

Discrete Time Observers and LQG Control ¹

The ZOH Plant Model: We consider a continuous LTI plant with state equations

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned}\tag{1}$$

that is to be regulated by a discrete-time controller with sampling and command updates at intervals T . In addition we assume that a *zero-order hold* (ZOH) at the controller output will produce a piecewise constant command $\mathbf{u}(t) = \mathbf{u}(kT)$ for $kT \leq t < (k+1)T$, $k = 0, 1, \dots, \infty$. The response of the system at time T will be

$$\begin{aligned}\mathbf{x}(T) &= e^{\mathbf{A}T}\mathbf{x}(0) + \int_0^T e^{\mathbf{A}(T-\tau)}\mathbf{B}\mathbf{u}(0)d\tau \\ &= e^{\mathbf{A}T}\mathbf{x}(0) + \left[e^{\mathbf{A}T} \int_0^T e^{-\mathbf{A}\tau}d\tau\mathbf{B} \right] \mathbf{u}(0) \\ &= \mathbf{\Phi}\mathbf{x}(0) + \mathbf{\Gamma}\mathbf{u}(0)\end{aligned}\tag{2}$$

where $\mathbf{\Phi} = e^{\mathbf{A}T}$ and $\mathbf{\Gamma} = e^{\mathbf{A}T} \int_0^T e^{-\mathbf{A}\tau}d\tau$ are constant matrices. Equation (2) predicts the response of the system at time T from the initial state and the zero-order input. This solution may be repeated piecewise on each interval, using the prior solution as the initial state, generating the difference equation description of the system:

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k)\end{aligned}\tag{3}$$

where the sampling interval T has been made implicit. The block diagram representation is shown in Fig. 1. We note that, for a zero-order input with updates at intervals T , the response of the discrete-time system in Eq. (3) will be identical to the continuous plant of Eq. (1) at the sampling times.

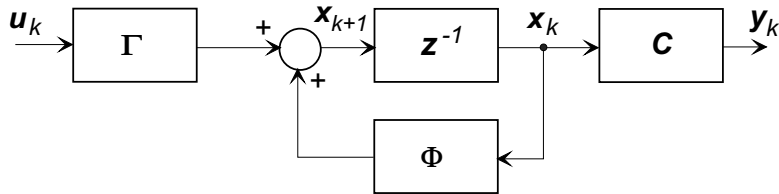


Figure 1: ZOH discrete-time equivalent system

Taking the z transform of Eqs. (3) Gives the pulse transfer-function matrix

$$\mathbf{H}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{\Gamma}\tag{4}$$

¹D. Rowell 12/13/04

Discrete-Time Controller Design: The properties of controllability and observability transfer between the discrete and continuous representations. The discrete-time model may therefore be used to design controllers for a controllable system described by Eqs. (1). For example in a pole-placement regulator with control law $\mathbf{u}(n) = -\mathbf{K}\mathbf{x}(n)$ the closed-loop state equations will be

$$\mathbf{x}(n+1) = (\mathbf{\Phi} - \mathbf{\Gamma}\mathbf{K})\mathbf{x}(n) \quad (5)$$

and the eigenvalues may be placed arbitrarily in the z -plane using the same techniques as for the continuous design. (Note that the effect of pole placement in the z -plane must be interpreted differently from the s -plane).

Discrete-Time Full Observer: The discrete-time model of Eqs. (3) may be used to generate an observer based controller for the continuous plant, with the assumption of zero-order command. We assume that the system is observable, and that we desire estimates $\hat{\mathbf{x}}(k)$ of the state $\mathbf{x}(k)$ based on observation of the output \mathbf{y} alone. As in the case of the continuous observer, we use feedback of the estimator error $\mathbf{y}(k) - \mathbf{C}\hat{\mathbf{x}}(k)$ as a correction term

$$\hat{\mathbf{x}}(k+1|k) = \mathbf{\Phi}\hat{\mathbf{x}}(k|k-1) + \mathbf{\Gamma}\mathbf{u}(k) + \mathbf{K}_e[\mathbf{y}(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1)] \quad (6)$$

where \mathbf{K}_e is the observer gain matrix, and the notation $\hat{\mathbf{x}}(k+1|k)$ denotes that the estimate (or prediction) of $\mathbf{x}(k+1)$ is made using measurements available at time k .

