MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.151 Advanced System Dynamics and Control

Linear Graph Modeling: One-Port Elements[1]

1 Introduction

In the previous handout *Energy and Power Flow in State Determined Systems* we examined elementary physical phenomena in five separate energy domains and used concepts of energy flow, storage and dissipation to define a set of lumped *elements*. These primitive elements form a set of building blocks for system modeling and analysis, and are known generically as *lumped one-port elements*, because they represent the spatial locations (ports) in a system at which energy is transferred. For each of the domains with the exception of thermal systems, we defined three passive elements, two of which store energy and a third dissipative element. In addition in each domain we defined two active *source* elements which are time varying sources of energy.

System dynamics provides a unified framework for characterizing the dynamic behavior of systems of interconnected one-port elements in the different energy domains, as well as in non-energetic systems. In this handout the one-port element descriptions are integrated into a common description by recognizing similarities between the elemental behavior in the energy domains, and by defining analogies between elements and variables in the various domains. The formulation of a unified framework for the description of elements in the energy domains provides a basis for development of unified methods of modeling systems which span several energy domains.

The development of a unified modeling methodology requires us to draw analogies between the variables and elements in different energy domains. Several different types of analogs may be defined. In this text we have chosen to relate elements using the concepts of generalized "through" and "across" variables associated with a *linear graph* system representation introduced by F.A. Firestone [1] and H.M. Trent [2], and described in detail in several texts [3-5]. This set of analogs allows us to develop modeling methods that are similar to well known techniques for electrical circuit analysis. The set of analogies we have selected is not unique, for example another widely used analogy is based on the concepts of "effort" and "flow" variables in *bond graph* modeling methods, developed by H.M. Paynter [6] and described in D.C. Karnopp, et al. [7]. These two methods lead to different analogies both of which are valid. For example, in this text we consider forces and electrical currents to be analogous, while in the bond graph method forces and electrical voltages are considered to be similar.

2 Generalized Through and Across Variables

Figure 1 shows a schematic representation of a single one-port element, in this case a mechanical spring, as a generic element with two "terminals" through which power flows, either to be stored, supplied, or dissipated by the element. This *two-terminal* representation may be thought of as a mechanical analog of an electrical element, in this case an inductor, with two connecting "wires". If all system elements are represented in this form, the interconnection of elements may be expressed in a common "circuit" structure and a unified method of modeling and analysis may be derived

[1] D. Rowell - Revised: 9/16/04

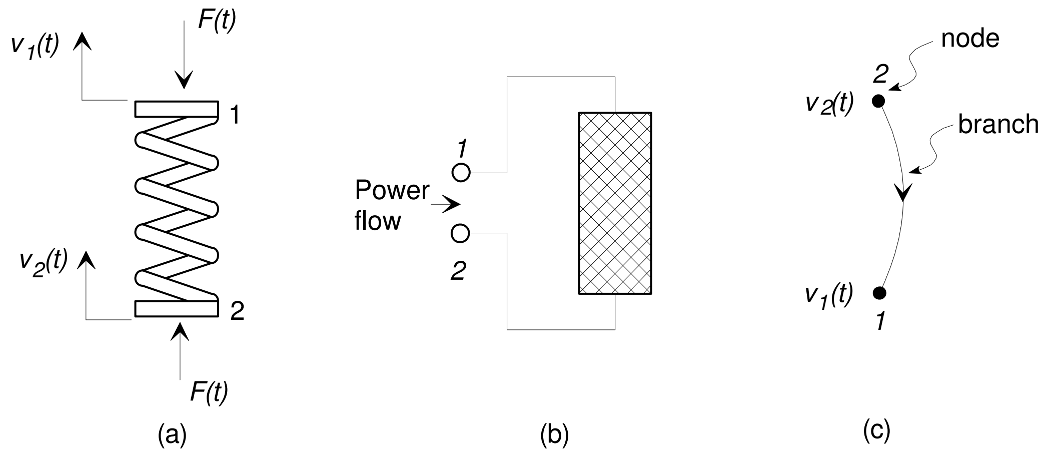


Figure 1: Schematic representation of a typical one-port element (a) a translational spring, (b) as a two-terminal element, and (c) as a linear graph element.

for this form known as a *linear graph.* In Fig. 1(c) the linear graph representation of the spring element is shown as a branch connecting two nodes.

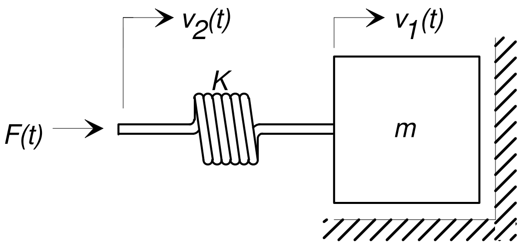
With the two-terminal representation, one of the two variables associated with the element is a physical quantity which may be considered to be measured "across" the terminals of the element, and the other variable represents a physical quantity which passes "through" the element. For example, in the case of mechanical elements such as the spring in Fig. 1 the two defined variables are v, the velocity, and F, the force associated with the element. The velocity associated with a mechanical element is defined to be the differential (or relative) velocity as measured between the two terminals of the element, that is $v = v_2 - v_1$ in Fig. 1; notice that it must be measured *across* the element. Figure 2 shows a simple system with the same spring connected between a mass m and an applied force source $F(t)$. In Fig. 2b the connection has been broken so that the forces acting on the spring and the mass may be examined. Assume that the force transmitted to the mass is $F_m(t)$. Because the spring element is assumed to be massless, Newton's laws of motion require that the sum of all external forces acting on it must sum to zero, or

$$F(t) - F_m(t) = 0.$$

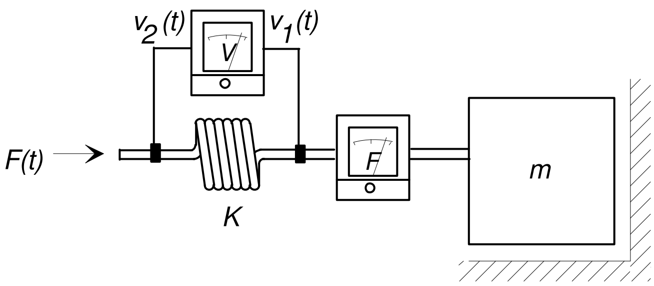
In other words $F_m(t) = F(t)$, and the external force applied to the spring element is transmitted *through* the spring to the mass element connected to the other side. Another way of looking at this is to say that in order to measure the force (tension) in a mechanical element, the element must be broken and a sensing device, such as a spring balance, inserted in series with the element as in Fig. 2b. Such arguments lead us to define elemental velocity v to be an *across-variable*, and force F to be a *through-variable* in mechanical systems.

Figure 3 shows a simple electrical circuit consisting of a battery and a resistor. The elemental variables in the electrical domain are current i and voltage drop v. In order to measure the current flowing in the resistor the electrical circuit must be broken, and an ammeter inserted so that the current flows *through* it. To measure the voltage drop associated with the resistor a voltmeter is connected directly *across* its terminals. Current is defined as the through-variable for electrical systems, and voltage drop is the across-variable.

We may extend the concept of through and across variables to all of the energy domains described in the handout *Energy and Power Flow in State Determined Systems.* Of the two variables defined for each domain, one is defined to be an across-variable because it is a *relative* quantity



(a) Mass and spring elements driven by an external force.



(b) Force is measured by inserting an instrument in series with the elements, velocity is measured by connecting an instrument across an element.

Figure 2: Definition of through and across variables in a simple mechanical system.

that must be measured as a difference between values at the two terminals of a network element. The other is designated as a through-variable that is continuous through any two-terminal element. Once the choice of this pair of variables has been established, generalized modeling and analysis techniques may be developed without regard to the particular energy domains associated with a system.

The through and across-variables for each energy domain discussed in this book are defined below:

Mechanical Systems: In both translational and rotational mechanical systems the *velocity drop* of an element is the velocity difference across its terminals. In the case of a translational mass or rotary inertia one terminal is always assumed to be connected to a constant velocity inertial reference frame. The force or torque associated with

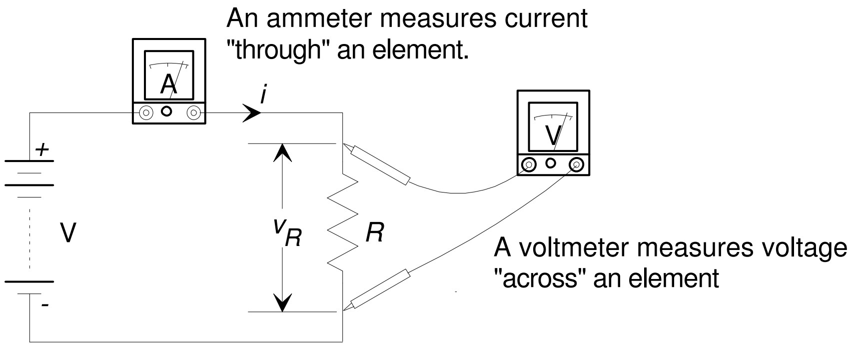


Figure 3: Definition of through and across variables in an electrical system.

an element is assumed to pass through the element. The elemental across-variable is therefore defined to be the relative velocity of the two terminals, and the elemental through-variable is defined to be the force or torque associated with the element.

Electrical Systems: In an electrical element, for example a capacitor, at any instant a potential (or voltage) difference exists between the terminals and a current flows through the element. The across-variable is therefore defined to be the voltage drop across the element, and the through-variable is defined to be the current flowing through the element.

Fluid Systems: In the fluid domain the pressure difference across an element satisfies the definition of an across-variable, while the volume flow rate through the element is a natural choice for the through-variable.

Thermal Systems: While not strictly analogous to the other domains, thermal systems may be analyzed by defining heat flow rate as the through-variable, and the temperature difference across an element as the across-variable.

The definitions of across and through-variables for all the energy domains are summarized in Table 1. In describing generic systems, without regard to a specific energy domain, it is convenient to define a set of *generalized variables*. The generalized across and through-variables are introduced as:

Generalized across-variable:	v
Generalized integrated across-variable:	$\mathsf{x} = \int_0^t \mathsf{v}dt + \mathsf{x}(0)$
Generalized through-variable:	f
Generalized integrated through-variable:	$\mathsf{h} = \int_0^t \mathsf{f}dt + \mathsf{h}(0)$

With the exception of thermal elements, the power $\mathcal{P}$ passing into a lumped one-port element in terms of the generalized variables is:

$$\mathcal{P} = \mathsf{f}\mathsf{v} \tag{1}$$

and the work performed by the system on the element over time period $0 \le t \le T$ may be expressed in terms of the generalized variables as:

$$W = \int_0^T \mathcal{P}dt = \int_0^T \mathsf{f}\mathsf{v}dt. \tag{2}$$

For thermal elements while an across-variable, temperature T, and a through variable, heat flow rate Q, may also be defined, the product is not power since Q is a power variable itself.

3 Generalization of One-Port Elements

In each of the energy domains, several primitive elements are defined: one or two ideal energy storage elements, a dissipative element, and a pair of source elements. For one of the energy storage elements, the energy is a function of its across-variable (for example an ideal mass element stores energy as a function of its velocity; $\mathcal{E} = \frac{1}{2}mv^2$), while in the other energy storage element the stored energy is a function of the through-variable; in a translational spring the stored energy is $\mathcal{E} = \frac{1}{2}KF^2$. The dissipative elements, which store no energy, and the source elements, which may supply energy or power continuously, complete the set of one-port elements. In this section these elements are classified into generic groups.

3.1 A-Type Energy Storage Elements

Energy storage elements in which the stored energy is a function of the across-variable are defined to be *A-type elements*, and are collectively designated as *generalized capacitances*. All A-type energy storage elements have constitutive equations of the form:

$$\mathsf{h} = \mathcal{F}(\mathsf{v}) \tag{3}$$

where h is the generalized integrated through-variable, v is the generalized across-variable, and $\mathcal{F}()$ designates a single-valued, monotonic function. In general Eq. (3) may represent a non-linear relationship, but a linear (or ideal) A-type element has a linear form of Eq. (3):

$$\mathsf{h} = \mathsf{C}\mathsf{v}, \tag{4}$$

where the constant of proportionality C is defined to be the ideal *generalized capacitance* of the element. Differentiation of Eq. (4) gives the generalized A-type elemental equation:

$$\mathsf{f} = \mathsf{C}\frac{d\mathsf{v}}{dt}. \tag{5}$$

The definition of the lumped elements shows that the capacitive elements are the translational mass, rotational inertia, electrical capacitance, fluid capacitance and thermal capacitance. The collection of A-type elements are shown in Fig. 4, and their elemental relationships are summarized in Table 2.

The two-terminal representation of A-type elements in systems often requires a connection to a known *reference value* of the across-variable. Figure 5 shows two A-type elements, a translational mass and a fluid capacitance. In a Newtonian mechanical system the momentum of a mass element m is measured with respect to a nonaccelerating inertial reference frame

$$h = m\left(v_m - v_{ref}\right).$$

where v_m is the mass of the element and v_{ref} is the velocity of the reference frame. Then differentiation gives

$$F = \frac{dh}{dt} = m\frac{d}{dt}\left(v_m - v_{ref}\right) = m\frac{dv_m}{dt}$$

since v_{ref} is constant. There is an implied connection to the reference velocity that defined the momentum, and one terminal must be connected to this reference value (usually assumed to be zero velocity). Similarly the angular velocity of an rotary inertia must be measured with respect to a nonaccelerating rotating reference frame.

In the case of the fluid capacitance defined by a vertical walled tank, the constitutive relationship relating volume to pressure is

$$V = C_f\left(P - P_{ref}\right)$$

where P_{ref} is the constant external pressure at the fluid surface and P is the pressure at the base of the tank. The elemental equation may be written

$$Q = \frac{dV}{dt} = C_f\frac{d}{dt}\left(P - P_{ref}\right) = C_f\frac{dP}{dt}$$

if P_{ref} is constant. The two terminal representation requires an implicit connection to the reference pressure.

System	Across-Variable (v)	Through-Variable (f)
Translational	velocity difference (v)	force (F)
Rotational	angular velocity difference (Ω)	torque (T)
Electrical	voltage drop (v)	current (i)
Fluid	pressure difference (P)	volume flow rate (Q)
Thermal	temperature difference (T)	heat flow rate (q)

System	Integrated Across-Variable (x)	Integrated Through-Variable (h)
Translational	linear displacement (x)	momentum (p)
Rotational	angular displacement (Θ)	angular momentum (h)
Electrical	flux linkage (λ)	charge (q)
Fluid	pressure difference momentum (Γ)	volume (V)
Thermal	(not defined)	heat (H)

Table 1: Definition of across and through-variables in the various energy domains.

