MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.151 Advanced System Dynamics and Control

Impedance-Based Modeling Methods[1]

1 Introduction

The terms *impedance* and *admittance* are commonly used in electrical engineering to describe algebraically the dynamic relationship between the current and voltage in electrical elements. In this chapter we extend the definition to relationships between generalized across and through-variables within an element, or a connection of elements, in any of the energy modalities described in this note. The algebraic impedance based modeling methods may be developed in terms of either transfer functions, linear operators, or the Laplace transform. All three methods rationalize the algebraic manipulation of differential relationships between system variables. In this note we have adopted the transfer function as the definition of impedance based relationships between system variables.

The impedance based relationships between system variables associated with a single element may be combined to generate algebraic relationships between variables in different parts of a system [1–3]. In this note we develop methods for using impedance based descriptions to derive input/output transfer functions, and hence system differential equations directly in the classical input/output form.

2 Driving Point Impedances and Admittances

Figure 1 shows a linear system driven by a single ideal source, either an across-variable source or a through-variable source. At the input port the dynamic relationship between the across and through-variables depends on both the nature of the source, and the system to which it is connected. If the across-variable V_{in} is defined by the source, the resulting source through-variable F_{in} depends

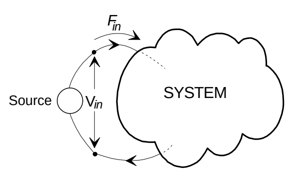

Figure 1: Definition of the driving point impedance of a port in a system.

on the structure of the system; conversely if the through-variable F_{in} is prescribed by the source, the across-variable V_{in} at the port is defined by the system. In either case a differential equation may be written to describe the dynamics of the resulting variable, and it is possible to define a *transfer function* that expresses the dynamic relationship between the dependent and independent input variables. For a system driven by an across-variable source $V_{in}(s)e^{st}$ the resulting particular solution $F_{in}(s)e^{st}$ for the through-variable is defined by the transfer function $Y(s)$:

$$F_{in}(s) = Y(s)V_{in}(s), \tag{1}$$

[1] D. Rowell 2/28/03

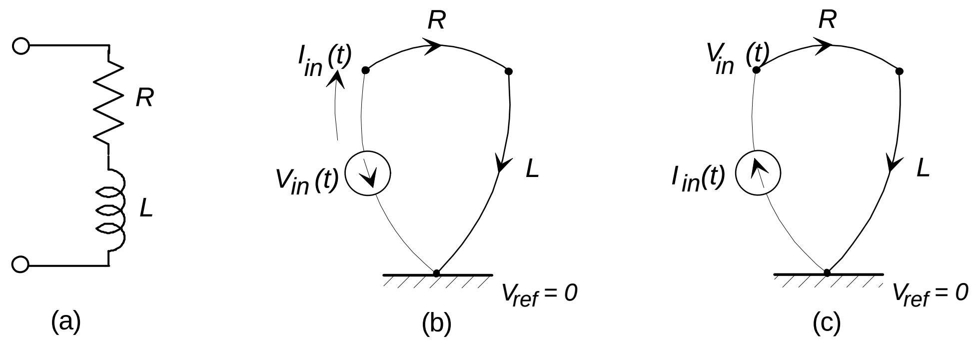

Figure 2: A series connected inductor and resistor, (b) driven by a voltage source, and (c) driven by a current source.

where $Y(s)$ is defined to be the *generalized driving-point admittance*, the *input admittance*, or simply the admittance of the system. Similarly for a system driven by a through-variable source, a transfer function, written $Z(s)$, defines the resulting across-variable particular solution:

$$V_{in}(s) = Z(s)F_{in}(s). \tag{2}$$

The transfer function $Z(s)$ is defined to be the *generalized driving-point impedance*, the *input impedance*, or more usually the impedance, of the system. Both $Z(s)$ and $Y(s)$ are properties of the system, and can be used to define a differential equation relationship between V_{in} and F_{in}. From Eqs. (1) and (2) it can be seen that

$$Z(s) = \frac{V_{in}(s)}{F_{in}(s)} \qquad \text{and} \qquad Y(s) = \frac{F_{in}(s)}{V_{in}(s)}, \tag{3}$$

and while they have been defined in terms of different causalities, the impedance and admittance are simply reciprocals

$$Y(s) = \frac{1}{Z(s)}. \tag{4}$$

Any transfer function is a rational function in the complex variable s, therefore if

$$Z(s) = \frac{P(s)}{Q(s)}$$

where $P(s)$ and $Q(s)$ are polynomials in s, then the admittance is

$$Y(s) = \frac{Q(s)}{P(s)}.$$

■ Example

Find the input impedance and admittance of the first-order electrical circuit consisting of a series connected inductor L and resistor R.

Solution: Assume that the system is driven by a voltage source $V_{in}(t)$, as indicated in the linear graph shown in Fig. 2b. The continuity condition applied to the input node requires that $I_{in}(t) = i_R(t)$, and the resulting differential equation is:

$$\frac{L}{R}\frac{dI_{in}}{dt} + I_{in} = \frac{1}{R}V_{in}(t). \tag{5}$$

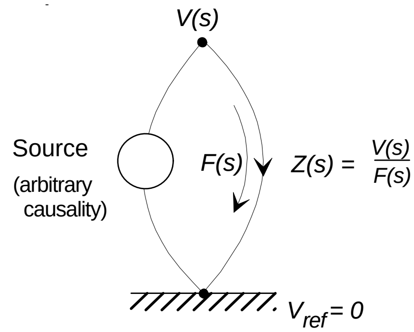

Figure 3: Definition of the impedance of a single generalized ideal element

The admittance transfer function is found by substituting s for the derivative and rearranging,

$$Y(s) = \frac{I_{in}(s)}{V_{in}(s)} = \frac{1/R}{(L/R)s + 1}. \tag{6}$$

Similarly, if the system is assumed to be driven by a current source $I_s(t)$, compatibility around the loop shown in Fig. 2c shows that the terminal voltage $V_{in}(t)$ is

$$V_{in} = v_R + v_L = L\frac{dI_{in}}{dt} + RI_{in}, \tag{7}$$

and the impedance transfer function is

$$Z(s) = \frac{V_{in}(s)}{I_{in}(s)} = Ls + R. \tag{8}$$

It is easy to show that $Z(s)$ and $Y(s)$ are reciprocals.

2.1 The Impedance of Ideal Elements

Consider a system that consists of a single ideal element connected to an ideal source as shown in Fig. 3. The elemental relationship between the across and through variables, may be used to define the impedance or admittance of the element.

The Generalized Capacitance: The elemental relationship for an A-Type element, or generalized capacitance C, in any energy domain is:

$$C\frac{dv_c}{dt} = f_c \tag{9}$$

where v_c is the across-variable on the capacitance, and f_c is the through-variable. Because $V_{in} = v_c$, and $F_{in} = f_c$, the definition of the impedance as a transfer function results in

$$Z(s) = \frac{V_{in}(s)}{F_{in}(s)} = \frac{1}{sC}, \tag{10}$$

and the admittance of the capacitance is

$$Y(s) = sC. \tag{11}$$

Energy modality:	Capacitance	Inductance	Resistance
Translational	$1/sm$	s/K	$1/B$
Rotational	$1/sJ$	s/K_r	$1/B_r$
Electrical	$1/sC$	sL	R
Fluid	$1/sC_f$	sI	R_f
Thermal	$1/sC_t$	—	R_t

Table 1: The impedance of ideal elements. The admittance is the reciprocal of the value given.

The Generalized Inductance: For a T-Type element, or generalized inductance L, the elemental equation is:

$$L\frac{df_L}{dt} = v_L \tag{12}$$

which gives the impedance and admittance transfer functions:

$$Z(s) = sL, \tag{13}$$

$$Y(s) = \frac{1}{sL}. \tag{14}$$

The Generalized Resistance: For any dissipative D-type element, R, the elemental equation relating the through and across variables is an algebraic relationship, $v = Rf$, so that the impedance and admittance transfer functions are also static or algebraic functions:

$$Z(s) = R, \tag{15}$$

$$Y(s) = \frac{1}{R}. \tag{16}$$

Table 1 summarizes the elemental impedances of the A-type, T-type, and D-type elements within each of the energy domains.

