MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.151 Advanced System Dynamics and Control

Modal Decomposition and the Time-Domain Response of Linear Systems [1]

In a previous handout we examined the response of the linear state determined system in terms of the state-transition matrix $\mathbf{\Phi}(t)$. In this handout we examine how the eigenvectors affect the response, and develop a geometrical interpretation that can give insight into system behavior.

We assume that we have a system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

with real, distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ determined from the the characteristic equation $\det[\lambda\mathbf{I} - \mathbf{A}] = 0$, and an associated set of eigenvectors $\mathbf{m}_1, \ldots, \mathbf{m}_n$. As before, we define the system modal matrix

$$\mathbf{M} = [\mathbf{m}_1 \mid \mathbf{m}_2 \mid \ldots \mid \mathbf{m}_n]$$

If the eigenvalues are distinct the eigenvectors are linearly independent, and form a *basis* for the vector state space. Consider the homogeneous response of the system to a set of initial conditions $\mathbf{x}(0)$:

$$\mathbf{x}_h(t) = \mathbf{\Phi}(t)\mathbf{x}(0) = e^{\mathbf{A}t}\mathbf{x}(0) \tag{1}$$

where the state transition matrix $\mathbf{\Phi}(t)$ may be expressed in terms of the eigenvalues/eigenvectors as

$$\mathbf{\Phi}(t) = \mathbf{M}e^{\mathbf{\Lambda}t}\mathbf{M}^{-1}. \tag{2}$$

where $e^{\mathbf{\Lambda}t}$ is a square matrix $(n \times n)$ with terms $e^{\lambda_i t}$ on the leading diagonal. Define

$$\mathbf{R} = \mathbf{M}^{-1} = \begin{bmatrix} \mathbf{r}_1 \\ \hline \mathbf{r}_2 \\ \hline \vdots \\ \hline \mathbf{r}_n \end{bmatrix},$$

where r_i is a row vector $(1 \times n)$. Then the homogeneous response may be written

$$\begin{aligned} \mathbf{x}_h(t) &= \left(\mathbf{M}e^{\mathbf{\Lambda}t}\right)(\mathbf{R}\mathbf{x}(0)) \\ &= \sum_{i=1}^{n} e^{\lambda_i t}\mathbf{m}_i\left(\mathbf{r}_i\mathbf{x}(0)\right) \\ &= \sum_{i=1}^{n} c_i e^{\lambda_i t}\mathbf{m}_i \end{aligned} \tag{3}$$

where the scalar coefficients $c_i = \mathbf{r}_i\mathbf{x}(0)$. Note that c_i are the inner (scalar or dot) product $c_i = \langle \mathbf{r}_i^T \mathbf{x}(0) \rangle$.

> Equation (3) states that the homogeneous response may be represented as a weighted superposition of the *system modes*, $e^{\lambda_i t}\mathbf{m}_i$, where the initial conditions $\mathbf{x}(0)$ affect the strength of excitation of each mode. This is known as *modal decomposition*.

[1] D. Rowell 10/16/04

■ Example 1

Find the response of the homogeneous system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x}$$

to the initial conditions $\mathbf{x}(0) = [2 \quad 3]^T$.

Solution: The system has eigenvalues $\lambda_{1,2} = -1, -2$ and a pair of associated eigenvectors $\mathbf{m_1} = [-1 \quad 1]^T$ and $\mathbf{m}_2 = [-1 \quad 2]^T$. The modal matrix is

$$\mathbf{M} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$$

and

$$\mathbf{R} = \mathbf{M}^{-1} = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix}$$

so that $\mathbf{r}_1 = [-2 \quad -1]$ and $\mathbf{r}_2 = [-1 \quad -1]$.

$$\begin{aligned} \mathbf{x}_h(t) &= \sum_{i=1}^{n} (\mathbf{r}_i \mathbf{x}(0))\, e^{\lambda_i t} \mathbf{m}_i \\ &= -7e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 5e^{-2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 7e^{-t} - 5e^{-2t} \\ -7e^{-t} + 10e^{-2t} \end{bmatrix} \end{aligned}$$

