Time-Domain Response of Linear Time-Invariant State Equations[1]

1 Introduction

In this note we examine the responses of linear, time-invariant (LTI) models expressed in the standard state equation form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}. \tag{2}$$

The solution proceeds in two steps; first the state variable response $\mathbf{x}(t)$ is determined by solving the set of first-order state equations, Eq. (1), and then the state response is substituted into the algebraic output equations, Eq. (2) in order to compute $\mathbf{y}(t)$.

Tthe total system state response $\mathbf{x}(t)$ is considered in two parts: the homogeneous solution $\mathbf{x}_h(t)$ that describes the response to an arbitrary set of initial conditions $\mathbf{x}(0)$, and a particular solution $\mathbf{x}_p(t)$ that satisfies the state equations for the given input $\mathbf{u}(t)$. The two components are then combined to form the total response.

The solution methods encountered in this chapter rely heavily on matrix algebra. In order to keep the treatment simple we attempt wherever possible to introduce concepts using a first-order system, in which the $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ matrices reduce to scalar values, and then to generalize results by replacing the scalars with the appropriate matrices.

2 State-Variable Response of Linear Systems

2.1 The Homogeneous State Response

The state-variable response of a system described by Eq. (1) with zero input and an arbitrary set of initial conditions $\mathbf{x}(0)$ is the solution of the set of n homogeneous first-order differential equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}. \tag{3}$$

To derive the homogeneous response $\mathbf{x}_h(t)$, we begin by considering the response of a first-order (scalar) system with state equation

$$\dot{x} = ax + bu \tag{4}$$

with initial condition $x(0)$. The homogeneous response $x_h(t)$ has an exponential form defined by the system time constant $\tau = -1/a$, or:

$$x_h(t) = e^{at}x(0). \tag{5}$$

The exponential term e^{at} in Eq. (5) may be expanded as a power series, to give:

$$x_h(t) = \left(1 + at + \frac{a^2 t^2}{2!} + \frac{a^3 t^3}{3!} + \ldots + \frac{a^k t^k}{k!} + \ldots\right) x(0), \tag{6}$$

where the series converges for all finite t.

Let us now assume that the homogeneous response $\mathbf{x}_h(t)$ of the state vector of a higher order linear time-invariant system, described by Eq. (3), can also be expressed as an infinite power series, similar in form to Eq. (6), but in terms of the square matrix $\mathbf{A}$, that is we assume:

$$\mathbf{x}_h(t) = \left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \ldots + \frac{\mathbf{A}^k t^k}{k!} + \ldots\right)\mathbf{x}(0) \tag{7}$$

where $\mathbf{x}(0)$ is the initial state. Each term in this series is a matrix of size $n \times n$, and the summation of all terms yields another matrix of size $n \times n$. To verify that the homogeneous state equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is satisfied by Eq. (7), the series may be differentiated term by term. Matrix differentiation is defined on an element by element basis, and because each system matrix $\mathbf{A}^k$ contains only constant elements:

$$
\begin{aligned}
\dot{\mathbf{x}}_h(t) &= \left(\mathbf{0} + \mathbf{A} + \mathbf{A}^2 t + \frac{\mathbf{A}^3 t^2}{2!} + \ldots + \frac{\mathbf{A}^k t^{k-1}}{(k-1)!} + \ldots\right)\mathbf{x}(0) \\
&= \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \ldots + \frac{\mathbf{A}^{k-1} t^{k-1}}{(k-1)!} + \ldots\right)\mathbf{x}(0) \\
&= \mathbf{A}\mathbf{x}_h(t).
\end{aligned}
\tag{8}
$$

Equation (8) shows that the assumed series form of the solution satisfies the homogeneous state equations, demonstrating that Eq. (7) is in fact a solution of Eq. (3). *The homogeneous response to an arbitrary set of initial conditions* $\mathbf{x}(0)$ *can therefore be expressed as an* infinite sum of time dependent matrix functions, involving only the system matrix $\mathbf{A}$. Because of the similarity of this series to the power series defining the scalar exponential, it is convenient to define the *matrix exponential* of a square matrix A as

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \ldots + \frac{\mathbf{A}^k t^k}{k!} + \ldots \tag{9}$$

which is itself a square matrix the same size as its defining matrix $\mathbf{A}$. The matrix form of the exponential is recognized by the presence of a matrix quantity in the exponent. The system homogeneous response $\mathbf{x}_h(t)$ may therefore be written in terms of the matrix exponential

$$\mathbf{x}_h(t) = e^{\mathbf{A}t}\mathbf{x}(0) \tag{10}$$

which is similar in form to Eq. (5). The solution is often written as

$$\mathbf{x}_h(t) = \Phi(t)\mathbf{x}(0) \tag{11}$$

where $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ is defined to be the *state transition matrix* $[1-5]$. Equation (11) gives the response at any time t to an arbitrary set of initial conditions, thus computation of $e^{\mathbf{A}t}$ at any t yields the values of all the state variables $\mathbf{x}(t)$ directly.

■ Example

Determine the matrix exponential, and hence the state transition matrix, and the homogeneous response to the initial conditions $x_1(0) = 2$, $x_2(0) = 3$ of the system with state equations:

$$
\begin{aligned}
\dot{x}_1 &= -2x_1 + u \\
\dot{x}_2 &= x_1 - x_2.
\end{aligned}
$$

Solution: The system matrix is

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}.$$

From Eq. (9) the matrix exponential (and the state transition matrix) is

$$
\begin{aligned}
\mathbf{\Phi}(t) &= e^{\mathbf{A}t} \\
&= \left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \ldots + \frac{\mathbf{A}^k t^k}{k!} + \ldots \right) \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} t + \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix} \frac{t^2}{2!} \\
&\quad + \begin{bmatrix} -8 & 0 \\ 7 & -1 \end{bmatrix} \frac{t^3}{3!} + \ldots \\
&= \begin{bmatrix} 1 - 2t + \dfrac{4t^2}{2!} - \dfrac{8t^3}{3!} + \ldots & 0 \\[2ex] 0 + t - \dfrac{3t^2}{2!} + \dfrac{7t^3}{3!} + \ldots & 1 - t + \dfrac{t^2}{2!} - \dfrac{t^3}{3!} + \ldots \end{bmatrix}. \quad (12)
\end{aligned}
$$

The elements ϕ_{11} and ϕ_{22} are simply the series representation for e^{-2t} and e^{-t} respectively. The series for ϕ_{21} is not so easily recognized but is in fact the first four terms of the the expansion of $e^{-t} - e^{-2t}$. The state transition matrix is therefore

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} \quad (13)$$

and the homogeneous response to initial conditions $x_1(0)$ and $x_2(0)$ is

$$\mathbf{x}_h(t) = \mathbf{\Phi}(t)\mathbf{x}(0) \quad (14)$$

or

$$
\begin{aligned}
x_1(t) &= x_1(0)e^{-2t} & (15) \\
x_2(t) &= x_1(0)\left(e^{-t} - e^{-2t}\right) + x_2(0)e^{-t}. & (16)
\end{aligned}
$$

With the given initial conditions the response is

$$
\begin{aligned}
x_1(t) &= 2e^{-2t} & (17) \\
x_2(t) &= 2\left(e^{-t} - e^{-2t}\right) + 3e^{-t} \\
&= 5e^{-t} - 2e^{-2t}. & (18)
\end{aligned}
$$

In general the recognition of the exponential components from the series for each element is difficult and is not normally used for finding a closed form for the state transition matrix.

Although the sum expressed in Eq. (9) converges for all $\mathbf{A}$, in many cases the series converges slowly, and is rarely used for the direct computation of $\mathbf{\Phi}(t)$. There are many methods for computing the elements of $\mathbf{\Phi}(t)$, including one presented below, that are much more convenient than the direct series definition. [1,5,6]

2.2   The Forced State Response of Linear Systems

We now consider the complete response of a linear system to an input $\mathbf{u}(t)$. Consider first a first-order system with a state equation $\dot{x} = ax + bu$ written in the form

$$\dot{x}(t) - ax(t) = bu(t). \tag{19}$$

If both sides are multiplied by an integrating factor e^{-at}, the left-hand side becomes a perfect differential

$$e^{-at}\dot{x} - e^{-at}ax = \frac{d}{dt}\left(e^{-at}x(t)\right) = e^{-at}bu \tag{20}$$

which may be integrated directly to give

$$\int_0^t \frac{d}{d\tau}\left(e^{-a\tau}x(\tau)\right)d\tau = e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)\,d\tau \tag{21}$$

and rearranged to give the state variable response explicitly:

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\,d\tau. \tag{22}$$

The development of the expression for the response of higher order systems may be performed in a similar manner using the matrix exponential $e^{-\mathbf{A}t}$ as an integrating factor. Matrix differentiation and integration are defined to be element by element operations, so that if the state equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ are rearranged, and all terms pre-multiplied by the square matrix $e^{-\mathbf{A}t}$:

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) = \frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{x}(t)\right) = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t). \tag{23}$$

Integration of Eq. (23) gives

$$\int_0^t \frac{d}{d\tau}\left(e^{-\mathbf{A}\tau}\mathbf{x}(\tau)\right)d\tau = e^{-\mathbf{A}t}\mathbf{x}(t) - e^{-\mathbf{A}0}\mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{24}$$

and because $e^{-\mathbf{A}0} = \mathbf{I}$ and $[e^{-\mathbf{A}t}]^{-1} = e^{\mathbf{A}t}$ the complete state vector response may be written in two similar forms

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{25}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau. \tag{26}$$

The full state response, described by Eq. (25) or Eq. (26) consists of two components: the first is a term similar to the system homogeneous response $\mathbf{x}_h(t) = e^{\mathbf{A}t}\mathbf{x}(0)$ that is dependent only on the system initial conditions $\mathbf{x}(0)$. The second term is a *convolution integral* that is the particular solution for the input $\mathbf{u}(t)$ with zero initial conditions.