3 The Impedance of Interconnected Elements

For a system of interconnected lumped parameter elements, the system input impedance (or admittance) may be found by using a set of simple rules for combining impedances (or admittances) directly from the system linear graph.

3.1 Series Connection of Elements

Elements sharing a common through-variable are said to be connected in series. For example, a linear graph shown in Figure 4 consists of a through-variable source F_{in} connected to three branches in series. The driving point impedance $Z(s)$ for the complete system is specified by the across-variable at the input port V_{in} and the corresponding through-variable F_{in}. The continuity condition applied to any node in the graph requires that all elements, including the source, share a common through variable, or $f_1 = f_2 = f_3 = F_{in}$. The compatibility condition applied to the single loop in the graph requires that

$$V_{in} = v_1 + v_2 + v_3. \tag{17}$$

The across-variable v_i on each branch may be written in terms of the elemental impedance $Z_i(s)$ and the common through-variable

$$\begin{aligned} V_{in}(s) &= f_1(s)Z_1(s) + f_2(s)Z_2(s) + f_3(s)Z_3(s) \\ &= F_{in}(s)\left[Z_1(s) + Z_2(s) + Z_3(s)\right] \\ &= F_{in}(s)Z(s) \end{aligned} \tag{18}$$

where

$$Z(s) = Z_1(s) + Z_2(s) + Z_3(s) \tag{19}$$

is the system driving point impedance.

In general, if N branches in a linear graph are connected in series, the equivalent impedance of the group of branches is the sum of the individual branch impedances:

$$Z(s) = \sum_{i=1}^{N} Z_i(s). \tag{20}$$

The equivalent admittance of the series combination can also be found directly from Eq. (17) since

$$\begin{aligned} V_{in}(s) &= \frac{f_1(s)}{Y_1(s)} + \frac{f_2(s)}{Y_2(s)} + \frac{f_3(s)}{Y_3(s)} \\ &= F_{in}(s)\left[\frac{1}{Y_1(s)} + \frac{1}{Y_2(s)} + \frac{1}{Y_3(s)}\right] \\ &= \frac{F_{in}(s)}{Y(s)} \end{aligned} \tag{21}$$

so that the equivalent admittance of the series elements is

$$\frac{1}{Y(s)} = \frac{1}{Y_1(s)} + \frac{1}{Y_2(s)} + \frac{1}{Y_3(s)}, \tag{22}$$

and in general for N elements connected in series

$$\frac{1}{Y(s)} = \sum_{i=1}^{N} \frac{1}{Y_i(s)}. \tag{23}$$

When two elements are connected in series, Eq. (23) reduces to the convenient form:

$$Y(s) = \frac{Y_1(s)Y_2(s)}{Y_1(s) + Y_2(s)}. \tag{24}$$

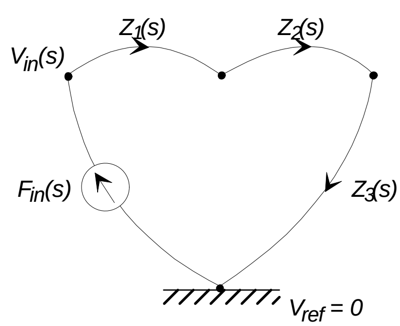

Figure 4: Linear graph of a system with three series connected elements.

■ Example

The second-order mechanical system shown in Fig. 5 is driven by a velocity source $V_{in}(t)$. Use impedance methods to derive a differential equation relating the force at the input to the input velocity.

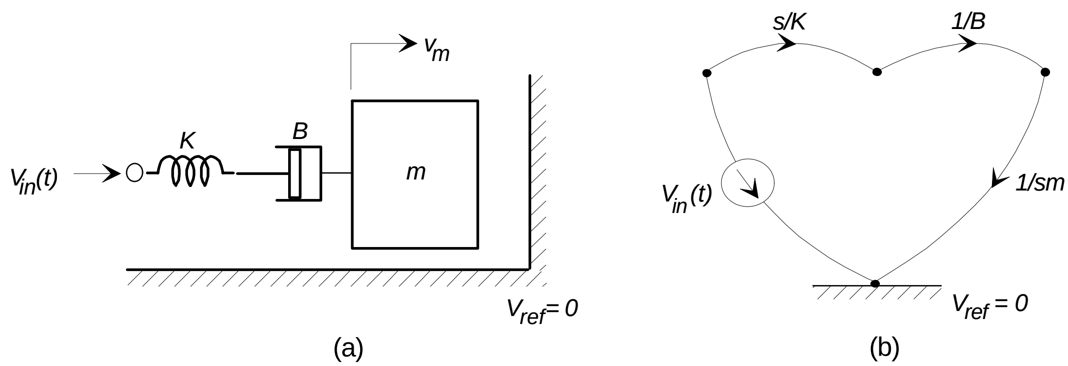

Figure 5: Series connected mechanical system.

Solution: The linear graph in Fig. 5 shows that the three elements are connected in series and share a common through-variable. The relationship between the source force and velocity is given by

$$F_{in}(s) = Y(s)V_{in}(s)$$

where $Y(s)$ is the system driving point admittance.

For the mass, spring, and dashpot elements the elemental impedances are (from Table 1):

$$Z_m(s) = 1/ms, \quad Z_K(s) = s/K, \quad \text{and} \quad Z_B(s) = 1/B$$

so that the total impedance at the input port is

$$\begin{aligned} Z(s) &= Z_m(s) + Z_K(s) + Z_B(s) \\ &= 1/ms + s/K + 1/B \\ &= \frac{s^2 + (K/B)s + k/m}{Ks}. \end{aligned} \tag{25}$$

The overall admittance could be computed directly from Eq. (22), but we note that

$$Y(s) = \frac{1}{Z(s)} = \frac{Ks}{s^2 + (K/B)s + K/m}. \tag{26}$$

From the definition of the admittance $F_{in}(s) = Y(s)V_{in}(s)$, so that

$$\left(s^2 + (K/B)s + K/m\right) F_{in}(s) = KsV_{in}(s)$$

which generates the differential equation

$$\frac{d^2F_{in}}{dt^2} + \frac{K}{B}\frac{dF_{in}}{dt} + \frac{K}{m}F_{in} = K\frac{dV_{in}}{dt}. \tag{27}$$

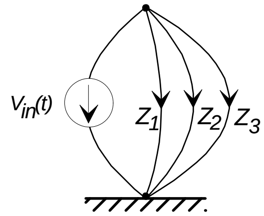

Figure 6: A system containing three parallel connected elements.

3.2 The Impedance of Parallel Connected Elements

The linear graph in Fig. 6 shows an across-variable source $V_{in}(t)$ connected to a parallel combination of three elements. The compatibility equation for any loop in the graph requires that all elements have a common across-variable, that is $v_1 = v_2 = v_3 = V_{in}$, while the continuity condition at the top node requires

$$F_{in} = f_1 + f_2 + f_3. \tag{28}$$

Using the impedance relationship $F(s) = V(s)/Z(s)$ (Eq. (2)) for each of the passive branches:

$$\begin{aligned} F_{in}(s) &= \frac{v_1(s)}{Z_1(s)} + \frac{v_2(s)}{Z_2(s)} + \frac{v_3(s)}{Z_3(s)} \\ &= V_{in}(s)\left[\frac{1}{Z_1(s)} + \frac{1}{Z_2(s)} + \frac{1}{Z_3(s)}\right] \\ &= \frac{V_{in}(s)}{Z(s)}, \end{aligned} \tag{29}$$

so that the equivalent driving point impedance of the system $Z(s)$ is

$$\frac{1}{Z(s)} = \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)} + \frac{1}{Z_3(s)}. \tag{30}$$

Equation 30 may be generalized to a parallel connection of N branches in a linear graph:

$$\frac{1}{Z(s)} = \sum_{i=1}^{N} \frac{1}{Z_i(s)}. \tag{31}$$

As in the case of series admittances, a convenient form of Eq. (31) can be written for two parallel impedances:

$$Z(s) = \frac{Z_1(s)Z_2(s)}{Z_1(s) + Z_2(s)}. \tag{32}$$

The admittance of a set of N parallel branches in a linear graph may be found by substituting into Eq. (28), with the result

$$Y(s) = \sum_{i=1}^{N} Y_i(s). \tag{33}$$

■ Example

Find the impedance and the admittance of the electrical system shown in Fig. 7.