Consider the product

$$\mathbf{RM} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix. Expansion of the product in terms of the rows and columns, and element by element comparison with the identity matrix shows

$$\mathbf{r}_i \mathbf{m}_i = 1 \qquad (4)$$

$$\text{and} \quad \mathbf{r}_i \mathbf{m}_j = 0 \qquad (i \neq j). \qquad (5)$$

If the initial conditions $\mathbf{x}(0)$ lie on an extension of an eigenvector, that is $\mathbf{x}(0) = q\mathbf{m}_j$, Eqs. (4) and (5) show that the coefficients $c_i = \mathbf{r}_i \mathbf{x}(0) = q\mathbf{r}_i \mathbf{m}_j = 0$ for all $i \neq j$ and Eq, (3) reduces to

$$\mathbf{x}_h(t) = q e^{\lambda_j t} \mathbf{m}_j. \qquad (6)$$

When an initial conditions $\mathbf{x}(0)$ lie on an extension of an eigenvector $\mathbf{m}_j$, only the mode $e^{\lambda_j t}\mathbf{m}_j$ associated with that eigenvector appears in the system response. The trajectory of the response $\mathbf{x}_h(t)$ in the state-space $\mathbf{x}$ is a straight line along the eigenvector $\mathbf{m}_j$. For a stable system this trajectory will approach the origin $\mathbf{x} = \mathbf{0}$ as $t \to \infty$.

■ **Example 2**

Find the response of the homogeneous system in the Example 1 to the initial conditions $\mathbf{x}(0) = [-3 \quad 6]^T$.

Solution: From the previous example the system has eigenvalues $\lambda_{1,2} = -1, -2$ and eigenvectors $\mathbf{m_1} = [-1 \quad 1]^T$, and $\mathbf{m}_2 = [-1 \quad 2]^T$. The matrix $\mathbf{R}$ has rows $\mathbf{r}_1 = [-2 \quad -1]$ and $\mathbf{r}_2 = [-1 \quad -1]$.

With the given initial conditions, $\mathbf{r}_1\mathbf{x}(0) = 0$, and $\mathbf{r}_2\mathbf{x}(0) = 3$. From Eq. (3) the homogeneous response is

$$\mathbf{x}_h(t) = 3e^{-2t}\mathbf{m}_2 = \begin{bmatrix} 3e^{-2t} \\ -6e^{-2t} \end{bmatrix}$$

indicating that the modal component e^{-t} has not been excited by these particular initial conditions. In this case $\mathbf{x}(0) = -3\mathbf{v}_2$, that is the initial condition lies along the eigenvector $\mathbf{m}_2$.

The n linearly independent eigenvectors form a *basis* in the state-space, therefore any point $\mathbf{x}$ may be expressed as a linear combination of the eigenvectors

$$\begin{aligned} \mathbf{x} &= q_1\mathbf{m}_1 + q_2\mathbf{m}_2 + \ldots + q_n\mathbf{m}_n \\ &= \sum_{i=1}^{n} q_i\mathbf{m}_i \end{aligned} \tag{7}$$

The components q_i may be found by noting that Eq. (7) may be written $\mathbf{x} = \mathbf{Mq}$ where $\mathbf{q} = [q_1 \ q_2 \ \ldots \ q_n]^T$, so that

$$\mathbf{q} = \mathbf{M}^{-1}\mathbf{x} \qquad \text{or} \qquad q_i = \mathbf{r}_i\mathbf{x} \tag{8}$$

From Eq. (6), the principle of superposition therefore allows the homogeneous response from initial conditions

$$\mathbf{x}(0) = q_1\mathbf{m}_1 + q_2\mathbf{m}_2 + \ldots + q_n\mathbf{m}_n = \sum_{i=1}^{n}(\mathbf{r}_i\mathbf{x}(0))\mathbf{m}_i$$

to be written as the sum of the responses to each modal component, that is Similarly, the system time-domain response $\mathbf{x}(t)$ may be written as a superposition of trajectories *along the eigenvectors*

$$\mathbf{x}(t) = \sum_{i=1}^{n} q_i e^{\lambda_i t}\mathbf{m}_i \tag{9}$$

The n modes of the system are excited by the components q_i of the initial conditions *that lie along the corresponding eigenvector* $\mathbf{m}_i$. Note that each component of the system response $q_i e^{\lambda_i t}\mathbf{m}_i$, when expressed as in Eq. (8), is a *straight line* path in the state-space, aligned with its eigenvector. Each component approaches the origin at a rate determined by the term $e^{\lambda_i t}$, generating a curved trajectory in state space.

■ **Example 3**

Use decomposition of the initial condition vector along the eigenvectors to solve the problem posed in Example 1.