Define the estimation error $\tilde{\mathbf{x}}$ as

$$\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}, \quad (7)$$

subtraction of Eq. (3) from Eq. (6) gives the error difference equation:

$$\begin{aligned} \tilde{\mathbf{x}}(k+1|k) &= \mathbf{\Phi}\tilde{\mathbf{x}}(k|k-1) - \mathbf{K}_e[\mathbf{y}(k) - \mathbf{C}\tilde{\mathbf{x}}(k|k-1)] \\ &= [\mathbf{\Phi} - \mathbf{K}_e\mathbf{C}]\tilde{\mathbf{x}}(k|k-1) \end{aligned} \quad (8)$$

which allows arbitrary z -plane placement of the observer poles, as in the continuous case.

The observer described by Eq. (6) is known as a *prediction* observer, based on the fact that it predicts the state values at the next time step from past measurements. This means that control based on $\tilde{\mathbf{x}}(k)$ does not depend on the most current measurements, which might lead to performance degradation. It is useful to construct an alternative observer formulation that provides a *current* estimator $\bar{\mathbf{x}}(k)$ based on the most recent measurements of \mathbf{y} . Modify Eq. (6) to estimate

$$\bar{\mathbf{x}}(k) = \mathbf{\Phi}\hat{\mathbf{x}}(k-1) + \mathbf{\Gamma}\mathbf{u}(k-1) \quad (9)$$

and

$$\hat{\mathbf{x}}(k|k) = \bar{\mathbf{x}}(k) + \mathbf{K}_e[\mathbf{y}(k) - \mathbf{C}\bar{\mathbf{x}}(k|k-1)] \quad (10)$$

where $\bar{\mathbf{x}}(k)$ is the estimate based on the prediction from the previous time step. Substitution of Eq. (10) into Eq. (9) and advancing one time-step gives

$$\bar{\mathbf{x}}(k+1|k) = \mathbf{\Phi}\bar{\mathbf{x}}(k|k-1) + \mathbf{\Gamma}\mathbf{u}(k) + \mathbf{\Phi}\mathbf{K}_e[\mathbf{y}(k) - \mathbf{C}\bar{\mathbf{x}}(k|k-1)] \quad (11)$$

In this case the error dynamics are described by

$$\tilde{\mathbf{x}}(k+1|k) = [\mathbf{\Phi} - \mathbf{K}_e\mathbf{C}\mathbf{\Phi}]\tilde{\mathbf{x}}(k|k-1) \quad (12)$$

The block diagram of the current observer is shown in Fig. 3. The question then becomes which of the two estimators should be used in closed-loop feedback design? In general,

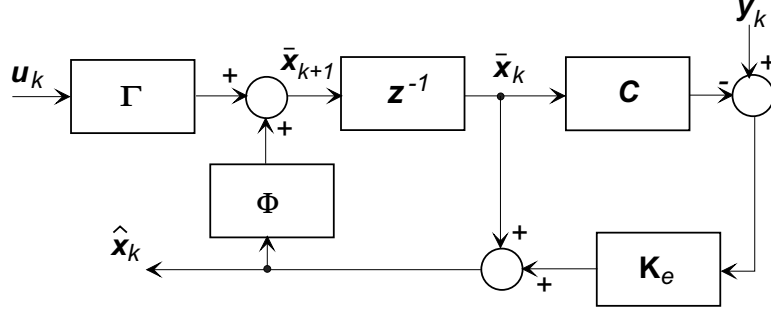


Figure 2: Current discrete observer

because it is based on the most recent sample, one should use the current estimate because it provides the fastest response to unknown disturbances or measurement errors.

The use of the current observer in a state-feedback controller with a continuous plant is shown in Fig. 3.

