = 0$, and $y(0^+) = q_1/\tau$. The steady-state response y_{ss} is

$$y_{ss} = \lim_{t\to\infty} y(t) = q_0. \tag{28}$$

The output moves from the initial value to the final value with a time constant τ.

1.3.2 The Input/Output Impulse Response

The characteristic impulse response $y_\delta(t)$ found in Eq. (19) is

$$y_\delta(t) = \frac{1}{\tau}e^{-t/\tau} \qquad \text{for} t \geq 0$$

Input $u(t)$	Characteristic Response	Input/Output Response $y(t)$ for $t \geq 0$
$u(t) = 0$		$y(t) = y(0)e^{-t/\tau}$
$u(t) = u_r(t)$	$y_r(t) = t - \tau \left(1 - e^{-t/\tau}\right)$	$y(t) = \left[q_0 t + (q_1 - q_0\tau)\left(1 - e^{-t/\tau}\right)\right]$
$u(t) = u_s(t)$	$y_s(t) = y_s(t) = 1 - e^{-t/\tau}$	$y(t) = \left[q_0 - \left(q_0 - \frac{q_1}{\tau}\right) e^{-t/\tau}\right]$
$u(t) = \delta(t)$	$y_\delta(t) = \frac{1}{\tau} e^{-t/\tau}$	$y(t) = \frac{q_1}{\tau}\delta(t) + \left(\frac{q_0}{\tau} - \frac{q_1}{\tau^2}\right) e^{-t/\tau}$

Table 2: The response of the first-order linear system $\tau\dot{y} + y = q_1\dot{u} + q_0 u$ for the singularity inputs.

with a discontinuity at time $t = 0$. Substituting into Eq. (25)

$$\begin{aligned} y(t) &= q_1 \frac{dy_\delta}{dt} + q_0 y_\delta(t) \\ &= \frac{q_1}{\tau}\delta(t) + \left(\frac{q_0}{\tau} - \frac{q_1}{\tau^2}\right) e^{-t/\tau}, \end{aligned} \tag{29}$$

where the impulse is generated by the discontinuity in $y_\delta(t)$ at $t = 0$ as shown in Eq. (26).

1.3.3 The Input/Output Ramp Response

The characteristic response to a unit ramp $r(t) = t$ is

$$y_r(t) = t - \tau \left(1 - e^{-t/\tau}\right)$$

and using Eq. (21) the response is:

$$\begin{aligned} y(t) &= q_1 \frac{d}{dt}\left[\left(t - \tau\left(1 - e^{-t/\tau}\right)\right) u_s(t)\right] + q_0 \left(t - \tau\left(1 - e^{-t/\tau}\right)\right) u_s(t) \\ &= \left[q_0 t + (q_1 - q_0\tau)\left(1 - e^{-t/\tau}\right)\right] u_s(t). \end{aligned} \tag{30}$$

1.4 Summary of Singularity Function Responses

Table 2 summarizes the homogeneous and forced responses of the first-order linear system described by the classical differential equation

$$\tau \frac{dy}{dx} + y = q_1 \frac{du}{dt} + q_0 u \tag{31}$$

for the three commonly used singularity inputs.

The response of a system with a non-zero initial condition, $y(0)$, to an input $u(t)$ is the sum of the homogeneous component due to the initial condition, and a forced component computed with zero initial condition, that is

$$y_{total}(t) = y(0)e^{-t/\tau} + y_u(t), \tag{32}$$

where $y_u(t)$ is the response of the system to the given input $u(t)$ if the system was originally at rest.

The response to an input that is a combination of inputs for which the response is known may be found by adding the individual component responses using the principle of superposition. The following examples illustrate the use of these solution methods.

■ Example 3

A mass $m = 10$ kg is at rest on a horizontal plane with viscous friction coefficient $B = 20$ N-s/m, as shown in Fig. 9. A short impulsive force of amplitude 200 N and duration 0.01 s is applied. Determine how far the mass travels before coming to rest, and how long it takes for the velocity to decay to less than 1% of its initial value.

Solution: The differential equation relating the velocity of the mass to the applied

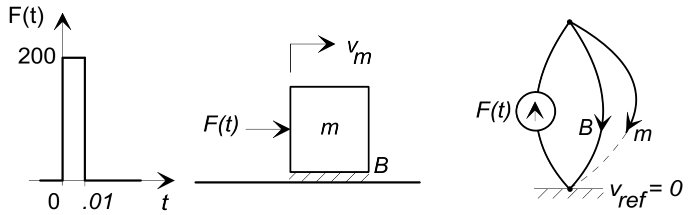

Figure 9: A mass element subjected to an impulsive force.

force is

$$\frac{m}{B}\frac{dv_m}{dt} + v_m = \frac{1}{B}F_{in}(t) \tag{i}$$

The system time constant is $\tau = m/B = 10/20 = 0.5$ seconds. The duration of the force pulse is much less than the time constant, and so it is reasonable to approximate the input as an impulse of strength (area) $200 \times .01 = 2$ N-s. The system impulse response (Eq. (29) is

$$v_m(t) = \frac{1}{m}e^{-Bt/m} \tag{ii}$$

so that if $u(t) = 2\delta(t)$ N-s the response is

$$v_m(t) = 0.2e^{-2t}. \tag{iii}$$

The distance x traveled may be computed by integrating the velocity

$$x = \int_0^\infty 0.2e^{-2t}dt = 0.1 \text{ m}. \tag{iv}$$

The time T for the velocity to decay to less than 1% of its original value is found by solving $v_m(T)/v_m(0) = 0.01 = e^{-2T}$, or $T = 2.303$ seconds.

■ Example 4

A disk flywheel J of mass 8 Kg and radius 0.5 m is driven by an electric motor that produces a constant torque of $T_{in} = 10$ N-m. The shaft bearings may be modeled as viscous rotary dampers with a damping coefficient of $B_R = 0.1$ N-m-s/rad. If the flywheel is at rest at $t = 0$ and the power is suddenly applied to the motor, compute and plot the variation in speed of the flywheel, and find the maximum angular velocity of the flywheel.

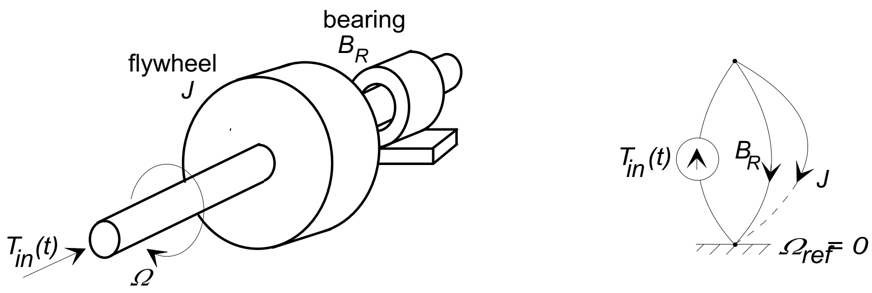

Figure 10: Rotary flywheel system and its linear graph

Solution: The state equation for the system may be found directly from the linear graph in Fig. 10:

$$\frac{d\Omega_J}{dt} = -\frac{B_R}{J}\Omega_J + \frac{1}{J}T_{in}(t), \tag{i}$$

which in the standard form is

$$\frac{J}{B_R}\frac{d\Omega_J}{dt} + \Omega_J = \frac{1}{B_R}T_{in}(t). \tag{ii}$$

For the flywheel $J = mr^2/2 = 1$ kg-m^2, and the time constant is

$$\tau = \frac{J}{B_R} = 10 \text{ s}. \tag{iii}$$

The characteristic response to a unit step in the forcing function is

$$y_s(t) = 1 - e^{-t/10} \tag{iv}$$

and by the principle of superposition, when the forcing function is scaled so that $f(t) = (T_{in}/B_R)u_s(t)$, the output is similarly scaled:

$$\Omega_J(t) = \frac{T_{in}}{B_R}\left(1 - e^{-(B_R/J)t}\right) = 100\left(1 - e^{-t/10}\right). \tag{v}$$

The steady-state angular velocity is

$$\Omega_{ss} = \lim_{t\to\infty}\Omega_J(t) = T_{in}/B_R = 100 \text{ rad/s} \tag{vi}$$

and the angular velocity reaches 98% of this value in $t = 4\tau = 40$ seconds. The step response is shown in Fig. 11.

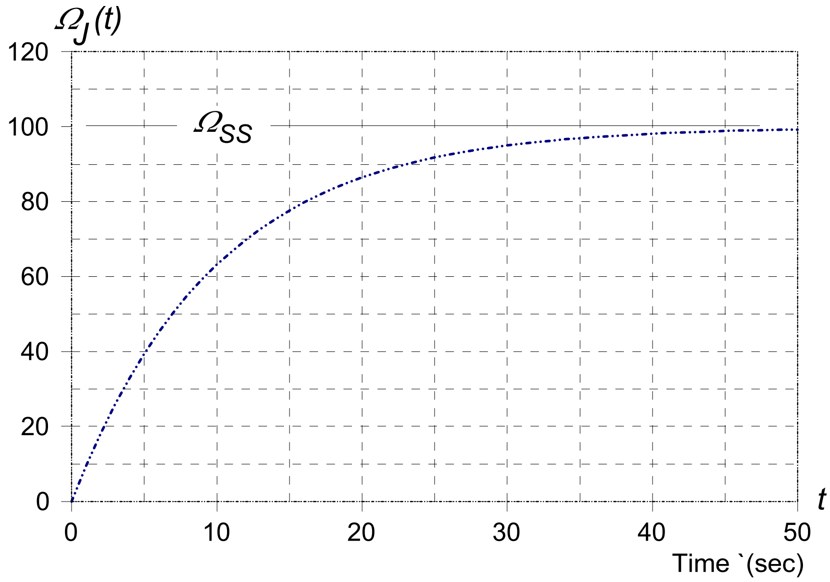

Figure 11: Response of the rotary flywheel system to a constant torque input, with initial condition $\Omega_J(0) = 0$, in Example 4

■ Example 5

During normal operation the flywheel drive system described in Example 4 is driven by a programmed torque source that produces a torque profile as shown in Fig. 12. The torque is ramped up to a maximum of 20 N-m over a period of 100 seconds, held at a constant value for 25 seconds and then reduced to zero. Find the resulting angular velocity of the shaft.

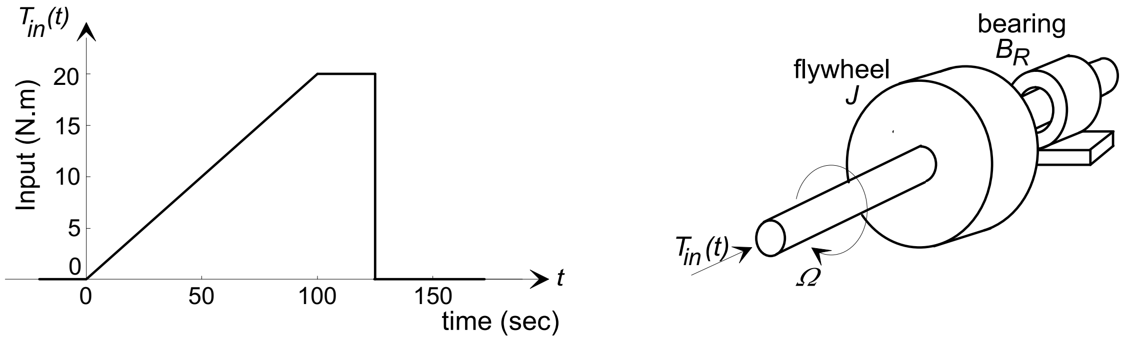

Figure 12: Rotary flywheel system and the input torque function specified in Example 5.