Evaluation of the integral in Eq. (26) involves matrix integration. For a system of order n and with r inputs, the matrix $e^{\mathbf{A}t}$ is $n \times n$, $\mathbf{B}$ is $n \times r$ and $\mathbf{u}(t)$ is an $r \times 1$ column vector. The product $e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)$ is therefore an $n \times 1$ column vector, and solution for each of the state equations involves a single scalar integration.

■ Example

Find the response of the two state variables of the system

$$\dot{x}_1 = -2x_1 + u$$
$$\dot{x}_2 = x_1 - x_2.$$

to a constant input $u(t) = 5$ for $t > 0$, if $x_1(0) = 0$, and $x_2 = 0$.

Solution: This is the same system described in Example 1. The state transition matrix was shown to be

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix}$$

With zero initial conditions, the forced response is (Eq. (25)):

$$\mathbf{x}(t) = e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}\tau} \mathbf{B}\mathbf{u}(\tau)d\tau. \tag{27}$$

Matrix integration is defined on an element by element basis, so that

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{2\tau} & 0 \\ e^{\tau} - e^{2\tau} & e^{\tau} \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} d\tau \tag{28}$$

$$= \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} \begin{bmatrix} \int_0^t 5e^{2\tau}d\tau \\ \int_0^t 5e^{\tau} - 5e^{2\tau}d\tau \end{bmatrix} \tag{29}$$

$$= \begin{bmatrix} \frac{5}{2} - \frac{5}{2}e^{-2t} \\ \frac{5}{2} - 5e^{-t} + \frac{5}{2}e^{-2t} \end{bmatrix} \tag{30}$$

3 The System Output Response

For either the homogeneous or forced responses, the system output response $\mathbf{y}(t)$ may be found by substitution of the state variable response into the algebraic system output equations

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}. \tag{31}$$

In the case of the homogeneous response, where $\mathbf{u}(t) = 0$, Eq. (31) becomes

$$\mathbf{y}_h(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0), \tag{32}$$

while for the forced response substitution of Eq. (26) into the output equations gives

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t). \tag{33}$$

■ Example

Find the response of the output variable

$$y = 2x_1 + x_2$$

in the system described by state equations

$$\dot{x}_1 = -2x_1 + u$$
$$\dot{x}_2 = x_1 - x_2.$$

to a constant input $u(t) = 5$ for $t > 0$, if $x_1(0) = 0$, and $x_2 = 0$.

Solution: This is the same system described in Example 1 with the same input and initial conditions as used in Example 2. The state variable response is (Example 2):

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{5}{2} - \frac{5}{2}e^{-2t} \\ \frac{5}{2} - 5e^{-t} + \frac{5}{2}e^{-2t} \end{bmatrix} \tag{34}$$

The output response is

$$\begin{aligned} y(t) &= 2x_1(t) + x_2(t) \\ &= \frac{15}{2} - \frac{5}{2}e^{-2t} - 5e^{-t}. \end{aligned} \tag{35}$$

4 The State Transition Matrix

4.1 Properties of the State Transition Matrix

Table 1 shows some of the properties that can be derived directly from the series definition of the matrix exponential $e^{\mathbf{A}t}$. For comparison the similar properties of the scalar exponential e^{at} are also given. Although the sum expressed in Eq. (9) converges for all $\mathbf{A}$, in many cases the series converges slowly, and is rarely used for the direct computation of $\mathbf{\Phi}(t)$. There are many methods for computing the elements of $\Phi(t)$, including one presented in the next section, that are much more convenient than the series definition. The matrix exponential representation of the state transition matrix allows some of its properties to be simply stated:

(1) $\mathbf{\Phi}(0) = \mathbf{I}$, which simply states that the state response at time $t = 0$ is identical to the initial conditions.

(2) $\mathbf{\Phi}(-t) = \mathbf{\Phi}^{-1}(t)$. The response of an unforced system before time $t = 0$ may be calculated from the initial conditions $\mathbf{x}(0)$,

$$\mathbf{x}(-t) = \mathbf{\Phi}(-t)\mathbf{x}(0) = \mathbf{\Phi}^{-1}(t)\mathbf{x}(0) \tag{36}$$

and the inverse always exists.

Scalar exponential:	Matrix exponential:
$e^{at} = 1 + at + \dfrac{a^2 t^2}{2!} + \dfrac{a^3 t^3}{3!} + \ldots$	$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \dfrac{\mathbf{A}^2 t^2}{2!} + \dfrac{\mathbf{A}^3 t^3}{3!} + \ldots$
$e^{a0} = 1$	$e^{\mathbf{A}0} = \mathbf{I}$
$e^{-at} = \dfrac{1}{e^{at}}$	$e^{-\mathbf{A}t} = \left[e^{\mathbf{A}t} \right]^{-1}$
$e^{a(t_1 + t_2)} = e^{at_1} e^{at_2}$	$e^{\mathbf{A}(t_1 + t_2)} = e^{\mathbf{A}t_1} e^{\mathbf{A}t_2}$
$e^{(a_1 + a_2)t} = e^{a_1 t} e^{a_2 t}$	$e^{(\mathbf{A}_1 + \mathbf{A}_2)t} = e^{\mathbf{A}_1 t} e^{\mathbf{A}_2 t}$ only if $\mathbf{A}_1 \mathbf{A}_2 = \mathbf{A}_2 \mathbf{A}_1$
$\dfrac{d}{dt} e^{at} = a e^{at} = e^{at} a$	$\dfrac{d}{dt} e^{\mathbf{A}t} = \mathbf{A} e^{\mathbf{A}t} = e^{\mathbf{A}t} \mathbf{A}$
$\displaystyle\int_0^t e^{at} dt = \dfrac{1}{a} \left[e^{at} - 1 \right]$	$\displaystyle\int_0^t e^{\mathbf{A}t} dt = \mathbf{A}^{-1} \left[e^{\mathbf{A}t} - \mathbf{I} \right] = \left[e^{\mathbf{A}t} - \mathbf{I} \right] \mathbf{A}^{-1}$
	if $\mathbf{A}^{-1}$ exists. Otherwise defined by the series.

Table 1: Comparison of properties of the scalar and matrix exponentials.

(3) $\mathbf{\Phi}(t_1)\mathbf{\Phi}(t_2) = \mathbf{\Phi}(t_1 + t_2)$. With this property the state response at time t may be defined from the system state specified at some time other than $t = 0$, for example at $t = t_0$. Using Property (2), the response at time $t = 0$ is

$$\mathbf{x}(0) = \mathbf{\Phi}(-t_0)\mathbf{x}(t_0) \tag{37}$$

and using the properties in Table 1,

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) = \mathbf{\Phi}(t)\mathbf{\Phi}(-t_0)\mathbf{x}(t_0) \tag{38}$$

or

$$\mathbf{x}_h(t) = \mathbf{\Phi}(t - t_0)\mathbf{x}(t_0). \tag{39}$$

(4) If $\mathbf{A}$ is a diagonal matrix then $e^{\mathbf{A}t}$ is also diagonal, and each element on the diagonal is an exponential in the corresponding diagonal element of the $\mathbf{A}$ matrix, that is $e^{a_{ii}t}$. This property is easily shown by considering the terms $\mathbf{A}^n$ in the series definition and noting that any diagonal matrix raised to an integer power is itself diagonal.

4.2 Determining the State Transition Matrix from the Laplace Transform

The state transition matrix may be found from the Laplace transform of the homogeneous state equations

$$\dot{\mathbf{x}}_h = \mathbf{A}\mathbf{x_h}$$

with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$. Using the derivative property of the Laplace transform

$$s\mathbf{X}_h(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}_h(s) \tag{40}$$

or

$$[s\mathbf{I} - \mathbf{A}]\,\mathbf{X}_h(s) = \mathbf{x}(0). \tag{41}$$

7

Then
$$\mathbf{X}_h(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) = \frac{\mathrm{adj}\,[s\mathbf{I} - \mathbf{A}]}{\det\,[s\mathbf{I} - \mathbf{A}]}\mathbf{x}(0). \tag{42}$$

Taking the inverse Laplace transform

$$\mathbf{x}_h(t) = \mathcal{L}^{-1}\left\{\frac{\mathrm{adj}\,[s\mathbf{I} - \mathbf{A}]}{\det\,[s\mathbf{I} - \mathbf{A}]}\right\}\mathbf{x}(0), \tag{43}$$

where $\mathcal{L}^{-1}\{\}$ indicates the inverse Laplace transform, from which it can be seen that

$$\mathbf{\Phi}(t) = \mathcal{L}^{-1}\left\{\frac{\mathrm{adj}\,[s\mathbf{I} - \mathbf{A}]}{\det\,[s\mathbf{I} - \mathbf{A}]}\right\}. \tag{44}$$

■ Example

Find the state transition matrix for a system having an $\mathbf{A}$ matrix:

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}$$

Solution: For this system

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+2 & -1 \\ -2 & s+3 \end{bmatrix}.$$

Then

$$\det\,[s\mathbf{I} - \mathbf{A}] = s^2 + 5s + 4, \quad \text{and} \quad \mathrm{adj}\,[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+3 & 1 \\ 2 & s+2 \end{bmatrix}$$

so that

$$\mathbf{\Phi}(t) = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{s+3}{s^2+5s+4} & \dfrac{1}{s^2+5s+4} \\[2ex] \dfrac{2}{s^2+5s+4} & \dfrac{s+2}{s^2+5s+4} \end{bmatrix}\right\} = \begin{bmatrix} \dfrac{2}{3}e^{-t} + \dfrac{1}{3}e^{-4t} & \dfrac{1}{3}e^{-t} - \dfrac{1}{3}e^{-4t} \\[2ex] \dfrac{2}{3}e^{-t} - \dfrac{2}{3}e^{-4t} & \dfrac{1}{3}e^{-t} + \dfrac{2}{3}e^{-4t} \end{bmatrix}$$

Another method for computing $\mathbf{\Phi}(t)$ is described in Section 4.4.