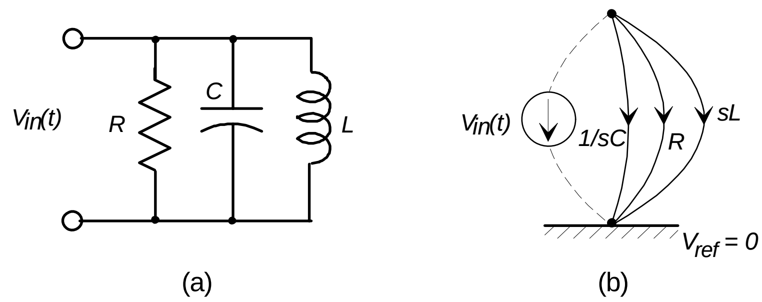

Figure 7: An electrical system with parallel connected elements.

Solution: For the capacitor, the inductor, and the resistor elements the elemental impedances are (from Table 1):

$$Z_C(s) = 1/sC, \quad Z_L(s) = sL, \quad \text{and} \quad Z_R(s) = R$$

so that the overall impedance is:

$$\begin{aligned} \frac{1}{Z(s)} &= \frac{1}{Z_C(s)} + \frac{1}{Z_L(s)} + \frac{1}{Z_R(s)} \\ &= sC + 1/sL + 1/R \\ &= \frac{s^2 + (1/RC)s + 1/LC}{s/C}, \end{aligned} \tag{34}$$

or

$$Z(s) = \frac{s/C}{s^2 + (1/RC)s + 1/LC}. \tag{35}$$

The admittance of the parallel combination is simply the sum of the individual admittances.

$$\begin{aligned} Y(s) &= Y_C(s) + Y_L(s) + Y_R(s) \\ &= sC + 1/sL + 1/R \\ &= \frac{s^2 + (1/RC)s + 1/LC}{s/C}. \end{aligned} \tag{36}$$

which is the reciprocal of the impedance found in Eq. (ii).

3.3 General Interconnected Impedances

Impedances that are not elemental impedances, but which represent combinations of lumped elements, may be combined and reduced to a single equivalent impedance using the above rules for combining series and parallel combinations.

For example, the linear graph in Fig. 8 contains four branches and a single source. Each branch in this graph is described by an impedance, and may represent a single element or a combination of elements. The two parallel impedances may be combined using Eq. (30), and the two resulting

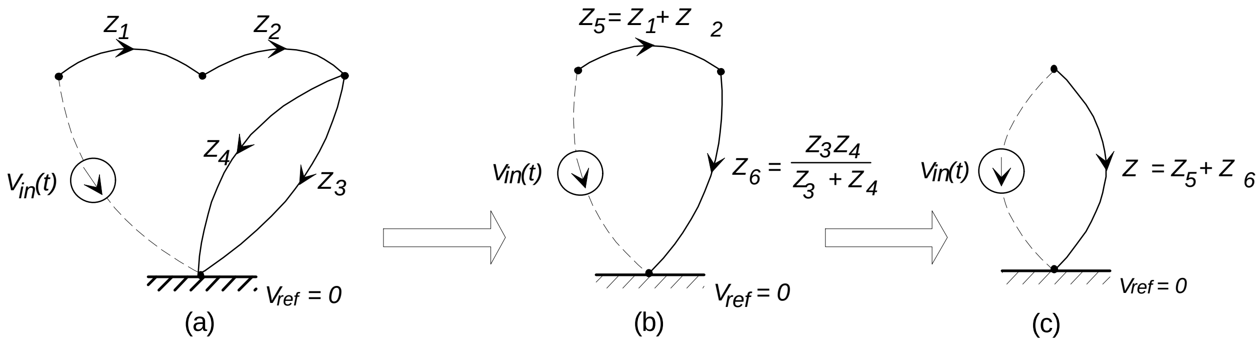

Figure 8: System Reduction by combining series and parallel impedances

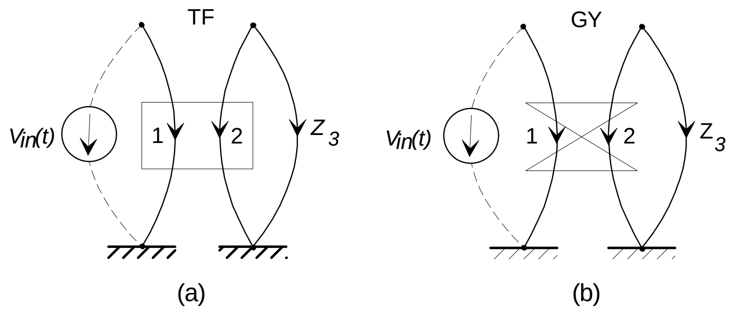

Figure 9: The input impedance of two-port elements with a single load element; (a) a transformer, and (b) a gyrator.

series branches may then be combined using Eq. (20) to give an equivalent system consisting of two series impedances $Z_5(s)$ and $Z_6(s)$ as shown in Fig. 8b, where

$$\begin{aligned} Z_5(s) &= Z_1(s) + Z_2(s) \\ Z_6(s) &= \frac{Z_3(s)\,Z_4(s)}{Z_3(s) + Z_4(s)}. \end{aligned}$$

These two impedances may be then combined as in Fig. 8c, giving the equivalent system input impedance $Z(s)$:

$$\begin{aligned} Z(s) &= Z_5(s) + Z_6(s) \\ &= \frac{(Z_1(s) + Z_2(s))(Z_3(s) + Z_4(s)) + Z_3(s)\,Z_4(s)}{Z_3(s) + Z_4(s)}. \end{aligned} \tag{37}$$

3.4 Impedance Relationships for Two-Port Elements

Energy conserving two-port elements, are used as transducers between different energy domains. The impedance of a sub-system on one side of a transforming or gyrating two-port element may be "reflected" to the other side using the two-port constitutive relationships. Figure 9 shows a transformer and a gyrator, each with a single impedance element Z_3 connected to one side. The equivalent impedance for the two cases, as seen from the input port may be derived as follows:

Transformer: Let the transformer shown in Fig. 9a have a ratio *TF*, defined by the constitutive relationship:

$$\begin{bmatrix} v_1 \\ f_1 \end{bmatrix} = \begin{bmatrix} TF & 0 \\ 0 & -1/TF \end{bmatrix} \begin{bmatrix} v_2 \\ f_2 \end{bmatrix}. \tag{38}$$

From the linear graph in Fig. 9, the compatibility and continuity conditions give $V_{in} = v_1$, $F_{in} = f_1$, $v_3 = v_2$, and $f_3 = -f_2$. Substitution of these conditions into Eq. (38) gives:

$$\begin{bmatrix} v_1(s) \\ f_1(s) \end{bmatrix} = \begin{bmatrix} TF & 0 \\ 0 & -1/TF \end{bmatrix} \begin{bmatrix} Z_3(s)f_3(s) \\ -f_3(s) \end{bmatrix}. \tag{39}$$

and since by definition $Z_1(s) = v_1(s)/f_1(s)$,

$$\begin{aligned} Z_1(s) &= \frac{(TF)Z_3(s)f_3(s)}{f_3(s)/(TF)} \\ &= (TF)^2 Z_3(s). \end{aligned} \tag{40}$$

The input impedance of a two-port element is therefore a factor of $(TF)^2$ times the impedance of the system on the other side.