Solution: From Example 4 the differential equation describing the system is

$$\frac{J}{B_R}\frac{d\Omega_J}{dt} + \Omega_J = \frac{1}{B_R}T_{in}(t), \qquad \text{(i)}$$

and with the values given ($J = 1$ Kg-m^2 and $B_R = 0.1$ N-m-s/rad)

$$10\frac{d\Omega_J}{dt} + \Omega_J = 10T_{in}(t), \tag{ii}$$

The torque input shown in Fig. 12 may be written as a sum of unit ramp and step singularity functions

$$T_{in}(t) = 0.2u_r(t) - 0.2u_r(t-100) - 200u_s(t-125). \tag{iii}$$

The response may be determined in three time intervals

(1) Initially $0 \le t < 100$ when the input is effectively $T_{in}(t) = 0.2u_r(t)$,

(2) for $100 \le t < 125$ seconds when the input is $T_{in}(t) = 0.2u_r(t) - 0.2u_r(t-100)$, and

(3) for $t \ge 125$ when $T_{in}(t) = 0.2u_r(t) - 0.2u_r(t-100) - 20u_s(t-125)$.

From Table 2 the response in the three intervals may be written

$0 \le t < 100$ s:

$$\Omega_J(t) = 2\left[t - 10\left(1 - e^{-t/10}\right)\right] \text{ rad/s},$$

$100 \le t < 125$ s:

$$\begin{aligned}\Omega_J(t) &= 2\left[t - 10\left(1 - e^{-t/10}\right)\right] \\ &\quad -2\left[(t-100) - 10\left(1 - e^{-(t-100)/10}\right)\right] \text{ rad/s},\end{aligned}$$

$t > 125$ s:

$$\begin{aligned}\Omega_J(t) &= 2\left[t - 10\left(1 - e^{-t/10}\right)\right] \\ &\quad -2\left[(t-100) - 10\left(1 - e^{-(t-100)/10}\right)\right] \text{ rad/s}, \\ &\quad -200\left[1 - \left(1 - e^{-(t-125)/10}\right)\right]\end{aligned}$$

The total response is plotted in Fig. 13.

■ Example 6

The first-order electrical circuit shown in Fig. 14 is known as a "lead" network and is commonly used in electronic control systems. Find the response of the system to an input pulse of amplitude 1 volt and duration 10 ms if $R_1 = R_2 = 10,000$ ohms and $C = 1.0$ μfd. Assume that at time $t = 0$ the output voltage is zero. **Solution:** From the linear graph the state variable is the voltage on the capacitor $v_c(t)$, and the output is the voltage across R_2. The state equation for the system is

$$\frac{dv_c}{dt} = -\frac{R_1 + R_2}{R_1 R_2 C}v_c + \frac{1}{R_2 C}V_{in}(t) \tag{i}$$

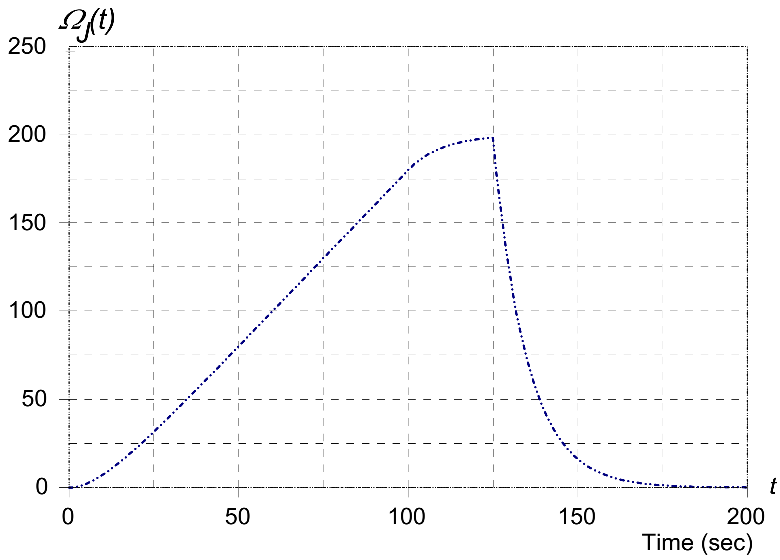

Figure 13: Response of the rotary flywheel system to the torque input profile $T_{in}(t) = 0.2u_r(t) - 0.2u_r(t-100) - 20u_s(t)$ N-m, with initial condition $\Omega_J(0) = 0$ rad/s.

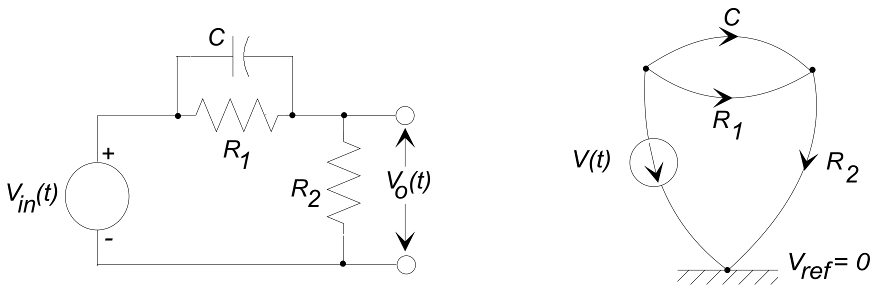

Figure 14: Electrical lead network and its linear graph.

and the output equation is

$$v_o(t) = v_{R_2} = -v_c + V_{in}(t), \tag{ii}$$

The input/output differential equation is

$$\frac{R_1 R_2 C}{R_1 + R_2}\frac{dv_o}{dt} + v_o = \frac{R_1 R_2 C}{R_1 + R_2}\frac{dV_{in}}{dt} + \frac{R_1}{R_1 + R_2}V_{in}. \tag{iii}$$

with the system time constant $\tau = R_1 R_2 C/(R_1 + R_2) = 5 \times 10^{-3}$ seconds.

The input pulse duration (10 ms) is comparable to the system time constant, and therefore it is not valid to approximate the input as an impulse. The pulse input can, however, be written as the sum of two unit step functions

$$V_{in}(t) = u_s(t) - u_s(t - 0.01) \tag{iv}$$

and the response determined in two separate intervals (1) $0 \leq t < 0.01$ s where the input is $u_s(t)$, and (2) $t \geq 0.01$ s, where both components contribute.

The input/output unit step response is given by Eq. (27),

$$\begin{aligned} v_o(t) &= \frac{R_2}{R_1+R_2} - \left(\frac{R_2}{R_1+R_2} - 1\right) e^{-t/\tau} \\ &= \frac{R_2}{R_1+R_2} + \frac{R_1}{R_1+R_2} e^{-t/\tau} \\ &= \left(0.5 + 0.5e^{-t/0.005}\right) \qquad \text{for } t \geq 0. \end{aligned} \tag{v}$$

At time $t = 0^+$ the initial response is $v_o(0^+) = 1$ volt, and the steady-state response $(v_0)_{ss} = 0.5$ volt. The settling time is approximately 4τ, or about 20 ms.

The response to the 10 ms duration pulse may be found from Eqs. (iv) and (v) by using the principle of superposition:

$$v_{pulse}(t) = v_o(t) - v_0(t - .01). \tag{vi}$$

In the interval $0 \leq t < 0.01$, the initial condition is zero and the response is:

$$v_{pulse}(t) = \left(0.5 + 0.5e^{-t/0.005}\right), \tag{vii}$$

in the second interval $t \geq .01$, when the input is $V_{in} = u_s(t) - u_s(t - .01)$, the response is the sum of two step responses:

$$\begin{aligned} v_{pulse}(t) &= \left(0.5 + 0.5e^{-t/0.005}\right) - \left(0.5 + 0.5e^{-(t-0.01)/0.005}\right) \\ &= 0.5\left(e^{t/.005} - e^{-(t-.01)/.005}\right) \\ &= 0.5e^{t/0.005}\left(1 - e^2\right) = -3.195e^{-t/.005} \text{ V}. \end{aligned} \tag{viii}$$

The step response (Eq. (v)) and the pulse response described by Eqs. (vii) and (viii) are plotted in Fig. 15.

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2 Second-Order System Transient Response

Second-order state determined systems are described in terms of two state variables. Physical second-order system models contain two independent energy storage elements which exchange stored energy, and may contain additional dissipative elements; such models are often used to represent the exchange of energy between mass and stiffness elements in mechanical systems; between capacitors and inductors in electrical systems, and between fluid inertance and capacitance elements in hydraulic systems. In addition second-order system models are frequently used to represent the exchange of energy between two independent energy storage elements in different energy domains coupled through a two-port element, for example energy may be exchanged between a mechanical mass and a fluid capacitance (tank) through a piston, or between an electrical inductance and mechanical inertia as might occur in an electric motor. Engineers often use second-order system models in the preliminary stages of design in order to establish the parameters of the energy storage and dissipation elements required to achieve a satisfactory response.

Second-order systems have responses that depend on the dissipative elements in the system. Some systems are oscillatory and are characterized by decaying, growing, or continuous oscillations.

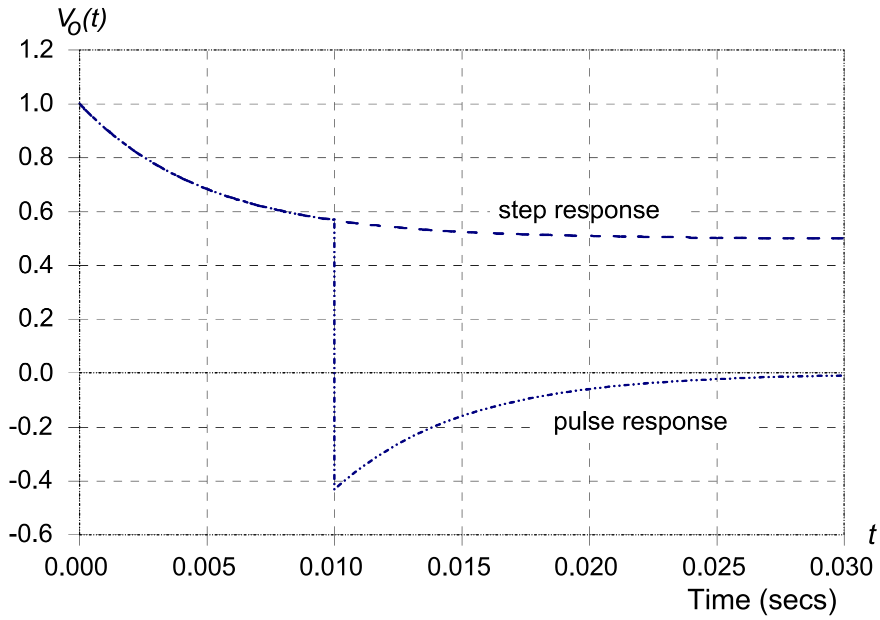

Figure 15: Response of the electrical lead network to a unit step in input voltage and to a unit amplitude pulse of duration 10 ms.

Other second order systems do not exhibit oscillations in their responses. In this section we define a pair of parameters that are commonly used to characterize second-order systems, and use them to define the the conditions that generate non-oscillatory, decaying or continuous oscillatory, and growing (or unstable) responses.

In the following sections we transform the two state equations into a single differential equation in the output variable of interest, and then express this equation in a standard form.