4.3   System Eigenvalues and Eigenvectors

In Example 2.1 each element of $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ is a sum of scalar exponential terms, and the homogeneous response of each of the two state variables is a sum of exponential terms. The homogeneous response of first and second-order systems is exponential in form, containing components of the form $e^{\lambda t}$, where λ is a root of the characteristic equation. The first-order homogeneous output response is of the form $y(t) = Ce^{\lambda t}$ where C is determined from the initial condition $C = y(0)$, and the second order response consists of two exponential components $y(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$ where the constants C_1 and C_2 are determined by a pair of initial conditions, usually the output and its derivative.

8

It is therefore reasonable to conjecture that for an nth order system the homogeneous response of each of the n state variables $x_i(t)$ consists of a weighted sum of n exponential components:

$$x_i(t) = \sum_{j=1}^{n} m_{ij} e^{\lambda_j t} \tag{45}$$

where the m_{ij} are constant coefficients that are dependent on the system structure and the initial conditions $\mathbf{x}(0)$. The proposed solution for the complete state vector may be written in a matrix form

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}
=
\begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix}
\begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}
\tag{46}
$$

or

$$
\mathbf{x_h}(t) = \mathbf{M}
\begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}
\tag{47}
$$

where $\mathbf{M}$ is an $n \times n$ matrix of the coefficients m_{ij}.

To determine the conditions under which Eq. (47) is a solution of the homogeneous state equations, the suggested response is differentiated and substituted into the state equation. From Eq. (45) the derivative of each conjectured state response is

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \lambda_j m_{ij} e^{\lambda_j t} \tag{48}$$

or in the matrix form

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}
=
\begin{bmatrix} \lambda_1 m_{11} & \lambda_2 m_{12} & \dots & \lambda_n m_{1n} \\ \lambda_1 m_{21} & \lambda_2 m_{22} & \dots & \lambda_2 m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 m_{n1} & \lambda_2 m_{n2} & \dots & \lambda_n m_{nn} \end{bmatrix}
\begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}.
\tag{49}
$$

Equations (46) and (49) may be substituted into the homogeneous state equation, Eq. (3),

$$
\begin{bmatrix} \lambda_1 m_{11} & \lambda_2 m_{12} & \dots & \lambda_n m_{1n} \\ \lambda_1 m_{21} & \lambda_2 m_{22} & \dots & \lambda_2 m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 m_{n1} & \lambda_2 m_{n2} & \dots & \lambda_n m_{nn} \end{bmatrix}
\begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}
= \mathbf{A}
\begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix}
\begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}
\tag{50}
$$

and if a set of m_{ij} and λ_i can be found that satisfy Eq. (50) the conjectured exponential form is a solution to the homogeneous state equations.

It is convenient to write the $n \times n$ matrix $\mathbf{M}$ in terms of its columns, that is to define a set of n column vectors $\mathbf{m}_j$ for $j = 1, 2, \dots, n$ from the matrix $\mathbf{M}$

$$
\mathbf{m}_j =
\begin{bmatrix} m_{1j} \\ m_{2j} \\ \vdots \\ m_{nj} \end{bmatrix}
$$

9

so that the matrix $\mathbf{M}$ may be written in a partitioned form

$$\mathbf{M} = \left[\; \mathbf{m}_1 \;\middle|\; \mathbf{m}_2 \;\middle|\; \ldots \;\middle|\; \mathbf{m}_n \;\right]. \tag{51}$$

Equation (50) may then be written

$$\left[\; \lambda_1\mathbf{m}_1 \;\middle|\; \lambda_2\mathbf{m}_2 \;\middle|\; \ldots \;\middle|\; \lambda_n\mathbf{m}_n \;\right] \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} = \mathbf{A} \left[\; \mathbf{m}_1 \;\middle|\; \mathbf{m}_2 \;\middle|\; \ldots \;\middle|\; \mathbf{m}_n \;\right] \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} \tag{52}$$

and for Eq.(52) to hold the required condition is

$$\begin{aligned} \left[\; \lambda_1\mathbf{m}_1 \;\middle|\; \lambda_2\mathbf{m}_2 \;\middle|\; \ldots \;\middle|\; \lambda_n\mathbf{m}_n \;\right] &= \mathbf{A} \left[\; \mathbf{m}_1 \;\middle|\; \mathbf{m}_2 \;\middle|\; \ldots \;\middle|\; \mathbf{m}_n \;\right] \\ &= \left[\; \mathbf{A}\mathbf{m}_1 \;\middle|\; \mathbf{A}\mathbf{m}_2 \;\middle|\; \ldots \;\middle|\; \mathbf{A}\mathbf{m}_n \;\right]. \end{aligned} \tag{53}$$

The two matrices in Eq. (53) are square $n \times n$. If they are equal then the corresponding columns in each must be equal, that is

$$\lambda_i\mathbf{m}_i = \mathbf{A}\mathbf{m}_i \qquad i = 1, 2, \ldots, n. \tag{54}$$

Equation (54) states that the assumed exponential form for the solution satisfies the homogeneous state equation provided a set of n scalar quantities λ_i, and a set of n column vectors $\mathbf{m}_i$ can be found that each satisfy Eq. (54).

Equation (54) is a statement of the classical *eigenvalue/eigenvector* problem of linear algebra. Given a square matrix $\mathbf{A}$, the values of λ satisfying Eq. (54) are known as the *eigenvalues*, or *characteristic values*, of $\mathbf{A}$. The corresponding column vectors $\mathbf{m}$ are defined to be the *eigenvectors*, or *characteristic vectors*, of $\mathbf{A}$. The homogeneous response of a linear system is therefore determined by the eigenvalues and the eigenvectors of its system matrix $\mathbf{A}$.

Equation (54) may be written as a set of homogeneous algebraic equations

$$\left[\lambda_i\mathbf{I} - \mathbf{A}\right]\mathbf{m}_i = 0 \tag{55}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. The condition for a non-trivial solution of such a set of linear equations is that

$$\Delta(\lambda_i) = \det\left[\lambda_i\mathbf{I} - \mathbf{A}\right] = 0. \tag{56}$$

which is defined to be the *characteristic equation* of the $n \times n$ matrix $\mathbf{A}$. Expansion of the determinant generates a polynomial of degree n in λ, and so Eq. (56) may be written

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_1\lambda + a_0 = 0 \tag{57}$$

or in factored form in terms of its n roots $\lambda_1, \ldots, \lambda_n$

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) = 0. \tag{58}$$

For a physical system the n roots are either real or occur in complex conjugate pairs. The eigenvalues of the matrix $\mathbf{A}$ are the roots of its characteristic equation, and these are commonly known as the system eigenvalues.

■ Example

Determine the eigenvalues of a linear system with state equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -9 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t).$$

Solution: The characteristic equation is $\det [\lambda \mathbf{I} - \mathbf{A}] = 0$ or

$$\det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 10 & 9 & \lambda + 4 \end{bmatrix} = 0 \tag{59}$$

$$\lambda^3 + 4\lambda^2 + 9\lambda + 10 = 0 \tag{60}$$

$$(\lambda + 2)(\lambda + (1 + j2))(\lambda + (1 - j2)) = 0 \tag{61}$$

The three eigenvalues are therefore $\lambda_1 = -2$, $\lambda_2 = -1 + j2$, and $\lambda_2 = -1 - j2$.

For each eigenvalue of a system there is an eigenvector, defined from Eq. (55). If the eigenvalues are distinct, the eigenvectors are linearly independent, and the matrix $\mathbf{M}$ is non-singular. In the development that follows it is assumed that $\mathbf{M}$ has an inverse, and therefore only applies to systems without repeated eigenvalues.

An eigenvector $\mathbf{m}_i$ associated with a given eigenvalue λ_i is found by substituting into the equation

$$[\lambda_i \mathbf{I} - \mathbf{A}] \mathbf{m}_i = 0. \tag{62}$$

No unique solution exists, however, because the definition of the eigenvalue problem, Eq. (54), shows that if $\mathbf{m}$ is an eigenvector then so is $\alpha \mathbf{m}$ for any non-zero scalar value α. The matrix $\mathbf{M}$, which is simply a collection of eigenvectors, defined in Eq. (51) therefore is not unique and some other information must be used to fully specify the homogeneous system response.

■ Example

Determine the eigenvalues and corresponding eigenvectors associated with a system having an $\mathbf{A}$ matrix:

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}.$$

Solution: The characteristic equation is $\det [\lambda \mathbf{I} - \mathbf{A}] = 0$ or

$$\det \begin{bmatrix} \lambda + 2 & -1 \\ -2 & \lambda + 3 \end{bmatrix} = 0$$

$$\lambda^2 + 5\lambda + 4 = 0$$

$$(\lambda + 4)(\lambda + 1) = 0. \tag{63}$$

11

The two eigenvalues are therefore $\lambda_1 = -1$, and $\lambda_2 = -4$. To find the eigenvectors these values are substituted into the equation $[\lambda_i \mathbf{I} - \mathbf{A}]\, \mathbf{m}_i = 0$. For the case $\lambda_1 = -1$ this gives

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{64}$$

Both of these equations give the same result; $m_{11} = m_{21}$. The eigenvector cannot be further defined, and although one particular solution is

$$\mathbf{m}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

the general solution must be defined in terms of an unknown scalar multiplier α_1

$$\mathbf{m}_1 = \begin{bmatrix} \alpha_1 \\ \alpha_1 \end{bmatrix} \tag{65}$$

provided $\alpha_1 \neq 0$.