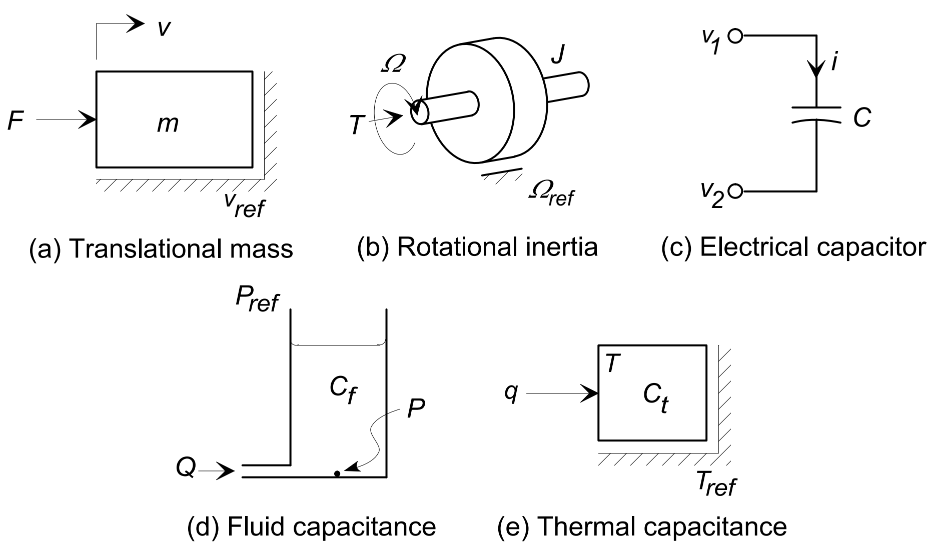


Figure 4: The A-type elements in the five energy domains described in this handout.

Element	Constitutive equation	Elemental equation	Energy
Generalized A-type	$\mathsf{h} = \mathsf{C}\mathsf{v}$	$\mathsf{f} = \mathsf{C}\frac{d\mathsf{v}}{dt}$	$\mathcal{E} = \frac{1}{2}\mathsf{C}\mathsf{v}^2$
Translational mass	$p = mv$	$F = m\frac{dv}{dt}$	$\mathcal{E} = \frac{1}{2}mv^2$
Rotational inertia	$h = J\Omega$	$T = J\frac{d\Omega}{dt}$	$\mathcal{E} = \frac{1}{2}J\Omega^2$
Electrical capacitance	$q = Cv$	$i = C\frac{dv}{dt}$	$\mathcal{E} = \frac{1}{2}Cv^2$
Fluid capacitance	$V = C_f P$	$Q = C_f\frac{dP}{dt}$	$\mathcal{E} = \frac{1}{2}C_f P^2$
Thermal capacitance	$H = C_t T$	$q = C_t\frac{dT}{dt}$	$\mathcal{E} = C_t T$

Table 2: Summary of elemental relationships for ideal A-type elements.

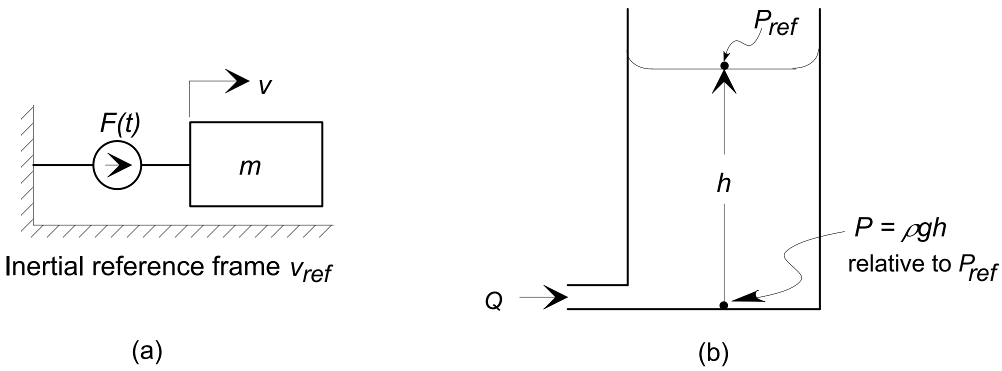


Figure 5: Implicit connection of typical A-type elements to a reference node, (a) a translational mass, and (b) a fluid capacitance.

Similarly the temperature associated with a thermal capacitance is measured with respect to a fixed reference temperature. The electrical capacitor, however, does not require connection to a fixed reference voltage and may have its two terminals connected to points of arbitrary voltage.

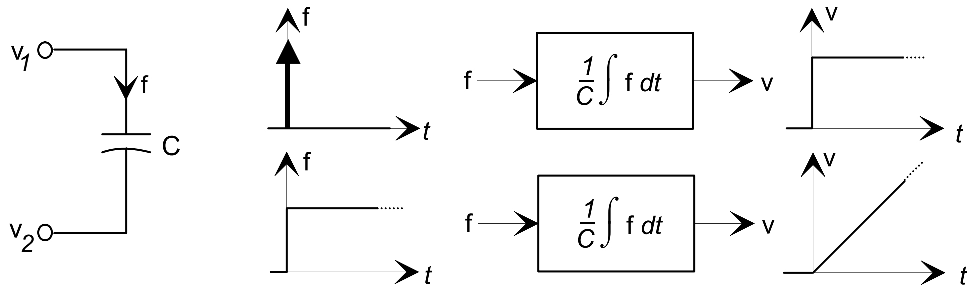
With the exception of the thermal capacitance, the energy stored in a pure or ideal A-type element is given by:

$$\mathcal{E} = \int_0^{\mathsf{h}} \mathsf{v}\, d\mathsf{h}. \tag{6}$$

For an ideal element with a constitutive equation given by Eq. (6), the stored energy can be expressed as:

$$\mathcal{E} = \int_0^{\mathsf{h}} \frac{\mathsf{h}}{\mathsf{C}} d\mathsf{h} = \frac{1}{2}\frac{\mathsf{h}^2}{\mathsf{C}} = \frac{1}{2}\mathsf{C}\mathsf{v}^2 \tag{7}$$

resulting in a form in which the energy is a direct function of the across-variable. For the ideal thermal capacitance the energy is simply $\mathcal{E} = H = C_t T$ and is a function of the across-variable.



(a) Generalized capacitance

(b) Response of capacitance to discontinuous changes in the through variable.

Figure 6: Across and through-variable relationships in an ideal A-type element.

■ Example

Show that that an A-type element is capable of both absorbing and supplying power.

Solution: For an A-type element the instantaneous power flow is

$$\mathcal{P} = \mathsf{f}\mathsf{v} = \mathsf{C}\frac{d\mathsf{v}}{dt}\mathsf{v}. \tag{8}$$

Our sign convention is that if $\mathcal{P} > 0$, power is flowing into the element, while if $\mathcal{P} < 0$ power is flowing from the element. Thus the direction of power flow is defined by Eq. (i); if v and $d\mathsf{v}/dt$ have the same sign the element is absorbing power and storing energy, while if the signs are opposite the element is returning stored energy to the system.

Consider a mechanical mass element. Equation (i) states that the element is accumulating energy whenever it is accelerated in the direction of its travel, and returns energy as it is decelerated.

Equation (7) shows that any change in the stored energy in an A-type element results from a change in the across-variable. In order to change the energy in a step-wise fashion the across-variable must change instantaneously. Eq. (5) shows that the through-variable is proportional to the derivative of the across-variable, therefore an instantaneous change in the stored energy requires an impulse in the through-variable. The stored energy in any A-type element cannot change instantaneously unless infinite power is available in the form of an impulse in force, torque, current or volume flow. Physical energy sources are generally *power limited*, and are therefore incapable of providing an instantaneous change in the across-variable or stored energy of an A-type element. Fig. 6 shows the relationships between across and through-variables for an A-type element.

■ Example

A satellite circling the earth every 90 minutes is subjected to cyclic heating by the sun as it passes in and out of the earth's shadow. Measurements have shown that it is reasonable to model the net solar heat flow rate $Q(t)$ into the satellite as a cosinusoidal function with the orbital period, assuming that at time $t = 0$ the satellite is at the position of peak sunlight. Find the time in the orbit at which the internal temperature within the satellite is a maximum.

Solution: Let the time t be measured in seconds, so that the the heat flow rate is

$$Q(t) = Q_{max} \cos \left(\frac{2\pi}{90 \times 60} t \right). \tag{9}$$

where Q_{max} is the peak heat flow rate (joules/sec). The satellite is modeled as a lumped thermal capacitance C_t and stores energy as an A-type thermal element. For a general A-type element the elemental equation is

$$\mathsf{f} = \mathsf{C} \frac{d\mathsf{v}}{dt}, \tag{10}$$

and for the thermal capacitance the relationship is

$$Q = C_t \frac{dT}{dt}. \tag{11}$$

In this case we require the value of $T(t)$, given $Q(t)$, so that Eq. (iii) must be written in integral form:

$$\begin{aligned} T(t) &= \frac{1}{C_t} \int_0^t Q dt + T(0) && (12) \\ &= \frac{1}{C_t} \int_0^t Q_{max} \cos \left(\frac{2\pi}{5400} t \right) dt + T(0) && (13) \\ &= \frac{5400 Q_{max}}{2\pi C_t} \sin \left(\frac{2\pi}{5400} t \right) + T(0) && (14) \end{aligned}$$

The system input $Q(t)$ and the response $T(t)$ are shown in Fig. 7. The temperature and

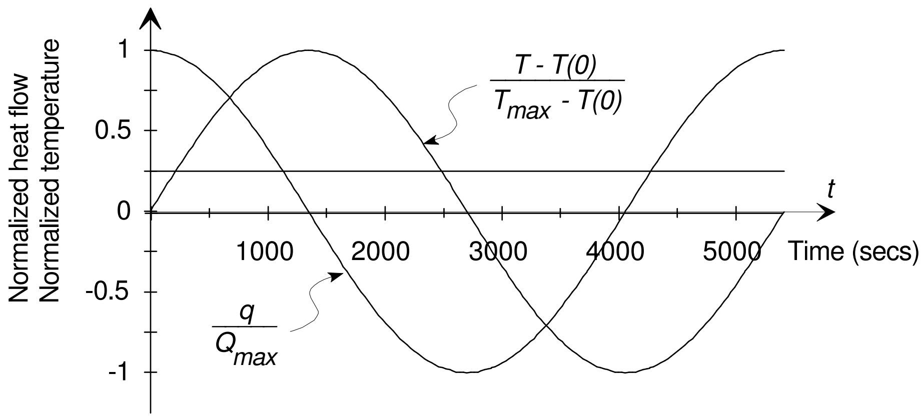


Figure 7: The input heat flow rate $Q(t)$ and the temperature response $T(t)$ of the satellite.

the input heat flow are not in synchrony; the response *lags* the input by one quarter of a cycle. Since $\sin\theta$ is a maximum when $\theta = \pi/2$, $T(t)$ is a maximum when $2\pi t/5400 = \pi/2$, or when $t = 1350$ sec. The satellite therefore reaches its maximum temperature 22.5 minutes after passing the point of maximum brightness, and the maximum temperature is:

$$T_{max} = \frac{5400 Q_{max}}{2\pi C_t} + T(0) \tag{15}$$

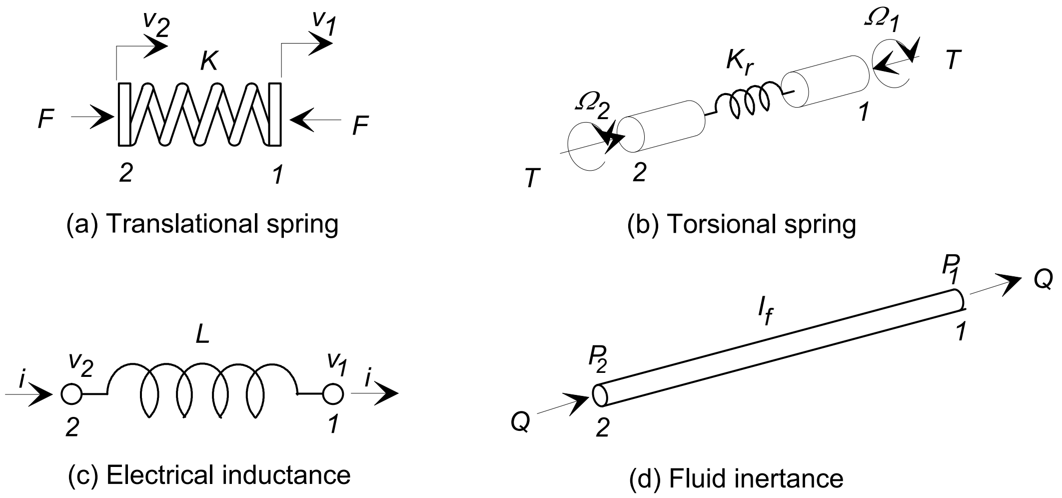


Figure 8: The T-type elements in the energy domains described in this handout. There is no T-type element for the thermal domain.

3.2 T-Type Energy Storage Elements

Energy storage elements in which the stored energy may be expressed as a function of the through-variable are designated as T-type elements, and are collectively known as *generalized inductances*. The T-type energy storage elements are defined by generalized constitutive equations of the form:

$$\mathsf{x} = \mathcal{F}(\mathsf{f}) \tag{16}$$

where x is the generalized integrated across-variable, f is the generalized through-variable, and $\mathcal{F}()$ designates a single-valued, monotonic function. For a linear, or ideal, T-type element the constitutive relationship Eq. (16) reduces to a simple linear equation

$$\mathsf{x} = \mathsf{L}\mathsf{f} \tag{17}$$

where the constant of proportionality L is defined to be the ideal *generalized inductance*. Differentiation of the constitutive equation gives the generalized elemental equation:

$$\mathsf{v} = \mathsf{L}\frac{d\mathsf{f}}{dt}. \tag{18}$$

Figure 8 shows the four T-type elements; there is no known thermal energy storage phenomenon that defines a T-type element for thermal systems. The generalized inductance is equivalent to the *reciprocal* of the mechanical translational and rotational spring constants, and is equivalent to the electrical inductance and the fluid inertance. Table 3 summarizes the elemental relationships for T-type elements.

The energy stored in a T-type pure or ideal element is given by:

$$\mathcal{E} = \int_0^{\mathsf{x}} \mathsf{f}\, d\mathsf{x} \tag{19}$$

For an ideal element, with a constitutive equation of Eq. (19), the energy is a direct function of the through-variable f:

$$\mathcal{E} = \int_0^{\mathsf{x}} \frac{\mathsf{x}}{\mathsf{L}} d\mathsf{x} = \frac{1}{2}\frac{\mathsf{x}^2}{\mathsf{L}} = \frac{1}{2}\mathsf{L}\mathsf{f}^2. \tag{20}$$

Element	Constitutive equation	Elemental equation	Energy
Generalized T-type	$\mathsf{x} = \mathsf{L}\mathsf{f}$	$\mathsf{v} = \mathsf{L}d\mathsf{f}/dt$	$\mathcal{E} = \frac{1}{2}\mathsf{L}\mathsf{f}^2$
Translational spring	$x = \frac{1}{K}F$	$v = \frac{1}{K}\frac{dF}{dt}$	$\mathcal{E} = \frac{1}{2K}F^2$
Torsional spring	$\Theta = \frac{1}{K_r}T$	$\Omega = \frac{1}{K_r}\frac{dT}{dt}$	$\mathcal{E} = \frac{1}{2K_r}T^2$
Electrical inductance	$\lambda = Li$	$v = L\frac{di}{dt}$	$\mathcal{E} = \frac{1}{2}Li^2$
Fluid inertance	$\Gamma = I_f Q$	$P = I_f\frac{dQ}{dt}$	$\mathcal{E} = \frac{1}{2}I_f Q^2$

Table 3: Summary of elemental relationships for ideal T-type elements.