2.0.1 Transformation of State Equations to a Single Differential Equation

The state equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ for a linear second-order system with a single input are a pair of coupled first-order differential equations in the two state variables:

$$\left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array} \right] = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] + \left[\begin{array}{c} b_1 \\ b_2 \end{array} \right] u. \tag{33}$$

or

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + b_1 u \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_1 + b_2 u. \end{aligned} \tag{34}$$

The state-space system representation may be transformed into a single differential equation in either of the two state-variables. Taking the Laplace transform of the state equations

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{B}\mathbf{U}(s) \\ \mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \\ &= \frac{1}{\det[s\mathbf{I} - \mathbf{A}]} \left[\begin{array}{cc} s - a_{22} & a_{12} \\ a_{21} & s - a_{11} \end{array} \right] \left[\begin{array}{c} b_1 \\ b_2 \end{array} \right] U(s) \end{aligned}$$

$$\det\left[s\mathbf{I}-\mathbf{A}\right]\mathbf{X}(s) = \begin{bmatrix} s-a_{22} & a_{12} \\ a_{21} & s-a_{11} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} U(s)$$

from which

$$\frac{d^2x_1}{dt^2} - (a_{11}+a_{22})\frac{dx_1}{dt} + (a_{11}a_{22}-a_{12}a_{21})\,x_1 = b_1\frac{du}{dt} + (a_{12}b_2 - a_{22}b_1)u. \tag{35}$$

and

$$\frac{d^2x_2}{dt^2} - (a_{11}+a_{22})\frac{dx_2}{dt} + (a_{11}a_{22}-a_{12}a_{21})\,x_2 = b_2\frac{du}{dt} + (a_{21}b_1 - a_{11}b_2)u. \tag{36}$$

which can be written in terms of the two parameters ω_n and ζ

$$\frac{d^2x_1}{dt^2} + 2\zeta\omega_n\frac{dx_1}{dt} + \omega_n^2 x_1 = b_1\frac{du}{dt} + (a_{12}b_2 - a_{22}b_1)u \tag{37}$$

$$\frac{d^2x_2}{dt^2} + 2\zeta\omega_n\frac{dx_2}{dt} + \omega_n^2 x_2 = b_2\frac{du}{dt} + (a_{21}b_1 - a_{11}b_2)u. \tag{38}$$

where ω_n is defined to be the *undamped natural frequency* with units of radians/second, and ζ is defined to be the system (dimensionless) *damping ratio*. These definitions may be compared to Eqs. (35) and (36), to give the following relationships:

$$\omega_n = \sqrt{a_{11}a_{22}-a_{12}a_{21}} \tag{39}$$

$$\begin{aligned} \zeta &= -\frac{1}{2\omega_n}\left(a_{11}+a_{22}\right) \\ &= \frac{-\left(a_{11}+a_{22}\right)}{2\sqrt{a_{11}a_{22}-a_{12}a_{21}}}. \end{aligned} \tag{40}$$

The undamped natural frequency and damping ratio play important roles in defining second-order system responses, similar to the role of the time constant in first-order systems, since they completely define the system homogeneous equation.

■ Example 7

Determine the differential equations in the state variables $x_1(t)$ and $x_2(t)$ for the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u. \tag{i}$$

Find the undamped natural frequency ω_n and damping ratio ζ for this system. **Solution:**

For this system

$$[sI - A] = \begin{bmatrix} s+1 & 2 \\ -2 & s+3 \end{bmatrix} \tag{ii}$$

and

$$\det\left[sI-A\right] = s^2 + 4s + 7$$

and therefore for state variable $x_1(t)$:

$$\frac{d^2x_1}{dt^2} + 4\frac{dx_1}{dt} + 7x_1 = \frac{du}{dt} + 3u. \tag{iii}$$

and for $x_2(t)$:

$$\frac{d^2x_2}{dt^2} + 4\frac{dx_2}{dt} + 7x_2 = 2u. \quad \text{(iv)}$$

By inspection of either Eq. (iii) or Eq. (iv), $\omega_n^2 = 7$, and $2\zeta\omega_n = 4$, giving $\omega_n = \sqrt{7}$ rad/s, and $\zeta = 2/\sqrt{7} = 0.755$.

2.0.2 Generation of a Differential Equation in an Output Variable

The output equation $\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$ for any system variable is a single algebraic equation:

$$\begin{aligned} y(t) &= \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [d]\, u(t) \\ &= c_1 x_1(t) + c_2 x_2(t) + du(t). \end{aligned} \quad (41)$$

and in the Laplace domain

$$\begin{aligned} Y(s) &= \Big(\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\Big)\, U(s) \\ &= \frac{1}{\det[s\mathbf{I}-\mathbf{A}]} \left(\mathbf{C}\mathrm{adj}(s\mathbf{I}-\mathbf{A}) + \det[s\mathbf{I}-\mathbf{A}]\,\mathbf{D}\right) \end{aligned}$$

The determinants may be expanded and the resulting equation written as a differential equation:

$$\frac{d^2y}{dt^2} - (a_{11}+a_{22})\frac{dy}{dt} + (a_{11}a_{22} - a_{12}a_{21})\, y = q_2\frac{d^2u}{dt^2} + q_1\frac{du}{dt} + q_0 u \quad (42)$$

or in terms of the standard system parameters

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = q_2\frac{d^2u}{dt^2} + q_1\frac{du}{dt} + q_0 u \quad (43)$$

where the coefficients q_0, q_1, and q_2 are

$$\begin{aligned} q_0 &= c_1(-a_{22}b_1 + a_{12}b_2) + c_2(-a_{11}b_2 + a_{21}b_1) + d\,(a_{11}a_{22} - a_{12}a_{21}) \\ q_1 &= c_1 b_1 + c_2 b_2 - d\,(a_{11}+a_{22}) \\ q_2 &= d. \end{aligned} \quad (44)$$

Notice that the left hand side of the differential equation is the same for all system variables, and that the only difference between any of the differential equations describing any system variable is in the constant coefficients q_2, q_1 and q_0 on the right hand side.

■ Example 8

A rotational system consists of an inertial load J mounted in viscous bearings B, and driven by an angular velocity source $\Omega_{in}(t)$ through a long light shaft with significant torsional stiffness K, as shown in the Fig. 16. Derive a pair of second-order differential equations for the variables Ω_J and Ω_K.

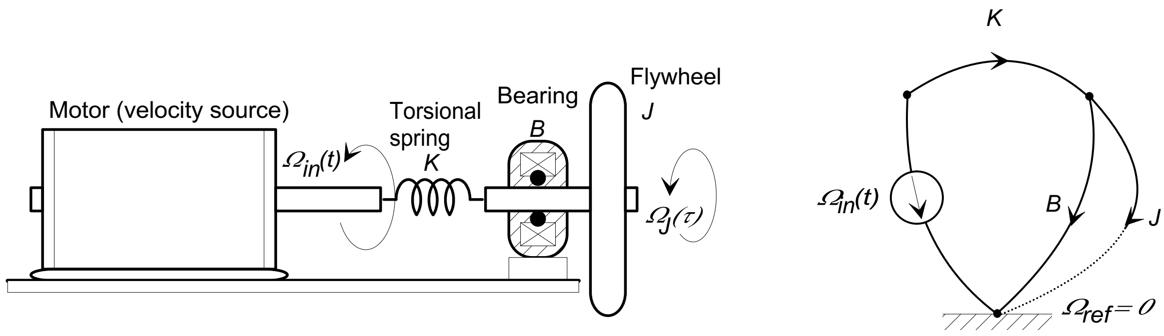

Figure 16: Rotational system for Example 8.

Solution: The state variables are Ω_J, and T_K, and the state and output equations are

$$\begin{bmatrix} \dot{\Omega}_j \\ \dot{T}_k \end{bmatrix} = \begin{bmatrix} -B/J & 1/J \\ -K & 0 \end{bmatrix} \begin{bmatrix} \Omega_J \\ T_K \end{bmatrix} + \begin{bmatrix} 0 \\ K \end{bmatrix} \Omega_{in}. \tag{i}$$

$$\begin{bmatrix} \Omega_J \\ \Omega_K \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \Omega_J \\ T_K \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{ii}$$

In this case there are two outputs and the transfer function matrix is

$$\begin{aligned} \mathbf{H}(s) &= \mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B} + \mathbf{D} \\ &= \frac{\mathbf{C}\text{adj}\left[s\mathbf{I} - \mathbf{A}\right]\mathbf{B} + \mathbf{D}}{\det\left[s\mathbf{I} - \mathbf{A}\right]} \\ &= \begin{bmatrix} \dfrac{K/J}{s^2 + (B/J)s + K/J} \\ \dfrac{s^2 + (B/J)s}{s^2 + (B/J)s + K/J} \end{bmatrix} \end{aligned} \tag{iii}$$

The required differential equations are therefore

$$\frac{d^2\Omega_J}{dt^2} + \frac{B}{J}\frac{d\Omega_J}{dt} + \frac{K}{J}\Omega_J = \frac{K}{J}\Omega_{in}. \tag{iv}$$

and

$$\frac{d^2\Omega_K}{dt^2} + \frac{B}{J}\frac{d\Omega_K}{dt} + \frac{K}{J}\Omega_K = \frac{d^2\Omega_{in}}{dt^2} + \frac{B}{J}\frac{d\Omega_{in}}{dt}. \tag{v}$$

The undamped natural frequency and damping ratio are found from either differential equation. For example, from Eq. (v) $\omega_n^2 = K/J$ and $2\zeta\omega_n = B/J$. From these relationships

$$\omega_n = \sqrt{\frac{K}{J}} \quad \text{and} \quad \zeta = \frac{B/J}{2\sqrt{K/J}} = \frac{B}{2\sqrt{KJ}}. \tag{vi}$$

2.1 Solution of the Homogeneous Second-Order Equation

For any system variable $y(t)$ in a second-order system, the homogeneous equation is found by setting the input $u(t) \equiv 0$ so that Eq. (43) becomes

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = 0. \tag{45}$$

The solution, $y_h(t)$, to the homogeneous equation is found by assuming the general exponential form

$$y_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{46}$$

where C_1 and C_2 are constants defined by the initial conditions, and the eigenvalues λ_1 and λ_2 are the roots of the characteristic equation

$$\det[s\mathbf{I} - \mathbf{A}] = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0, \tag{47}$$

found using the quadratic formula:

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \tag{48}$$

If $\zeta = 1$, the two roots are equal ($\lambda_1 = \lambda_2 = \lambda$), a modified form for the homogeneous solution is necessary:

$$y_c(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t} \tag{49}$$

In either case the homogeneous solution consists of two independent exponential components, with two arbitrary constants, C_1 and C_2, whose values are selected to make the solution satisfy a given pair of initial conditions. In general the value of the output $y(0)$ and its derivative $\dot{y}(0)$ at time $t = 0$ are used to provide the necessary information.