Similarly, for the second eigenvalue, $\lambda_2 = -4$, the equations are

$$\begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{66}$$

which both state that $-2m_{12} = m_{22}$. The general solution is

$$\mathbf{m}_2 = \begin{bmatrix} \alpha_2 \\ -2\alpha_2 \end{bmatrix} \tag{67}$$

for $\alpha_2 \neq 0$. The following are all eigenvectors corresponding to $\lambda_2 = -4$:

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} 15 \\ -30 \end{bmatrix} \qquad \begin{bmatrix} -7 \\ 14 \end{bmatrix}.$$

Assume that the system matrix $\mathbf{A}$ has no repeated eigenvalues, and that the n distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define the *modal matrix* $\mathbf{M}$ by an arbitrary set of corresponding eigenvectors $\mathbf{m}_i$:

$$\mathbf{M} = \begin{bmatrix} \mathbf{m}_1 & | & \mathbf{m}_2 & | & \ldots & | & \mathbf{m}_n \end{bmatrix}. \tag{68}$$

From Eq. (47) the homogeneous system response may be written

$$\begin{aligned} \mathbf{x_h}(t) &= \mathbf{M} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1 \mathbf{m_1} & | & \alpha_2 \mathbf{m_2} & | & \ldots & | & \alpha_n \mathbf{m_n} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} \end{aligned} \tag{69}$$

12

for any non-zero values of the constants α_i. The rules of matrix manipulation allow this expression to be rewritten:

$$
\begin{aligned}
\mathbf{x}_h(t) &= \left[\ \mathbf{m_1}\ \middle|\ \mathbf{m_2}\ \middle|\ \dots\ \middle|\ \mathbf{m}_n\ \right]
\begin{bmatrix}
\alpha_1 e^{\lambda_1 t} \\
\alpha_2 e^{\lambda_2 t} \\
\vdots \\
\alpha_n e^{\lambda_n t}
\end{bmatrix} \\
&= \left[\ \mathbf{m_1}\ \middle|\ \mathbf{m_2}\ \middle|\ \dots\ \middle|\ \mathbf{m}_n\ \right]
\begin{bmatrix}
e^{\lambda_1 t} & 0 & \dots & 0 \\
0 & e^{\lambda_2 t} & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & e^{\lambda_n t}
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n
\end{bmatrix} \\
&= \mathbf{M}e^{\boldsymbol{\Lambda}t}\alpha
\end{aligned}
\tag{70}
$$

where α is a column vector of length n containing the unknown constants α_i, and $e^{\boldsymbol{\Lambda}t}$ is an $n \times n$ diagonal matrix containing the *modal responses* $e^{\lambda_i t}$ on the leading diagonal

$$
e^{\boldsymbol{\Lambda}t} =
\begin{bmatrix}
e^{\lambda_1 t} & 0 & \dots & 0 \\
0 & e^{\lambda_2 t} & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & e^{\lambda_n t}
\end{bmatrix}.
\tag{71}
$$

At time $t = 0$ all of the diagonal elements in $e^{\boldsymbol{\Lambda}0}$ are unity, so that $e^{\boldsymbol{\Lambda}0} = \mathbf{I}$ is the identity matrix and Eq. (70) becomes

$$
\mathbf{x}(0) = \mathbf{M}\mathbf{I}\alpha.
\tag{72}
$$

For the case of distinct eigenvalues the modal matrix $\mathbf{M}$ is non-singular, and the vector α may be found by pre-multiplying each side of the equation by $\mathbf{M}^{-1}$:

$$
\alpha = \mathbf{M}^{-1}\mathbf{x}(0)
\tag{73}
$$

specifying the values of the unknown α_i in terms of the system initial conditions $\mathbf{x}(0)$. The complete homogeneous response of the state vector is

$$
\mathbf{x}_h(t) = \left[\mathbf{M}e^{\boldsymbol{\Lambda}t}\mathbf{M}^{-1}\right]\mathbf{x}(0).
\tag{74}
$$

Comparison with Eq. (11) shows that the state transition matrix may be written as

$$
\boldsymbol{\Phi}(t) = \mathbf{M}e^{\boldsymbol{\Lambda}t}\mathbf{M}^{-1}
\tag{75}
$$

leading to the following important result:

> *Given a linear system of order n, described by the homogeneous equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, and where the matrix $\mathbf{A}$ has n distinct eigenvalues, the homogeneous response of any state variable in the system from an arbitrary set of initial conditions $\mathbf{x}(0)$ is a linear combination of n modal components $e^{\lambda_i t}$, where the λ_i are the eigenvalues of the matrix* $\mathbf{A}$.

4.4 A Time Domain Method For Determining the State Transition Matrix

A Laplace domain method of deriving $\mathbf{\Phi}(t)$ was described in Section 4.2. In this section we examine another method based on the eigenvalues. Equation (75) provides the basis for the determination of the state transition matrix for systems with distinct eigenvalues:

(1) Substitute numerical values into the system $\mathbf{A}$ matrix, and compute the system eigenvalues, a modal matrix $\mathbf{M}$, and its inverse $\mathbf{M}^{-1}$. A computer based linear algebra package provides a convenient method for doing this.

(2) Form the diagonal matrix $e^{\mathbf{\Lambda} t}$ by placing the modal components $e^{\lambda_i t}$ on the leading diagonal.

(3) The state transition matrix $\mathbf{\Phi}(t)$ is $\mathbf{\Phi}(t) = \mathbf{M}e^{\mathbf{\Lambda} t}\mathbf{M}^{-1}$.

■ Example

Determine the state transition matrix for the system discussed in Example 2.2, and find its homogeneous response to the initial conditions $x_1(0) = 1$, and $x_2(0) = 2$.

Solution: The system is described by the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}$$

and in Example 2.2 the eigenvalues are shown to be $\lambda_1 = -1$, and $\lambda_2 = -4$, and a pair of corresponding eigenvectors are

$$\mathbf{m_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{m_2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

A modal matrix $\mathbf{M}$ is therefore

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

and its inverse $\mathbf{M}^{-1}$ is

$$\mathbf{M}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

The matrix $e^{\mathbf{\Lambda} t}$ is found by placing the modal responses e^{-t} and e^{-4t} on the diagonal:

$$e^{\mathbf{\Lambda} t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-4t} \end{bmatrix}. \tag{76}$$

The state transition matrix $\mathbf{\Phi}(t)$ is

$$\mathbf{\Phi}(t) = \left[\mathbf{M}e^{\mathbf{\Lambda} t}\mathbf{M}^{-1} \right] = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2e^{-t} + e^{-4t} & e^{-t} - e^{-4t} \\ 2e^{-t} - 2e^{-4t} & e^{-t} + 2e^{-4t} \end{bmatrix}. \tag{77}$$

The homogeneous response for the given pair of initial conditions is

$$\mathbf{x}_h(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$$

or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2e^{-t} + e^{-4t} & e^{-t} - e^{-4t} \\ 2e^{-t} - 2e^{-4t} & e^{-t} + 2e^{-4t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad (78)$$

or

$$x_1(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t} \qquad (79)$$

$$x_2(t) = \frac{4}{3}e^{-t} + \frac{2}{3}e^{-4t}. \qquad (80)$$

4.5   Systems with Complex Eigenvalues

The modal components of the response are defined by the eigenvalues of the matrix $\mathbf{A}$, as found from the roots of the characteristic equation

$$\det\left[\lambda\mathbf{I} - \mathbf{A}\right] = 0$$

which is a polynomial of degree n in λ, with constant and real coefficients. Any polynomial equation with real coefficients must have roots that are real, or which appear in complex conjugate pairs. The eigenvalues of $\mathbf{A}$ are therefore either real, or occur as pairs of the form $\lambda_{i,i+1} = \sigma \pm j\omega$, where $j = \sqrt{-1}$. In such cases there are modal response components in the matrix $e^{\mathbf{\Lambda}t}$ of the form $e^{(\sigma \pm j\omega)t} = e^{\sigma t}e^{\pm j\omega t}$. However, when the state transition matrix is computed the conjugate components combine, and the Euler relationships

$$\sin\omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \qquad (81)$$

$$\cos\omega t = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \qquad (82)$$

may be used to express the elements of the state transition matrix in a purely real form $e^{\sigma t}\sin(\omega t)$ or $e^{\sigma t}\cos(\omega t)$.

■ **Example**

Determine the state transition matrix for the undamped mechanical oscillator, with a mass $m = 1$ Kg suspended on a spring with stiffness $K = 100$ N/m, as shown with its linear graph in Fig. 1.

Solution: The state equations for this system are

$$\begin{bmatrix} \dot{F}_K \\ \dot{v}_m \end{bmatrix} = \begin{bmatrix} 0 & K \\ -1/m & 0 \end{bmatrix} \begin{bmatrix} F_K \\ v_m \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F_s(t). \qquad (83)$$

With the values given the state equations become

$$\begin{bmatrix} \dot{F}_K \\ \dot{v}_m \end{bmatrix} = \begin{bmatrix} 0 & 100 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} F_K \\ v_m \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F_s(t). \qquad (84)$$
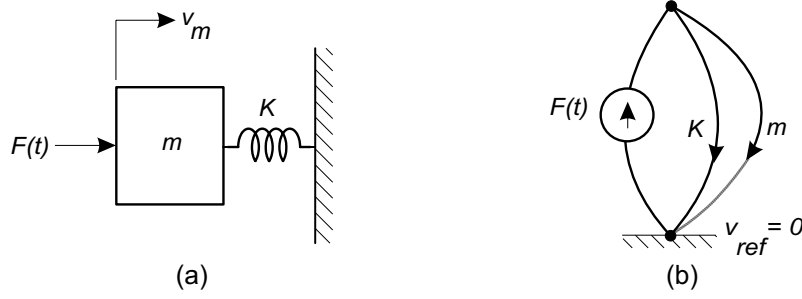
15

Figure 1: Simple spring-mass mechanical oscillator.