As in the case of an A-type element, it is not possible to change the stored energy or the through-variable in a T-type element instantaneously without an infinite source of power.

■ Example

It is commonly observed in electrical circuits containing inductances that when a switch is opened a brief electrical arc may develop across the air gap, causing the switch contacts to become pitted. In severe cases arcing may occur between the turns of the coil itself causing breakdown of the electrical insulation and perhaps destruction of the inductor. Explain why this arcing occurs.

Solution: Consider the circuit shown in Fig. 9. An inductor is a T-type element, and

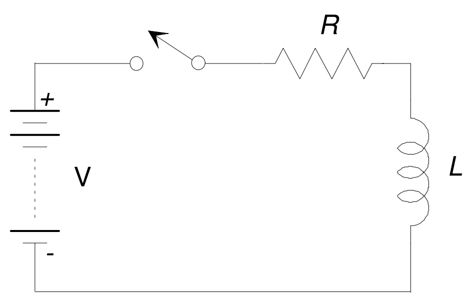


Figure 9: An electrical circuit containing an inductance.

has an elemental equation

$$v = L\frac{di}{dt}. \tag{21}$$

If a current i is flowing just before the switch is opened, the energy stored in the magnetic field of the inductor is $\mathcal{E} = \frac{1}{2}Li^2$. When the current is interrupted the magnetic field "collapses" and the stored energy must be either returned to the system or dissipated. The rapid change in the magnetic flux as the field decays generates a large inductive voltage in the coil. This *induced* voltage is sufficient to cause the arc, a short current pulse across the gap, that is potentially damaging to the switch and the coil itself. The inductive back emf (electromotive force) attempts to maintain the current through the coil so as to dissipate the stored energy.

This phenomenon may be described in terms of the elemental equation Eq. (i). An attempt to decrease the current instantaneously creates a large negative value of the derivative di/dt, generating a correspondingly large value of the across-variable v. The arcing allows the current to continue briefly after the switch is opened and therefore to decay in a finite time. In practice engineers often connect semiconductor diodes or capacitors across inductors to provide an alternate current path and reduce inductive voltage spikes and arcing.

3.3 D-Type Dissipative Elements

The elements that dissipate energy are collectively known as D-type elements. They are defined by an algebraic relationship between the across and through-variables of the form:

$$\mathsf{v} = \mathcal{F}(\mathsf{f}) \qquad \text{or} \qquad \mathsf{f} = \mathcal{F}^{-1}(\mathsf{v}) \tag{22}$$

where f and v are the generalized through and across variables respectively. For linear (ideal) dissipative elements the relationship is commonly expressed in two forms:

$$\mathsf{v} = \mathsf{R}\mathsf{f} \qquad \text{or} \qquad \mathsf{f} = \frac{1}{\mathsf{R}}\mathsf{v} \tag{23}$$

where R is defined to be the *generalized ideal resistance*. It is also common to define the *conductance* $\mathsf{G} = 1/\mathsf{R}$ as the reciprocal of the resistance and to write Eqs. (23) as

$$\mathsf{f} = \mathsf{G}\mathsf{v} \qquad \text{or} \qquad \mathsf{v} = \frac{1}{\mathsf{G}}\mathsf{f}. \tag{24}$$

The generalized resistances are equivalent to the reciprocals of the mechanical and rotational damping constants, and are equivalent to the electrical, fluid and thermal resistances. For all D-type elements, except the thermal resistance element, power supplied to the element is converted into heat and dissipated. For the ideal elements the power may be expressed as:

$$\mathcal{P} = \mathsf{R}\mathsf{f}^2 = \frac{1}{\mathsf{R}}\mathsf{v}^2. \tag{25}$$

The power $\mathcal{P}$ is always a positive quantity, and flows into a D-type element.

In the thermal D-type element power is not dissipated. In this case, because the through-variable is power, the element simply acts to impede heat flow. Table 4 summarizes the algebraic D-type relationships for resistances.

The dissipative elements store no energy and instantaneous changes in the power dissipated by the elements are associated with instantaneous changes in the through and across-variables as indicated by the ideal elemental equation in which the through and across-variables are directly related by the constant R.

Element	Elemental equations		Power dissipated
Generalized D-type	$\mathsf{f} = \frac{1}{\mathsf{R}}\mathsf{v}$	$\mathsf{v} = \mathsf{Rf}$	$\mathcal{P} = \frac{1}{\mathsf{R}}\mathsf{v}^2 = \mathsf{Rf}^2$
Translational damper	$F = Bv$	$v = \frac{1}{B}F$	$\mathcal{P} = Bv^2 = \frac{1}{B}F^2$
Rotational damper	$T = B_r\Omega$	$\Omega = \frac{1}{B_r}T$	$\mathcal{P} = B_r\Omega^2 = \frac{1}{B_r}T^2$
Electrical resistance	$i = \frac{1}{R}v$	$v = Ri$	$\mathcal{P} = \frac{1}{R}v^2 = Ri^2$
Fluid resistance	$Q = \frac{1}{R_f}P$	$P = R_fQ$	$\mathcal{P} = \frac{1}{R_f}p^2 = Q^2R_f$
Thermal resistance	$q = \frac{1}{R_t}T$	$T = R_tq$	

Table 4: Summary of elemental relationships for ideal D-type elements.

■ Example

An electrical resistance of value R is connected to a voltage source that supplies a sinusoidal voltage of the form $V(t) = V_m \sin(\omega t)$, as shown in Fig. 10. Find the *average* power dissipated in the resistor over one period of the voltage input.

Solution: The sinusoidal applied voltage $V(t)$ repeats itself with a period $T = 2\pi/\omega$

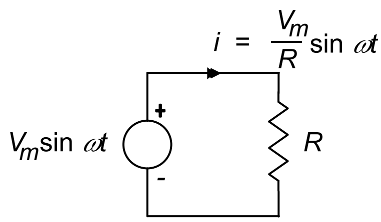


Figure 10: An electrical resistance.

seconds. The instantaneous power dissipated in the resistance is

$$\mathcal{P}(t) = \frac{v^2(t)}{R} = \frac{V_m^2}{R}\sin^2(\omega t). \tag{26}$$

The average power dissipated over one period T is found by integrating the power over one period and dividing by the period:

$$\begin{aligned} \mathcal{P}_{avg} &= \frac{1}{T}\int_0^T \mathcal{P}(t)dt = \frac{2\pi}{\omega}\int_0^{2\pi/\omega} \frac{V_m^2}{R}\sin^2(\omega t)dt \\ &= \frac{V_m^2\omega}{2\pi R}\left(\frac{\pi}{\omega}\right) = \left(\frac{V_m}{\sqrt{2}}\right)^2 \frac{1}{R}. \end{aligned} \tag{27}$$

This expression shows that the average power dissipated in R over one period of the sinusoidal voltage is the same as would be dissipated by a constant applied voltage of value $v = V_m/\sqrt{2} = 0.707V_m$ volts.

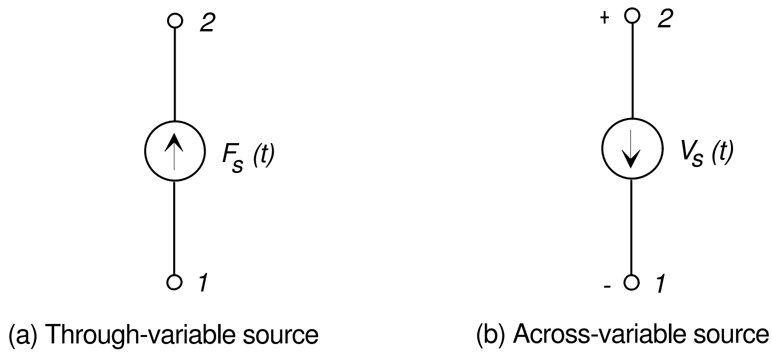


Figure 11: Idealized source elements.

3.4 Ideal Sources

In each energy domain two general types of idealized sources may be defined:

- the ideal *Across-Variable Source* in which the generalized across-variable is a specified function of time $f(t)$
$$\mathsf{V}_s(t) = f(t),$$
and is independent of the through-variable, and

- the ideal *Through-Variable Source* in which the generalized through-variable is a specified function of time
$$\mathsf{F}_s(t) = f(t)$$
and is independent of the across-variable.

An example of an through-variable source is an idealized positive *displacement pump* in a fluid system, in which the flow rate is a prescribed function of time and is independent of the pressure required to maintain the flow, while an example of an across-variable source is a regulated laboratory electrical power supply in which the output voltage is independent of the current drawn by the circuit to which it is connected. The ideal sources are not power or energy limited and theoretically may supply infinite power and energy.

The symbols for the ideal sources are shown in Fig. 11 where in the through-variable source the arrow designates the assumed positive direction of through-variable flow, and in the across-variable source the arrow designates the assumed direction of the across-variable decrease or drop. For each source type one variable is an independently specified function of time.

The value of the complementary variable of each source is determined by the system to which the source is connected. A source may provide power and energy to a system, or may absorb power and energy, depending upon the sign of the complementary source variable. Table 5 defines the source types in each of the energy domains.

■ **Example**

A force source is used to accelerate and deaccelerate a mass in a cyclic motion as shown

Energy Domain	Across-variable source	Through-variable source
Generalized	Across-Variable $\mathsf{V}_s(t)$	Through variable $\mathsf{F}_s(t)$
Mechanical translational	Velocity source $V_s(t)$	force source $F_s(t)$
Mechanical rotational	Angular velocity source $\Omega_s(t)$	Torque source $T_s(t)$
Electrical	Voltage source $V_s(t)$	current source $I_s(t)$
Fluid	Pressure source $P_s(t)$	Flow source $Q_s(t)$
Thermal	Temperature source $T_s(t)$	Heat flow source $Q_s(t)$

Table 5: Definition of ideal sources.

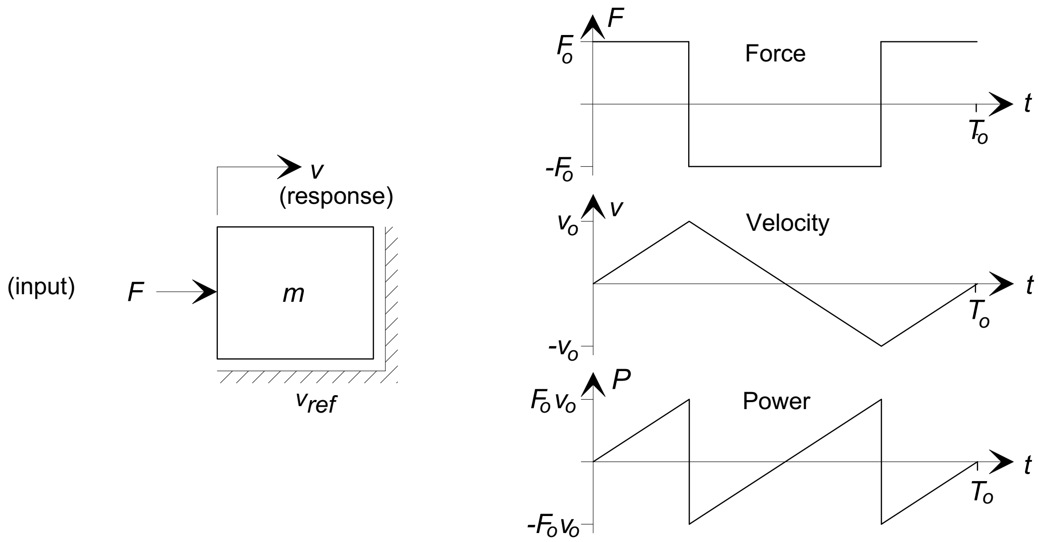


Figure 12: A mass element driven by a force source.

in Fig. 12.

The force source provides a square wave in force cycling between values of $+F_o$ and $-F_o$ with a total cycle time of T_o. In this example the velocity of the mass as a function of time and the power flow into the mass as a function of time are to be determined. As shown in Fig. 12, the mass velocity is defined as positive when the force is positive. The velocity of the mass m is determined from the elemental equation

$$F_s = m\frac{dv}{dt} \tag{28}$$

The problem solution may be found by solving Eq. (i) in each fraction of the total time. Over the time period $0 \le t < T_o/4$, the elemental equation may be expressed as:

$$F_o dt = m dv \tag{29}$$

and integrated to yield:

$$v = \frac{1}{m}F_o t \tag{30}$$

Over the period $T_o/4 \leq t < 3T_o/4$, the equation may also be integrated, noting that at time $t = T_o/4$ the mass has velocity v_o to yield:

$$(-F_o)dt = mdv \tag{31}$$

and integrated to yield:

$$v(t) = v_o - \frac{F_o}{m}(t - To/4) \tag{32}$$

Over the period of time $3T_o/4 \leq t < T_o$, the equation may be integrated, noting that at time $t = 3To/4$, the velocity is $-v_o$:

$$F_o dt = mdv \tag{33}$$

and integrated to yield:

$$v = -v_o + \frac{F_o}{m}(t - 3T_o/4) \tag{34}$$

Using the results from integration the elemental equation the velocity of the mass is plotted over one period of time T_o in Fig. 12. For subsequent periods of time the velocity may be determined in a similar fashion. The velocity curve is a sawtooth function, that is an alternating series of linear curves with positive and negative slopes with values of $\pm F_o/m$, the mass acceleration. The maximum and minimum velocities are:

$$v_o = \pm\frac{0.25F_oT_o}{m} \tag{35}$$

The power delivered to the mass is:

$$\mathcal{P} = F_s v \tag{36}$$

and may be determined by multiplying the force and velocity curves together as shown in Fig. 12. During the period 0 to $T_o/4$, the source provides positive power to the mass, accelerating it in the positive direction. In the period $T_o/4$ to $T_o/2$, the force source opposes the motion of the mass, absorbing power and decreasing its velocity, and then in the period $T_o/2$ to $3T_o/4$, the negative force results in a velocity which has increasingly negative values and again supplies power to the mass. During the period $3T_o/4$ to T_o, the force is in the positive direction and the mass velocity is negative, so that the source absorbs power from the mass.

Over a full cycle of period T_o, the integral of the power supplied by the source is zero; for half of the cycle the source supplies energy while for the other half it absorbs the kinetic energy stored in the mass.

---

4 Causality

Each of the primitive elements is defined by an elemental equation that relates its through and across-variables. This equation represents a *constraint* between the across-variable and the through-variable that must be satisfied at any instant. An immediate consequence is that the across-variable and the through-variable cannot both be independently specified at the same time. One variable must be considered to be defined by the system or an external input, the other variable is defined by the elemental equation. This is known as *causality.*

In the energy storage elements the constraint is expressed as a differential or integral relationship, that defines the element as having *integral* or *derivative* causality. For example, a mass element m has an elemental relationship that is normally written in the form

$$F = m\frac{dv}{dt}.$$

If a mass element is driven by an defined velocity $v(t)$ the required force F is determined by the above elemental equation; solution for the through variable $F(t)$ requires differentiation of the velocity v, and the element is said to be in *derivative causality.* On the other hand, if the element is driven by a specified force $F(t)$, its resulting velocity is determined by rewriting the elemental equation:

$$\frac{dv}{dt} = \frac{1}{m}F \qquad \text{or} \qquad v(t) = \frac{1}{m}\int_0^t F dt + v(0),$$

which is known as the *integral causality* form. In Example 3.1 the thermal capacitance of the satellite is in integral causality because the heat flow-rate is specified by the solar flux.