The initial conditions for the output variable may be specified directly as part of the problem statement, or they may have to be determined from knowledge of the state variables $x_1(0)$ and $x_2(0)$ at time $t = 0$. The homogeneous output equation may be used to compute $y(0)$ directly from elements of the **A** and **C** matrices,

$$y(0) = c_1 x_1(0) + c_2 x_2(0), \tag{50}$$

and the value of the derivative $\dot{y}(0)$ may be determined by differentiating the output equation and substituting for the derivatives of the state variables from the state equations:

$$\begin{aligned} \dot{y}(0) &= c_1\dot{x}_1(0) + c_2\dot{x}_2(0) \\ &= c_1\left(a_{11}x_1(0) + a_{12}x_2(0)\right) + c_2\left(a_{21}x_1(0) + a_{22}x_2(0)\right). \end{aligned} \tag{51}$$

To illustrate the influence of damping ratio and natural frequency on the system response, we consider the response of an unforced system output variable with initial output conditions of $y(0) = y_0$, and $\dot{y}(0) = 0$. If the roots of the characteristic equation are distinct, imposing these initial conditions on the general solution of Eq. (46) gives:

$$\begin{aligned} y(0) = y_0 &= C_1 + C_2 \\ \left.\frac{dy}{dt}\right|_{t=0} = 0 &= \lambda_1 C_1 + \lambda_2 C_2. \end{aligned} \tag{52}$$

With the result that

$$C_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} y_0 \quad \text{and} \quad C_2 = \frac{\lambda_1}{\lambda_1 - \lambda_2} y_0. \tag{53}$$

For this set of initial conditions the homogeneous solution is therefore

$$y_h(t) = y_0\left[\left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)e^{\lambda_1 t} + \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right)e^{\lambda_2 t}\right] \tag{54}$$

$$= y_0\frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left[\frac{1}{\lambda_1}e^{\lambda_1 t} - \frac{1}{\lambda_2}e^{\lambda_2 t}\right]. \tag{55}$$

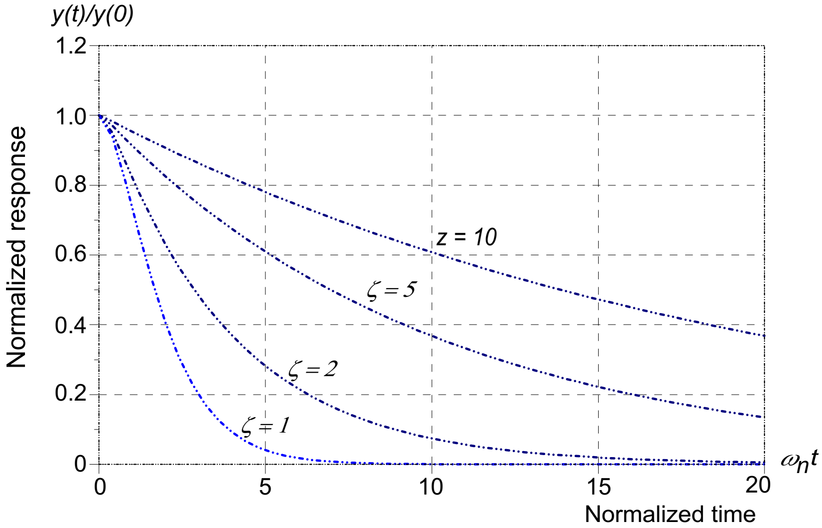

Figure 17: Homogeneous response of an overdamped and critically damped second-order system for the initial condition $y(0) = 1$, and $\dot{y}(0) = 0$.

If the roots of the characteristic equation are identical $\lambda_1 = \lambda_2 = \lambda$, the solution is based on Eq. (49) and is:

$$y_h(t) = y_0 \left[e^{\lambda t} - \lambda t e^{\lambda t} \right]. \tag{56}$$

The system response depends directly on the values of the damping ratio ζ and the undamped natural frequency ω_n. Four separate cases are described below:

Overdamped System ($\zeta > 1$): When the damping ratio ζ is greater than one, the two roots of the characteristic equation are real and negative:

$$\lambda_1, \lambda_2 = \omega_n \left(-\zeta \pm \sqrt{\zeta^2 - 1} \right). \tag{57}$$

From Eq. (55) the homogeneous response is

$$y_h(t) \quad = \quad y_0 \left[\frac{-\zeta + \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}} e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} - \frac{-\zeta - \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}} e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} \right] \tag{58}$$

which is the sum of two decaying real exponentials, each with a different decay rate that defines a time constant

$$\tau_1 = -\frac{1}{\lambda_1}, \quad \tau_2 = -\frac{1}{\lambda_2}. \tag{59}$$

The response exhibits no overshoot or oscillation, and is known as an *overdamped* response. Figure 17 shows this response as a function of ζ using a normalized time scale of $\omega_n t$.

Critically Damped System ($\zeta = 1$): When the damping ratio $\zeta = 1$ the roots of the characteristic equation are real and identical,

$$\lambda_1 = \lambda_2 = -\omega_n. \tag{60}$$

The solution to the initial condition response is found from Eq. (56):

$$y_h(t) = y_0 \left[e^{-\omega_n t} + \omega_n t e^{-\omega_n t} \right] \tag{61}$$

which is shown in Figure 17. This response form is known as a *critically damped* response because it marks the transition between the non-oscillatory overdamped response and the oscillatory response described in the next paragraph.

Underdamped System ($0 \le \zeta < 1$): When the damping ratio is greater than or equal to zero but less than 1, the two roots of the characteristic equation are complex conjugates with negative real parts:

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d \tag{62}$$

where $j = \sqrt{-1}$, and where ω_d is defined to be the *damped natural frequency*:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{63}$$

The response may be determined by substituting the values of the roots in Eq. (62) into Eq. (55):

$$\begin{aligned} y_h(t) &= y_0\left[\left(\frac{-\zeta\omega_n - j\omega_d}{-2j\omega_d}\right)e^{(-\zeta\omega_n + j\omega_d)t} + \left(\frac{-\zeta\omega_n + j\omega_d}{2j\omega_d}\right)e^{(-\zeta\omega_n - j\omega_d)t}\right] \\ &= y_0 e^{-\zeta\omega_n t}\left[\frac{e^{+j\omega_d t} + e^{-j\omega_d t}}{2} + \left(\frac{\zeta\omega_n}{\omega_d}\right)\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{2j}\right]. \end{aligned} \tag{64}$$

When the Euler identities $\cos\alpha = \left(e^{+j\alpha} + e^{-j\alpha}\right)/2$ and $\sin\alpha = \left(e^{+j\alpha} - e^{-j\alpha}\right)/2j$ are substituted the solution is:

$$\begin{aligned} y_h(t) &= y_0 e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right] \\ &= y_0\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \psi) \end{aligned} \tag{65}$$

where the phase angle ψ is

$$\psi = \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}. \tag{66}$$

The initial condition response for an *underdamped* system is a damped cosine function, oscillating at the damped natural frequency ω_d with a phase shift ψ, and with the rate of decay determined by the exponential term $e^{-\zeta\omega_n t}$. The response for underdamped second-order systems are plotted against normalized time $\omega_n t$ for several values of damping ratio in Figure 18.

For damping ratios near unity, the response decays rapidly with few oscillations, but as the damping is decreased, and approaches zero, the response becomes increasingly oscillatory. When the damping is zero, the response becomes a pure oscillation

$$y_h(t) = y_0\cos(\omega_n t), \tag{67}$$

and persists for all time. (The term "undamped natural frequency" for ω_n is derived from this situation, because a system with $\zeta = 0$ oscillates at a frequency of ω_n.) As the damping ratio increases from zero, the frequency of oscillation ω_d decreases, as shown by Eq. (63), until at a damping ratio of unity, the value of $\omega_d = 0$ and the response consists of a sum of real decaying exponentials.

The decay rate of the amplitude of oscillation is determined by the exponential term $e^{-\zeta\omega_n t}$. It is sometimes important to determine the ratio of the oscillation amplitude from one cycle to the next. The cosine function is periodic and repeats with a period $T_p = 2\pi/\omega_d$, so that if the response

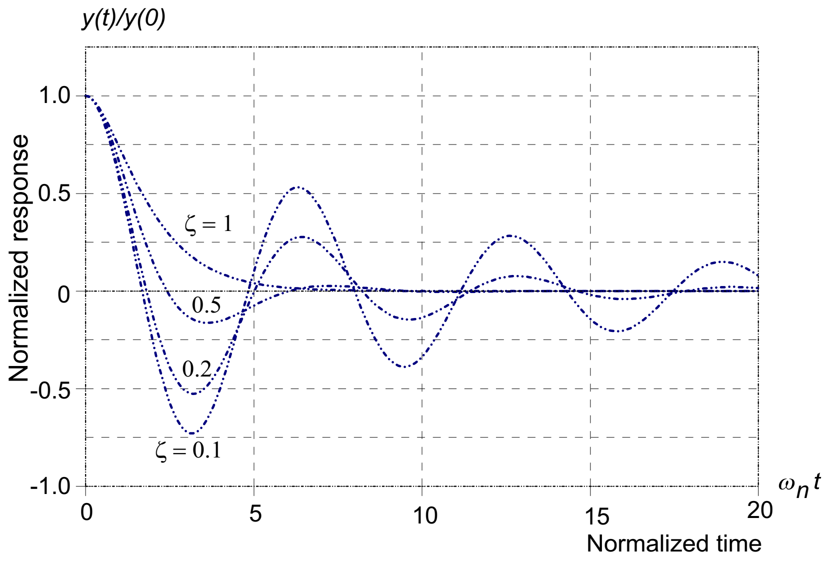

Figure 18: Normalized initial condition response of an underdamped second-order system as a function of the damping ratio ζ.

at an arbitrary time t is compared with the response at time $t + T_p$, an amplitude decay ratio DR may be defined as:

$$\begin{aligned} \text{DR} &= \frac{y(t+T_p)}{y(t)} \qquad \text{provided } y(t) \neq 0 \\ &= \frac{e^{-\zeta\omega_n(t+2\pi/\omega_d)}}{e^{-\zeta\omega_n t}} \\ &= e^{-2\pi\zeta/\sqrt{1-\zeta^2}} \end{aligned} \tag{68}$$

The decay ratio is unity if the damping ratio is zero, and decreases as the damping ratio increases, reaching a value of zero as the damping ratio approaches unity.

Unstable System ($\zeta < 0$): If the damping ratio is negative, the roots to the characteristic equation have positive real parts, and the real exponential term in the solution, Eq. (46), grows in an unstable fashion. When $-1 < \zeta < 0$, the response is oscillatory with an overall exponential growth in amplitude, as shown in Figure 19, while the solution for $\zeta < -1$ grows as a real exponential.

■ Example 9

Many simple mechanical systems may be represented by a mass coupled through spring and damping elements to a fixed position as shown in Figure 20. Assume that the mass has been displaced from its equilibrium position and is allowed to return with no external forces acting on it. We wish to (1) find the response of the system model from an initial displacement so as to determine whether the mass returns to its equilibrium position with no overshoot, (2) to determine the maximum velocity that it reaches. In addition we wish (3) to determine which system parameter we should change in order to guarantee no overshoot in the response. The values of system parameters are $m = 2$ kg, $K = 8$ N/m, $B = 1.0$ N-s/m and the initial displacement $y_0 = 0.1$ m.

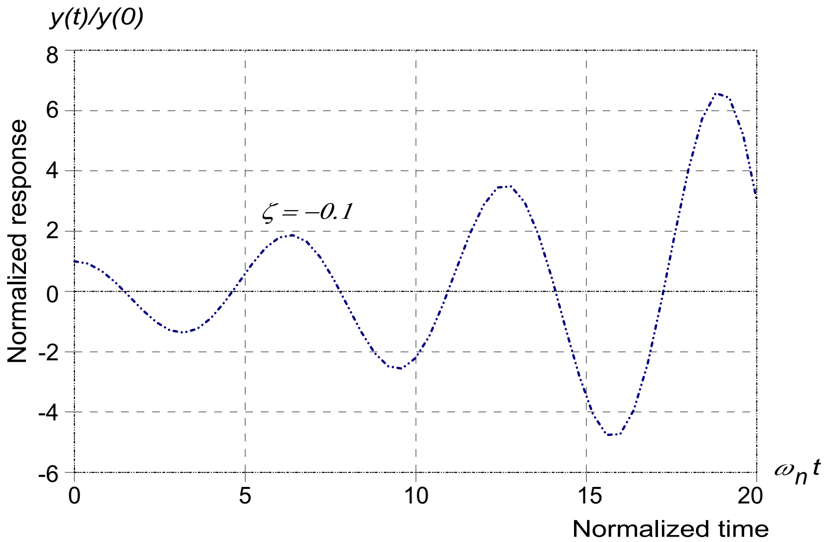

Figure 19: A typical unstable oscillatory response of a second-order system when the damping ratio ζ is negative.