The characteristic equation $\det [\lambda \mathbf{I} - \mathbf{A}] = 0$ is

$$\lambda^2 + 100 = 0 \tag{85}$$

so that the eigenvalues are purely imaginary: $\lambda_1 = j10$ and $\lambda_2 = -j10$. A pair of corresponding eigenvectors is found by substituting these values back into the equation $[\lambda_i \mathbf{I} - \mathbf{A}] \mathbf{m}_i = \mathbf{0}$ and choosing an arbitrary scaling constant:

$$\mathbf{m}_1 = \begin{bmatrix} 1 \\ 0.1j \end{bmatrix}, \qquad \mathbf{m}_2 = \begin{bmatrix} 1 \\ -0.1j \end{bmatrix}.$$

The modal matrix $\mathbf{M}$ and its inverse are

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 0.1j & -0.1j \end{bmatrix} \qquad \mathbf{M}^{-1} = \frac{1}{-0.2j} \begin{bmatrix} -0.1j & -1 \\ -0.1j & 1 \end{bmatrix}$$

and the state transition matrix is

$$\begin{aligned}
\mathbf{\Phi}(t) &= \frac{1}{-0.2j} \begin{bmatrix} 1 & 1 \\ 0.1j & -0.1j \end{bmatrix} \begin{bmatrix} e^{j10t} & 0 \\ 0 & e^{-j10t} \end{bmatrix} \begin{bmatrix} -0.1j & -1 \\ -0.1j & 1 \end{bmatrix} \\[2ex]
&= \begin{bmatrix} \dfrac{e^{j10t} + e^{-j10t}}{2} & 10\dfrac{e^{j10t} - e^{-j10t}}{2j} \\[2ex] -0.1\dfrac{e^{j10t} - e^{-j10t}}{2j} & \dfrac{e^{j10t} + e^{-j10t}}{2} \end{bmatrix} \\[2ex]
&= \begin{bmatrix} \cos(10t) & 10\sin(10t) \\ -0.1\sin(10t) & \cos(10t) \end{bmatrix}. \tag{86}
\end{aligned}$$

The homogeneous response to arbitrary initial values of F_K and v_m is therefore

$$\begin{bmatrix} F_K(t) \\ v_m(t) \end{bmatrix} = \begin{bmatrix} \cos(10t) & 10\sin(10t) \\ -0.1\sin(10t) & \cos(10t) \end{bmatrix} \begin{bmatrix} F_K(0) \\ v_m(0) \end{bmatrix} \tag{87}$$

and the individual state responses are

$$\begin{aligned}
F_K(t) &= F_K(0)\cos(10t) + 10v_m(0)\sin(10t) \tag{88} \\
v_m(t) &= -0.1F_K(0)\sin(10t) + v_m(0)\cos(10t) \tag{89}
\end{aligned}$$

which are purely real responses, despite the imaginary system eigenvalues.

4.6 Systems with Repeated Eigenvalues

The method for deriving the state transition matrix presented in the previous section is dependent on the existence of the inverse of the modal matrix $\mathbf{M}$, that is it must be non-singular. In general if there are two or more eigenvalues with the same value, the eigenvectors are not linearly independent and $\mathbf{M}^{-1}$ does not exist. It is possible to extend the method to handle repeated eigenvalues, as described in references on linear system theory.

When there is a pair of repeated eigenvalues, instead of a linear combination of simple exponential terms, the state response may have a pair of components $e^{\lambda_i t}$ and $te^{\lambda_i t}$ corresponding to the two identical eigenvalues. This does not imply that the state transition matrix $\mathbf{\Phi}(t)$ does not exist for systems with repeated eigenvalues; it may be computed by other methods. It may however lose the simple exponential form assumed throughout this note. For further information refer to more advanced texts [1–5].

4.7 Stability of Linear Systems

A system is said to be *asymptotically stable* if the homogeneous response of the state vector $\mathbf{x}(t)$ returns to the origin of the state-space from any arbitrary set of initial conditions $\mathbf{x}(0)$ as time $t \to \infty$. This definition of stability is equivalent to stating that the homogeneous response of all state variables must decay to zero in the absence of any input to the system, or

$$\lim_{t \to \infty} x_i(t) = 0 \tag{90}$$

for all $i = 1, \ldots, n$. This condition may be rewritten in terms of the state transition matrix

$$\lim_{t \to \infty} \mathbf{\Phi}(t)\mathbf{x}(0) = \mathbf{0} \tag{91}$$

for any $\mathbf{x}(0)$. All of the elements of the state transition matrix are linear combinations of the modal components $e^{\lambda_i t}$, therefore the stability of a system depends on all such components decaying to zero with time. For real eigenvalues this requires that $\lambda_i < 0$, since any positive eigenvalue generates a modal response that increases exponentially with time. If eigenvalues appear in complex conjugate pairs $\lambda_{i,i+1} = \sigma \pm j\omega$ the state homogeneous response contains components of the form $e^{\sigma t}\sin(\omega t)$ or $e^{\sigma t}\cos(\omega t)$. If $\sigma > 0$ these components grow exponentially with time and the system is by definition unstable. The requirements for system stability may be therefore summarized:

> *A linear system, described by state equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{u}$, is asymptotically stable if and only if all eigenvalues of the matrix $\mathbf{A}$ have negative real parts.*

Three other separate conditions should be considered:

1. If one or more eigenvalues, or pair of conjugate eigenvalues, has a positive real part there is at least one corresponding modal component that increases exponentially without bound from any finite initial condition, violating the definition of stability.

2. Any pair of conjugate eigenvalues that are purely imaginary, $\lambda_{i,i+1} = \pm j\omega$ with a real part $\sigma = 0$, generate an *undamped* oscillatory component in the state response. The magnitude of the homogeneous system response neither decays or grows, but continues to oscillate for all time at a frequency ω. Such a system is defined to be *marginally stable*.

3. An eigenvalue with a value $\lambda = 0$ generates a modal component e^{0t} that is a constant. The system response neither decays nor grows, and again the system is defined to be marginally stable.

■ Example

Discuss the stability of an inverted pendulum consisting of a mass m on the end of a long light rod of length l which is mounted in a rotational bearing that exhibits a viscous rotational drag, as shown in Fig. 2.
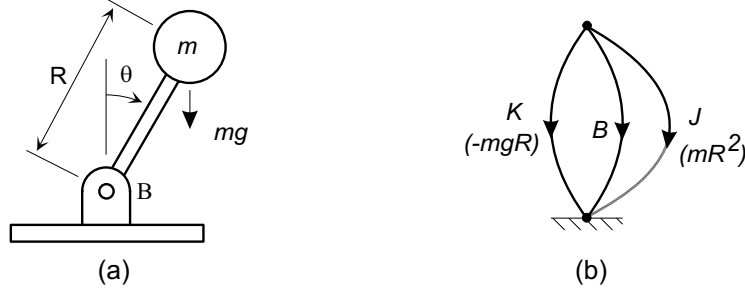

Figure 2: An inverted pendulum.

Solution: The system may be treated as a rotational system. The moment of inertia J of the mass is $J = ml^2$, and when the rod is displaced from the vertical, gravity exerts a torque $mgl \sin\theta$ about the bearing as shown. This system is inherently non-linear because of the angular dependence of the torque, but it may be linearized by using the small angle approximation $\sin\theta \approx \theta$ for small θ. Then the restoring torque after a displacement θ is $-mgl\theta$. This is the constitutive relationship of an ideal T-type element, and allows the gravitational effect to be modeled as a torsional spring with a negative spring constant $K = -mgl$. Let the frictional drag about the bearing be represented by the damping coefficient B. The linear graph for the system is shown in Fig. 2. Define the states to be the torque in the spring T_K and the angular velocity of the shaft Ω. The linearized state equations for this homogeneous system are:

$$\begin{bmatrix} \dot{T}_K \\ \dot{\Omega}_J \end{bmatrix} = \begin{bmatrix} 0 & K \\ -1/J & -B/J \end{bmatrix} \begin{bmatrix} T_K \\ \Omega_J \end{bmatrix}. \tag{92}$$

The characteristic equation is

$$\det\left[\lambda \mathbf{I} - \mathbf{A}\right] = \det \begin{bmatrix} \lambda & -K \\ 1/J & \lambda + B/J \end{bmatrix} = 0 \tag{93}$$

or

$$\begin{aligned} \lambda^2 + \frac{B}{J}\lambda + \frac{K}{J} &= 0 \\ \lambda^2 + \frac{B}{J}\lambda - \frac{g}{R} &= 0. \end{aligned} \tag{94}$$

The quadratic equation may be used to determine the two eigenvalues

$$\lambda_{1,2} = -\frac{B}{2J} \pm \frac{1}{2}\sqrt{\left(\frac{B}{J}\right)^2 + \frac{4g}{R}}. \tag{95}$$

The following may be noted directly from Eq. (v).

1. The quantity under the radical is positive, therefore both eigenvalues are real.

2.
$$\frac{1}{2}\sqrt{\left(\frac{B}{J}\right)^2 + \frac{4g}{R}} > \frac{B}{2J} \tag{96}$$

so that there is always a positive eigenvalue.

We conclude, therefore, that the inverted pendulum is an unstable system and in its linearized form will exhibit an exponential growth in angular displacement from the vertical for any finite initial offset or angular velocity.

4.8   Transformation of State Variables

The choice of a set of state variables used to represent a system is not unique. It is possible to define an infinite number of different representations by *transforming* the state vector by linear operations. If a system is described by a state vector $\mathbf{x}$, a new set of state variables $\mathbf{x}'$ may be generated from any linearly independent combination of the original state variables $x_i(t)$, for example:

$$x_i'(t) = p_{i1}x_1(t) + p_{i2}x_2(t) + \cdots + p_{in}x_n(t) \tag{97}$$

where the p_{ij} are constants. This linear transformation may be written in matrix form

$$\mathbf{x} = \mathbf{P}\mathbf{x}' \tag{98}$$

where $\mathbf{P}$ is a non-singular $n \times n$ square matrix. (With this definition the elements p_{ij} in Eq. (90) above are elements of $\mathbf{P}^{-1}$.) The state equations in the new state variables become

$$\dot{\mathbf{x}} = \mathbf{P}\dot{\mathbf{x}}' = \mathbf{A}\mathbf{P}\mathbf{x}' + \mathbf{B}\mathbf{u} \tag{99}$$

and pre-multiplying each side by the inverse of $\mathbf{P}$, the new set of state equations gives

$$\dot{\mathbf{x}}' = \left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)\mathbf{x}' + \left(\mathbf{P}^{-1}\mathbf{B}\right)\mathbf{u}. \tag{100}$$

The output equation must be similarly transformed

$$\mathbf{y} = (\mathbf{C}\mathbf{P})\mathbf{x}' + \mathbf{D}\mathbf{u}. \tag{101}$$

The system is now represented by modified $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ matrices. The state variable representation is an *internal* system representation that is made observable to the system environment through the output equations. The input-output system dynamic behavior should be independent of the internal system representation. For the transformed state equations the characteristic equation is defined in terms of the transformed $\mathbf{A}$ matrix

$$\det\left[\lambda\mathbf{I} - \left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)\right] = 0. \tag{102}$$

If the substitution $\mathbf{I} = \mathbf{P}^{-1}\mathbf{I}\mathbf{P}$ is made:

$$\det\left[\lambda\mathbf{P}^{-1}\mathbf{P} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right] = 0 \tag{103}$$

$$\det\left[\mathbf{P}^{-1}\left[\lambda\mathbf{I} - \mathbf{A}\right]\mathbf{P}\right] = 0. \tag{104}$$

19

Since the determinant of a product is the product of the determinants, the characteristic equation is

$$\det\left[\mathbf{P}^{-1}\right]\det\left[\lambda\mathbf{I}-\mathbf{A}\right]\det\left[\mathbf{P}\right]=0 \tag{105}$$

and because $\mathbf{P}$ is not singular, $\det\left[\mathbf{P}\right]\neq 0$ and $\det\left[\mathbf{P}^{-1}\right]\neq 0$, the transformed characteristic equation is

$$\det\left[\lambda\mathbf{I}-\mathbf{A}\right]=0 \tag{106}$$

which is the same as that of the original state equations, leading to the following important conclusion:

> The characteristic equation, and hence the eigenvalues λ_i and the modal response components $e^{\lambda_i t}$, are invariant under a linear transformation of the system state variables. These quantities are properties of the system itself and are not affected by the choice of state variables used to describe the system dynamics.