Gyrator: The input impedance of a gyrator connected to a system with a known impedance is derived in a similar manner; with the difference that for a gyrator there is a proportionality between the across-variable on one side and the through-variable on the other side. Let the gyrator shown in Fig. 9b have a ratio GY, defined by the constitutive relationship:

$$\begin{bmatrix} v_1 \\ f_1 \end{bmatrix} = \begin{bmatrix} 0 & GY \\ -1/GY & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ f_2 \end{bmatrix}. \tag{41}$$

The compatibility and continuity conditions yield $V_{in} = v_1$, $F_{in} = f_1$, $v_3 = v_2$, and $f_3 = -f_2$ with the result

$$\begin{bmatrix} v_1(s) \\ f_1(s) \end{bmatrix} = \begin{bmatrix} 0 & GY \\ -1/GY & 0 \end{bmatrix} \begin{bmatrix} Z_3(s)f_3(s) \\ -f_3(s) \end{bmatrix}, \tag{42}$$

and since $Z_1(s) = v_1(s)/f_1(s)$

$$\begin{aligned} Z_1(s) &= \frac{-GY f_3(s)}{-Z_3(s)f_3(s)/GY} \\ &= (GY)^2 \frac{1}{Z_3} \\ &= (GY)^2 Y_3. \end{aligned} \tag{43}$$

The input impedance at one side of a gyrator is therefore a factor of $(GY)^2$ times the *admittance* of the load connected to the other side. The gyrator effectively changes the nature of the apparent load. For example, a capacitive element C with an impedance $Z = 1/sC$ connected to one side of a gyrator appears as an equivalent inductive element with impedance $Z = s(GY)^2C$ when reflected to the other side.

■ Example

A permanent magnet d.c. motor, modeled as shown in Fig. 10, drives a load that is a pure inertia J. If the motor produces torque $T_m = -K_m i_m$, and generates a back emf $v_m = K_m \Omega_m$, find the equivalent electrical impedance at the motor terminals.

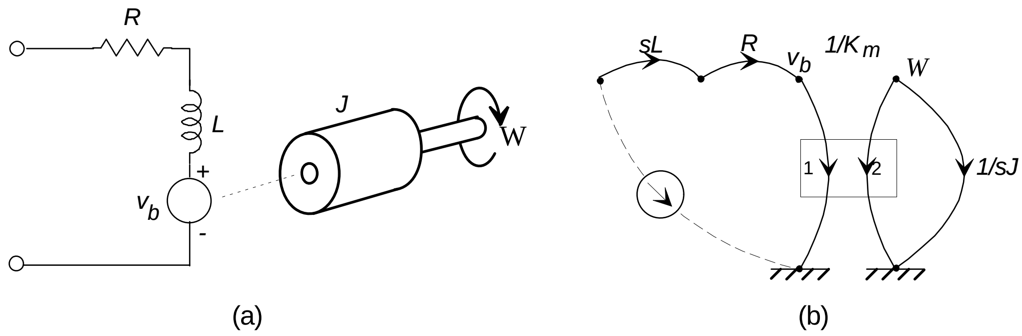

Figure 10: A second-order system consisting of a d.c. motor with an inertial load.

Solution: The motor is a transforming transducer between the electrical and rotational domains, and with the linear graph shown shown in Fig. 10 has a ratio $TF = 1/K_m$

$$\begin{bmatrix} v_m \\ i_m \end{bmatrix} = \begin{bmatrix} K_m & 0 \\ 0 & -1/K_m \end{bmatrix} \begin{bmatrix} \Omega_m \\ T_m \end{bmatrix}. \tag{44}$$

With the substitution $T_J = -T_m$, and $\Omega_m = T_J Z_J$ where $Z_J = 1/sJ$, the impedance of the two-port referred to the electrical side is

$$Z_m = \frac{v_m}{i_m} = \frac{K_m^2}{sJ} \tag{45}$$

The overall electrical impedance of the motor, as seen from the terminals is therefore

$$\begin{aligned} Z &= R + sL + \frac{K_m^2}{sJ} \\ &= \frac{LJs^2 + RJs + K_m^2}{Js} \end{aligned} \tag{46}$$

■ Example

Find the mechanical impedance as reflected to the piston rod in the translational/hydraulic system shown in Fig. 11. Assume that the piston has area A and that the pipe in the hydraulic system has inertance I_f, and frictional losses modeled as fluid resistance R_f.

Solution: For the piston $v_p = -q_p/A$ and $F = Ap$ which define a gyrator relationship with ratio $-1/A$. The hydraulic system impedance is

$$\begin{aligned} Z_f &= R_f + sI_f + 1/sC_f \\ &= \frac{I_f C_f s^2 + R_f C_f s + 1}{sC_f}. \end{aligned} \tag{47}$$

From Eq. (34) the mechanical impedance is

$$\begin{aligned} Z_m &= A^2 Y_f \\ &= \frac{A^2 C_f s}{I_f C_f s^2 + R_f C_f s + 1}. \end{aligned} \tag{48}$$

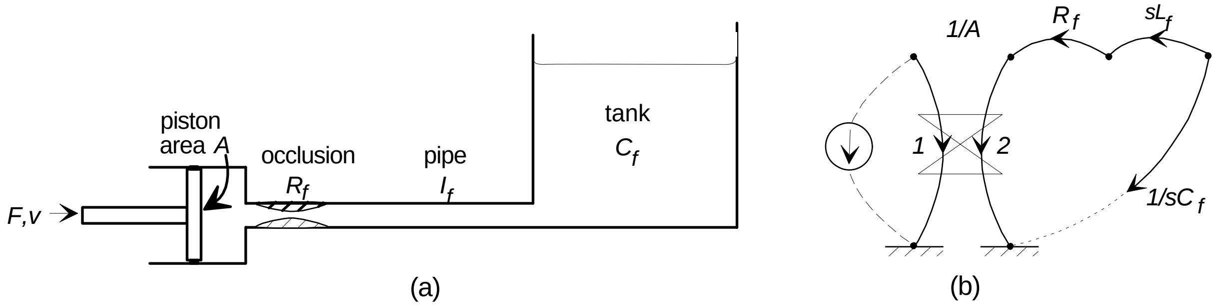

Figure 11: A fluid system containing a gyrator.

4 Transfer Function Generation Using Impedances

Impedance based modeling methods provide a convenient way of generating the transfer function of a linear system directly from its linear graph. Consider a single-input single-output system, with a linear graph containing N branches described by impedances $Z_i(s)$, $i = 1, \ldots, N$ and a single source, either an across-variable source V_s or a through variable-source F_s. There are a total of $2N$ system variables associated with the passive branches; the N across-variables v_i, and the N through-variables f_i. In each branch the impedance ,or admittance, provides an algebraic relationship between the through and across-variable,

$$v_i(s) = Z_i(s) f_i(s) \qquad or \qquad f_i(s) = Y_i(s) v_i(s). \tag{49}$$

In order to solve the system N additional independent equations are needed.

If a *tree* containing K branches is constructed from the linear graph, the remaining $N - K$ branches form the set of links. Compatibility and continuity equations based upon the tree define the additional required equations. It is not necessary that the tree be the system's *normal* tree, the only restriction is on the location of the input source: if the system contains an across-variable source it must be represented as a branch tree, and a through-variable source must be contained in the links. The N independent linear equations may be formed by the following steps:

(1) On each branch in the graph define either the across-variable or the through-variable as a *primary* variable. The set of equations is expressed in terms of these N variables, the remaining N *secondary* variables are eliminated in the next steps.

(2) Generate $N - K$ compatibility equations by replacing the links into the tree one at a time, and write the resulting loop equation in terms of the across-variable drops around the loop. If any of the across-variables in a compatibility equation are not primary variables, the impedance relationships of Eq. 49 are used to eliminate the secondary across-variable.

(3) Generate K continuity equations by applying the principle of extended continuity to each open node in the tree, and form an equation in terms of the through-variables entering the closed volume around the node. If any through-variable is not a primary variable, it is eliminated by substitution using the admittance relationship of Eq. 49.