Solution: From the linear graph model in Figure 20 the two state variables are the

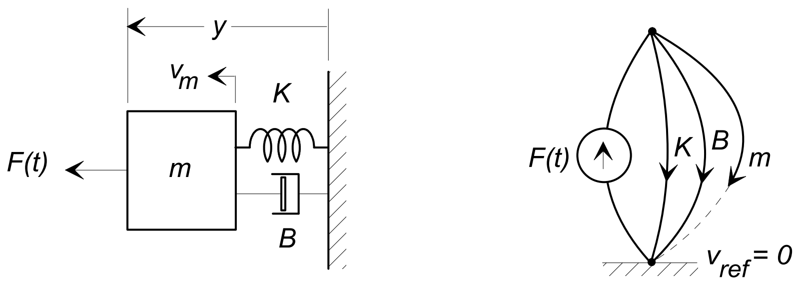

Figure 20: Second-order mechanical system.

velocity of mass $x_1 = v_m$, and the force in the spring $x_2 = F_K$. The state equations for the system, with an input force $F_{in}(t)$ acting on the mass are:

$$\begin{bmatrix} \dot{v}_m \\ \dot{F}_K \end{bmatrix} = \begin{bmatrix} -B/m & -1/m \\ K & 0 \end{bmatrix} \begin{bmatrix} v_m \\ F_K \end{bmatrix} + \begin{bmatrix} 1/m \\ 0 \end{bmatrix} F_{in}(t). \quad \text{(i)}$$

The output variable y is the position of the mass, which can be found from the constitutive relation for the force in the spring $F_K = Ky$ and therefore the output equation:

$$y\,(t) = (0)\,v_m + \left(\frac{1}{K}\right) F_K + (0)\,F_{in}(t). \quad \text{(ii)}$$

The characteristic equation is

$$\det\left[\lambda \mathbf{I} - \mathbf{A}\right] = \det \begin{bmatrix} \lambda + B/m & 1/m \\ -K & \lambda \end{bmatrix} = 0, \quad \text{(iii)}$$

or

$$\lambda^2 + \frac{B}{m}\lambda + \frac{K}{m} = 0. \quad \text{(iv)}$$

The undamped natural frequency and damping ratio are therefore

$$\omega_n = \sqrt{\frac{K}{m}}, \quad \text{and} \quad \zeta = \frac{B}{2m\omega_n} = \frac{B}{2\sqrt{Km}}. \tag{v}$$

With the given system parameters, the undamped natural frequency and damping ratio are

$$\omega_n = \sqrt{\frac{8}{2}} = 2 \text{ rad/s}, \quad \zeta = \frac{1}{4 \times 2} = 0.125.$$

Because the damping ratio is positive but less than unity, the system is stable but underdamped; the response $y_h(t)$ is oscillatory and therefore exhibits overshoot. The solution is given directly by Eq. (65):

$$y_h(t) = y_0 \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos(\omega_d t - \psi), \tag{vi}$$

and when the computed values of ω_d and ψ are substituted,

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 2\sqrt{1-(.125)^2} = 1.98 \text{ rad/s},$$

and

$$\psi = \tan^{-1} \frac{0.125}{\sqrt{1-(.125)^2}} = 0.125 \text{ r},$$

the response is:

$$y_h(t) = 0.101 e^{-.25t} \cos(1.98t - 0.125) \text{ m}. \tag{vii}$$

The response is plotted in Fig. 21a, where it can be seen that the mass displacement response $y(t)$ overshoots the equilibrium position by almost 0.1 m, and continues to oscillate for several cycles before settling to the equilibrium position.

The velocity of the mass $v_m(t)$ is related to the displacement $y(t)$ by differentiation of Eq. (vi),

$$v_m(t) = \frac{d}{dt} y_h(t) = -\frac{y_0\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t. \tag{viii}$$

The velocity response is plotted in Figure 21b, where the maximum value of the velocity is found to be -0.17 m/s at a time of 0.75 s.

In order to achieve a displacement response with no overshoot, an increase in the system damping is required to make $\zeta \geq 1$. Since the damping ratio ζ is directly proportional to B, the value of the viscous damping parameter B would have to be increased by a factor of 8, that is to $B = 8$ N-s/m to achieve critical damping. With this value the response is given by Eq. (61):

$$y(t) = 0.1 \left(e^{-2t} + 2te^{-2t} \right) \tag{ix}$$

The critically damped displacement response is also plotted in Figure 21a, showing that there is no overshoot.

As before, the velocity of the mass may be found by differentiating the position response

$$v(t) = 0.1 \left(-2e^{-2t} + 2e^{-2t} + 4te^{-2t} \right) = 0.4te^{-2t} \tag{x}$$

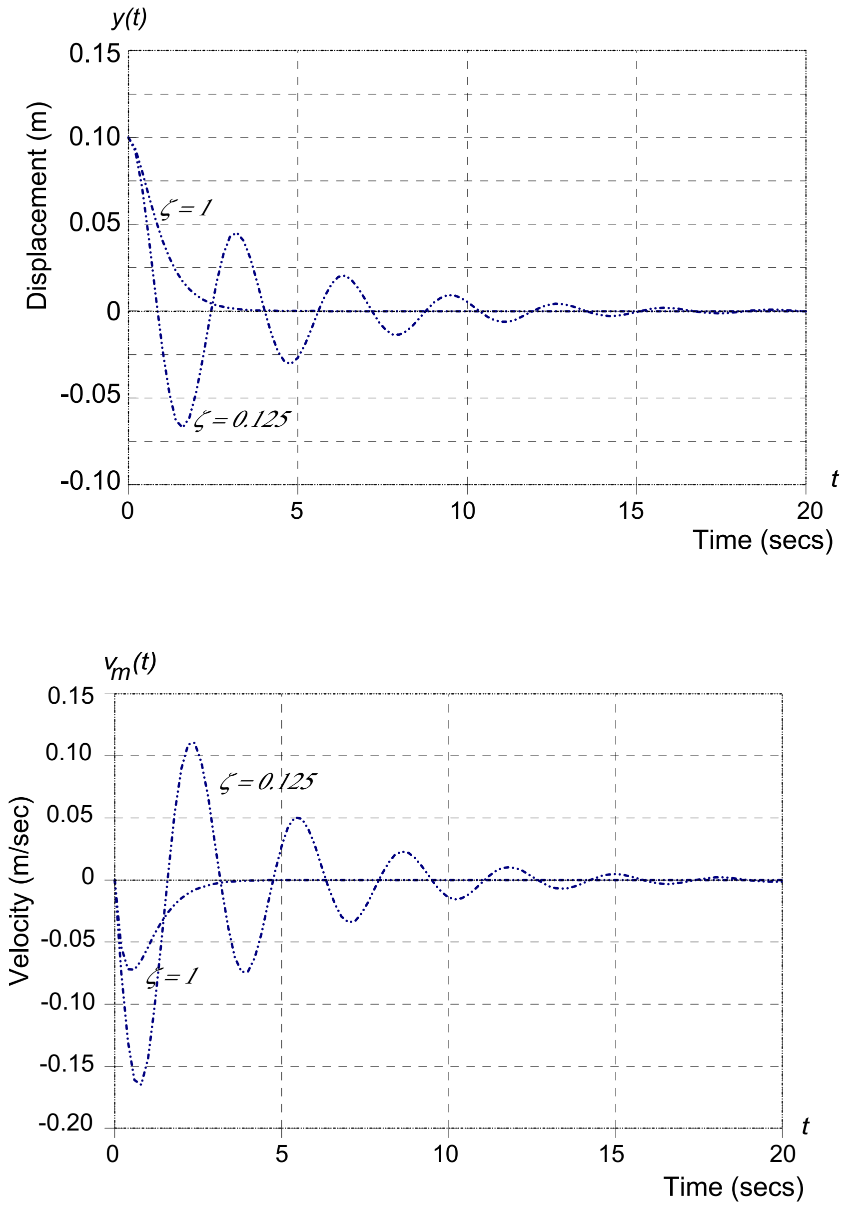

Figure 21: The displacement (a) and velocity (b) response of the mechanical second-order system.

The velocity response is plotted in Figure 21b where it can be seen that it reaches a maximum value of 0.075 m/s at a time of 0.5 s. The maximum velocity in the critically damped case is less than 45% of the maximum velocity when the damping ratio $\zeta = 0.125$.

2.2 Characteristic Second-Order System Transient Response

2.2.1 The Standard Second-Order Form

The input-output differential equation in any variable $y(t)$ in a linear second-order system is given by Eq. (43):

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = q_2\frac{d^2u}{dt^2} + q_1\frac{du}{dt} + q_0 u,$$

where the coefficients q_0, q_1, and q_2 are defined in Eqs. (44). Because the input $u(t)$ is a known function of time, a forcing function

$$f(t) = q_2\frac{d^2u}{dt^2} + q_1\frac{du}{dt} + q_0 u \tag{69}$$

may be defined. The forced response of a second-order system described by Eq. (43) may be simplified by considering in detail the behavior of the system to various forms of the forcing function $f(t)$. We therefore begin by examining the response of the system

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = f(t). \tag{70}$$

The response of this standard system form defines a characteristic response for any variable in the system. The derivative, scaling, and superposition properties of linear systems allow the response of any system variable $y_i(t)$ to be derived directly from the response $y(t)$:

$$y_i(t) = q_2\frac{d^2y}{dt^2} + q_1\frac{dy}{dt} + q_0 y(t). \tag{71}$$

In the sections that follow, the response of the standard form to the unit step, ramp, and impulse singularity functions are derived with the assumption that the system is at rest at time $t = 0$, that is $y(0) = 0$ and $\dot{y}(0) = 0$. The generalization of the results to responses of systems with derivatives on the right hand side is straightforward.

2.2.2 The Step Response of a Second-Order System

We start by deriving the response $y_s(t)$ of the standard system, Eq. (70), to a step of unit amplitude. The forced differential equation is:

$$\frac{d^2y_s}{dt^2} + 2\zeta\omega_n\frac{dy_s}{dt} + \omega_n^2 y_s = u_s(t), \tag{72}$$

where $u_s(t)$ is the unit step function.

The solution to Eq. (71) is the sum of the homogeneous response and a particular solution. For the case of distinct roots of the characteristic equation, λ_1 and λ_2, the total solution is

$$\begin{aligned} y_s(t) &= y_h(t) + y_p(t) \\ &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + y_p(t). \end{aligned} \tag{73}$$

The particular solution may be found using the method of undetermined coefficients, we take $y_p(t) = K$ and substitute into the differential equation giving

$$\omega_n^2 K = 1 \tag{74}$$

or

$$y_s(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \frac{1}{\omega_n^2}. \tag{75}$$

The constants C_1 and C_2 are chosen to satisfy the two initial conditions:

$$y_s(0) \quad = \quad C_1 + C_2 + \frac{1}{\omega_n^2} = 0 \tag{76}$$

$$\left.\frac{dy_s}{dt}\right|_{t=0} \quad = \quad C_1\lambda_1 + C_2\lambda_2 = 0, \tag{77}$$

which may be solved to give:

$$C_1 = \frac{\lambda_2}{\omega_n^2(\lambda_1 - \lambda_2)}, \qquad C_2 = \frac{\lambda_1}{\omega_n^2(\lambda_2 - \lambda_1)}. \tag{78}$$

The solution for the unit step response when the roots are distinct is therefore:

$$y_s(t) \quad = \quad \frac{1}{\omega_n^2}\left[1 - \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_1 t} + \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2 t}\right)\right] \tag{79}$$

$$\quad = \quad \frac{1}{\omega_n^2}\left[1 - \frac{\lambda_2\lambda_1}{\lambda_2 - \lambda_1}\left(\frac{1}{\lambda_1} e^{\lambda_1 t} - \frac{1}{\lambda_2} e^{\lambda_2 t}\right)\right] \tag{80}$$

It can be seen that the second and third terms in Eq. (80) are identical to those in the homogeneous response, Eq. (55), so that the solution may be written for the overdamped case as:

$$y_s(t) = \frac{1}{\omega_n^2}\left[1 - \frac{1}{\tau_2 - \tau_1}\left(\tau_2 e^{-t/\tau_2} - \tau_1 e^{-t/\tau_1}\right)\right] \qquad \text{for } \zeta > 1, \tag{81}$$

where $\tau_1 = -1/\lambda_1$ and $\tau_2 = -1/\lambda_2$ are time constants as previously defined.