Dissipative elements always operate in algebraic causality because the through and across-variables are related by algebraic equations.

The concept of causality becomes important in developing models of systems of interconnected elements. When an element is part of an interconnected system its causality is determined by the system structure. It will be shown later that all *independent* energy storage elements in a system can be expressed in integral causality.

5 Linearization of Nonlinear Elements

In many physical systems the constitutive relations used to define model elements are inherently nonlinear. The analysis of systems containing such elements is a much more difficult task than that for a system containing only linear (ideal) elements, and for many such systems of interconnected nonlinear elements there may be no exact analysis technique. In engineering studies it is often convenient to approximate the behavior of nonlinear pure elements by equivalent linear elements that are valid over a limited range of operation.

In many practical situations an element operates at a nominal, non-zero, value of its through or across-variable and is subjected to small deviations about this equilibrium value. For example the springs in the suspension of an automobile may be inherently nonlinear over the full range of operation, but in normal use they are subjected to a nominal load force of the weight of the car, with "small" perturbation forces superimposed by the normal road conditions. We may, with care, use a linearized model of the spring that is valid over a limited range of operation. While any dynamic analyses based upon such models is at best an approximation to the behavior of the real system, for preliminary analyses such models frequently capture the dominant features of the overall system response.

Assume that a pure element is operating with an equilibrium value v_0 of its across-variable, or f_0 of its through-variable. For small deviations about these values a pair of *incremental* variables v^* and f^* may be defined

$$\mathsf{v}^* = \mathsf{v} - \mathsf{v}_0 \tag{37}$$
$$\mathsf{f}^* = \mathsf{f} - \mathsf{f}_0. \tag{38}$$

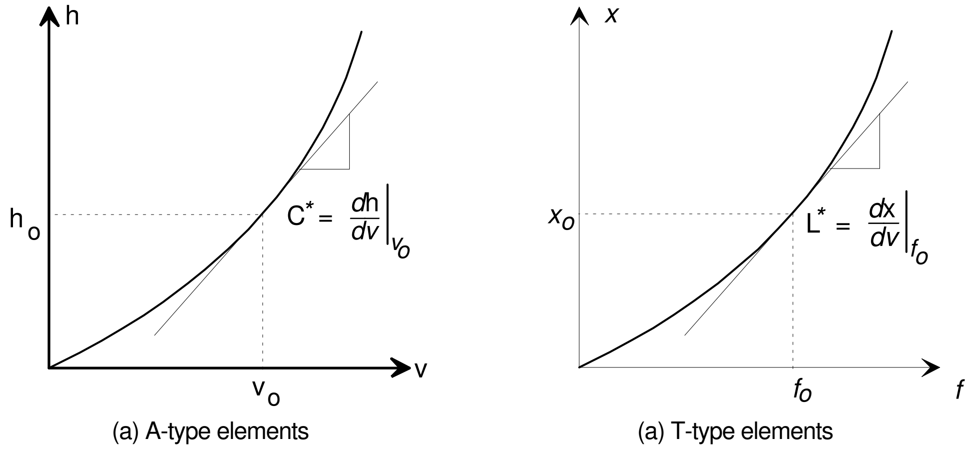


Figure 13: Linearization of constitutive relationships for A-type and T-type elements.

Similarly, if under equilibrium conditions one or both of the integrated through or across variables is constant with a value h_0 and x_0 respectively, incremental values may be defined as perturbations from the nominal values:

$$\mathsf{h}^* = \mathsf{h} - \mathsf{h}_0 \tag{39}$$

$$\mathsf{x}^* = \mathsf{x} - \mathsf{x}_0. \tag{40}$$

The linearized elemental behavior is defined in terms of these incremental variables.

5.1 A-Type Elements

The A-type element defined in Eq. (3) has a single-valued, monotonic relationship between the integrated through-variable and the across-variable, that is

$$\mathsf{h} = \mathcal{F}(\mathsf{v}). \tag{41}$$

Under equilibrium conditions both h and v are constant with values h_0 and v_0. When v is perturbed from equilibrium, the nonlinear function $\mathcal{F}(\mathsf{v})$ may be expressed as a Taylor series about v_0:

$$\begin{aligned} \mathsf{h} &= \mathcal{F}(\mathsf{v})\big|_{\mathsf{v}=\mathsf{v}_0} + \left.\frac{d\mathcal{F}(\mathsf{v})}{d\mathsf{v}}\right|_{\mathsf{v}=\mathsf{v}_0} (\mathsf{v}-\mathsf{v}_0) + \frac{1}{2!}\left.\frac{d^2\mathcal{F}(\mathsf{v})}{d\mathsf{v}^2}\right|_{\mathsf{v}=\mathsf{v}_0} (\mathsf{v}-\mathsf{v}_0)^2 + \cdots \\ &= \mathsf{h}_0 + \left.\frac{d\mathcal{F}(\mathsf{v})}{d\mathsf{v}}\right|_{\mathsf{v}=\mathsf{v}_0} \mathsf{v}^* + \frac{1}{2!}\left.\frac{d^2\mathcal{F}(\mathsf{v})}{d\mathsf{v}^2}\right|_{\mathsf{v}=\mathsf{v}_0} \mathsf{v}^{*2} + \cdots. \end{aligned} \tag{42}$$

For small changes in v, v^* is small, and higher order terms in the series may be neglected. If second and higher terms may be neglected, only the first two terms of the series are retained and an approximate linear relationship results:

$$\mathsf{h} - \mathsf{h}_0 \approx \left.\frac{d\mathcal{F}(\mathsf{v})}{d\mathsf{v}}\right|_{\mathsf{v}=\mathsf{v}_0} \mathsf{v}^*, \tag{43}$$

or

$$\mathsf{h}^* = \mathsf{C}^* v^* \tag{44}$$

where

$$\mathsf{C}^* = \left. \frac{d\mathcal{F}(\mathsf{v})}{d\mathsf{v}} \right|_{\mathsf{v}=\mathsf{v}_0}. \tag{45}$$

Equation (45) is a constitutive relationship for an ideal A-type element with capacitance C^* and represents the elemental behavior of the nonlinear element in the region of the equilibrium point. The linearized generalized capacitance C^* is the slope of the constitutive characteristic at the operating point, as shown in Fig. 13a. This linear approximation is used to define the elemental equation of an equivalent linear A-type element in the region of the equilibrium point by differentiation

$$\mathsf{f}^* = \frac{d\mathsf{h}^*}{dt} \approx \mathsf{C}^* \frac{d\mathsf{v}^*}{dt}. \tag{46}$$

The linearized elemental equation may be used as an approximation to the behavior of the nonlinear element.

■ Example

A conical tank with angle 60^{o} at the base drains through an orifice to the atmosphere. In normal operation the tank contains a fluid volume V_0. Find an expression for a linearized fluid capacitance that may be used to represent the tank for small deviations about its nominal operating point.

Solution: Consider an elemental disk of fluid of width dh at a height h above the base.

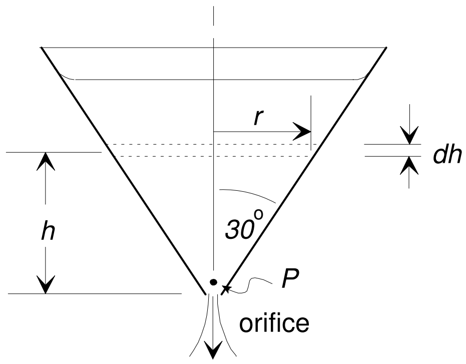


Figure 14: A nonlinear fluid system and its linear graph.

The radius of the disk is $r = h\tan(\pi/6) = h/\sqrt{3}$. Its volume dV is:

$$dV = \pi r^2 dh = \frac{\pi}{3} h^2 dh. \tag{47}$$

If the tank is filled to height h, the total volume of fluid V stored is:

$$V = \int_0^h \frac{\pi}{3} h^2 dh = \frac{\pi}{9} h^3 \tag{48}$$

and the pressure at the outlet is $P = \rho g h$, where ρ is the density of the fluid and g is the acceleration due to gravity. Then

$$P = \left(\frac{9}{\pi}\right)^{\frac{1}{3}} \rho g V^{\frac{1}{3}} \tag{49}$$

or

$$V = \frac{\pi}{9\left(\rho g\right)^3} P^3. \tag{50}$$

which is the constitutive relationship of a *pure* but *non-ideal* A-type fluid element.

At the operating point $V = V_0$, and the corresponding pressure at the base of the tank is P_0, which may be found directly from Eq. (iii). The equivalent linear fluid capacitance C^* is found by differentiating Eq. (iv):

$$\begin{aligned} C^* &= \left.\frac{dV}{dp}\right|_{P=P_0} = 3\frac{\pi}{9\left(\rho g\right)^3} P_0^2 \\ &= \frac{3}{\rho g}\left(\frac{\pi}{9}\right)^{\frac{1}{3}} V_0^{\frac{2}{3}}. \end{aligned} \tag{51}$$

The equivalent linear elemental equation is:

$$Q^* = C^* \frac{dP^*}{dt}. \tag{52}$$

5.2 T-Type Elements

Nonlinear pure T-type elements may be linearized in a similar manner. Equation (16) defines a T-type element as a single-valued, monotonic relationship between the integrated across-variable and the through-variable:

$$\mathsf{x} = \mathcal{F}\left(\mathsf{f}\right). \tag{53}$$

If there is a nominal operating point defined by x_0 and f_0 the nonlinear constitutive relationship may be expressed as a Taylor series about that equilibrium point

$$\mathsf{x} = \mathsf{x}_0 + \left.\frac{d\mathcal{F}\left(\mathsf{f}\right)}{d\mathsf{f}}\right|_{\mathsf{f}=\mathsf{f}_0} \mathsf{f}^* + \frac{1}{2!}\left.\frac{d^2\mathcal{F}\left(\mathsf{f}\right)}{d\mathsf{f}^2}\right|_{\mathsf{f}=\mathsf{f}_0} \mathsf{f}^{*2} + \cdots \tag{54}$$

the first two terms may be used to define an approximate linear relationship:

$$\mathsf{x}^* = \mathsf{x} - \mathsf{x}_0 \approx \left.\frac{d\mathcal{F}\left(\mathsf{f}\right)}{d\mathsf{f}}\right|_{\mathsf{f}=\mathsf{f}_0} \mathsf{f}^*. \tag{55}$$

An elemental relationship may be found by differentiating both sides:

$$\mathsf{v}^* \approx \left.\frac{d\mathcal{F}\left(\mathsf{f}\right)}{d\mathsf{f}}\right|_{\mathsf{f}=\mathsf{f}_0} \frac{d\mathsf{f}^*}{dt} = \mathsf{L}^* \frac{d\mathsf{f}^*}{dt} \tag{56}$$

where

$$\mathsf{L}^* = \left.\frac{d\mathcal{F}\left(\mathsf{f}\right)}{d\mathsf{f}}\right|_{\mathsf{f}=\mathsf{f}_0} \tag{57}$$

is a linearized generalized inductance representing the elemental behavior of the pure element at the equilibrium point. Figure 13b shows the linearizing approximation of the constitutive relationship at the operating point.

■ Example

The measured force-extension characteristic of a spring has been found to closely approximate $F = 0.125 \times 10^6 x^3$. In its normal operating mode the spring is subjected to a static load F_0 with a small sinusoidal force superimposed. Find the equivalent linearized stiffness of the spring.

Solution: The stiffness of a spring is the reciprocal of the generalized inductance. The constitutive relation may be rewritten

$$x = 2 \times 10^{-2} F^{\frac{1}{3}}.$$

Then

$$\begin{aligned} \frac{1}{K^*} &= \left.\frac{dx}{dF}\right|_{F=F_0} && (58) \\ &= \frac{2}{3} \times 10^{-2} F_0^{-\frac{2}{3}} && (59) \end{aligned}$$

or

$$K^* = 1.5 \times 10^2 F_0^{\frac{2}{3}}. \qquad (60)$$

5.3 D-Type Elements

D-type elements are characterized by an algebraic relationship between the the across and through-variables:

$$\mathsf{v} = \mathcal{F}(\mathsf{f}) \qquad (61)$$

The nonlinear function may be expanded as a Taylor series and the linear terms retained to form an approximation to the elemental behavior

$$\mathsf{v}^* = \mathsf{v} - \mathsf{v}_0 \approx \left.\frac{d\mathcal{F}(\mathsf{f})}{d\mathsf{x}}\right|_{\mathsf{f}=\mathsf{f}_0} \mathsf{f}^*. \qquad (62)$$

Then

$$\mathsf{v}^* \approx R^* \mathsf{f}^*. \qquad (63)$$

where

$$R^* = \left.\frac{d\mathcal{F}(\mathsf{f})}{d\mathsf{v}}\right|_{\mathsf{f}=\mathsf{f}_0} \qquad (64)$$

is a linearized resistance. An expression for a linearized conductance G^* may be developed similarly.

The linearization of lumped elements is summarized in Table 5.3

■ Example

A set of measurements made on a test vehicle traveling along a straight road showed that the aerodynamic drag force is approximately described by a quadratic relationship

$$F_d = c_0 v^2. \qquad (65)$$

Element	Constitutive	Linearized Elemental Equations	Elemental Value
A-Type	$\mathsf{h} = \mathcal{F}(\mathsf{v})$	$\mathsf{f}^* = \mathsf{C}^* \dfrac{d\mathsf{v}^*}{dt}$	$\mathsf{C}^* = \left. \dfrac{d\mathcal{F}(\mathsf{v})}{d\mathsf{v}} \right\|_{\mathsf{v}=\mathsf{v}_0}$
T-Type	$\mathsf{x} = \mathcal{F}(\mathsf{f})$	$\mathsf{v}^* = \mathsf{L}^* \dfrac{d\mathsf{f}^*}{dt}$	$\mathsf{L}^* = \left. \dfrac{d\mathcal{F}(\mathsf{f})}{d\mathsf{f}} \right\|_{\mathsf{f}=\mathsf{f}_0}$
D-Type	$\mathsf{v} = \mathcal{F}(\mathsf{f})$	$\mathsf{v}^* = \mathsf{R}^* \mathsf{f}^*$	$\mathsf{R}^* = \left. \dfrac{d\mathcal{F}(\mathsf{f})}{d\mathsf{f}} \right\|_{\mathsf{f}=\mathsf{f}_0}$

Table 6: Summary of linearized lumped parameter elements

where c_0 is an overall drag coefficient and v is the velocity. In its normal operation the vehicle is known to travel at a nominal speed v_0 but is subjected to small variations in this speed. Find a linearized D-type element that approximates the behavior of the drag force for vehicle speeds that are close to v_0.