Each of the N equations generated in the above steps is a linear algebraic equation in the N primary variables. While there is considerable freedom in choosing whether the across or the through-variable should be the primary variable, it is often convenient (but not essential) to select the system output variable as a primary variable. The equations may be rearranged and written in the form

$$\mathbf{Z}\mathbf{x} = \mathbf{U} \tag{50}$$

where $\mathbf{Z}$ is a square $n \times n$ matrix of impedance based coefficients, $\mathbf{x}$ is a column vector of the n primary variables, and $\mathbf{U}$ is a column vector with elements related to the system input. Any of the standard algebraic methods for solving a system of linear equations may be used to find the transfer function between one of the primary variables and the input.

■ Example

Find the transfer function relating the capacitor voltage v_c to the input voltage V_{in} in the electrical circuit shown in Fig. 12.

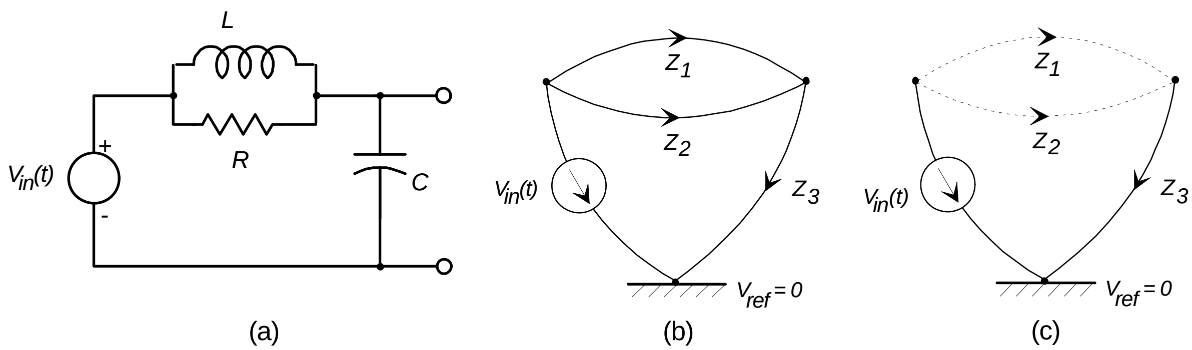

Figure 12: Second-order electrical circuit, its linear graph, and tree.

Solution: The three impedances shown in the linear graph in Fig. 12b are

$$Z_1 = R, \qquad Z_2 = sL, \quad \text{and} \quad Z_3 = 1/sC.$$

The choice of the three primary variables is somewhat arbitrary. In this case select the voltage v_3 (because it is the output variable), and the currents i_1, i_2. From the tree given in Fig. 12c the two compatibility equations are

$$\begin{aligned} v_1 + v_3 - V_{in} &= 0 && (51) \\ v_2 + v_3 - V_{in} &= 0 && (52) \end{aligned}$$

and the single continuity equation is

$$i_1 + i_2 - i_3 = 0. \tag{53}$$

Eqs. (i), (ii), and (iii) may be written in terms of the primary variables using the impedance relationships:

$$\begin{aligned} Z_1 i_1 + v_3 &= V_{in} && (54) \\ Z_2 i_2 + v_3 &= V_{in} && (55) \\ i_1 + i_2 - Y_3 v_3 &= 0 && (56) \end{aligned}$$

where the s dependence has been omitted for convenience. The three equations may be written in matrix form

$$\begin{bmatrix} Z_1 & 0 & 1 \\ 0 & Z_2 & 1 \\ 1 & 1 & -Y_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} V_{in} \\ V_{in} \\ 0 \end{bmatrix}, \tag{57}$$

and Cramer's Rule (Appendix A) may be used to solve for $v_3(s)$:

$$v_3 = \frac{\det \begin{bmatrix} Z_1 & 0 & V_{in} \\ 0 & Z_2 & V_{in} \\ 1 & 1 & 0 \end{bmatrix}}{\det \begin{bmatrix} Z_1 & 0 & 1 \\ 0 & Z_2 & 1 \\ 1 & 1 & -Y_3 \end{bmatrix}} \tag{58}$$

$$= \frac{(Z_1 + Z_2)}{Z_1 Z_2 Y_3 + Z_1 + Z_2} V_{in}. \tag{59}$$

The required transfer function is:

$$H(s) = \frac{v_c(s)}{V_{in}(s)} = \frac{v_3(s)}{V_{in}(s)} \tag{60}$$

$$= \frac{Z_1 + Z_2}{Z_1 Z_2 Y_3 + Z_1 + Z_2} \tag{61}$$

$$= \frac{R + sL}{RLCs^2 + R + sL} \tag{62}$$

$$= \left(\frac{1}{RLC}\right) \frac{R + sL}{s^2 + (1/RC)s + 1/LC} \tag{63}$$

and the system differential equation is

$$\frac{d^2 v_c}{dt^2} + \frac{1}{RC}\frac{dv_c}{dt} + \frac{1}{LC} v_c = \frac{1}{RC}\frac{dV_{in}}{dt} + \frac{1}{LC} V_{in}. \tag{64}$$

A system containing N passive elements requires the solution of N linear equations if each element is represented as a discrete branch in the graph. There is, however, no restriction on the form of the impedances that may be represented in the branches of a linear graph used to generate the compatability and continuity equations. The rules for combining series and parallel elements, described in Section 3, may be used to reduce the number of branches in a linear graph, and therefore the number of equations to be solved. Care should be taken, however, not to mask the output variable through any graph reduction. If the output variable is a through-variable the branch specifying that variable should not be eliminated; if the output is an across-variable the nodes related to that variable should be retained.

■ Example

The hydraulic system shown in Fig. 13 has a pump, characterized as a Thevenin source

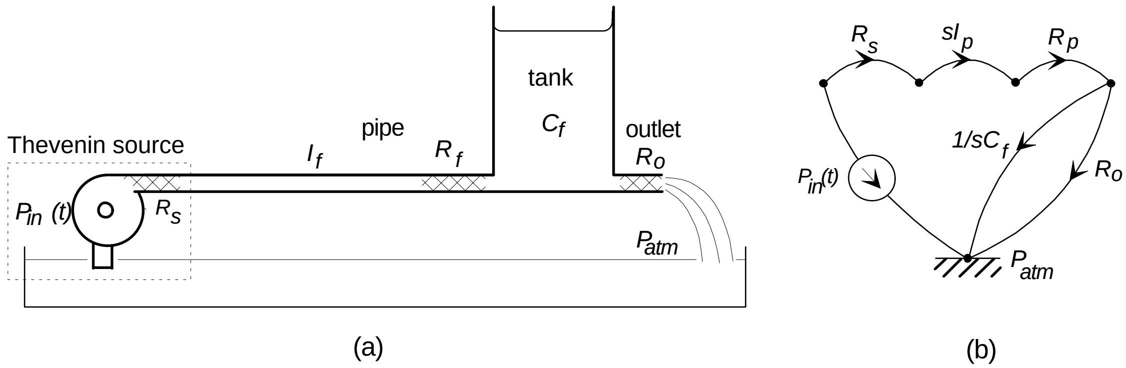

Figure 13: A fluid system and its linear graph.

with a pressure source P_{in} and a series resistance R_s, connected to a long pipe with lumped inertance I_p and resistance R_p and a vertical walled tank C_f. A discharge valve is partially opened and is modeled as a linear resistance R_o. Find the transfer function relating the pressure at the bottom of the tank p_c to the source pressure P_{in}.

Solution: The system as shown has five passive elements and would require the solution of a set of five linear equations to generate the transfer function directly. If however, the three series elements R_s, R_p, and I_p are combined into a single impedance

$$Z_1 = R_s + R_p + sI_p$$

and the two parallel elements, C_f and R_o, are combined into an equivalent impedance

$$Z_2 = \frac{R_o}{R_o Cs + 1}$$

the system may be represented by a reduced linear graph containing just two passive elements, as shown in Fig. 14.