For the underdamped case, when $\lambda_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$ and $\lambda_1 = \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$, from Eq. (65) the solution is:

$$y_s(t) = \frac{1}{\omega_n^2}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \psi)\right] \qquad \text{for } 0 < \zeta < 1. \tag{82}$$

where as before the phase angle $\psi = \tan^{-1}\left(\zeta/\sqrt{1-\zeta^2}\right)$.

When the roots of the characteristic equation are identical ($\zeta = 1$) and $\lambda_1 = \lambda_2 = -\omega_n$, the homogeneous solution has a modified form, and the total solution is:

$$y_s(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t} + \frac{1}{\omega_n^2}. \tag{83}$$

The solution which satisfies the initial conditions is:

$$y_s(t) \quad = \quad \frac{1}{\omega_n^2}\left[1 - e^{\lambda t} + \lambda t e^{\lambda t}\right]$$

$$\quad = \quad \frac{1}{\omega_n^2}\left[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}\right] \qquad \text{for } \zeta = 1. \tag{84}$$

In all three cases the response settles to a steady equilibrium value as time increases. We define the steady-state response as

$$y_{ss} = \lim_{t \to \infty} y_s(t) = \frac{1}{\omega_n^2}. \tag{85}$$

The second-order system step response is a function of both the system damping ratio ζ and the undamped natural frequency ω_n. The step responses of stable second-order systems are plotted in Figure 22 in terms of non-dimensional time $\omega_n t$, and normalized amplitude $y(t)/y_{ss}$.

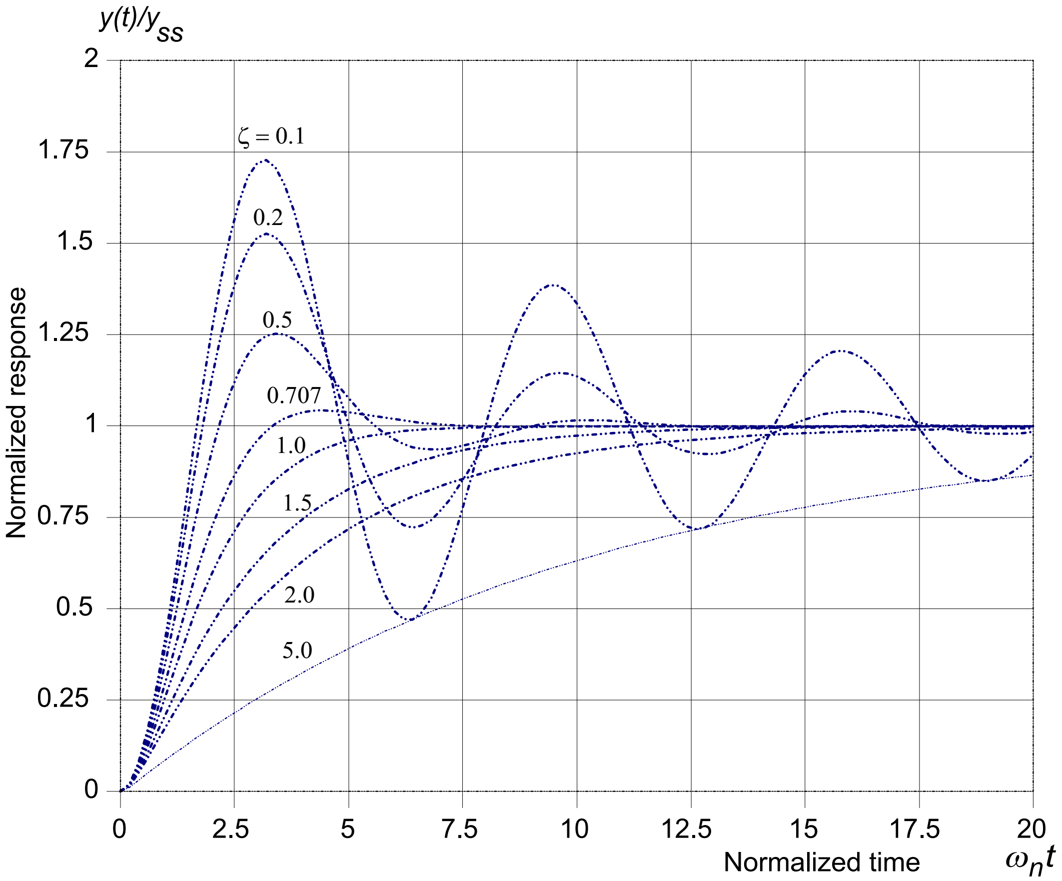

Figure 22: Step response of stable second-order systems with the differential equation $\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = u(t)$.

For damping ratios less than one, the solutions are oscillatory and overshoot the steady-state response. In the limiting case of zero damping the solution oscillates continuously about the steady-state solution y_{ss} with a maximum value of $y_{max} = 2y_{ss}$ and a minimum value of $y_{min} = 0$, at a frequency equal to the undamped natural frequency ω_n. As the damping is increased, the amplitude of the overshoot in the response decreases, until at *critical* damping, $\zeta = 1$, the response reaches steady-state with no overshoot. For damping ratios greater than unity, the response exhibits no overshoot, and as the damping ratio is further increased the response approaches the steady-state value more slowly.

■ **Example 10**

The electric circuit in Figure 23 contains a current source driving a series inductive and resistive load with a shunt capacitor across the load. The circuit is representative of motor drive systems and induction heating systems used in manufacturing processes. Excessive peak currents during transients in the input could damage the inductor. We therefore wish to compute response of the current through the inductor to a step in the input current to ensure that the manufacturers stated maximum current is not exceeded during start up. The circuit parameters are $L = 10^{-4}$ h, $C = 10^{-8}$ fd, and $R = 50$ ohms. Assume that the maximum step in the input current is to be 1.0 amp.

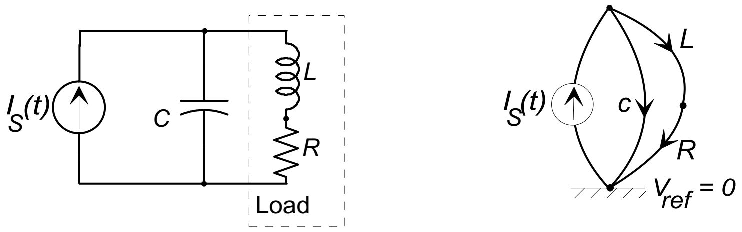

Figure 23: A second-order electrical system.

Solution: From the linear graph in Figure 23 the state variables are the voltage across the capacitor $v_C(t)$, and the current in the inductor $i_L(t)$. The state equations for the system are:

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} I_s. \tag{i}$$

The differential equation relating the current i_L to the source current I_s is found by Cramer's rule:

$$\det \begin{bmatrix} S & 1/C \\ -1/L & S + R/L \end{bmatrix} \{i_L\} = \det \begin{bmatrix} S & 1/C \\ -1/L & 0 \end{bmatrix} \{I_s\} \tag{ii}$$

or

$$\frac{d^2 i_L}{dt^2} + \frac{R}{L}\frac{di_L}{dt} + \frac{1}{LC} i_L = \frac{1}{LC} I_s, \tag{iii}$$

and the undamped natural frequency ω_n and damping ratio ζ are:

$$\omega_n = \frac{1}{\sqrt{LC}} = 10^6 \text{ rad/s} \tag{iv}$$

$$\zeta = \frac{(R/L)}{2/\sqrt{LC}} = \frac{R}{2}\sqrt{\frac{C}{L}} = 0.25. \tag{v}$$

The system is underdamped ($\zeta < 1$) and oscillations are expected in the response. The differential equation is similar to the standard form and therefore has a unit step response in the form of Eq. (2.2.2):

$$i_L(t) = \left(\omega_n^2\right) \frac{1}{\omega_n^2} \left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos(\omega_d t - \psi)\right] \tag{vi}$$

$$= 1 - 1.033 e^{-0.25\times 10^6 t} \cos\left(0.968 \times 10^6 t - .2527\right) \tag{vii}$$

which is plotted in Fig. 24. The step response shows that the peak current is 1.5 amp, which is approximately 50% above the steady-state current.

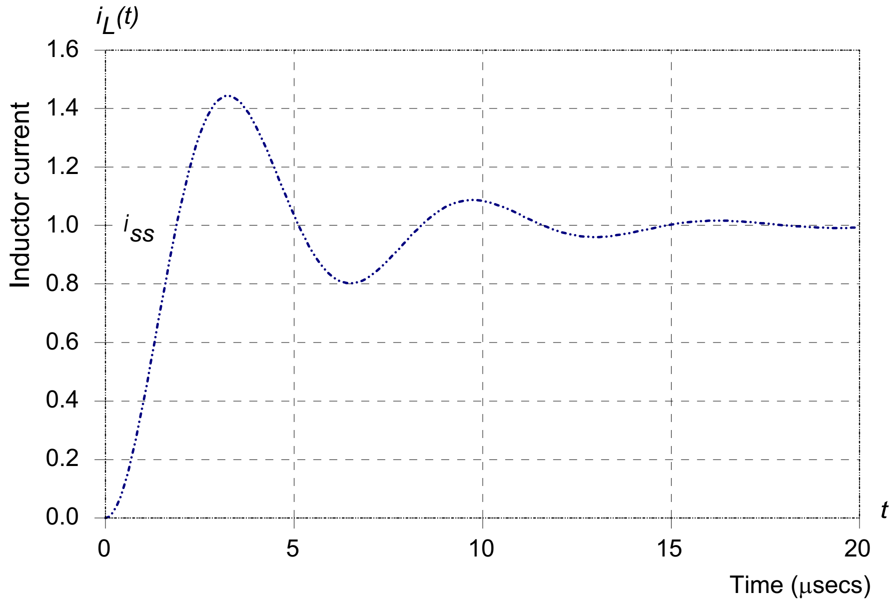

Figure 24: Response of the inductor current $i_L(t)$ to a 1 amp step in the input current I_s.