Solution: The aerodynamic drag is a *pure* dissipative element, which may be expressed as an equivalent nonlinear damper

$$F_d = \mathcal{F}(v) = c_0 v^2. \tag{66}$$

The linearized representation of this damper is

$$F_d^* = B v^* \tag{67}$$

where $v^* = v - v_0$, and $F_d^* = F_d - F_0$ represent excursions from the nominal operating point. The value of the equivalent linear damper coefficient B^* is

$$B^* = \left. \frac{dF_d}{dv} \right|_{v=v_0} \tag{68}$$

$$= 2c_0 v_0. \tag{69}$$

The value of the linearized damper coefficient B^* is directly proportional to the equilibrium velocity and at high velocities is relatively large while at low velocities it is relatively small.

The value of the drag force computed by Eq. (iii) is the *excursion* from the nominal operating value, and the total drag force acting on the vehicle is given by:

$$\begin{aligned} F_d &\approx F_0 + B^* v^* \\ &\approx c_0 v_0^2 + b^* (v - v_0) \end{aligned} \tag{70}$$

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6 Introduction to Linear Graph Models

Graphical techniques are widely used to aid in the formulation and representation of models of dynamic systems. Linear graphs represent the topological relationships of lumped element inter-

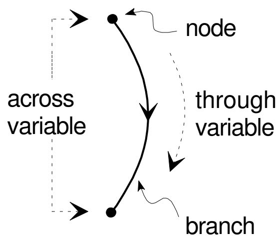

Figure 15: Linear graph representation of a single passive element as a directed line segment.

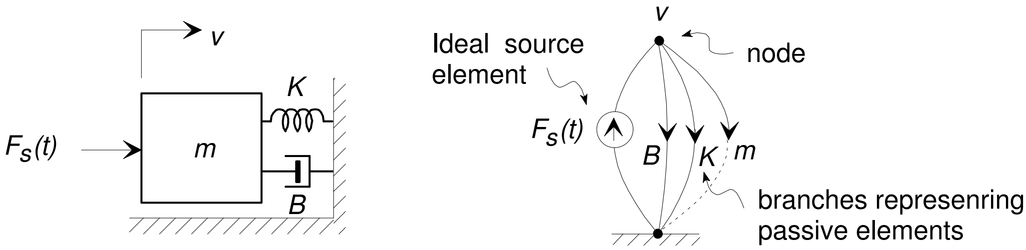

Figure 16: Linear graph representation of a simple mechanical system.

connections within a system. The term *linear* in this context denotes a graphical lineal (or line) segment representation as shown in Figure 15, and is not related to the concept of mathematical linearity. Linear graphs are used to represent system structure in many energy domains, and are a unified method of representing systems that involve more that one energy medium. They are similar in form to electrical circuit diagrams. A graph is constructed from:

1. A set of *branches* that each represent an energy port associated with a passive or source system element. Each branch is drawn as an oriented line segment.

2. A set of *nodes* (designated by dots) that represent the points of interconnection of the lumped elements. All graph branches terminate at nodes. The nodes define points in the system where distinct across-variable values may be measured (with respect to a reference node), for example points with distinct velocities in a mechanical system or points in an electrical system that have distinct voltages.

A typical complete linear graph, representing a simple mechanical system with a single source and three one-port elements, is shown in Fig. 16. In this case there are three nodes, representing points in the system at which distinct velocities may be measured. In practice it is common, but not necessary, to designate one of the nodes as a *reference node*, and to draw this node as a horizontal line (sometimes cross-hatched) as shown. In mechanical systems the reference node is usually selected to be the velocity of the inertial reference frame, while in electrical systems it commonly represents the system "ground" or zero-voltage point. In fluid systems the reference node designates the reference pressure (often atmospheric pressure) from which all system pressures are measured. Apart from this special interpretation the reference node behaves identically to all other nodes in the graph.

In a linear graph one-port elements are represented in a two-terminal form. Each element generates a branch in the graph and is drawn as a line segment between the two appropriate nodes. Associated with each branch is an elemental through-variable, assumed to pass through the line segment, and an elemental across-variable which is the difference between the across-variable values at the two nodes. Each linear graph branch thus represents the functional relationship between its

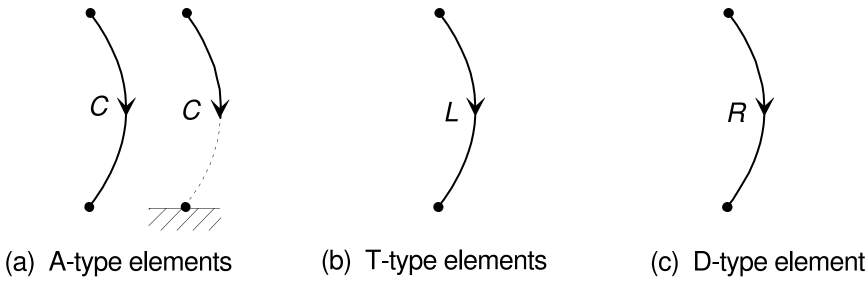

Figure 17: Linear graph representation of generalized one-port passive elements.

across and through-variables as defined by the elemental equation. Linear graph segments may be used to represent pure or ideal elements.

7 Linear Graph Representation of One-Port Elements

Graph branches that represent one-port elements are drawn as *oriented* line segments, with an arrow that designates a sign convention adopted for the through and across-variables. Figure 17 shows branches for the generalized passive energy storage and dissipation elements. Each branch is labeled with the generalized element type, and the across and through-variables in the branch are related by the elemental equation for the element. For the three generalized ideal (linear) elements the relationships are:

- For a generalized ideal A-type element (capacitance) C:

$$\frac{d\mathsf{v}}{dt} = \frac{1}{\mathsf{C}}\mathsf{f}. \tag{71}$$

- For a generalized ideal T-type element (inductance) L:

$$\frac{d\mathsf{f}}{dt} = \frac{1}{\mathsf{L}}\mathsf{v}. \tag{72}$$

- For a generalized ideal D-type element (resistance) R:

$$\mathsf{v} = \mathsf{R}\mathsf{f}, \qquad \text{or} \qquad \mathsf{f} = \frac{1}{\mathsf{R}}\mathsf{v} \tag{73}$$

where for energy storage elements the equations are expressed with the derivative on the left-hand side.

As described previously, A-type elements (with the exception of electrical capacitors) must have their across-variable defined with respect to a constant reference value. For example, the velocity difference on a mass element is defined with respect to a constant velocity inertial reference frame. The branches representing these A-type elements therefore *must have one end connected to the reference node*. Some authors use a dotted line to show this implicit connection to ground, as shown in Fig. 17. Apart from this notational difference, A-type branches are treated identically to all other branches.

Each branch contains an arrow that designates the sign convention associated with the across and through-variables. The arrow on the graph element is drawn in the direction for which:

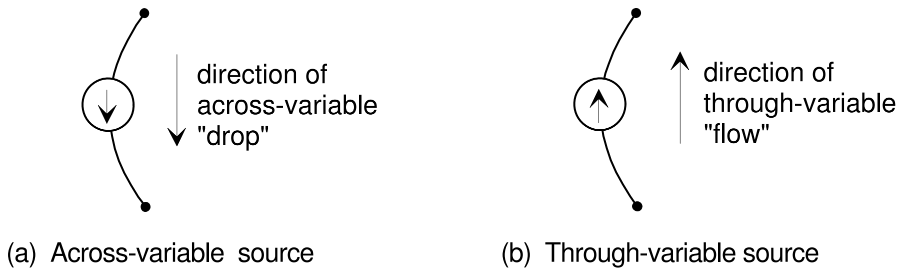

Figure 18: Linear graph representation of ideal source elements.

- v, the across-variable associated with the branch is defined to be *decreasing*, that is in the direction of the assumed across-variable "drop"
- the through-variable f, is defined as having a positive value.

With this convention, when the elemental across and through-variables have the same direction (or sign) power $\mathcal{P} = \mathsf{fv}$ is positive and flows into the element.

The choice of arrow direction on passive branches simply establishes a convention to define positive and negative values of the through and across-variables and is arbitrary. The arrow direction does not affect the equation formulation procedures, or any subsequent system analyses; the effect of reversing an arrow direction is simply to reverse the sign of the defined across and through variable on the element. The choice of sign convention is discussed more fully later.

Ideal source elements are represented by linear graph segments containing a circle as shown in Figure 18. In all source elements one variable, either the across or through-variable, is a prescribed independent function of time. For source elements the arrow associated with the branch designates the sign associated with the source variable:

1. For a through-variable source the arrow designates the direction defined for positive through-variable flow, and
2. For an across-variable source the arrow designates the direction defined for the across variable *drop*.

The arrow on an across-variable source branch is commonly drawn toward the reference node, since that is usually the direction of the assumed drop in across-variable value.

8 Element Interconnection Laws

Linear graphs represent the structure of a system model and specify the manner in which elements are connected. The general interconnection laws for linear graph elements are derived in this section, with one set of laws relating across-variables, and a second set relating through-variables, following the developments of several authors [1-3].

8.1 Compatibility

The *compatibility* law represents a set of constraints on across-variables in the graph that may be related to physical laws that govern the interconnection of lumped elements. It may be stated:

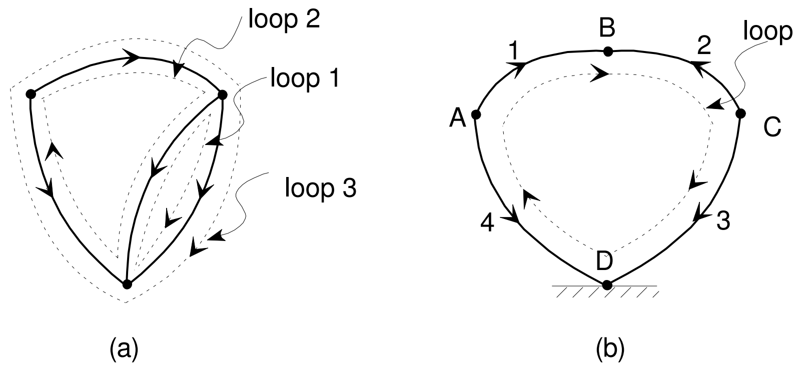

Figure 19: Compatibility equations defined from loops in a linear graph, (a) some possible loops in a graph, and (b) a loop containing four nodes and four branches.

> The sum of the across-variable *drops* on the branches around any closed loop in a linear graph is identically zero, or:
> $$\sum_{i=1}^{N} \mathsf{v}_i = 0 \tag{74}$$
> for any N elements forming a closed loop in the graph.

A compatibility equation may be written for any closed loop in a graph, including inner loops or outer loops, as shown in Fig. 19. Because the arrows on the branches indicate the direction of the across-variable drop, they are used to assign the sign to terms in the summation; if the loop traverses a branch in the direction of an arrow the term in the summation is positive, while if a branch is traversed against an arrow the term in the sum is assigned a negative value.

Figure 19b shows a single loop with four branches and four nodes. With the arrow directions as shown the compatibility equation for this loop is

$$\sum_{i=1}^{4} \mathsf{v}_i = \mathsf{v}_1 - \mathsf{v}_2 + \mathsf{v}_3 - \mathsf{v}_4. \tag{75}$$

We can demonstrate the compatibility law using the loop in Fig. 19b. The across-variable drop on an element is the difference between the value of the across-variable at the two nodes to which it is connected, for example $\mathsf{v}_1 = \mathsf{v}_A - \mathsf{v}_B$ is the drop associated with element 1. If all of the nodal values are substituted into Eq. (75), then

$$\sum_{i=1}^{4} \mathsf{v}_i = (\mathsf{v}_A - \mathsf{v}_B) - (\mathsf{v}_C - \mathsf{v}_B) + (\mathsf{v}_C - \mathsf{v}_D) - (\mathsf{v}_A - \mathsf{v}_D) = 0. \tag{76}$$

The physical interpretation of the compatibility law in the various energy domains is:

Mechanical systems: The velocity drops across all elements sum to zero around any closed path in a linear graph. Compatibility in mechanical systems is a geometric constraint which ensures that all elements remain in contact as they move.

Electrical systems: The compatibility law is identical to Kirchoff's voltage law which states that the summation of all voltage drops around any closed loop in an electrical circuit is identically zero.

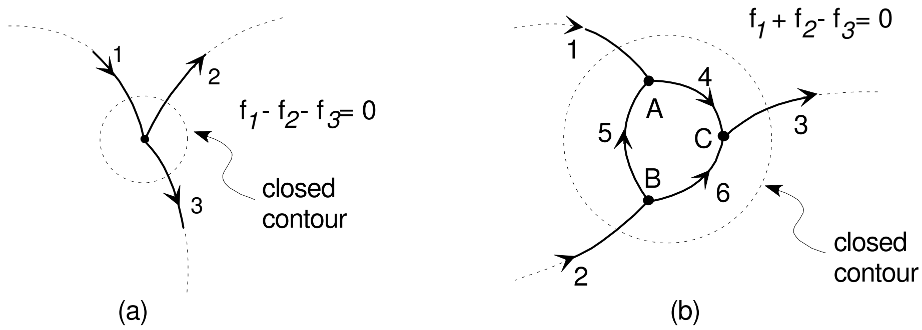

Figure 20: The definition of continuity conditions at (a) a single node in a linear graph, and (b) the extended principle of continuity applied to any closed contour on a graph.

Fluid systems: Pressure is a scalar potential which must sum to zero around any closed path in a fluid system.

Thermal systems: Temperature is a scalar potential which must sum to zero around any closed path in a thermal system.

8.2 Continuity

The *continuity* law specifies constraints on the through-variables in a linear graph that may be related to physical laws governing the interconnection of elements. It may be stated as follows:

> The sum of through-variables flowing *into* any closed contour drawn on a linear graph is zero, that is
>
> $$\sum_{i=1}^{N} \mathsf{f}_i = 0 \tag{77}$$
>
> for any N branches that intersect a closed contour on the graph.

Continuity is applied by drawing a closed contour on the linear graph and summing the through-variables of branches that intersect the contour, as shown in Fig. 20. The arrow direction on each branch is used to designate the sign of each term in the summation.

For the special case in which a contour is drawn around a single node, the continuity law states that the sum of through-variables flowing *into* any node in a linear graph is identically zero. The law of continuity at a single node is illustrated in Fig. 20a. In this case $\mathsf{f}_1 - \mathsf{f}_2 - \mathsf{f}_3 = 0$. The extended principle of continuity for a general contour may be demonstrated by considering the example containing three nodes shown in Fig. 20b. The continuity conditions at the three nodes are

$$\mathsf{f}_1 - \mathsf{f}_4 + \mathsf{f}_5 = 0 \qquad \text{at node A} \tag{78}$$

$$\mathsf{f}_2 - \mathsf{f}_5 - \mathsf{f}_6 = 0 \qquad \text{at node B} \tag{79}$$

$$-\mathsf{f}_3 - \mathsf{f}_4 + \mathsf{f}_6 = 0 \qquad \text{at node C.} \tag{80}$$

For the contour enclosing all three nodes, the sum of through variables into the contour is

$$\mathsf{f}_1 + \mathsf{f}_2 - \mathsf{f}_3 = (\mathsf{f}_4 - \mathsf{f}_5) + (\mathsf{f}_5 + \mathsf{f}_6) - (\mathsf{f}_4 + \mathsf{f}_6) = 0. \tag{81}$$

The principle of continuity applied to any node states that there can be no accumulation of the through-variable at that node. If the principle did not hold, it would imply that that the integrated

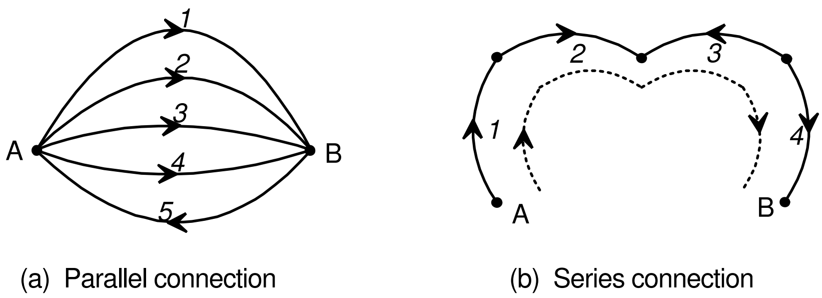

Figure 21: System elements connected in parallel and series.

through-variable was non-zero at the node, and the node would either store or dissipate energy, thus acting as one of the primitive elements described in previously.