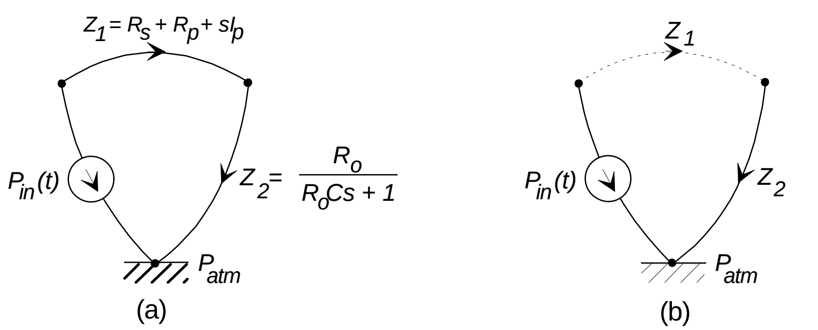

Figure 14: A reduced linear graph for the fluid system and its tree

The reduced system may be represented by just two linear equations. If the primary variables are selected as q_1 and p_2, the compatibility and continuity equations are:

$$q_1 Z_1 + p_2 = P_{in} \quad (65)$$
$$q_1 - p_2 Y_2 = 0 \quad (66)$$

which may be written in matrix form:

$$\begin{bmatrix} Z_1 & 1 \\ 1 & -Y_2 \end{bmatrix} \begin{bmatrix} q_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} P_{in} \\ 0 \end{bmatrix} \tag{67}$$

The required output variable $p_c = p_2$, and Cramer's Rule gives the solution

$$p_2(s) = \frac{\det \begin{bmatrix} Z_1 & P_{in} \\ 1 & 0 \end{bmatrix}}{\det \begin{bmatrix} Z_1 & 1 \\ 1 & -Y_2 \end{bmatrix}} \tag{68}$$

$$= \frac{P_{in}(s)}{Z_1 Y_2 + 1} \tag{69}$$

When the values of the impedance Z_1 and the admittance Y_2 are substituted

$$H(s) = \frac{p_c(s)}{P_{in}(s)} \tag{70}$$

$$= \frac{1}{(R_s + R_p + sI_p)(R_o C_f s + 1)/R_o + 1} \tag{71}$$

$$= \frac{R_o}{I_p C_f R_o s^2 + (I_p + (R_s + R_p) R_o C_f) s + (R_s + R_p + R_o)} \tag{72}$$

The two-port transducing elements may be incorporated into the procedure by using their constitutive relationships. For the transformer there is a direct algebraic relationship between the across-variables associated with the two branches, that is

$$\begin{bmatrix} v_1 \\ f_1 \end{bmatrix} = \begin{bmatrix} TF & 0 \\ 0 & -1/TF \end{bmatrix} \begin{bmatrix} v_2 \\ f_2 \end{bmatrix} \tag{73}$$

where TF is the transformer ratio. Thus while there are four variables associated with the two branches of the transformer, only two of them are independent. One across-variable and one through-variable should be chosen as the two primary variables, and the elemental relationships of Eq. (73) used to eliminate the secondary variables from the compatibility and continuity equations. The same rules as specified in the Linear Graph modeling method are applied in constructing the tree, namely that one and only one of the branches of a transformer should be included in the tree.

Similarly the constitutive equations for a gyrator are:

$$\begin{bmatrix} v_1 \\ f_1 \end{bmatrix} = \begin{bmatrix} 0 & GY \\ -1/GY & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ f_2 \end{bmatrix} \tag{74}$$

where GY is the gyrator ratio. In this case there is a direct algebraic relationship between the across-variable on one branch and the through-variable in the other branch. The two primary variables should therefore be selected as the two across-variables or the two through-variables. The relationships in Eq. (74) may be used to eliminate the secondary variables. The tree should contain either both branches of a gyrator or neither of the two branches.

■ **Example**

Figure 15 shows a model of a moving coil d.c. voltmeter. The principle of operation is similar to that of a permanent magnet d.c. motor. A coil is wound on a rotating armature and is mounted in a magnetic field so that it generates a torque proportional to the current. As the coil moves a back e.m.f. is generated that is proportional to the angular velocity of the armature. The armature is modeled as series lumped inductance and resistance elements in a manner similar to the motor. The mechanical side of the meter is modeled as an inertia J, representing the armature, and a rotational spring K. When a constant current is applied the steady-state deflection of the spring is directly proportional to the current. A series resistor is used to limit the current through the meter.

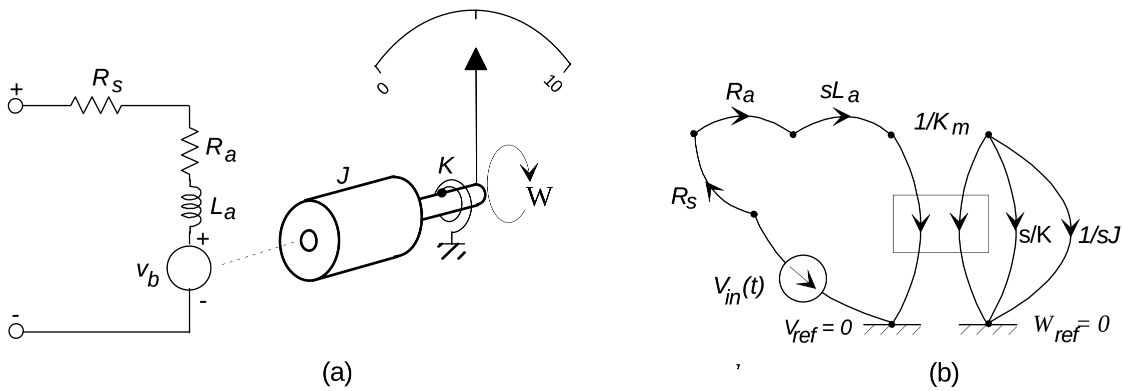

Figure 15: A dc voltmeter and its linear graph.

Assume that the meter is connected to a voltage source V_{in}. Find the transfer function relating the angular velocity of the armature Ω_J to the input voltage V_{in}.

Solution: The meter is modeled as a transformer between the electrical and rotational domains because there is a direct relationship between the through and across variables on the electrical and mechanical sides. The constitutive relationships are therefore

$$\begin{bmatrix} \Omega_m \\ T_m \end{bmatrix} = \begin{bmatrix} 1/K_m & 0 \\ 0 & -K_m \end{bmatrix} \begin{bmatrix} v_m \\ i_m \end{bmatrix}$$

where K_m is the meter torque constant and the transformer ratio $TF = 1/K_m$.

The three series electrical elements R_s, R_a, and L_a may be combined into a single impedance element

$$Z_1 = R_s + R_a + sL_a, \tag{75}$$

and the parallel mechanical elements J and K may also be reduced to an equivalent impedance

$$Z_4 = \frac{s}{Js^2 + K}, \tag{76}$$

generating the reduced linear graph and the tree shown in Fig. 16.