2.2.3 Impulse response of a Second-Order System

The derivative property of linear systems allows the impulse response $y_\delta(t)$ of any linear system to be found by differentiating the step response $y_s(t)$

$$y_\delta(t) = \frac{dy_s}{dt} \quad \text{because} \quad \delta(t) = \frac{d}{dt}u_s(t) \tag{86}$$

where $u_s(t)$ is the unit step function. For the standard system defined in Eq. (70) with $f(t) = \delta(t)$, the differential equation is

$$\frac{d^2y_\delta}{dt^2} + 2\zeta\omega_n\frac{dy_\delta}{dt} + \omega_n^2 y_\delta = \delta(t). \tag{87}$$

When the roots of the characteristic equation λ_1 and λ_2 are distinct, the impulse response is found by differentiating Eq. (80):

$$y_\delta(t) = \frac{1}{\omega_n^2}\frac{d}{dt}\left[1 - \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}e^{\lambda_1 t} + \frac{\lambda_1}{\lambda_1 - \lambda_2}e^{\lambda_2 t}\right)\right] \tag{88}$$

$$= \frac{1}{\omega_n^2}\frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)$$

$$= \frac{1}{\lambda_1 - \lambda_2}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right). \tag{89}$$

since $\omega_n^2 = \lambda_1\lambda_2$. For the case of real and distinct roots, $(\zeta > 1)$, $\lambda_1 = -\zeta\omega_n + \sqrt{\zeta^2 - 1}\omega_n$ and $\lambda_2 = -\zeta\omega_n - \sqrt{\zeta^2 - 1}\omega_n$, this reduces to

$$y_\delta(t) = \frac{1}{2\omega_n\sqrt{\zeta^2 - 1}}\left(e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} - e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t}\right)$$

$$= \frac{1}{2\omega_n\sqrt{\zeta^2-1}}\left(e^{-t/\tau_1} - e^{-t/\tau_2}\right) \tag{90}$$

where $\tau_1 = -1/\lambda_1$, and $\tau_2 = -1/\lambda_2$.

For the case of complex conjugate roots, $0 < \zeta < 1$, Eq. (90) reduces to

$$y_\delta(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t). \tag{91}$$

For a critically damped system ($\zeta = 1$), the impulse response may be found by differentiating Eq. (84), giving:

$$y_\delta(t) = te^{-\omega_n t}. \tag{92}$$

Figure 25 shows typical impulse responses for an overdamped, critically damped, and underdamped systems.

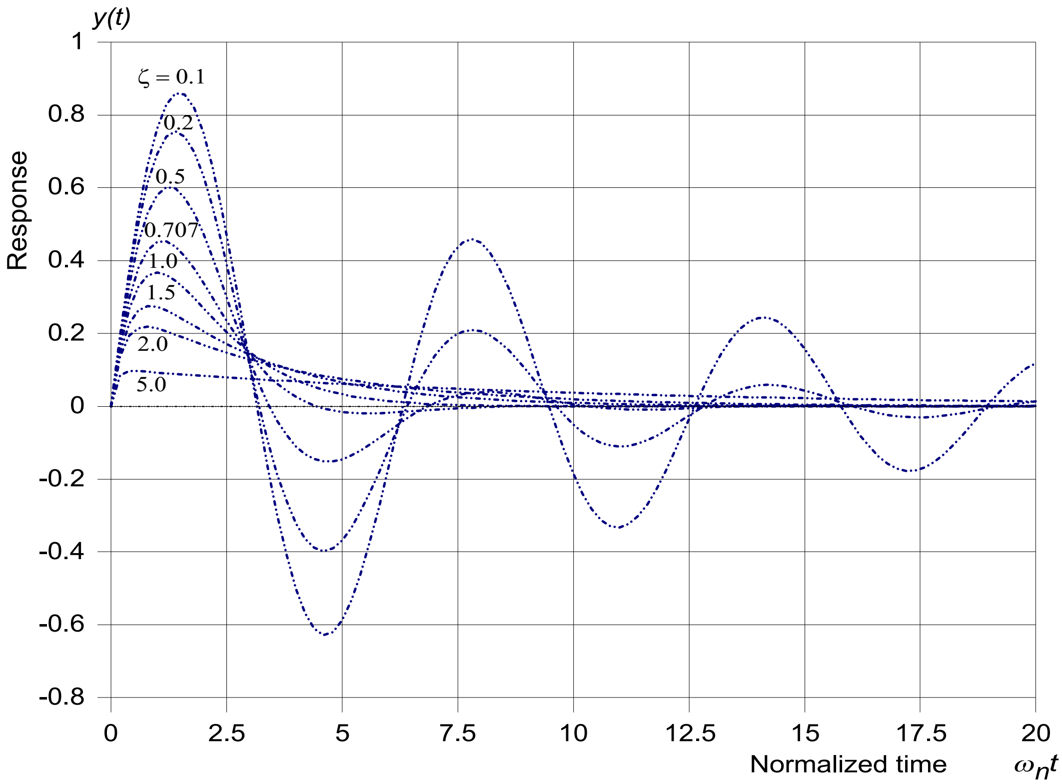

Figure 25: Typical impulse responses for overdamped, critically damped and underdamped second-order systems.

2.2.4 The Ramp Response of a Second-Order System:

The integral property of linear systems, allows the ramp response $y_r(t)$ to a forcing function $f(t) = t$ to be found by integrating the step response $y_s(t)$

$$y_r(t) = \int_0^t y_s(t)dt \quad \text{because} \quad r(t) = \int_0^t u_s(t)dt \tag{93}$$

where $u_s(t)$ is the unit step function. For the standard system defined in Eq. (70) with $f(t) = t$, the forced differential equation is

$$\frac{d^2y_r}{dt^2} + 2\zeta\omega_n\frac{dy_r}{dt} + \omega_n^2 y_r = t. \tag{94}$$

When the roots of the characteristic equation are distinct, the ramp response is found by integrating Eq. (80), that is

$$\begin{aligned} y_r(t) &= \frac{1}{\omega_n^2}\int_0^t \left[1 - \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left(\frac{1}{\lambda_1}e^{\lambda_1 t} - \frac{1}{\lambda_2}e^{\lambda_2 t}\right)\right] dt \\ &= \frac{1}{\omega_n^2}\left[t - \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left(\frac{1}{\lambda_1^2}\left[e^{\lambda_1 t} - 1\right] - \frac{1}{\lambda_2^2}\left[e^{\lambda_2 t} - 1\right]\right)\right] \\ &= \frac{1}{\omega_n^2}\left[t - \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left(\frac{1}{\lambda_1^2}e^{\lambda_1 t} - \frac{1}{\lambda_2^2}e^{\lambda_2 t}\right) - \frac{\lambda_1 + \lambda_2}{\lambda_1\lambda_2}\right] \end{aligned} \tag{95}$$

For an overdamped system with real distinct roots, $\lambda_1 = -\zeta\omega_n + \sqrt{\zeta^2 - 1}\omega_n$ and $\lambda_2 = -\zeta\omega_n - \sqrt{\zeta^2 - 1}\omega_n$, the ramp response may be found from Eq. (95) directly, or by making the partial substitutions for ζ and ω_n:

$$y_r(t) = \frac{1}{\omega_n^2}t - \frac{1}{2\omega_n\sqrt{1 - \zeta^2}}\left(\tau_1^2 e^{-t/\tau_1} - \tau_2^2 e^{-t/\tau_2}\right) - \frac{2\zeta}{\omega_n^3}. \tag{96}$$

which consists a term that is itself a ramp, a pair of decaying exponential terms, and a constant offset term. When the system is underdamped with complex conjugates roots, Eq. (95) may be written:

$$y_r(t) = \frac{1}{\omega_n^2}t + \frac{e^{-\zeta\omega_n t}}{\omega_n^3}\left(2\zeta\cos\omega_d t + \frac{2\zeta^2 - 1}{\sqrt{1 - \zeta^2}}\sin\omega_d t\right) - \frac{2\zeta}{\omega_n^3} \tag{97}$$

which consists of a ramp function, a damped oscillatory term and a constant offset.

When the roots are real and equal ($\zeta = 1$) the response is found by integrating Eq. (84):

$$\begin{aligned} y_r(t) &= \frac{1}{\omega_n^2}\int_0^t \left[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}\right] dt \\ &= \frac{1}{\omega_n^2}\left[t + \frac{2}{\omega_n}e^{-\omega_n t} + te^{-\omega_n t} - \frac{2}{\omega_n}\right] \end{aligned} \tag{98}$$

2.2.5 Summary of Singularity Function Responses

The characteristic responses of a linear system to the ramp, step, and impulse functions are summarized in Table 3.

2.3 Second-Order System Transient Response

The characteristic response defined in the previous section is the response to a forcing function $f(t)$ as defined in Eq. (69). The response of a system to an input $u(t)$ may be determined directly by superposition of characteristic responses. The complete differential equation

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = q_2\frac{d^2u}{dt^2} + q_1\frac{du}{dt} + q_0 u \tag{99}$$

in general involves a summation of derivatives of the input. The principle of superposition allows us to determine the system response to each component of the forcing function and to sum the

Damping ratio	Input $f(t)$	Characteristic Response $y(t)$
$0 \leq \zeta < 1$	$f(t) = u_r(t)$	$y_r(t) = \dfrac{1}{\omega_n^2}\left[t + \dfrac{e^{-\zeta\omega_n t}}{\omega_n}\left(2\zeta\cos\omega_d t + \dfrac{2\zeta^2 - 1}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) - \dfrac{2\zeta}{\omega_n}\right]$
	$f(t) = u_s(t)$	$y_s(t) = \dfrac{1}{\omega_n^2}\left[1 - \dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \psi)\right]$
	$f(t) = \delta(t)$	$y_\delta(t) = \dfrac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin(\omega_d t)$
$\zeta = 1$	$f(t) = u_r(t)$	$y_r(t) = \dfrac{1}{\omega_n^2}\left[t + \dfrac{2}{\omega_n}e^{-\omega_n t} + te^{-\omega_n t} - \dfrac{2}{\omega_n}\right]$
	$f(t) = u_s(t)$	$y_s(t) = \dfrac{1}{\omega_n^2}\left[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}\right]$
	$f(t) = \delta(t)$	$y_\delta(t) = te^{-\omega_n t}$
$\zeta > 1$	$f(t) = u_r(t)$	$y_r(t) = \dfrac{1}{\omega_n^2}\left[t + \dfrac{\omega_n}{2\sqrt{1-\zeta^2}}\left(\tau_1^2 e^{-t/\tau_1} - \tau_2^2 e^{-t/\tau_2}\right) - \dfrac{2\zeta}{\omega_n}\right]$
	$f(t) = u_s(t)$	$y_s(t) = \dfrac{1}{\omega_n^2}\left[1 - \dfrac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}\right)\right]$
	$f(t) = \delta(t)$	$y_\delta(t) = \dfrac{1}{2\omega_n\sqrt{\zeta^2-1}}\left(e^{-t/\tau_1} - e^{-t/\tau_2}\right)$

Notes:

1. The damped natural frequency $\omega_d = \sqrt{1-\zeta^2}\omega_n$ for $0 \leq \zeta < 1$.
2. The phase angle $\psi = \tan^{-1}\left(\zeta/\sqrt{1-\zeta^2}\right)$ for $0 \leq \zeta < 1$.
3. For over-damped systems ($\zeta > 1$) the time constants are $\tau_1 = 1/\left(\zeta\omega_n - \sqrt{\zeta^2-1}\omega_n\right)$, and $\tau_2 = 1/\left(\zeta\omega_n + \sqrt{\zeta^2-1}\omega_n\right)$.

Table 3: Summary of the characteristic transient responses of the system $\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = f(t)$ to the unit ramp $u_r(t)$, the unit step $u_s(t)$, and the impulse $\delta(t)$.

individual responses. In addition, the derivative property tells us that if the response to a forcing function $f(t) = u(t)$ is $y_u(t)$, the other components are derivatives of $y_u(t)$ and the total response is

$$y(t) = q_2 \frac{d^2 y_u}{dt^2} + q_1 \frac{dy_u}{dt} + q_0 y_u. \tag{100}$$

As in the case of first order systems, the derivatives must take into account discontinuities at time $t = 0$.