In each of the energy domains, the principle of continuity corresponds to the following physical constraints:

Mechanical systems: In a translational (or rotational) mechanical system continuity at a node arises as a direct expression of Newton's laws of motion, which require that the sum of forces (or torques) acting at any massless point must be identically zero.

Electrical systems: The principle of continuity at an electrical node is Kirchoff's current law, which states that the sum of currents flowing into any node (junction) in a circuit must be identically zero.

Fluid systems: A node represents a junction of elements in a fluid system. The continuity principle requires that the sum of volume flow rates into the junction must be zero; if this was not true then fluid would accumulate at the junction.

Thermal systems: In a thermal system the continuity of heat flow rate ensures that there is no accumulation of heat at any junction between elements.

8.3 Series and Parallel Connection of Elements

Figure 21 shows two possible connections of elements within a linear graph. In Fig. 21a several elements are connected in *parallel*, that is they are connected between a common pair of nodes. Compatibility equations may be written for the loop containing any pair of branches to show $\mathsf{v}_1 = \mathsf{v}_2 = \mathsf{v}_3 = \mathsf{v}_4 = -\mathsf{v}_5$. Similarly the continuity condition applied to node B shows that $\mathsf{f}_1 + \mathsf{f}_2 + \mathsf{f}_3 + \mathsf{f}_4 - \mathsf{f}_5 = 0$. In general elements connected in parallel share a common across-variable, and the through-variable divides among the elements at the two nodes.

Figure 21b shows four elements connected in *series*. In this configuration, with the arrows as indicated, the continuity condition may be applied to each of the internal nodes to show that $\mathsf{f}_1 = \mathsf{f}_2 = -\mathsf{f}_3 = \mathsf{f}_4$. If this series chain of elements is part of a loop, the compatibility condition requires that the across-variable drop across the chain is the sum of the individual drops of the branches, that is $\mathsf{v}_{AB} = \mathsf{v}_1 + \mathsf{v}_2 - \mathsf{v}_3 + \mathsf{v}_4$. Elements that are connected in series share a common through-variable.

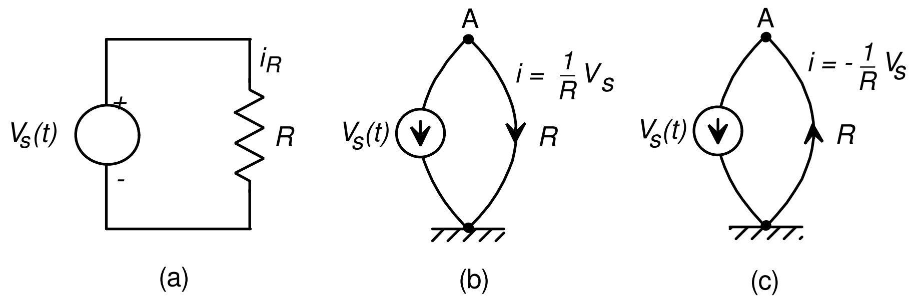

Figure 22: Illustration of passive element sign conventions using a simple electrical model.

9 Sign Conventions on One-Port System Elements

A simple electrical system consisting of a battery and resistor is shown in Fig. 22. The battery is modeled as an across-variable (voltage) source and the resistor is modeled as an ideal D-type element. The positive (+) and negative (-) battery terminals are indicated. This simple system has only two nodes; the voltage reference node, arbitrarily chosen as the battery's negative terminal, and a node corresponding to the battery's positive terminal, which is the only other distinct voltage in the system. Branches corresponding to the source and resistive elements are connected in parallel between these nodes. The sign convention for the source requires that the arrow point in the direction of the assumed *voltage drop*. We have assumed that positive voltage corresponds to a positive across-variable value, and therefore the arrow must point downward, that is from node A toward the reference node as shown. The sign convention for the resistor may be arbitrarily assigned, and in the figure the two possibilities are shown. In Figure 22b the arrow is aligned in the direction of the assumed voltage drop, that is directed toward the reference node. In this case the compatibility equation from the graph is

$$-V_s + v_R = 0, \tag{82}$$

which together with the D-type elemental equation for the resistor $v_R = Ri_R$ gives an expression for the current in the resistor:

$$i_R = \frac{1}{R} V_s. \tag{83}$$

In Figure 22c the same system graph is redrawn with the arrow reversed on the resistor branch. The compatibility equation then becomes

$$V_s + v_R = 0, \tag{84}$$

and the current through the resistor is therefore

$$i_R = -\frac{1}{R} V_s. \tag{85}$$

which is opposite in sign to the first case. The direction defined as positive current flow is opposite in the two systems. A positive value of a computed through-variable implies that the "flow" is in the direction of the arrow, a positive across-variable means that the "drop" is in the direction of the arrow. In this example the negative result implies that the direction of the current flow is opposite to that of the arrow. The results of both models are physically equivalent. The power flow into the resistor is positive regardless of the arrow direction.

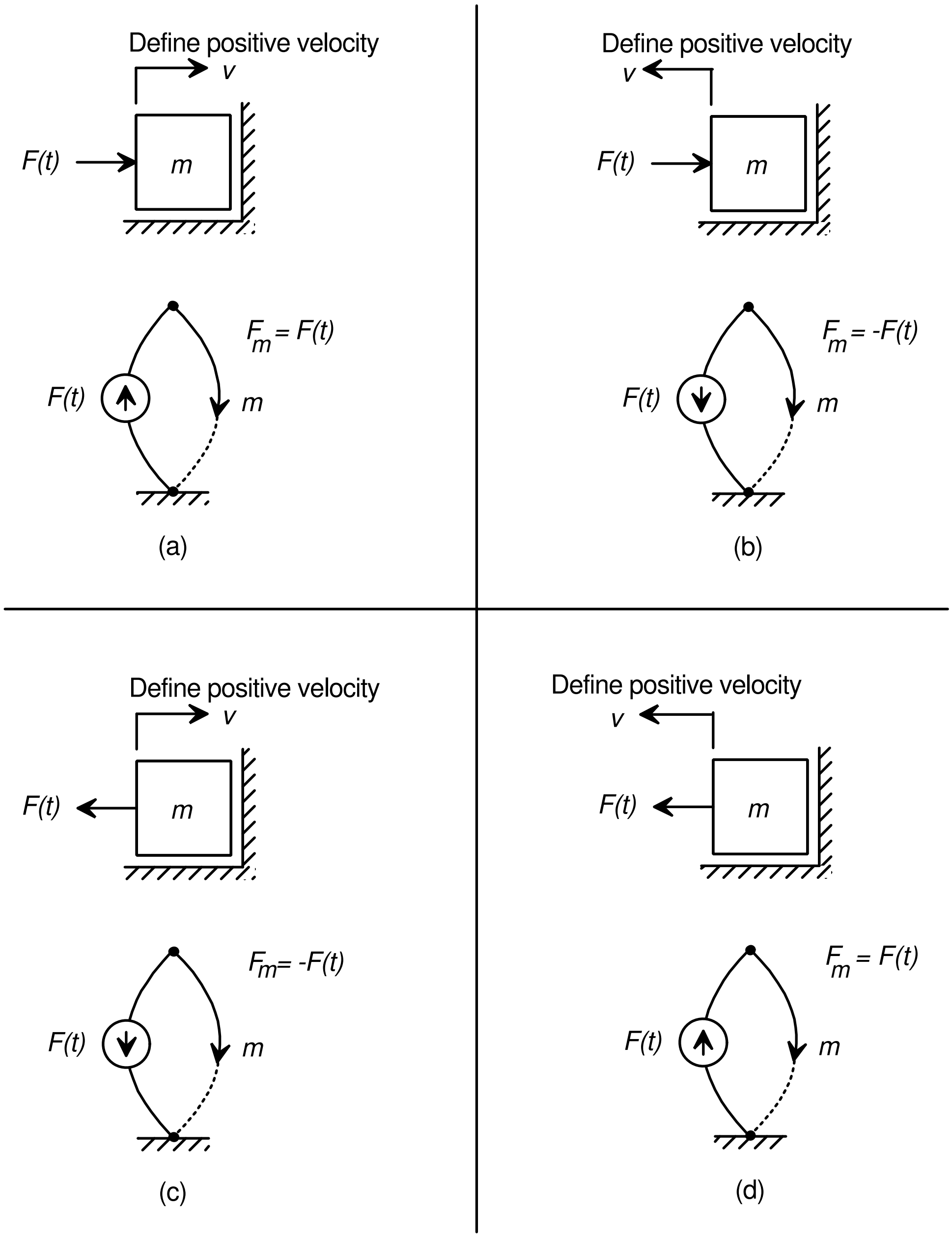

Figure 23: Possible force and velocity orientations for a simple translational mass.

Figure 23 shows a simple mechanical system consisting of a mass resting on a frictionless plane and moving under the influence of an external prescribed force source. Four possible assumed positive force and velocity conditions are shown together with the corresponding linear graphs. In each case the upper node represents the velocity of the mass in the defined direction. An increase in the value of the across-variable indicates an increase in velocity in that direction. The sign convention assigned to the force source defines whether a positive force increases or decreases the velocity of the mass. In cases (a) and (d) the force and velocity directions are aligned and a positive force accelerates the mass in the direction of the applied force.

In practice it is often convenient to adopt a convention directing all arrows on passive elements away from sources and toward the reference node, and then to assign a source convention that is compatible with the convention defined in the physical system.

10 Linear Graph Models of Systems of One-Port Elements

The representation of a physical system as a set of interconnected one-port linear graph elements is a *system graph.* The construction of a system graph usually requires a number of modeling decisions and engineering judgments. The general procedure may be summarized by the following steps:

1. Define the system boundary and analyze the physical system to determine the essential features that must be included in the model, including the system inputs, the outputs of interest, the energy domains involved, and the required elements.

2. Form a schematic, or pictorial, model of the physical system and establish a sign convention for the variables in the physical system.

3. Determine the necessary lumped parameter elements which represent the system sources, energy storage and dissipation elements.

4. Identify the across-variables that define the linear graph nodes, and draw a set of nodes.

5. Determine the appropriate nodes for each lumped element, and insert each element into the graph.

6. Select a set of sign conventions for the passive elements and draw the arrows on the graph.

7. Select the sign conventions for the system source elements to be consistent with the physical model and enter them in the graph.

The formulation of the model in steps 1–3 is perhaps the most difficult part of the modeling process, for it requires a detailed knowledge of the system configuration and the physics of the energy domain involved. Usually engineering approximations and assumptions are required in the model formulation. Care must be taken to include all of the essential elements so as to capture the required dynamic behavior of the physical system while not making the model overly complex. Whenever practicable, model responses should be verified against measurements made on the physical system, and the model modified if necessary to ensure fidelity of the response.

In the remainder of this section we develop modeling procedures to derive linear graphs in the five energy domains.

10.1 Mechanical Translational System Models

Translational system models utilize mass (A-type), spring (T-type), and damper (D-type) one-port passive elements, together with velocity (across-variable) and force (through-variable) ideal source elements. The graph nodes represent points of distinct velocity with respect to an inertial reference frame. All A-type (mass) elements in a mechanical system must be connected to the inertial reference node.

■ Example

A mass m supported on a cantilever beam and subjected to a prescribed force $F_s(t)$ is shown in Fig. 24a. In the figure positive velocity is defined as downward and is aligned

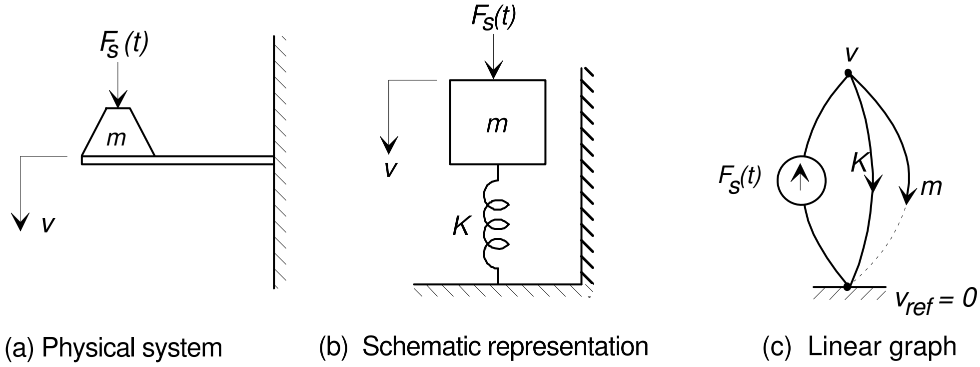

Figure 24: A mechanical system consisting of a mass element on a cantilevered beam

in the direction of the positive force. It is assumed that the displacement of the mass is small, so that the system may be represented as a translational system in which all velocities are in the vertical direction. The schematic model, shown in Fig. 24b may be represented with the following elements:

1. A force source $F_s(t)$ to represent the system input.
2. A mass element m to represent the mass.
3. The beam is assumed to be massless, and is represented by a spring element K that models the effective force-displacement characteristic of the end point.

There are only two nodes required in this example (the reference node, and a node representing the velocity of the mass). The elements are inserted in the graph by noting:

1. The velocity of the mass must be referenced to the fixed reference node.
2. The force source $F_s(t)$ acts on the mass, and acts with respect to the same fixed reference node.
3. One end of the spring moves with the velocity of the mass, the other end is connected to the zero velocity reference node.

The sign orientation of the force source $F_s(t)$ is chosen so that a positive force yields a positive mass velocity, as shown in the pictorial representation. The completed linear graph is shown in Fig. 24c.

The system graph indicates that all three branches are connected in parallel, with a compatibility condition indicating that:

$$v_m = v_K, \tag{86}$$

and at either node a continuity equation may be written to show

$$F_s - F_K - F_m = 0. \tag{87}$$

As with any parallel connection, the across-variable (velocity) of the mass and spring are identical, and the applied through-variable (force $F_s(t)$) divides between the mass and the spring.

10.2 Mechanical Rotational Systems

The construction of a linear graph model for a mechanical rotational system is similar to that for translational systems. Nodes on the graph represent points of distinct angular velocity, with respect to an inertial reference angular velocity, and the passive elements are rotary inertias (A-type), torsional springs (T-type), and rotary dampers (D-type). The across-variable source is an angular velocity source, and the through-variable source is a torque source. As in the case of the translational systems all A-type (inertia) elements are referenced to the inertial reference frame.