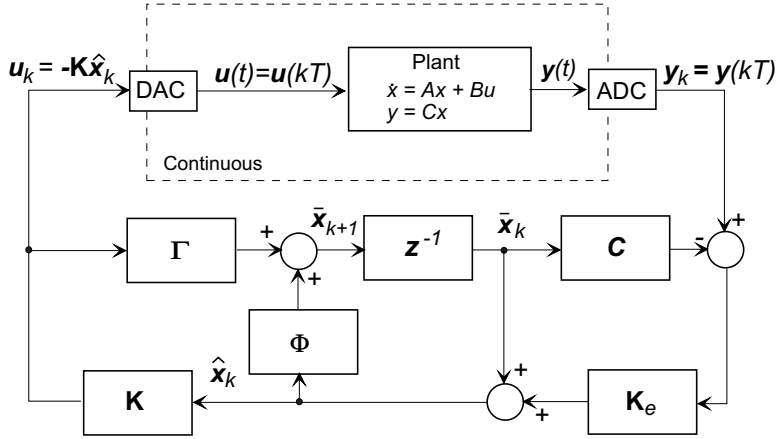


Figure 3: Discrete-time state feedback control of a continuous plant using a current discrete observer

Optimal Estimation – The Kalman-Filter: Optimal estimation provides an alternative rationale for the choice of observer gains in the current estimator. Instead of arguments based on the pole placement, the optimal estimator is based on observer performance in the presence of process noise and measurement errors. Suppose the discrete plant model of Eq. (3) is extended

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}\mathbf{u}(k) + \mathbf{\Gamma}_1\mathbf{w}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k) \end{aligned} \quad (13)$$

where the process noise $\mathbf{w}(k)$ and measurement noise $\mathbf{v}(k)$ are white Gaussian random sequences with zero mean, that is

$$\mathcal{E}\{\mathbf{w}(k)\} = 0 \quad \text{and} \quad \mathcal{E}\{\mathbf{v}(k)\} = 0$$

where $\mathcal{E}\{\}$ denotes the statistical expectation, and zero correlation (white)

$$\mathcal{E}\{\mathbf{w}(i)\mathbf{w}^T(j)\} = 0 \quad \text{and} \quad \mathcal{E}\{\mathbf{v}(i)\mathbf{v}^T(j)\} = 0 \quad \text{for} \quad i \neq j$$

and have covariances defined by

$$\mathcal{E} \left\{ \mathbf{w}(k) \mathbf{w}^T(k) \right\} = \mathbf{R}_w \quad \text{and} \quad \mathcal{E} \left\{ \mathbf{v}(k) \mathbf{v}^T(k) \right\} = \mathbf{R}_v$$

The optimization task is to determine a set of observer gains \mathbf{K}_e to minimize the variance of the estimation error, which is denoted $\mathbf{P}(k)$:

$$\mathbf{P}(k) = \mathcal{E} \left\{ (\mathbf{x}(k) - \hat{\mathbf{x}}(k)) (\mathbf{x}(k) - \hat{\mathbf{x}}(k))^T \right\},$$

in the presence of the process and measurement noise. The derivation is beyond the scope of this handout, we simply state that the solution is a structure the same as the full-state current observer with set of time-varying gains \mathbf{K}_o

$$\hat{\mathbf{x}}(k) = \Phi \bar{\mathbf{x}}(k) + \mathbf{K}_o(k) (\mathbf{y}(k) - \mathbf{C} \bar{\mathbf{x}}(k)) \quad (14)$$

where

$$\mathbf{K}_o(k) = \mathbf{P}(k) \mathbf{C}^T \mathbf{R}_v^{-1} \quad (15)$$

and where $\mathbf{P}(k)$ is found from

$$\mathbf{P}(k) = \mathbf{M}(k) - \mathbf{M}(k) \mathbf{C}^T (\mathbf{C} \mathbf{M}(k) \mathbf{C}^T + \mathbf{R}_v)^{-1} \mathbf{C} \mathbf{M}(k). \quad (16)$$