Solution: From Example 1, the system has eigenvalues $\lambda_{1,2} = -1, -2$ and modal matrix

$$\mathbf{M} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}.$$

With the initial conditions $\mathbf{x}(0) = [2 \;\; 3]^T$. Then

$$\mathbf{q} = \mathbf{M}^{-1}\mathbf{x}(0) = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

so that $q_1 = -7$ and $q_2 = 5$. From Eq. (9) the response is therefore

$$\mathbf{x}_h(t) = -7e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 5e^{-2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7e^{-t} - 5e^{-2t} \\ -7e^{-t} + 10e^{-2t} \end{bmatrix}$$

Complex Eigenvalues:

We now consider the case of a second-order system with pair of complex conjugate eigenvalues

$$\lambda_{1,2} = \alpha \pm j\beta.$$

In this case the eigenvectors are also complex conjugates, and the modal matrix may be written

$$\mathbf{M} = [\mathbf{m}_1 \mid \mathbf{m}_2] = [\mathbf{m}' + j\mathbf{m}'' \mid \mathbf{m}' - j\mathbf{m}'']$$

where $\mathbf{m}' = \Re\{\mathbf{m}_1\}$ and $\mathbf{m}'' = \Im\{\mathbf{m}_1\}$. Similarly

$$\mathbf{R} = \mathbf{M}^{-1} = \begin{bmatrix} \mathbf{r}_1 \\ \hline \mathbf{r}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{r}' + j\mathbf{r}'' \\ \hline \mathbf{r}' - j\mathbf{r}'' \end{bmatrix},$$

where $\mathbf{r}' = \Re\{\mathbf{r}_1\}$ and $\mathbf{r}'' = \Im\{\mathbf{r}_1\}$. As before $\mathbf{RM} = \mathbf{I}$, and the products involving the real and imaginary parts of $\mathbf{r}$ and $\mathbf{m}$ are

$$\begin{aligned} \mathbf{r}'\mathbf{m}' &= 0.5 \\ \mathbf{r}''\mathbf{m}'' &= -0.5 \\ \mathbf{r}'\mathbf{m}'' &= 0 \\ \mathbf{r}''\mathbf{m}' &= 0 \end{aligned} \tag{10}$$

From Eq. (3)

$$\begin{aligned} \mathbf{x}_h(t) &= e^{(\alpha+j\beta)t}(\mathbf{m}' + j\mathbf{m}'')\left((\mathbf{r}' + j\mathbf{r}'')\mathbf{x}(0)\right) + e^{(\alpha-j\beta)t}(\mathbf{m}' - j\mathbf{m}'')\left((\mathbf{r}' - j\mathbf{r}'')\mathbf{x}(0)\right) \\ &= e^{\alpha t}\left(2\cos(\beta t)\left((\mathbf{r}'\mathbf{x}(0))\mathbf{m}' - (\mathbf{r}''\mathbf{x}(0))\mathbf{m}''\right) - 2\sin(\beta t)\left((\mathbf{r}'\mathbf{x}(0))\mathbf{m}'' - (\mathbf{r}''\mathbf{x}(0))\mathbf{m}'\right)\right) \end{aligned} \tag{11}$$

which indicates that provided $\alpha < 0$, the homogeneous response of the state-variables $\mathbf{x}_{hi}(t)$ are decaying sinusoids with angular frequency β rad/s and with amplitudes and phases determined by the initial conditions $\mathbf{x}(0)$.

■ Example 4

Find the response of the homogeneous system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \mathbf{x}$$

to the initial conditions $\mathbf{x}(0) = [3 \quad 2]^T$.

Solution: The system has complex eigenvalues $\lambda_{1,2} = -1 + j1, -1 - j1$ and a pair of associated eigenvectors $\mathbf{m_1} = [-1 - j1 \quad 2]^T$ and $\mathbf{m}_2 = [-1 + j1 \quad 2]^T$. The modal matrix is

$$\mathbf{M} = \begin{bmatrix} -1 + j1 & -1 + j1 \\ 2 & 2 \end{bmatrix}$$

so that

$$\mathbf{m}' = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{m}'' = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Also

$$\mathbf{R} = \mathbf{M}^{-1} = \begin{bmatrix} 0.5j & 0.25 + 0.25j \\ -0.5j & 0.25 - 0.25j \end{bmatrix}$$

so that

$$\mathbf{r}' = \begin{bmatrix} 0 & 0.25 \end{bmatrix} \qquad \text{and} \qquad \mathbf{r}'' = \begin{bmatrix} 0.5 & 0.25 \end{bmatrix}.$$