■ Example

A power transmission driving a large flywheel is shown in Figure 25a. The flywheel is

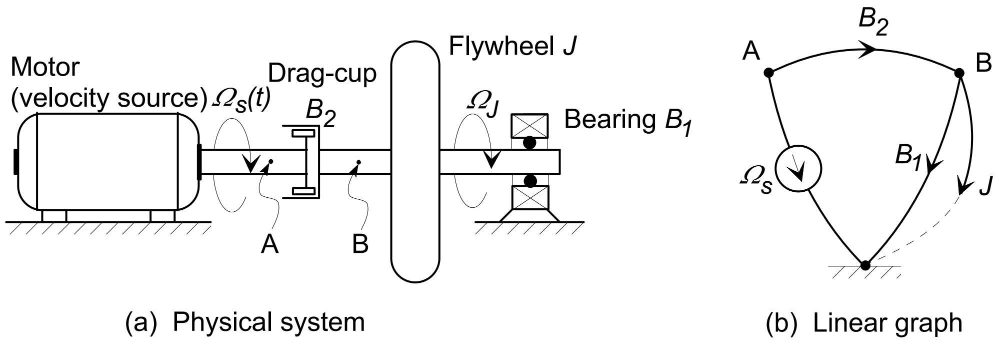

Figure 25: A rotational system consisting of flywheel driven through a drag-cup.

supported on bearings and is driven through a frictional drag-cup transmission by a motor that acts as an angular velocity source $\Omega_s(t)$. Clockwise angular rotations are defined as positive. The following elements are used to represent the system:

1. The system input from the motor is modeled as an angular velocity source $\Omega_s(t)$.
2. The flywheel is modeled as a rotary inertia J.
3. The shaft bearings are modeled as a rotary damper B_1 to account for energy dissipation due to friction as the shaft rotates.

4. The drag-cup transmission is modeled as a rotary damper B_2 connecting the motor to the flywheel.

It is assumed that the shafts are rigid and massless, so that they do not deflect and do not add significant rotational inertia to the system.

The schematic diagram shows that there are two distinct angular velocities with respect to the reference, labeled as points A and B in Fig. 25a, and therefore three nodes are necessary in the linear graph. The reference node is defined to be stationary, that is $\Omega_{ref} = 0$. The elements may be inserted into the graph by noting that:

1. The angular velocity Ω_J of the flywheel must be defined relative to the fixed reference node.
2. The inner bearing race rotates at the same angular velocity as the flywheel and the housing is fixed, thus the damper B_1 is inserted in parallel with the flywheel.
3. For the transmission drag-cup element B_2, one end rotates at the angular velocity of the input shaft, Ω_A, and the other end rotates at the angular velocity of the flywheel, $\Omega_B = \Omega_J$. It is therefore inserted between the nodes A and B.
4. The source angular velocity $\Omega_s(t)$ is defined with respect to the reference node.

The sign of the angular velocity source is selected to provide a positive angular velocity to the damper requiring the arrow to point toward the reference node. The completed linear graph is shown in Figure 25b.

10.3 Linear Graph Models of Electrical Systems

Electrical system models consist of capacitors (A-type), inductors (T-type), and resistors (D-type) as passive elements, and voltage (across-variable) and current (through-variable) ideal sources. Electrical circuits are usually easily translated to linear graphs because the topology of the linear graph is similar to the circuit diagram. The wires and connections between components in the circuit diagram are implicitly the nodes on the graph because they represent points of defined voltage. The following example illustrates the conversion of an electrical circuit to a linear graph form.

■ Example

Figure 26 shows an electrical filter designed to minimize the transmission of high frequency electrical noise from an alternator to sensitive electronic equipment. The linear graph is generated by the following steps:

1. The alternator is represented by an ideal voltage source, and the electrical noise is modeled as variations of the voltage about its nominal value.
2. The electronic instrument is modeled as a resistive load R_L. The value of the resistance is determined from the manufacturer's specification of the nominal operating voltage and current for the instrument.
3. The circuit diagram shown in Fig. 26b is used to generate the system model.

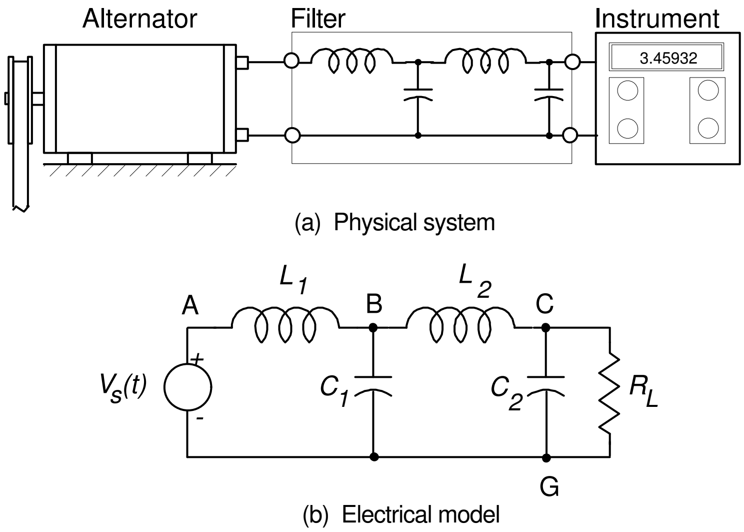

Figure 26: An electrical filter shown as (a) the physical system, and (b) an electrical equivalent modeling the source and the load.

4. The passive electrical elements, the two coils and two capacitors are each represented by single lumped elements.
5. The circuit diagram is labeled with four nodes, the reference ground node G and three others, labeled A, B, and C in Fig. 26b. Each node represents a point in the circuit where a distinct voltage could be measured.
6. The elements are inserted between the nodes as shown in Fig. 27.
7. Sign conventions for the passive elements are established by directing the arrows away from the source and toward the reference node.
8. The sign convention for the voltage source is established as shown in Fig. 27 to correspond with that shown for the source in Fig. 26b.

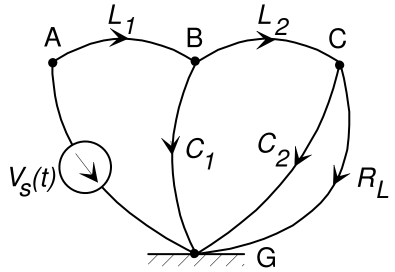

Figure 27: Linear graph representation of the electrical filter.

10.4 Fluid System Models

Linear graph models for fluid systems are based on pressure drop P as the across-variable, and volume flow rate Q as the through-variable. Nodes on the graph represent distinct points of fluid pressure with respect to a constant reference pressure, and the passive elements are fluid capacitances (A-type), fluid inertances (T-type), and fluid resistances (D-type). The across-variable source is a pressure source, and the through-variable source is a flow source. Fluid A-type elements are referenced to a fixed pressure node.

■ Example

A water storage system consisting of a large reservoir, two control valves and a tank is illustrated in Figure 28. The system is fed by rainfall. The system may be represented

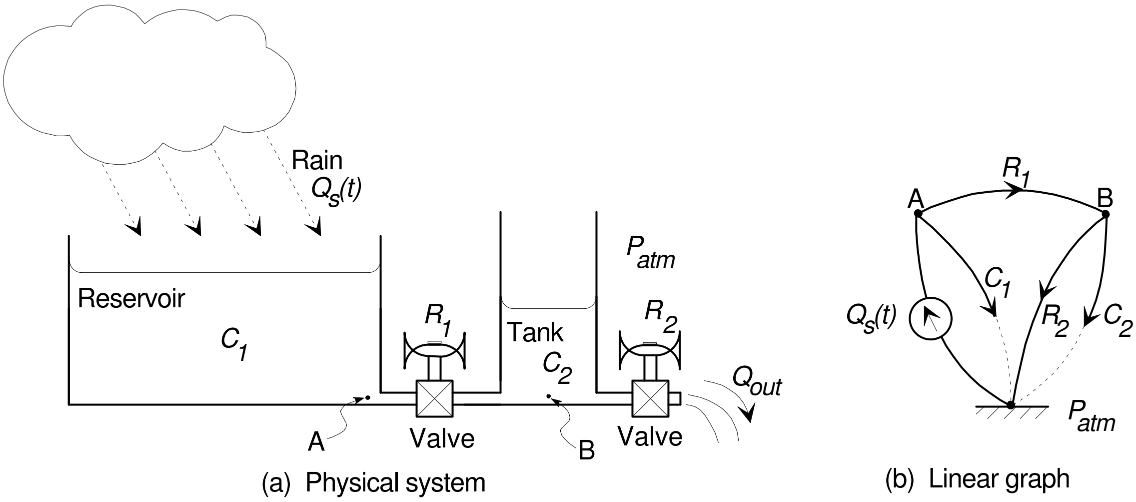

Figure 28: A fluid system with two storage tanks.

with the following elements

1. Rainfall – A flow source $Q_s(t)$
2. The reservoir – A fluid capacitor C_1
3. The two valves – In a partially open state the valves are modeled as linear fluid resistances, R_1 and R_2.
4. The storage tank – a fluid capacitor C_2.

It is assumed that the connecting pipes are sufficiently short so that pressure drops associated with piping resistances and fluid inertances may be neglected. The figure shows that there are two independent pressures in the system, at the base of the reservoir, point A, and at the base of the tank point, B. The graph therefore requires three nodes; the reference node representing atmospheric pressure and the two capacitance pressures.

The two fluid capacitances (A-type elements) are placed between the appropriate nodes and the reference node P_{atm}. The outlet valve R_2 discharges between the storage tank pressure P_A and the reference pressure P_{atm}, and so is connected in parallel with C_2. The pressure drop across valve R_1 is $P_A - P_B$ and so it is inserted between the two nodes A and B. Finally the flow source Q_s is inserted between the capacitance C_1 and the reference node. The sign convention in the flow source Q_s is selected to give an increase in reservoir pressure when the source flow is positive. Figure 28b shows the completed linear graph.

In the next example we examine a simple lumped equivalent model of the *distributed* inertance and resistance effects in a long pipe.

■ Example

In the system shown in Figure 29a fluid is pumped into a tank through a long pipe. The tank discharges to atmospheric pressure through a partially open valve. The model is formed to study the dynamic response of the flow through the outlet valve in response to changes in the pressure generated by the pump.

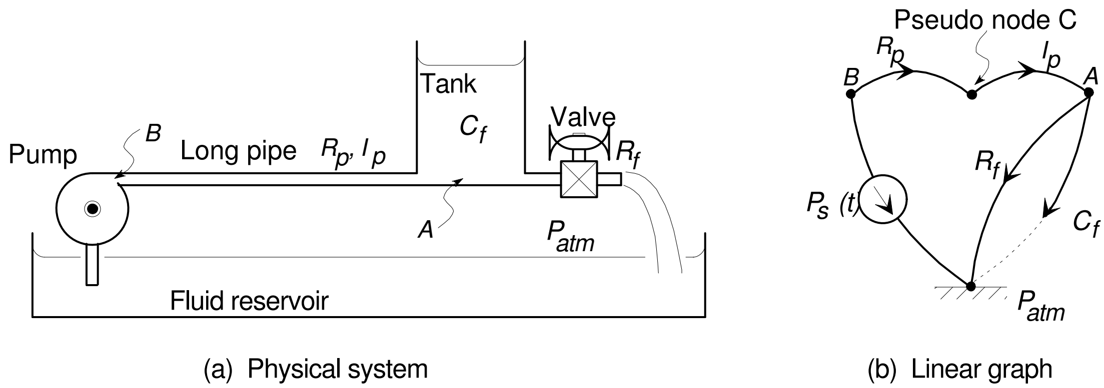

Figure 29: A fluid system that includes pipe effects in its model.

The pump is represented as a pressure source $P_s(t)$. The open tank is represented as a fluid capacitance C. The discharge valve is modeled as an ideal fluid resistance R_1.

In the previous example it was assumed that pressure drops associated with the connecting pipes could be ignored; in this example the pipe is of sufficient length that internal pressure drops need to be included in the model. The pipe is assumed to:

1. dissipate energy through frictional losses at the walls, and
2. to store energy associated with the motion of the fluid within the pipe.

While these two effects are distributed throughout the length of the pipe, they may be approximated by a combination of a single lumped resistance R_p and a fluid inertance I_p. The two elements have a common flow Q and are described by the elemental equations:

$$P_{R_P} = R_P Q \qquad \text{for the resistance, and} \qquad (88)$$

$$P_{I_P} = I_P \frac{dQ}{dt} \qquad \text{for the inertance.} \qquad (89)$$

It is reasonable to assume that the total pressure drop across the pipe is the sum of the two effects, and that the pipe should be modeled as a *series* connection of the elements.

A non-physical node is created in the linear graph to represent the point of connection of the two lumped elements that are used to model the effects of *distributed* resistance and inertance in the pipe.

With the addition of the pseudo-node the linear graph requires a total of four nodes, representing the reference pressure, the pressure at the base of the tank A, the pressure at the end of the long pipe B, and at the node C representing the junction of the pipe resistance and inertance elements. The fluid capacitance is inserted between node A and the reference node. The pipe elements are inserted in series between the tank A and the pump B; the order is arbitrary. The discharge resistance R_1 is connected to the reference node, indicating that the flow is to atmospheric pressure. The standard sign convention for passive elements is adopted, and the flow source direction is established to ensure that a positive flow from the pump establishes a positive pressure in the tank. Figure 29b shows the completed model.

10.5 Thermal System Models

Thermal systems are inherently different from the other energy domains because (1) the product of the across and through-variables (temperature and heat flow rate) is not power, (2) there is no defined T-type energy storage element, and (3) the D-type element does not dissipate energy. The two passive elements are a thermal capacitance (A-type) and a thermal resistance (D-type). The sources are a temperature source (across-variable source) and a heat flow source (through-variable source).

■ Example

A laboratory furnace used to heat cylindrical metal specimens is illustrated in Figure 30. The system model elements include:

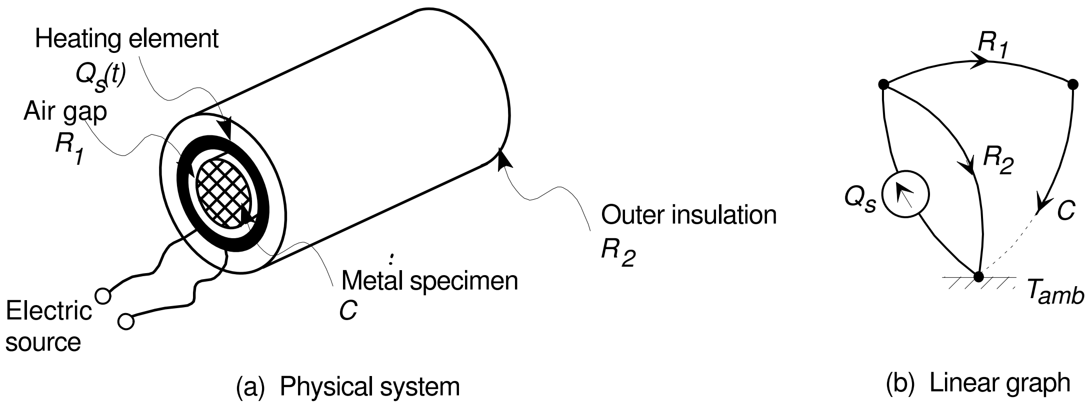

Figure 30: A laboratory furnace system.