The matrix $\mathbf{M}(k)$ is the covariance of the state estimates $\bar{\mathbf{x}}(k)$ before the measurement. The estimate $\bar{\mathbf{x}}(k)$ is found from $\hat{\mathbf{x}}(k-1)$ using Eq. (13) with $w(k-1) = 0$ because we know that this is the expected value since the expected value of the plant noise is zero. Thus

$$\bar{\mathbf{x}}(k) = \Phi \hat{\mathbf{x}}(k-1) + \Gamma \mathbf{u}(k-1) \quad (17)$$

This equation is known as the “time update”, whereas the change in the estimate from $\bar{\mathbf{x}}(k)$ to $\hat{\mathbf{x}}(k)$ after measuring $y(k)$ in Eq. (14) is known as the “measurement update”. The matrix $\mathbf{M}(k)$ is updated as

$$\mathbf{M}(k+1) = \Phi \mathbf{P}(k) \Phi^T + \Gamma_1 \mathbf{R}_w \Gamma_1^T \quad (18)$$

The solution starts with assumed initial values for $\mathbf{M}(0)$ and $\hat{\mathbf{x}}(0)$. At each time step the filter is updated in two steps:

At the measurement time: $k+1 \Rightarrow k$, sample $\mathbf{y}(k)$

$$\hat{\mathbf{x}}(k) = \bar{\mathbf{x}}(k) + \mathbf{K}_o(k) (\mathbf{y}(k) - \mathbf{C} \bar{\mathbf{x}}(k))$$

Between updates: Prepare for the next update

$$\begin{aligned} \bar{\mathbf{x}}(k+1) &= \Phi \hat{\mathbf{x}}(k) + \Gamma \mathbf{u}(k) \\ \mathbf{M}(k+1) &= \Phi \mathbf{P}(k) \Phi^T + \Gamma_1 \mathbf{R}_w \Gamma_1^T \\ \mathbf{P}(k+1) &= \mathbf{M}(k+1) - \mathbf{M}(k+1) \mathbf{C}^T (\mathbf{C} \mathbf{M}(k+1) \mathbf{C}^T + \mathbf{R}_v)^{-1} \mathbf{C} \mathbf{M}(k+1) \\ \mathbf{K}_o(k+1) &= \mathbf{P}(k+1) \mathbf{C}^T \mathbf{R}_v^{-1} \end{aligned}$$

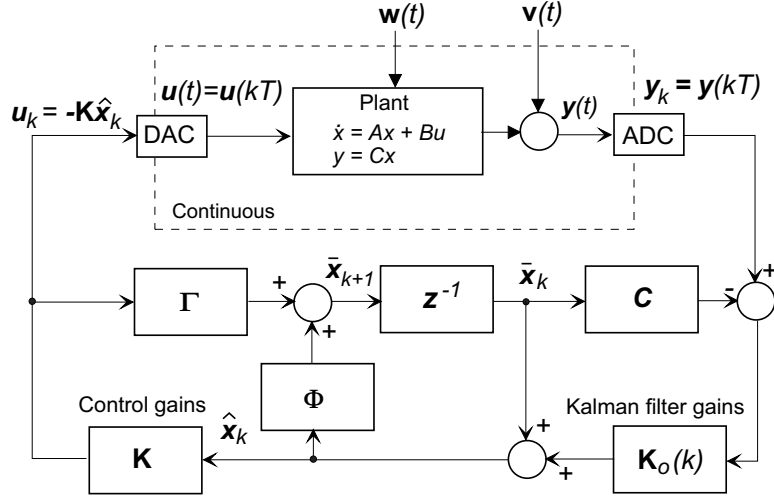


Figure 4: Control of a continuous plant with a Kalman-filter based observer

A block diagram for the closed-loop control of a continuous plant using a Kalman-filter estimator is shown in Fig. 4.