If a transformation of the state variable representation is made, it is necessary to similarly transform any initial conditions to the new representation using

$$\mathbf{x}'(0)=\mathbf{P}^{-1}\mathbf{x}(0). \tag{107}$$

4.8.1 Transformation to Diagonal Form

A transformation that results in a diagonal form of the system matrix $\mathbf{A}$ can provide insight into the internal structure of a system. Consider a system with distinct eigenvalues $\lambda_1,\ldots,\lambda_n$ and a modal matrix $\mathbf{M}$, formed by adjoining columns of eigenvectors as described in Section 4.3. Let $\mathbf{x}'$ be the transformed state vector, defined by $\mathbf{x}=\mathbf{M}\mathbf{x}'$, so that the new set of state and output equations are

$$\dot{\mathbf{x}}' = \left(\mathbf{M}^{-1}\mathbf{A}\mathbf{M}\right)\mathbf{x}'+\left(\mathbf{M}^{-1}\mathbf{B}\right)\mathbf{u} \tag{108}$$

$$\mathbf{y} = \left(\mathbf{C}\mathbf{M}\right)\mathbf{x}'+\mathbf{D}\mathbf{u}. \tag{109}$$

The new system matrix is $\left(\mathbf{M}^{-1}\mathbf{A}\mathbf{M}\right)$. As in Eq. (53), the product $\mathbf{A}\mathbf{M}$ may be written in terms of the eigenvalues and eigenvectors

$$\mathbf{A}\mathbf{M}=\left[\;\mathbf{A}\mathbf{m}_1\;\middle|\;\mathbf{A}\mathbf{m}_2\;\middle|\;\ldots\;\middle|\;\mathbf{A}\mathbf{m}_n\;\right]=\left[\;\lambda_1\mathbf{m}_1\;\middle|\;\lambda_2\mathbf{m}_2\;\middle|\;\ldots\;\middle|\;\lambda_n\mathbf{m}_n\;\right] \tag{110}$$

because $\mathbf{A}\mathbf{m}_i=\lambda_i\mathbf{m}_i$ is the relationship that defined the eigenvalue λ_i. Equation (108) can be rearranged and written

$$\begin{aligned}
\mathbf{A}\mathbf{M} &= \left[\;\mathbf{m}_1\;\middle|\;\mathbf{m}_2\;\middle|\;\ldots\;\middle|\;\mathbf{m}_n\;\right]\begin{bmatrix}\lambda_1 & 0 & \ldots & 0\\ 0 & \lambda_2 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \lambda_n\end{bmatrix}\\
&= \mathbf{M}\mathbf{\Lambda}
\end{aligned} \tag{111}$$

where $\mathbf{\Lambda}$ is the diagonal $n\times n$ square matrix containing the system eigenvalues on the leading diagonal

$$\mathbf{\Lambda}=\begin{bmatrix}\lambda_1 & 0 & \ldots & 0\\ 0 & \lambda_2 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \lambda_n\end{bmatrix}. \tag{112}$$

If both sides of Eq. (109) are pre-multiplied by $\mathbf{M}^{-1}$

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \mathbf{M}^{-1}\mathbf{M}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}, \tag{113}$$

the transformed state equations are

$$\dot{\mathbf{x}}' = \boldsymbol{\Lambda}\mathbf{x}' + \mathbf{B}'\mathbf{u}. \tag{114}$$

where $\mathbf{B}' = (\mathbf{M}^{-1}\mathbf{B})$. Equation (112) represents a set of n *uncoupled* first-order differential equations, each of which may be written

$$\dot{x}_i = \lambda_i x_i + \sum_{j=1}^{r} b'_{ij} u_j \tag{115}$$

and does not involve any cross coupling from other states. The homogeneous state equations $\dot{\mathbf{x}}' = \boldsymbol{\Lambda}\mathbf{x}'$ are simply

$$\dot{x}_i = \lambda_i x_i. \tag{116}$$

The state transition matrix for the diagonal system is $\boldsymbol{\Phi}(t) = e^{\boldsymbol{\Lambda}t}$ as given by Eq. (71)

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \tag{117}$$

and the homogeneous response $\mathbf{x}_h(t) = \boldsymbol{\Phi}(t)\mathbf{x}(0)$ has the simple uncoupled form

$$x_i(t) = x_i(0)e^{\lambda_i t} \tag{118}$$

for each state variable.

Systems with repeated eigenvalues may not be reducible to a diagonal form, but may be represented in a closely related form, known as the Jordan form [1–5].

■ Example

Transform the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

to diagonal form and find the homogeneous response to an arbitrary set of initial conditions.

Solution: The $\mathbf{A}$ matrix in this example is the same as that examined in Examples 2.2, and 3 that is

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}.$$

In the previous examples it was shown that for this system the eigenvalues are $\lambda_1 = -1$, and $\lambda_2 = -4$, and that a suitable modal matrix and its inverse are

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \qquad \mathbf{M}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

The transformed state variables are

$$\mathbf{x}' = \mathbf{M}^{-1}\mathbf{x} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{x} \tag{119}$$

or

$$x_1'(t) = \frac{2}{3}x_1(t) + \frac{1}{3}x_2(t) \tag{120}$$

$$x_2'(t) = \frac{1}{3}x_1(t) - \frac{1}{3}x_2(t). \tag{121}$$

The transformed state equations are

$$\dot{\mathbf{x}} = \left(\mathbf{M}^{-1}\mathbf{A}\mathbf{M}\right)\mathbf{x} + \left(\mathbf{M}^{-1}\mathbf{B}\right)\mathbf{u} \tag{122}$$

or

$$\begin{bmatrix} \dot{x_1}' \\ \dot{x_2}' \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}u(t) \tag{123}$$

which is clearly diagonal in form with the eigenvalues on the leading diagonal of the transformed $\mathbf{A}$ matrix. The state transition matrix is found by placing the modal components on the diagonal:

$$\mathbf{\Phi}(t) = e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \tag{124}$$

and the homogeneous state response in the transformed system representation is $\mathbf{x}_h'(t) = \mathbf{\Phi}(t)\mathbf{x}'(0)$:

$$x_1'(t) = x_1'(0)e^{-t} \tag{125}$$

$$x_2'(t) = x_2'(0)e^{-4t} \tag{126}$$

where the transformed initial states are found from Eqs. (ii) and (iii):

$$x_1'(0) = \frac{2}{3}x_1(0) + \frac{1}{3}x_2(0) \tag{127}$$

$$x_2'(0) = \frac{1}{3}x_1(0) - \frac{1}{3}x_2(0). \tag{128}$$

---

5 The Response of Linear Systems to the Singularity Input Functions

The singularity input functions (the impulse, step, and ramp functions) are a set of functions that are commonly used to characterize the transient response characteristics of linear time-invariant systems. The forced system response, given by Equations (25) and (26) is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{129}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau, \tag{130}$$

and may be used to derive simple expressions for the response of linear time-invariant systems to the individual singularity functions.

5.1 The Impulse Response

Assume that the input vector $\mathbf{u}(t)$ is a weighted series of impulse functions $\delta(t)$ applied to the r system inputs:

$$\mathbf{u}(t) = \mathbf{K}\delta(t) = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_r \end{bmatrix} \delta(t). \tag{131}$$

The vector $\mathbf{K}$ is used to distribute impulses among the r inputs. For example, if the response to a single impulse on the jth input is required, then $k_i = 0$ for all $k \neq j$, and $k_j = 1$. The state vector impulse response is found by substituting $\mathbf{u}(t) = \mathbf{K}\delta(t)$ into Eq. (26):

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{K}\delta(\tau)d\tau. \tag{132}$$

The sifting, or sampling, property of the delta function states that for any finite value of Δ

$$\int_{0-\Delta}^{0+\Delta} f(t)\delta(t)dt = f(0),$$

which allows Eq. (132) to be simplified

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\mathbf{B}\mathbf{K} \\ &= e^{\mathbf{A}t}\left(\mathbf{x}(0) + \mathbf{B}\mathbf{K}\right). \end{aligned} \tag{133}$$

The effect of impulsive inputs on the state response is similar to a set of initial conditions; the response for $t > 0$ is defined entirely by the state transition matrix. For an asymptotically stable linear system the impulse response decays to zero as time $t \to \infty$. The system output equations may be used to find the impulse response of any output variable

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\left(\mathbf{x}(0) + \mathbf{B}\mathbf{K}\right) + \mathbf{D}\mathbf{K}\delta(t) \tag{134}$$

which shows a direct feedthrough of the impulses for any system in which $\mathbf{D} \neq \mathbf{0}$.