1. The metal specimen, modeled as a thermal capacitance element C
2. The space between the specimen and the furnace coil element, modeled as a thermal resistance R_1

3. The heating element, modeled as a heat (through-variable) source Q_s,
4. The outer insulation around the element, modeled as a thermal resistance element R_2.

In the selection of these elements, the representation of the metal specimen as a single thermal capacitance assumes that temperature gradients between its surface and center may be neglected, that is the cylinder may be represented as a single lump at a uniform temperature. In addition the resistance R_1 between the heater coil and specimen represents the combined effects of the air gap and any insulation around the inner surface of the coil, while the resistance R_2 from the heater coil to the furnace exterior wall represents the heat losses to the environment and includes the total effective resistance due to the coil-insulation interface, the insulation itself, and the insulation-atmosphere interface.

The model contains two distinct temperatures with respect to the ambient environmental temperature T_{ref}; the temperature associated with the furnace heat source, and the temperature of the specimen itself. The graph therefore contains three nodes, including the reference node. The thermal capacitance C_T is referenced to the ambient temperature, and is connected to the source node through the resistive element R_1. Resistance R_2 represents direct heat loss to the environment through the outer insulation and is connected directly across the source node. The sign convention adopted for the heat source ensures an increase in the temperature of the capacitance for a positive heat flow. Figure 30b shows the linear graph.

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11 Physical Source Modeling

The ideal source elements are capable of supplying infinite power to a system. Physical energy sources, on the other hand, have a limit on the power that they can supply. For example, the terminal voltage of an electrical battery decreases as the current demand from the system increases. A battery is limited in the power it can supply even if the terminals are short circuited. For small current loads it may be satisfactory to model a battery as a voltage source, but in more demanding situations, with large and varying current requirements, the model of the battery must represent the variation of the terminal voltage. In general, physical energy sources are represented by a non-linear relationship between across and through-variables such as shown in Fig. 31, and only over a limited range of operation may a real source be represented by an ideal across or through-variable source.

The power-limited characteristic of a real source can often be approximated by coupling an ideal source element with a D-type resistive element. A typical power-limited source characteristic is represented in Fig. 32. It has a maximum value of its output across-variable $\mathsf{V_s}$ when the supplied through-variable is zero, corresponding to an open-circuit condition of an electrical source, and a maximum value of the supplied through-variable $\mathsf{F_s}$ when the across-variable is zero, corresponding to a short-circuited electrical source. If the characteristic is a straight line, with a slope $(-\mathsf{R})$, the relationship between the across and through-variable at the source terminals at any point on the characteristic may be expressed as a linear algebraic equation in either of the following two forms:

$$\mathsf{v} = \mathsf{V_s} - \mathsf{Rf} \tag{90}$$

$$\mathsf{f} = \mathsf{F_s} - \frac{1}{\mathsf{R}}\mathsf{v} \tag{91}$$

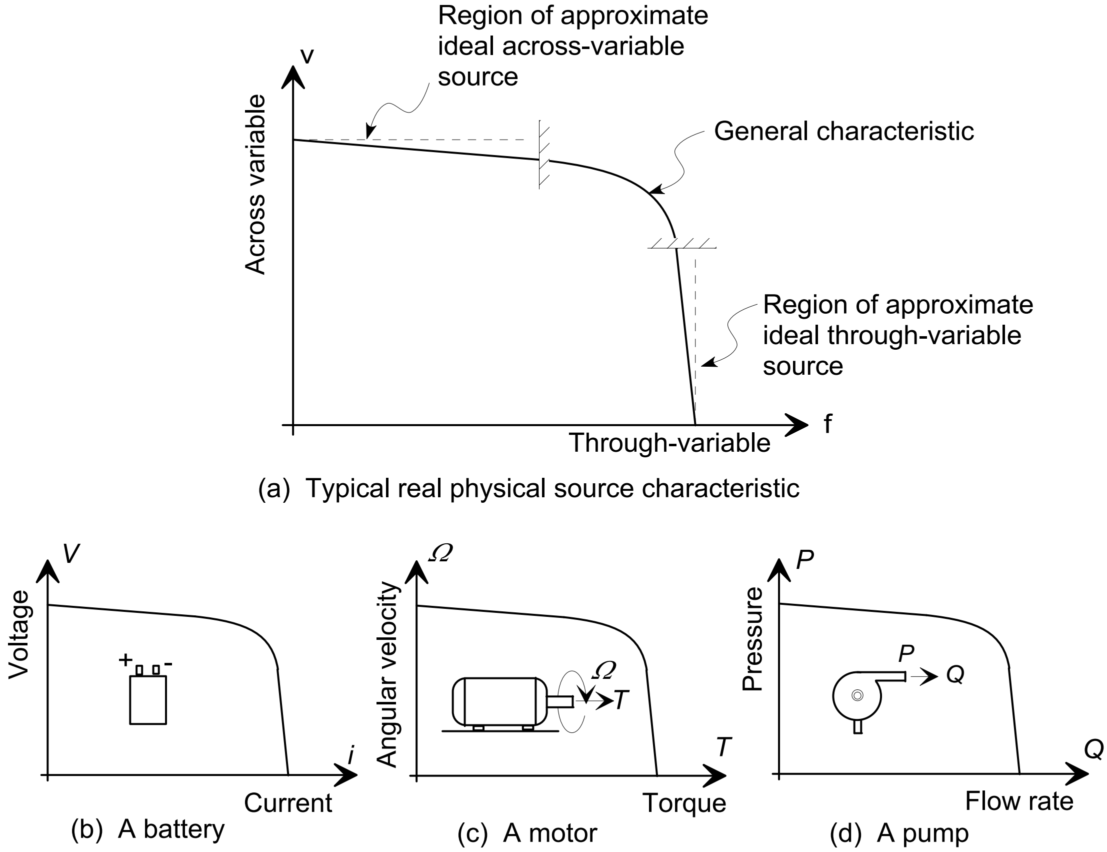


Figure 31: General characteristics of real sources.

where v is the source across-variable when it is supplying through-variable f to the system. The first form states that if $\mathsf{f} = 0$, the across-variable is equal to $\mathsf{V_s}$ and as f increases the output across-variable v decreases linearly. The second form states that if $\mathsf{v} = 0$, the output through-variable f is equal to $\mathsf{F_s}$ and as v increases the through-variable decreases linearly.

The two forms generate two possible models for a power-limited source with a linear characteristic:

1. Equation (90) may be implemented by an ideal across-variable source of value $\mathsf{V_s}$ in *series* with a resistance element with a value R as shown in Fig. 33a. This series equivalent source model is known as a *Thevenin equivalent* source.

2. Equation (91) may be implemented by an ideal through-variable source of value $\mathsf{F_s}$ in *parallel* with a resistance of value R as shown in Fig. 33b. This configuration is known as a *Norton equivalent* source model.

These two models of real sources are equivalent and have identical characteristics as measured at their terminals. Either may be used in the modeling of systems involving physical sources that may be approximated by a linear characteristic.

The load power $\mathcal{P}$ delivered by an equivalent source model depends on the across and through variables at the terminals. For the Thevenin source the power is:

$$\mathcal{P} = \mathsf{vf} = \mathsf{V_s f} - \mathsf{R f^2} \tag{92}$$

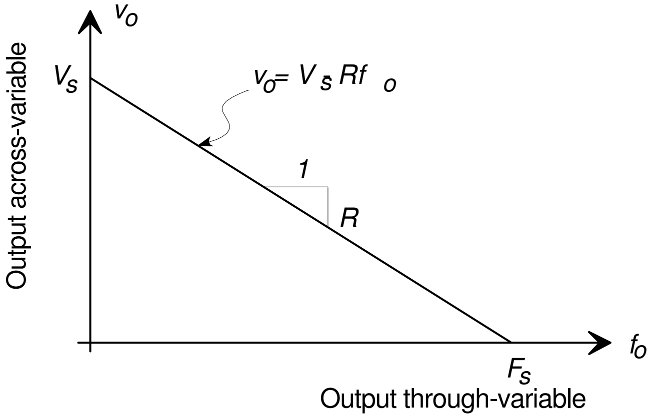


Figure 32: Characteristic of a simple linear source.

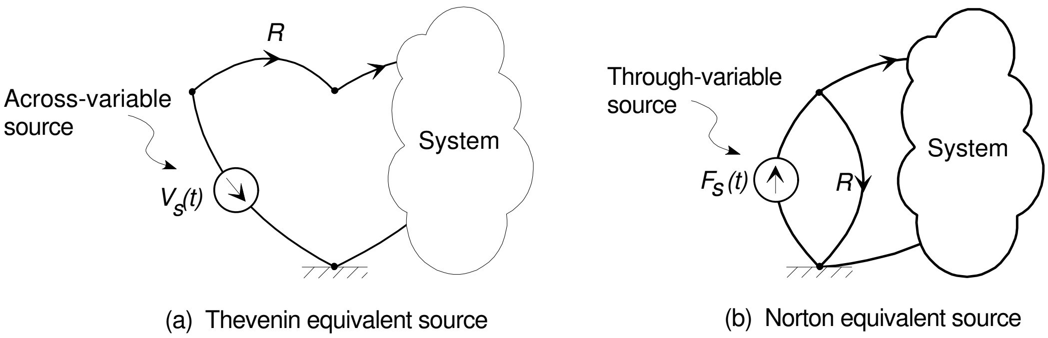


Figure 33: Thevenin and Norton models of power-limited physical sources.

and for the Norton source it is

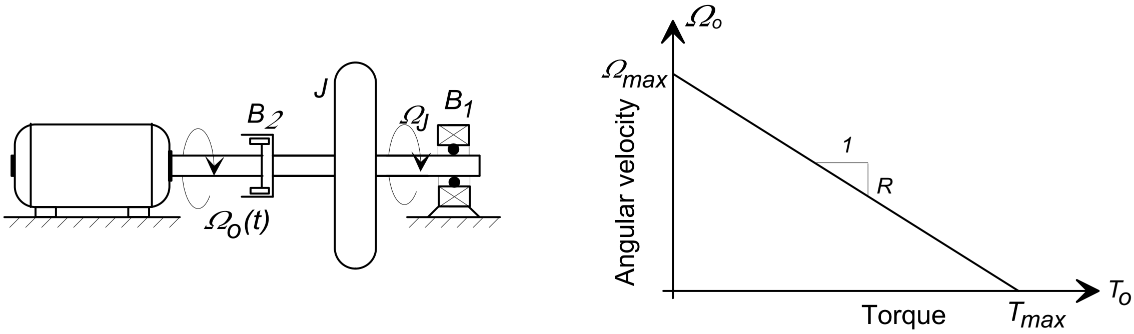
$$\mathcal{P} = \mathsf{vf} = \mathsf{vF_s} - \frac{1}{\mathsf{R}}\mathsf{v}^2 \tag{93}$$

The maximum power an equivalent source can provide is found by differentiating Eq. (92) with respect to f or Eq. (93) with respect to v and equating the derivative to zero. In either case the maximum power is supplied when $\mathsf{f} = \mathsf{V_s}/2\mathsf{R}$ and $\mathsf{v} = \mathsf{RF_s}/2$. The maximum power supplied is $P_{max} = \mathsf{V_s F_s}/4$.

■ Example

It has been found that the performance of the rotational flywheel drive model derived in Example 10.2, with the linear graph shown in Fig. 25, does not adequately reflect the dynamic response of the physical system over the full operating range of interest. Measurements on the system show that it is not valid to represent the motor as an ideal angular velocity source over the full speed range. With a fixed supply voltage and no load, the motor spins at an angular velocity of Ω_{max}, but as the torque load is increased the motor speed decreases linearly until the shaft is stationary and generates a torque T_{max}. Extend the linear graph model of Example 10.2 to include (a) a Thevenin and (b) a Norton source equivalent for the motor.

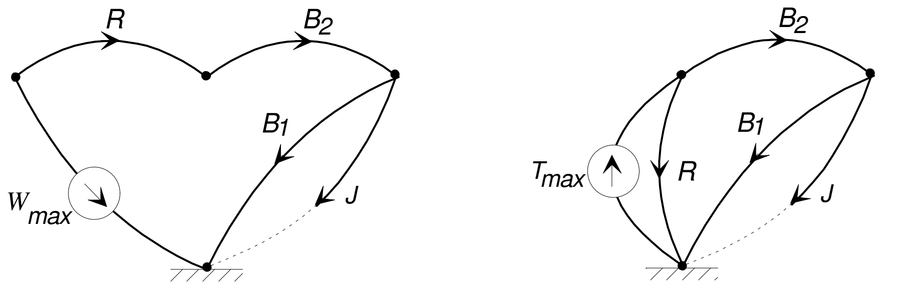
Solution: The measurements on the motor indicate that the source characteristic is as shown in Fig. 34.



(a) The motor drive system (b) The motor source characteristic

Figure 34: The source characteristic of the electric motor.

The equivalent source resistance is found from the slope of the characteristic, that is $\mathsf{R} = \Omega_{max}/T_{max}$. Figure 35 shows the two modified system linear graphs using (a) a Thevenin and (b) a Norton source equivalent model for the motor. Both are equivalent with respect to the dynamic behavior of the flywheel.



(a) Thevenin source based model (b) Norton source based model

Figure 35: The rotational system of Fig. 4.11 redrawn with (a) a Thevenin equivalent source, and (b) a Norton equivalent source.

Thevenin and Norton source models may also be used to approximate the behavior of nonlinear physical sources when the range of variation of the source variables is small. Consider a source with a nonlinear characteristic

$$\mathsf{v} = \mathcal{F}(\mathsf{f}) \tag{94}$$

and assume that the source normally operates with small excursions about a nominal operating point $\mathsf{v} = \mathsf{v}_o$ and $\mathsf{f} = \mathsf{f}_o$. If Eq. (94) can be expanded as a Taylor series and the first two terms are retained

$$\mathsf{v} \approx \mathsf{v}_o + \left.\frac{d\mathcal{F}(\mathsf{f})}{d\mathsf{f}}\right|_{\mathsf{f}=\mathsf{f}_o} (\mathsf{f} - \mathsf{f}_o). \tag{95}$$

If we define a D-type element

$$\mathsf{R}^* = -\left.\frac{d\mathcal{F}(\mathsf{f})}{d\mathsf{f}}\right|_{\mathsf{f}=\mathsf{f}_o} \tag{96}$$

we can write an approximate source characteristic

$$\begin{aligned} \mathsf{v} &= \mathsf{v_o} - \mathsf{R}^*(\mathsf{f} - \mathsf{f_o}) \\ &= \mathsf{V}_s - \mathsf{R}^*\mathsf{f} \end{aligned} \tag{97}$$

where $\mathsf{V}_s = \mathsf{v_o} + \mathsf{R}^*\mathsf{f_o}$. Equation (97) defines a Thevenin equivalent source with an ideal source V_s and a series resistance R^*. A linearized Norton source can also be expressed as a through-variable source $\mathsf{F_s} = \mathsf{f_o} + (1/\mathsf{R}^*)\mathsf{v_o}$ in parallel with a D-type element R^*.

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