Choose as primary variables i_1, i_2, Ω_3, and Ω_4, and note that the output variable is $\Omega_J = \Omega_4$. The tree shown in Fig. 16 generates the following compatibility and continuity

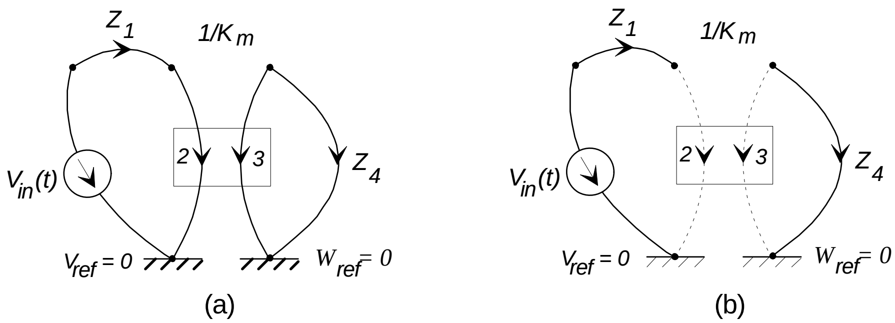

Figure 16: Reduced linear graph and tree for the moving coil meter.

equations:

$$
\begin{aligned}
v_1 + v_2 &= V_{in} && (77)\\
-\Omega_3 + \Omega_4 &= 0 && (78)\\
i_1 - i_2 &= 0 && (79)\\
-T_3 - T_4 &= 0 && (80)
\end{aligned}
$$

with the following constraint equations to eliminate the four secondary variables

$$
\begin{aligned}
v_1 &= Z_1 i_1 && (81)\\
v_2 &= K_m \Omega_3 && (82)\\
T_3 &= -K_m i_2 && (83)\\
T_4 &= Y_4 \Omega_4. && (84)
\end{aligned}
$$

If these constraints are substituted into Eqs. (iii) – (vi), the equations become:

$$
\begin{bmatrix} Z_1 & 0 & K_m & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & K_m & 0 & -Y_4 \end{bmatrix}
\begin{bmatrix} i_1 \\ i_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix}
=
\begin{bmatrix} V_{in} \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (85)
$$

Cramer's rule generates the solution

$$
\begin{aligned}
\Omega_4 &= \frac{\det \begin{bmatrix} Z_1 & 0 & K_m & V_{in} \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & K_m & 0 & 0 \end{bmatrix}}{\det \begin{bmatrix} Z_1 & 0 & K_m & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & K_m & 0 & -Y_4 \end{bmatrix}} && (86)\\
&= \frac{-K_m V_{in}}{-Z_1 Y_4 - K_m^2} && (87)\\
&= \frac{s K_m V_{in}}{(R_s + R_a + sL_a)(Js^2 + K) + sK_m^2} && (88)
\end{aligned}
$$

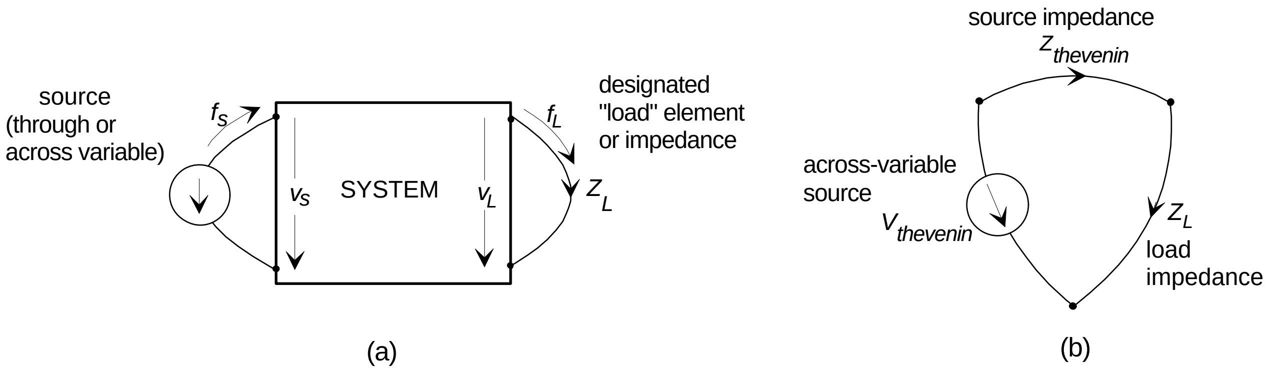

Figure 17: Thevenin equivalent of a system containing a single source and a load element Z_L.

so that

$$H(s) = \frac{\Omega_J}{V_{in}(s)} = \frac{\Omega_4}{V_{in}(s)} \tag{89}$$

$$= \frac{sK_m}{JL_a s^3 + J\left(R_s + R_a\right)s^2 + \left(KL_a + K_m^2\right)s + \left(R_s + R_a\right)K}. \tag{90}$$

---

5 Source Equivalent Models

The concept of Thevenin and Norton equivalent sources are introduced to account for the power limitation of physical sources. The observed "droop" in the characteristic of a physical source is modeled by creating an *equivalent source* containing an ideal source and a dissipative D-type element. The concept of Thevenin and Norton source models may be extended and used as an aid to modeling systems that have a defined *load impedance*, that is an impedance that defines the output variable.

5.1 Thevenin Equivalent System Model

Thevenin's theorem may be stated as follows:

> Any linear system of arbitrary complexity excited by a single active source, and driving an external load Z_L may be modeled as a single across-variable source V_s, connected in series with a single impedance element $Z_o(s)$.

Figure 17 shows the structure of the Thevenin model. Regardless of the internal complexity of the system the theorem allows the overall system to be reduced to just three elements; the source V_s, and two passive impedances Z_o, and the load Z_L.

The values of the equivalent source and the series impedance are found as follows:

1. The across-variable source V_s is the "no-load" across-variable at the output port, obtained when the load impedance Z_L is removed from the system. It may be found from the transfer function relating the output across-variable to the input when Z_L is disconnected from the system.

2. The series impedance Z_o is the system *output impedance*, found by setting any source (either through or across-variable) to zero and determining the driving-point impedance of the system at the output port.

An across-variable source is set to zero by replacing it with a “short-circuit”, that is the nodes to which the source is connected are joined together. Conversely a through-variable source is set to zero by removing the branch from the linear graph leaving the nodes intact, but creating an “open-circuit”.

The following example serves to demonstrate how Thevenin’s theorem may be used to derive a system model.

■ Example

A rotational power transmission system consists of a velocity source Ω_{in} coupled to a shaft through a flexible coupling with torsional stiffness K. The shaft is supported in a bearing with viscous frictional coefficient B and is connected to a machine tool, modeled as an unknown linear impedance $Z_L(s)$. Find the Thevenin equivalent source model for the shaft transmission system, and derive the transfer function for the complete model.

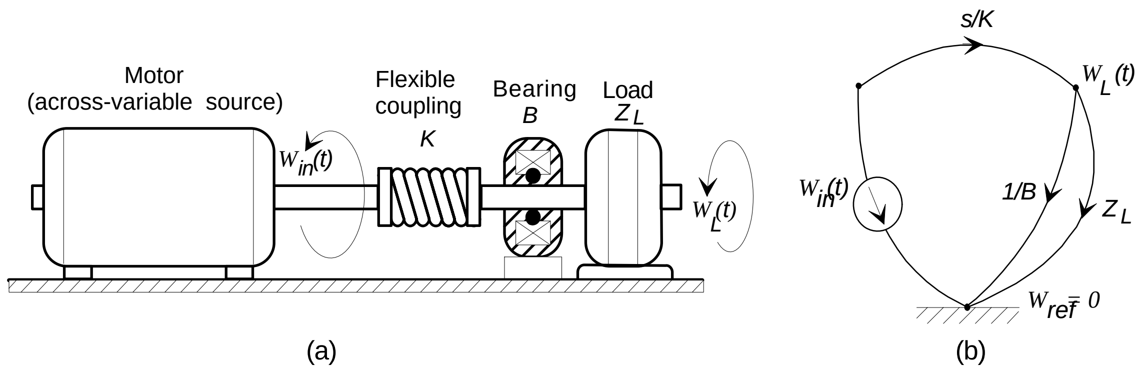

Figure 18: Rotational power transmission system; (a) the physical system, (b) the linear graph.

Solution: Figure 18 shows the power transmission system, its linear graph and the Thevenin equivalent system. The Thevenin across-variable source element Ω_s is found by removing the load Z_L from the output port and determining the “no-load” across variable, using the modified linear graph shown in Fig. 19a. The compatability and continuity conditions on this linear graph yield:

$$\Omega_s(s) = \frac{Z_B}{Z_B + Z_K}\Omega_{in}(s). \tag{91}$$

The output impedance Z_o is found from Fig. 19b by setting the source Ω_{in} to zero, and computing the impedance seen at the output port with Z_L removed. The output impedance is the impedance of the elements Z_K and Z_B in parallel, that is:

$$Z_o = \frac{Z_B Z_K}{Z_B + Z_K}. \tag{92}$$

The Thevenin equivalent system is shown in Fig. 19c.