■ Example

The electrical filter shown in Fig. 3 is installed in the power line of a computer in a factory environment to reduce the possibility of damage caused by impulsive voltage spikes created by other equipment. Measurements have shown that the transients are approximately 100 volts in amplitude and have a typical duration of 10 μsecs. The values of the filter components are $L_1 = 10$mH, $L_2 = 10$mH, and $C = 1\mu$fd. The load resistance is 50 ohms. Find the effect of the filter on the voltage spikes.
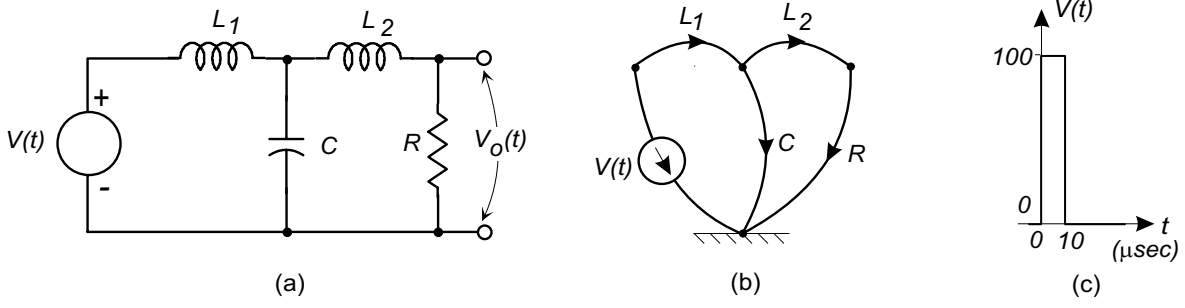
Figure 3: A third-order electrical filter, (b) its linear graph, and (c) a typical voltage pulse at its input.

Solution: The system output is the voltage v_R. The system has state variables i_{L_1}, i_{L_1}, and v_C. The state and output equations are:

$$
\begin{bmatrix} \dot{i}_{L_1} \\ \dot{i}_{L_2} \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1/L_1 \\ 0 & -R/L_2 & 1/L_2 \\ 1/C & -1/C & 0 \end{bmatrix} \begin{bmatrix} i_{L_1} \\ i_{L_2} \\ v_C \end{bmatrix} + \begin{bmatrix} 1/L_1 \\ 0 \\ 0 \end{bmatrix} V_s(t) \tag{135}
$$

$$
v_R = \begin{bmatrix} 0 & R & 0 \end{bmatrix} \begin{bmatrix} i_{L_1} \\ i_{L_2} \\ v_C \end{bmatrix}, \tag{136}
$$

and the input is modeled as an impulse of strength $100 \times 10^{-5} = 10^{-3}$ volt-sec, that is $V_s(t) = 10^{-3}\delta(t)$, as determined from the area under the typical voltage spike shown in Fig. 3c.

If the values of the system parameters are substituted into the matrices, they become:

$$
\mathbf{A} = \begin{bmatrix} 0 & 0 & -100 \\ 0 & -5,000 & 100 \\ 1,000,000 & -1,000,000 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix},
$$

$$
\mathbf{C} = \begin{bmatrix} 0 & 50 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{K} = [0.001].
$$

From Eq. (134) the output impulse response is :

$$
\begin{aligned}
y(t) &= \mathbf{C}e^{\mathbf{A}t}\mathbf{B}\mathbf{K} \tag{137} \\
&= (\mathbf{C}\mathbf{M})\, e^{\mathbf{\Lambda}t} \left(\mathbf{M}^{-1}\mathbf{B}\mathbf{K} \right), \tag{138}
\end{aligned}
$$

using Eq. (75). A linear algebra software package is used to find the system eigenvalues:

$$
\lambda_1 = -1210 + 13867j, \quad \lambda_2 = -1210 - 13867j, \quad \lambda_1 = -2581,
$$

and the modal matrix and its inverse:

$$
\mathbf{M} = \begin{bmatrix} 0.0006 + 0.0072j & 0.0006 - 0.0072j & 0.0388 \\ 0.0018 - 0.0067j & 0.0018 + 0.0067j & 0.0413 \\ 1 & 1 & 1 \end{bmatrix}
$$

$$
\mathbf{M}^{-1} = \begin{bmatrix} -6.2301 - 36.673j & -6.6450 + 35.4j & 0.5161 - 0.042j \\ -6.2301 + 36.673j & -6.6450 - 35.4j & 0.5161 + 0.042j \\ 12.4603 & 13.2900 & -0.0322 \end{bmatrix}.
$$

24

The matrices $\mathbf{M}^{-1}\mathbf{BK}$ and $\mathbf{CM}$ are

$$\mathbf{M}^{-1}\mathbf{BK} = \begin{bmatrix} -0.6230 - 3.667j \\ -0.6230 + 3.667j \\ 0.1246 \end{bmatrix},$$

$$\mathbf{CM} = \begin{bmatrix} 0.0917 - 0.3355j & 0.0917 + 0.3355j & 2.0666 \end{bmatrix}$$

so that the solution is:

$$v_R(t) = \mathbf{CM} \begin{bmatrix} e^{(-1210+13867j)t} & 0 & 0 \\ 0 & e^{(-1210+13867j)t} & 0 \\ 0 & 0 & e^{-2581t} \end{bmatrix} \mathbf{M}^{-1}\mathbf{BK} \quad (139)$$

$$= 2.575e^{-2581t} + (-1.2875 - 0.1273j)\, e^{(-1210+13867j)t}$$
$$+ (-1.2875 + 0.1273j)\, e^{(-1210-13867j)t} \quad (140)$$

$$= 2.575e^{-2581t}$$
$$+ e^{-1210t} \left(-2.575\cos(13867t) + 0.2546\sin(13867t) \right). \quad (141)$$

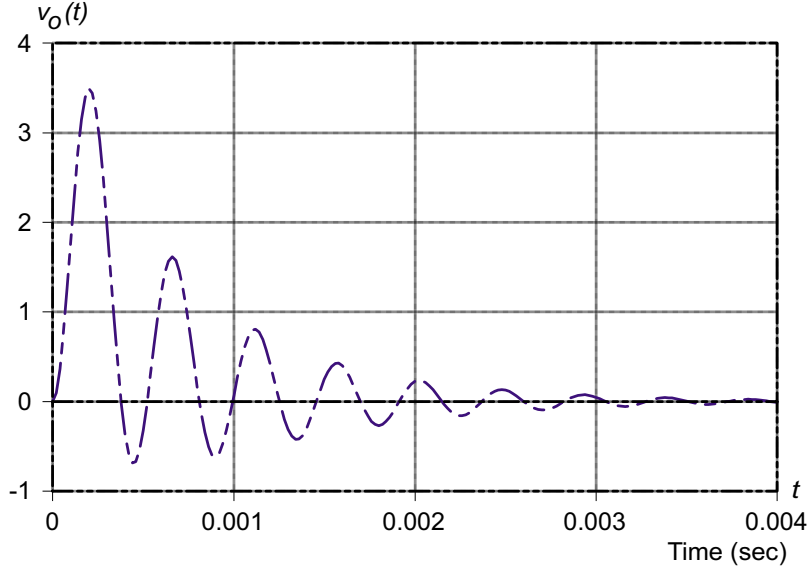The impulse response is plotted in Fig. 4. The maximum voltage reached is approxi-

Figure 4: Response of the filter to an impulsive input of 100 volts amplitude and 10^{-5} seconds duration.

mately 3 volts; a significant reduction from the input amplitude of 100 volts.

5.2 The Step Response

Assume that the input vector is a weighted series of unit step functions $u_s(t)$ applied to the r inputs, that is

$$\mathbf{u}(t) = \mathbf{K}u_s(t) = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_r \end{bmatrix} u_s(t). \quad (142)$$

25

The vector $\mathbf{K}$ is used to distribute the step among the r inputs, and if the response to a unit step on a single input channel is desired, for example the jth input, then $k_i = 0$ for all $k \neq j$, and $k_j = 1$. The state step response is found by substituting $\mathbf{u}(t) = \mathbf{K}u_s(t)$ into Eq. (26):

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{K}u_s(\tau)d\tau \tag{143}$$

and because $u_s(t) = 1$ for all $t > 0$, the integral may be rewritten

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{K}d\tau \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left[\int_0^t e^{-\mathbf{A}\tau}d\tau\right]\mathbf{B}\mathbf{K} \end{aligned} \tag{144}$$

where the order of the matrices must be preserved. If $\mathbf{A}^{-1}$ exists the element by element integration of the matrix exponential may be found using the integration property described in Table 1

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\mathbf{A}^{-1}\left[\mathbf{I} - e^{-\mathbf{A}t}\right]\mathbf{B}\mathbf{K} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}\left[e^{\mathbf{A}t} - \mathbf{I}\right]\mathbf{B}\mathbf{K}. \end{aligned} \tag{145}$$

since $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$, which may be shown directly from the series definition. The output response is

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \\ &= \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\mathbf{A}^{-1}\left[e^{\mathbf{A}t} - \mathbf{I}\right]\mathbf{B}\mathbf{K} + \mathbf{D}\mathbf{K}u_s(t). \end{aligned} \tag{146}$$

If $\mathbf{A}$ is non-singular, that is if $\mathbf{A}$ does not have an eigenvalue $\lambda_i = 0$, then the step response reaches a steady-state constant value $\mathbf{x}_{ss}$ as $t \to \infty$ and

$$\begin{aligned} \mathbf{x}_{ss} = \lim_{t \to \infty} \mathbf{x}(t) &= \lim_{t \to \infty}\left[e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}\left[e^{\mathbf{A}t} - \mathbf{I}\right]\mathbf{B}\mathbf{K}\right] \\ &= -\mathbf{A}^{-1}\mathbf{B}\mathbf{K} \end{aligned} \tag{147}$$

because $\lim_{t \to \infty}\left[e^{\mathbf{A}t}\right] = \mathbf{0}$. The steady-state output response is

$$\mathbf{y}_{ss} = \left[-\mathbf{C}\mathbf{A}^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{K}. \tag{148}$$

The steady-state response may be confirmed directly from the state equations. When steady-state is reached, all derivatives of the state variables are, by definition, identically zero:

$$\mathbf{0} = \mathbf{A}\mathbf{x_{ss}} + \mathbf{B}\mathbf{K} \tag{149}$$

giving the same result, $\mathbf{x}_{ss} = -\mathbf{A}^{-1}\mathbf{B}\mathbf{K}$.