Also

$$\mathbf{r}'\mathbf{x}(0) = \begin{bmatrix} 0 & 0.25 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 0.5 \qquad \text{and} \qquad \mathbf{r}''\mathbf{x}(0) = \begin{bmatrix} 0.5 & 0.25 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 2.$$

From Eq. (11) the response is

$$\begin{aligned} \mathbf{x}_h(t) &= e^{\alpha t} \left(2\cos(\beta t)\left((\mathbf{r}'\mathbf{x}(0))\mathbf{m}' - (\mathbf{r}''\mathbf{x}(0))\mathbf{m}''\right) - 2\sin(\beta t)\left((\mathbf{r}'\mathbf{x}(0))\mathbf{m}'' + (\mathbf{r}''\mathbf{x}(0))\mathbf{m}'\right)\right) \\ &= e^{-t} \left[2\cos(t)\left(0.5\begin{bmatrix} -1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) - 2\sin(t)\left(0.5\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)\right] \\ &= \begin{bmatrix} e^{-t}\left(3\cos(t) + 5\sin(t)\right) \\ e^{-t}\left(2\cos(t) - 8\sin(t)\right) \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{34}e^{-t}\cos(t + 1.0304) \\ \sqrt{68}e^{-t}\cos(t - 1.3258) \end{bmatrix} \end{aligned}$$

There are two modes associated with the pair of complex conjugate eigenvalues. If the initial conditions are chosen so that $\mathbf{x}(0) = \Re\{\mathbf{m}_1\} = \mathbf{m}'$ then

$$\begin{aligned} \mathbf{x}_h(t) &= e^{\alpha t}\left(2\cos(\beta t)\left((\mathbf{r}'\mathbf{m}')\mathbf{m}' - (\mathbf{r}''\mathbf{m}')\mathbf{m}''\right) - 2\sin(\beta t)\left((\mathbf{r}'\mathbf{m}'))\mathbf{m}'' - (\mathbf{r}''\mathbf{m}')\mathbf{m}'\right)\right) \\ &= e^{\alpha t}\left(\cos(\beta t))\mathbf{m}' - \sin(\beta t)\mathbf{m}''\right) \qquad \text{(Mode 1)} \end{aligned} \tag{12}$$

from Eqs. (10), and similarly if $\mathbf{x}(0) = \Im\{\mathbf{m}_1\} = \mathbf{m}''$ then

$$\begin{aligned} \mathbf{x}_h(t) &= e^{\alpha t}\left(2\cos(\beta t)\left((\mathbf{r}'\mathbf{m}'')\mathbf{m}' - (\mathbf{r}''\mathbf{m}'')\mathbf{m}''\right) - 2\sin(\beta t)\left((\mathbf{r}'\mathbf{m}''))\mathbf{m}'' - (\mathbf{r}''\mathbf{m}'')\mathbf{m}'\right)\right) \\ &= e^{\alpha t}\left(\sin(\beta t))\mathbf{m}' + \cos(\beta t)\mathbf{m}''\right) \qquad \text{(Mode 2).} \end{aligned} \tag{13}$$

These two modes are similar in form but differ in the amplitude and phase of the damped oscillations.

The Forced Response

For the forced single-input (SI) system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

the term $\mathbf{Bu}$ may be decomposed among the n eigenvectors $\mathbf{m}_i$ as

$$\mathbf{Bu} = \mathbf{f}(t) = \sum_{i=1}^{n} f_i(t)\mathbf{m}_i$$

where $f_i(t) = \mathbf{r}_i\mathbf{f}(t)$. The total response, consisting of the homogeneous component and the convolution integral, may then be written

$$\mathbf{x}(t) = \sum_{i=1}^{n}(\mathbf{r}_i\mathbf{x}(0))e^{\lambda_i t}\mathbf{m}_i + \int_0^t \sum_{i=1}^{n}(\mathbf{r}_i\mathbf{B}u(\tau))e^{\lambda_i(t-\tau)}\mathbf{m}_i d\tau \tag{14}$$

which indicates that the effect of the system input on each of the system modes may be considered independently. The amplitude of the excitation of the ith mode is given by

$$\int_0^t (\mathbf{r}_i\mathbf{B}u(\tau))e^{\lambda_i(t-\tau)}\mathbf{m}_i d\tau$$

If the input is chosen so that $\mathbf{B}u(t)$ always lies in the direction of one of the eigenvectors, then only one of the modes is excited by that input.