In an actual design problem, meaningful values can be assigned to \mathbf{R}_v , which is based on sensor noise which can often be found from the specifications. The same cannot be said of the process noise, which is often an mathematical artifact that is used to expedite the optimization. Physically, \mathbf{R}_w is used to account for unknown disturbances and uncertainties in the plant model. The disturbance noise model should be chosen to approximate any known disturbances, but the designer is often forced to use “acceptable” values based on simulation studies.

LCG Control – the Steady-State Kalman-Filter: In practice, the time-varying Kalman gains tend to steady-state values as k increases. In a control system that runs for a very long time, the limiting gains may be used to define a so-called *linear quadratic gaussian* (LQG) regulator. The structure is the same as the current observer based controller, but the difference is that instead of the design being based on observer pole placement, it is based upon minimization of estimate errors in the presence of process and measurement noise. Again the derivation is beyond the scope of this handout, suffice it to say that the steady-state Kalman-filter gains K_∞ are

$$\mathbf{K}_\infty = \mathbf{M}_\infty \mathbf{C}^T (\mathbf{C} \mathbf{M}_\infty \mathbf{C}^T + \mathbf{R}_v)^{-1} \quad (19)$$

where \mathbf{M}_∞ is derived from the stable eigenvectors of the estimator Hamiltonian

$$\mathcal{H}_e = \begin{bmatrix} \Phi^T + \mathbf{C}^T \mathbf{R}_v^{-1} \mathbf{C} \Phi^{-1} \Gamma_1 \mathbf{R}_w \Gamma_1^T & -\mathbf{H}^T \mathbf{R}_v^{-1} \mathbf{C} \Phi^{-1} \\ -\Phi^{-1} \Gamma_1 \mathbf{R}_w \Gamma_1^T & \Phi^{-1} \end{bmatrix}$$

and

$$\mathbf{M}_\infty = \Lambda \mathbf{X}^{-1}$$

where

$$\begin{bmatrix} \mathbf{X} \\ \Lambda \end{bmatrix}$$

are the stable eigenvectors.

LQG Design Example Using Matlab: Consider the coupled-mass system in Fig. (5) Under the influence of applied force $u(t)$. The system has equations of motion:

$$\begin{aligned} M\ddot{y} + B(\dot{y} - \dot{d}) + K(y - d) &= u \\ m\ddot{d} + B(\dot{d} - \dot{y}) + K(d - y) &= 0; \end{aligned}$$

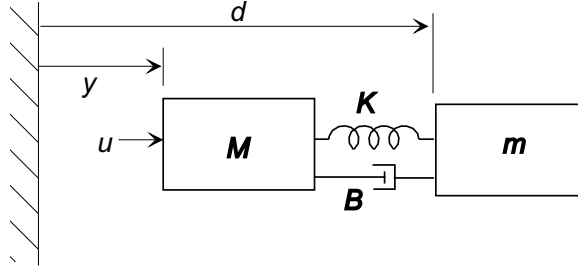


Figure 5: Compliant mass example

If we assume that the measurable outputs are the displacements d and y , the state and output equations are:

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/m & -B/m & K/m & B/m \\ 0 & 0 & 0 & 1 \\ K/M & B/M & -K/M & -B/M \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M \end{bmatrix} u \\ \mathbf{y} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} \end{aligned}$$

where $\mathbf{x} = [d \ \dot{d} \ y \ \dot{y}]^T$. The following parameters are used: $M = 1$, $m = 0.1$, $K = 0.091$, $B = 0.0036$. A sampling period of $T = 0.4$ s. is assumed. With these values the state equations are:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.91 & -0.036 & 0.91 & 0.036 \\ 0 & 0 & 0 & 1 \\ 0.091 & 0.0036 & -0.091 & -0.0036 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

and after converting to a ZOH discrete model we have:

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.9285 & 0.3876 & 0.0715 & 0.0124 \\ -0.3516 & 0.9146 & 0.3516 & 0.0854 \\ 0.0071 & 0.0012 & 0.9929 & 0.3988 \\ 0.0352 & 0.0085 & -0.0352 & -0.9915 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.0013 \\ 0.0124 \\ 0.0799 \\ 0.3988 \end{bmatrix} u(k)$$