■ Example

The hydraulic system shown in Fig. 5 is driven by a constant pressure pump. At time $t = 0$ the tank is empty and the pump is turned on at a constant pressure of 10 N/m^2. Find an expression for the flow in the inlet pipe for $t > 0$.
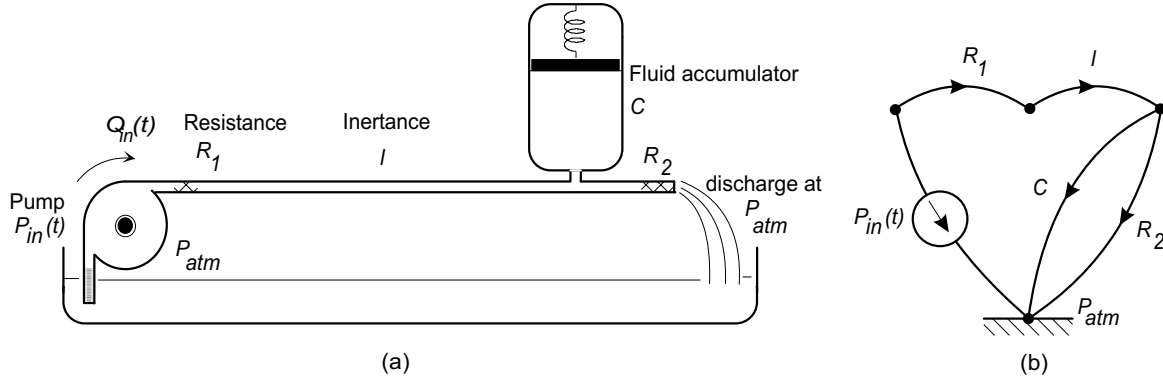
Figure 5: A hydraulic system (a), and (b) its linear graph.

Solution: The system is modeled as shown. Lumped fluid inertance I and resistance R_1 elements are used to account for pressure drops in the pipe. The tank is modeled as a fluid capacitance C, and the outlet valve is modeled as a linear fluid resistance. The following values are assumed: $I = 0.25$ N-sec^2/m^5, $R_1 = 1$ N-sec/m^5, $R_2 = 1$ N-sec/m^5, and $C = 1$ m^5/N. The system state equations and the output equation for the inlet flow are:

$$\begin{bmatrix} \dot{P}_C \\ \dot{Q}_I \end{bmatrix} = \begin{bmatrix} -1/R_2C & 1/C \\ -1/I & -R_1/I \end{bmatrix} \begin{bmatrix} P_C \\ Q_I \end{bmatrix} + \begin{bmatrix} 0 \\ 1/I \end{bmatrix} P_{in}(t) \tag{150}$$

$$Q_I = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} P_C \\ Q_I \end{bmatrix}. \tag{151}$$

With the values given

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ -4 & -4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The step response with zero initial conditions is

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{C}\mathbf{A}^{-1}\left[e^{\mathbf{A}t} - I\right]\mathbf{B}\mathbf{K} \\ &= \left(\mathbf{C}\mathbf{A}^{-1}\mathbf{M}\right)e^{\mathbf{\Lambda}t}\left(\mathbf{M}^{-1}\mathbf{B}\mathbf{K}\right) - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\mathbf{K}. \end{aligned} \tag{152}$$

The system eigenvalues are $\lambda_1 = -2.5 + 1.323j$, and $\lambda_2 = -2.5 - 1.323j$. The input weighting matrix $\mathbf{K} = [10]$, and

$$\mathbf{M} = \begin{bmatrix} -0.375 - 0.331j & -0.375 + 0.331j \\ 1 & 1 \end{bmatrix},$$

$$\mathbf{M}^{-1} = \begin{bmatrix} 1.512j & 0.5 + 0.567j \\ -1.512j & 0.5 - 0.567j \end{bmatrix}, \quad \mathbf{A}^{-1} = \begin{bmatrix} -0.5 & -0.125 \\ 0.5 & -0.125 \end{bmatrix}.$$

The following matrices are computed:

$$\mathbf{C}\mathbf{A}^{-1}\mathbf{M} = \begin{bmatrix} -0.3125 - 0.1654j & -0.3125 + 0.1654j \end{bmatrix},$$

$$\mathbf{M}^{-1}\mathbf{B}\mathbf{K} = \begin{bmatrix} -20 + 22.678j \\ -20 - 22.678j \end{bmatrix},$$

$$\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\mathbf{K} = \begin{bmatrix} -5.0 \end{bmatrix},$$

and used in the solution:

$$Q_I(t) = \begin{bmatrix} -0.3125 - 0.1654j & -0.3125 + 0.1654j \end{bmatrix} \tag{153}$$

$$\begin{bmatrix} e^{(-2.5+1.3223j)t} & 0 \\ 0 & e^{(-2.5-1.3223j)t} \end{bmatrix} \begin{bmatrix} -20 + 22.678j \\ -20 - 22.678j \end{bmatrix} + 5.0 \tag{154}$$

$$= 5.0 + e^{-2.5t} \left(-5 \cos \left(1.323t \right) + 20.8 \sin \left(1.323t \right) \right). \tag{155}$$
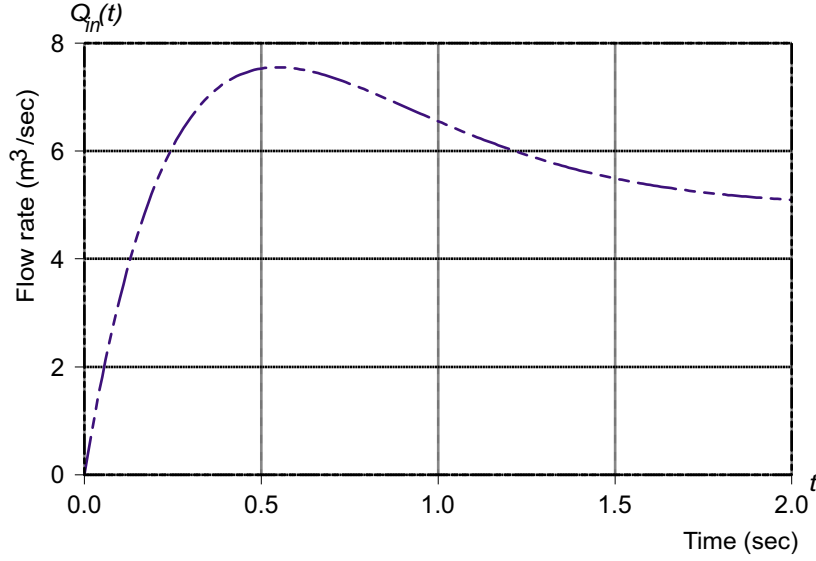
which is plotted in Fig. 6.

Figure 6: Response of the hydraulic system to a $10\text{N}/\text{m}^2$ step in pump pressure.

5.3 The Ramp Response

If the input vector is a weighted series of unit ramp functions distributed among the r inputs, that is

$$\mathbf{u}(t) = \mathbf{K}t = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_r \end{bmatrix} t. \tag{156}$$

The ramp response may be found without solving the full response equation Eq. (26) by the use of the integration property of linear systems, namely that if the response to an input $\mathbf{u}(t)$ is $\mathbf{y}(t)$, the forced response to an input $\int_0^t \mathbf{u}(t)dt$ is $\int_0^t \mathbf{y}(t)dt$. The ramp function t is the integral of the unit step, therefore the forced component of the ramp response is simply the integral of the forced component of the step response:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \mathbf{A}^{-1} \left[e^{\mathbf{A}\tau} - \mathbf{I} \right] \mathbf{B}\mathbf{K}d\tau$$

$$= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1} \left[\mathbf{A}^{-1} \left[e^{\mathbf{A}t} - I \right] - \mathbf{I}t \right] \mathbf{B}\mathbf{K}. \tag{157}$$

28

The output ramp response is found by substituting into the output equations:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\mathbf{A}^{-1}\left[\mathbf{A}^{-1}\left[e^{\mathbf{A}t} - I\right] - \mathbf{I}t\right]\mathbf{B}\mathbf{K} + \mathbf{D}\mathbf{K}t. \tag{158}$$

■ Example

Find the ramp response of a first-order system written in the standard form

$$\tau\frac{dx}{dt} + x = u(t)$$

where τ is the system time constant.

Solution: In state space form the first-order system is

$$\dot{x} = -\frac{1}{\tau}x + \frac{1}{\tau}u \tag{159}$$

The first-order ramp response (Eq. (157)) reduces to

$$x(t) = e^{at}x(0) + a^{-1}\left[a^{-1}\left[e^{at} - 1\right] - t\right]b, \tag{160}$$

where in this case $a = -1/\tau$ and $b = 1/\tau$. Substitution of these values gives

$$x(t) = e^{-t/\tau}x(0) + t - \tau\left(1 - e^{-t/\tau}\right) \tag{161}$$

which, if the initial condition $x(0) = 0$, is identical to the result given in Section 2.

■ References:

1. Schultz D. G., and Melsa J. L., *State Functions and Linear Control Systems*, McGraw-Hill, New York NY, 1967

2. Luenberger D. G., *Introduction to Dynamic Systems, Theory, Models, and Applications*, John Wiley and Sons Inc., New York NY, 1979

3. Skelton R. E., *Dynamic Systems Control - Linear Systems Analysis and Synthesis*, John Wiley and Sons Inc., New York NY, 1988

4. Chen C-T., *Linear System Theory and Design*, Holt, Rinehart and Winston, New York NY, 1984

5. Reid J. G., *Linear System Fundamentals*, McGraw-Hill, New York NY, 1983

6. Moler C., and Van Loan C., "Nineteen Dubious Ways to Compute the Exponential of a Matrix," *SIAM Review*, 20, 4, Oct 1978, 801-836