■ Example 11

Determine the response of a physical system with differential equation

$$\frac{d^2 y}{dt^2} + 8\frac{dy}{dt} + 16y = 3\frac{du}{dt} + 2u$$

to a step input $u(t) = 2$ for $t \geq 0$.

Solution: The characteristic equation is

$$\lambda^2 + 10\lambda + 16 = 0 \tag{i}$$

which has roots $\lambda_1 = -2$ and $\lambda_2 = -8$. For this system $\omega_n = 4$ rad/s and $\zeta = 1.25$; the system is overdamped. The characteristic response to a unit step is (from Table 3):

$$y_s(t) = \frac{1}{\omega_n^2}\left[1 - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}\right)\right] \tag{ii}$$

where $\tau_1 = 1/2$, and $\tau_2 = 1/8$, or

$$\begin{aligned} y_s(t) &= \frac{1}{16}\left[1 - \frac{8}{3}\left(\frac{1}{2}e^{-2t} - \frac{1}{8}e^{-8t}\right)\right] \\ &= \frac{1}{16} - \frac{1}{12}e^{-2t} + \frac{1}{48}e^{-8t} \end{aligned} \tag{iii}$$

The system response to a step of magnitude 2 is therefore

$$\begin{aligned} y(t) &= 2\left[3\frac{dy_s}{dt} + 2y_s\right] \\ &= 2\left[3\left(\frac{1}{6}e^{-2t} - \frac{1}{6}e^{-8t}\right) + 2\left(\frac{1}{16} - \frac{1}{12}e^{-2t} + \frac{1}{48}e^{-8t}\right)\right] \\ &= \frac{1}{4} - \frac{2}{3}e^{-2t} + \frac{11}{12}e^{-8t} \end{aligned} \tag{iv}$$

For systems in which $q_2 \neq 0$ a further simplification is possible. The system differential equation may be written in operational form

$$y(t) = \frac{q_2 S^2 + q_1 S + q_0}{S^2 + 2\zeta\omega_n S + \omega_n^2}\{u\} \tag{101}$$

and rearranged as

$$y(t) = q_2 \{u\} + \frac{(q_1 - 2b_2\zeta\omega_n) S + (q_0 - b_2\omega_n^2)}{S^2 + 2\zeta\omega_n S + \omega_n^2} \{u\} \tag{102}$$

The response is then found from the characteristic response and the input

$$y(t) = q_2 u(t) + (q_1 - 2b_2\zeta\omega_n) \frac{dy_c}{dt} + \left(q_0 - b_2\omega_n^2\right) y_c(t). \tag{103}$$

■ Example 12

Find the response of a physical system with the differential equation

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 4y = \frac{d^2u}{dt^2} + 2\frac{du}{dt} + u$$

to a step input $u(t) = 2$ for $t \geq 0$.

Solution: The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0 \tag{i}$$

which has a pair of coincident roots $\lambda_1 = \lambda_2 = -2$. The system is critically damped with $\omega_n = 2$ rad/s. The characteristic impulse response is (from Table 3):

$$y_\delta(t) = te^{-\omega_n t} = te^{-2t}. \tag{ii}$$

Because $q_2 \neq 0$ we may write the system response as

$$\begin{aligned} y(t) &= q_2\delta(t) + (q_1 - 2b_2\zeta\omega_n) \frac{dy_\delta}{dt} + \left(q_0 - b_2\omega_n^2\right) y_\delta(t) \\ &= \delta(t) - 4\frac{dy_\delta}{dt} - 2y_\delta \\ &= \delta(t) - te^{-2t} - 2e^{-2t} \end{aligned} \tag{iii}$$

■ Example 13

An electric motor is used to drive a large diameter fan through a coupling as shown in Fig. 26. The motor is not an ideal source, but exhibits a torque-speed characteristic that allows it to be modeled as a Thevenin equivalent source with an ideal angular velocity source $\Omega_s(t) = \Omega_0$ in series with a hypothetical rotary damper B_m. The motor is coupled to the fan through a flexible coupling with torsional stiffness K_r, and the fan impeller is modeled as an inertia J with the bearing and impeller aerodynamic loads modeled as an equivalent rotary damper B_r.

The response of the fan speed when the motor is energized is of particular interest since if the fan speed exceeds the design speed, the impeller can experience excessive stresses

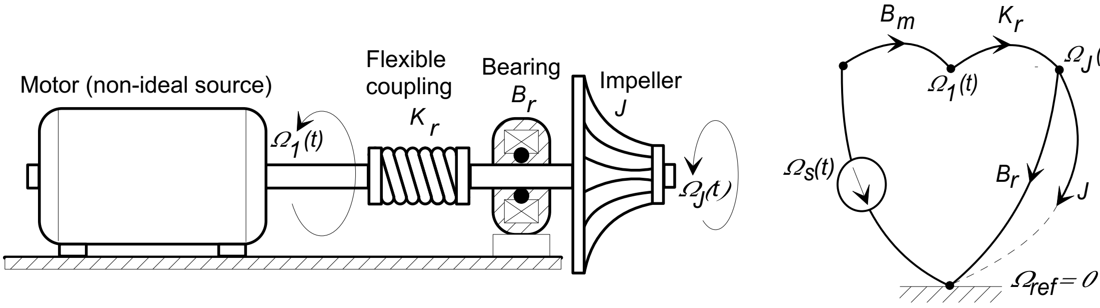

Figure 26: Electric motor fan drive system.

due to centrifugal forces. It is desired to select the system components so that the fan impeller reaches its operating speed with no overshoot. The torque in the coupling K_r during the start-up transient is also of interest because excessive torque could lead to failure. The motor specifications indicate that $\Omega_0 = 100$ rad/s, and $B_m = 1.0$ N-m-s/rad. The inertia of the fan impeller is $J = 1.0$ Kg-m^2 and the net drag of the bearings and aerodynamic load is $B_r = 1.0$ N-m-s/rad. The coupling stiffness to achieve an impeller response with no overshoot is to be determined, and the response of the system state variables is to be determined.

Solution: The state equations for the system may be expressed in term of the two state variables Ω_J, the fan impeller angular velocity, and T_K, the torque in the flexible coupling,

$$\begin{bmatrix} \dot{\Omega}_J \\ \dot{T}_K \end{bmatrix} = \begin{bmatrix} -B_r/J & 1/J \\ -K_r & -K_r/B_m \end{bmatrix} \begin{bmatrix} \Omega_J \\ T_K \end{bmatrix} + \begin{bmatrix} 0 \\ K_r \end{bmatrix} \Omega_s. \tag{i}$$

The system characteristic equation is

$$\det[\lambda \mathbf{I} - \mathbf{A}] = \lambda^2 + \left(\frac{B_r}{J} + \frac{K_r}{B_m}\right)\lambda + \frac{K_r}{J}\left(1 + \frac{B_r}{B_m}\right) = 0, \tag{ii}$$

and the undamped natural frequency and damping ratio are

$$\omega_n = \sqrt{\frac{K_r}{J}\left(1 + \frac{B_r}{B_m}\right)} \tag{iii}$$

$$\zeta = \frac{1}{2\omega_n}\left(\frac{B_r}{J} + \frac{K_r}{B_m}\right). \tag{iv}$$

Notice that for this system the values of the two damping coefficients influence both the natural frequency and the damping ratio.

1. The differential equation describing the fan speed is

$$\frac{d^2 y_1}{dt^2} + 2\zeta\omega_n \frac{dy_1}{dt} + \omega_n^2 y_1 = \frac{K_r}{J}\Omega_s \tag{v}$$

with a constant input $\Omega_s(t) = \Omega_0$. For no overshoot in the step response on starting the motor the system must be at least critically damped ($\zeta \geq 1$). Using Eqs. (ii) and (iii), the value of K_r required for critical damping may be found by setting $\zeta = 1$ in Eq. (iv), giving

$$2\sqrt{\frac{K_r}{J}\left(1 + \frac{B_r}{B_m}\right)} = \left(\frac{B_r}{J} + \frac{K_r}{B_m}\right), \tag{vi}$$

and with the system parameter values this equation gives

$$K_r = 5.83 \text{ N/m}. \tag{vii}$$

With this value of K_r the system parameters are $\zeta = 1$ and $\omega_n = 3.41$ rad/s. From Table 3 the unit characteristic step response is

$$y_s(t) = \frac{1}{\omega_n^2}\left[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}\right] \tag{viii}$$

and the impeller response to a step of 100 rad/s is

$$\begin{aligned} \Omega_J(t) &= \frac{100 q_0}{\omega_n^2}\left[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}\right] \\ &= 50\left[1 - e^{-3.41t} - 3.41 t e^{-\omega_n t}\right]. \end{aligned} \tag{ix}$$

The response in fan speed is similar to the non-dimensional form shown in Fig. 22, and is plotted in Fig 27. Note that the steady-state speed is 50 rad/s, which is one half of the motor no-load speed of 100 rad/s.

2. The differential equation relating the torque T_K to the source velocity is

$$\frac{d^2 T_K}{dt^2} + 2\zeta\omega_n \frac{dT_K}{dt} + \omega_n^2 T_K = K_r \frac{d\Omega_s}{dt} + \frac{K_r B_r}{J}\Omega_s \tag{x}$$

which contains both the input Ω_s and its derivative. Then

$$\begin{aligned} T_K(t) &= 100\left[K_r t e^{-\omega_n t} + \frac{K_r B_r}{\omega_n^2 J}\left(1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}\right)\right] \\ &= 50\left(1 - e^{-3.41t} + 8.22 t e^{-3.41t}\right) \text{ N-m}, \end{aligned} \tag{xi}$$

which is plotted in Fig. 27. Notice that in this case, although the system is critically damped ($\zeta = 1$), the response overshoots the steady-state value. This behavior is common for output variables that involve the derivative of the input in their differential equation.

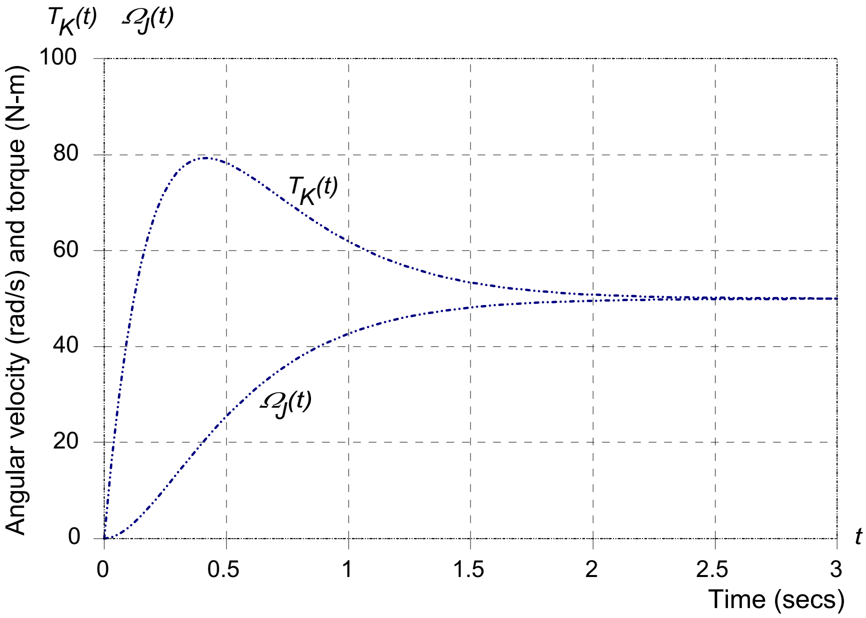

Figure 27: Step response of shaft coupling torque T_K, and fan angular velocity Ω_J.