The following Matlab script takes the plant model, discretizes it using a ZOH discrete model, chooses LQR gains, forms a Kalman-filter observer for state estimation, combines

the plant, controller, and observer models and plots some initial condition responses of the closed-loop system:

```

% 2.151: LQG regulator for coupled masses
% (See handout on discrete observers).
%-----
% Enter the continuous plant model
%
M = 1; m = 0.1; k = 0.091; b = 0.0036;
A = [ 0 1 0 0
      -k/m -b/m k/m b/m
      0 0 0 1
      k/M b/M -k/M -b/M];
B = [0; 0; 0; 1/M];
C = [1 0 0 0
      0 0 1 0
      0 0 0 0]; % get d, y and u as outputs
D = [0; 0; 1];
%-----
% Create the LTI continuous model
%
plant = ss(A, B, C, D, 'InputName', 'Force F (N)', 'OutputName', ...
           {'Displacement d (m)', 'Displacement y (m)', 'Force F (N)'});
set(plant, 'StateName', {'d', 'd_dot', 'y', 'y_dot'})
%-----
% Create the Discrete model with a sampling time of T=0.4 s.
%
T = 0.4;
ZOH = c2d(plant, T, 'zoh');
% Retrieve the matrices
[Phi, Gamma, Cd, Dd] = ssdata(ZOH);
%-----
% Design the regulator by computing the LQR Gain matrix K
% Set R = [1] arbitrarily and scale the Q matrix accordingly.
% Make Q diagonal
%
R = 1;
Q11 = input('LQR controller Q_11: ');
Q22 = input('LQR controller Q_22: ');
Q = diag([Q11 0 Q22 0]);
K = dlqr(Phi, Gamma, Q, R);
%-----
% Compute the Kalman filter gains
% Assume rms noise of 1% on each sensor channel
Rv = 0.01^2 * eye(2);
% Input Rw

```

```

Rw      = input('Enter estimator Rw: ');
sensors = [1, 2];           % d and y are sensed
known   = [1];             % force u
P       = ss(Phi, [Gamma Gamma], C, [D D], T);
[Observer, Ko] = kalman(P, Rw, Rv, [], sensors, known);
%-----
% Create the regulator and the closed-loop system
%
lqg_reg = lqgreg(Observer, K, 'current');
feedin  = [1];             % force u
feedout = [1, 2];         % d and y
Gcl     = feedback(ZOH, lqg_reg, feedin, feedout, +1);
%-----
% Compute and plot the initial condition response
% Set x_1(0) = 1, all others to zero.
x0      = zeros(8,1);
x0(1) = 1;
figure(1), clf
initial(Gcl, x0)
% Plot all states and compare actual values with Kalman estimates
[y, t, x] = initial(Gcl, x0, 10);
figure(2), clf
subplot(2,2,1), stairs(t,x(:, [1 5])), grid, legend('x_1=d', 'x_1hat', 0)
title('Response of states and predictive estimates to x_1(0) = 1'), ...
      xlabel('Time (s)')
subplot(2,2,2), stairs(t,x(:, [2 6])), grid, legend('x_2=ddot', 'x_2hat', 0)
xlabel('Time (s)')
subplot(2,2,3), stairs(t,x(:, [3 7])), grid, legend('x_3=y', 'x_3hat', 0)
xlabel('Time (s)')
subplot(2,2,4), stairs(t,x(:, [4 8])), grid, legend('x_4=ydot', 'x_4hat', 0)
xlabel('Time (s)')
%-----

```

References:

1. Franklin, G.F., Powell, J. D., Workman, M.L. *Digital Control of Dynamic Systems* (3rd Ed.), Addison Wesley
2. Philips, C.L., Nagle, H.T. *Digital Control System Analysis and Design* (2nd Ed.), Prentice Hall
3. Astrom, K.J., Wittenmark, B. *Computer-Controlled Systems*, Prentice Hall