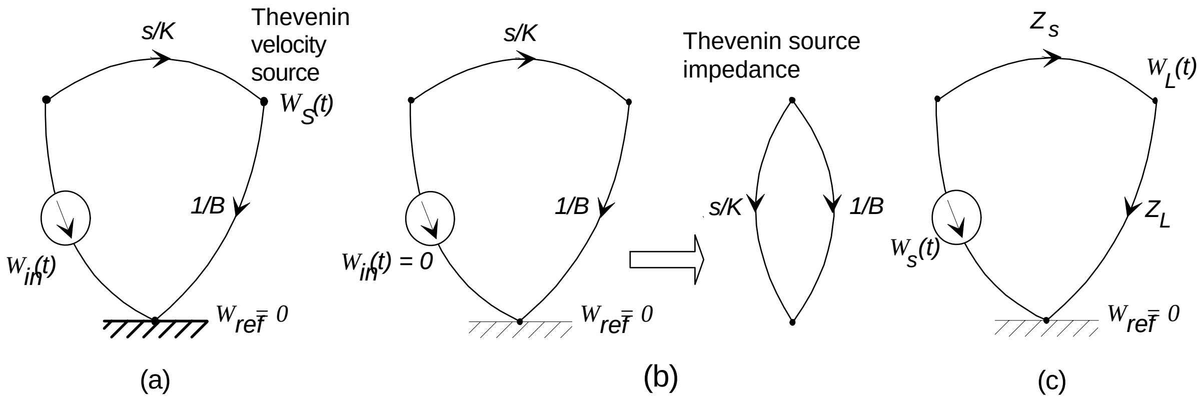

Figure 19: Reduced linear graphs for determination of (a) the Thevenin equivalent source Ω_s, and (b) the system output impedance Z_o, and (c) the Thevenin source equivalent system.

When the load impedance Z_L is connected, the output angular velocity Ω_L computed from the Thevenin model is:

$$\Omega_L(s) = \frac{Z_L}{Z_o + Z_L}\Omega_s(s) \tag{93}$$

$$= \left(\frac{Z_B(s)}{Z_B(s) + Z_K(s)}\right)\left(\frac{Z_L\left(Z_B + Z_K\right)}{Z_B Z_K + Z_L\left(Z_B + Z_K\right)}\right)\Omega_{in}(s) \tag{94}$$

$$= \frac{Z_L Z_B}{Z_B Z_K + Z_L Z_B + Z_L Z_K}\Omega_{in}(s) \tag{95}$$

which is the result we seek. Although the values $Z_B = 1/B$ and $Z_K = s/K$ may be substituted, the impedance of the load Z_L must be known in order to find the complete transfer function.

5.2 Norton Equivalent System Model

Norton's theorem, which is analogous to the Thevenin theorem, states:

> Any linear system connected to a single external load Z_L may be represented by an equivalent through-variable source F_o, connected in parallel with an impedance Z_o across the output port.

Figure 20 shows the structure of a Norton source equivalent model. The difference between the Norton and Thevenin source models lies only in the nature of the assumed source and the series/parallel connection of the impedance element. In all respects the systems are equivalent; no measurement at the output can distinguish between them.

The values of the source and impedance elements are found as follows:

1. The value of the through-variable source F_s is the value of the through-variable at the output port when the load impedance Z_L is reduced to zero. This may be considered the "short-circuit" output through-variable.

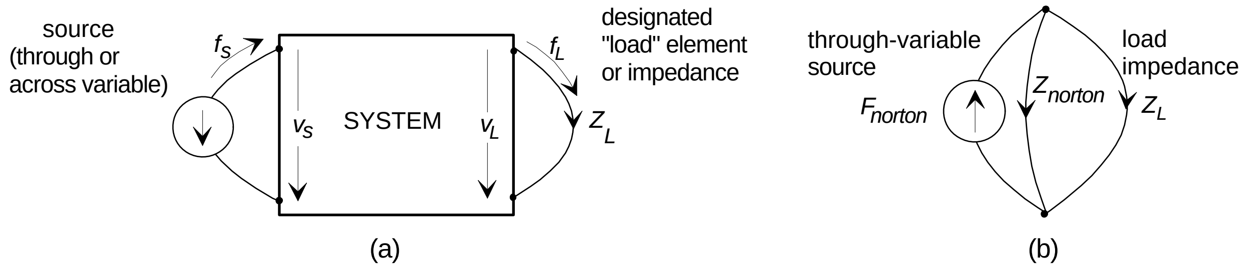

Figure 20: Norton equivalent of a system containing a single source and a load element Z_L.

2. The value of the parallel impedance element Z_o is identical to that of the Thevenin equivalent source; it is the system output impedance, found by setting all internal sources to zero and measuring the system impedance at the output port. In Norton source equivalent systems the output admittance $Y_o(s) = 1/Z_o(s)$ is often used.

The Norton and Thevenin models are equivalent descriptions of the system dynamic behavior as measured at the output port. However, neither model is a representation of the internal structure of the system.

■ Example

Find the Norton equivalent source model of the rotational power transmission system of Example 8, and show that it produces the same transfer function as the Thevenin

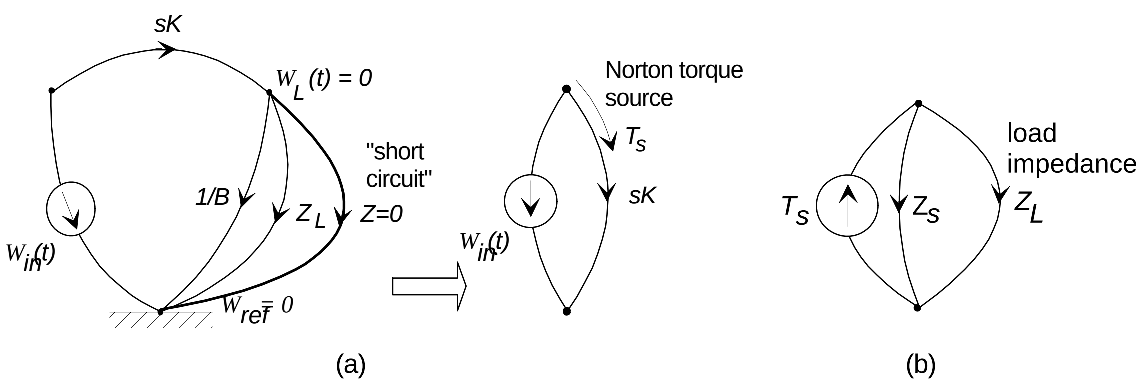

Figure 21: Reduced linear graphs and Norton equivalent for system in Example 8.

model.

Solution: The Norton through variable source is found by setting $Z_L = 0$, which effectively “short circuits” Z_B. Then the Norton source torque T_s, is equal to the through-variable T_L in the load branch

$$T_s = \frac{\Omega_{in}}{Z_K}. \tag{96}$$

It was shown in Example 8 that the system output impedance is

$$Z_o = \frac{Z_B Z_K}{Z_B + Z_K} \tag{97}$$

The Norton equivalent system is shown in Fig. 21. At the single node the continuity equation is

$$T_s - T_o - T_L = 0 \tag{98}$$

or

$$\frac{\Omega_{in}}{Z_K} - \Omega_o Z_o - \Omega_L Z_L = 0. \tag{99}$$

A compatibility condition shows that $\Omega_L = \Omega_o$, and substituting for Z_o gives the result

$$\Omega_L(s) = \frac{Z_L Z_B}{Z_B Z_K + Z_L Z_B + Z_L Z_K}\Omega_{in}(s) \tag{100}$$

which is the same as derived using the Thevenin method in Example 8.

■ References:

1. Shearer, J. L., Murphy A. T., Richardson H. H., *Introduction to System Dynamics*, Addison-Wesley Publishing Company, Reading MA, 1967

2. Reid, J. G., *Linear System Fundamentals*, McGraw-Hill, New York NY, 1983

3. Dorny, C. N., *Understanding Dynamic Systems*, Prentice Hall Inc., Englewood Cliffs NJ, 1993